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On the convergence of random polynomials and multilinear forms

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Abstract

We consider different kinds of convergence of homogeneous polynomials and multilinear forms in random variables. We show that for a variety of complex random variables, the almost sure convergence of the polynomial is equivalent to that of the multilinear form, and to the square summability of the coefficients. Also, we present polynomial Khintchine inequalities for complex gaussian and Steinhaus variables. All these results have no analogues in the real case. Moreover, we study the L_p-convergence of random polynomials and derive certain decoupling inequalities without the usual tetrahedral hypothesis. We also consider convergence on "full subspaces" in the sense of Sjögren, both for real and complex random variables, and relate it to domination properties of the polynomial or the multilinear form, establishing a link with the theory of homogeneous polynomials on Banach spaces.

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0. Introduction

In this article we study the convergence of polynomials and multilinear forms in random variables. If $\{X_i\}_{i\in\mathbb{N}}$ is a sequence of complex or real random variables, we relate different kinds of convergence (almost sure, in L_p, in full subspaces, etc.) of the k-homogeneous polynomial

$$\sum_{i_1,\dots,i_k} a_{i_1,\dots,i_k} X_{i_1}\cdots X_{i_k} \tag{1}$$

to properties of the coefficients $\{a_{i_1,\dots,i_k}\}_{i_1,\dots,i_k\in\mathbb{N}}$ (or mappings related to these coefficients). By the convergence of the k-homogeneous polynomial we understand the convergence of the random series (1), see the comments before Theorem 2.

For the real case, the almost sure convergence of the multilinear form

$$\sum_{i_1,\dots,i_k} a_{i_1,\dots,i_k} Y^{(1)}_{i_1}\cdots Y^{(k)}_{i_k}, \tag{2}$$

where $\{Y_i^{(1)}\}_{i\in\mathbb{N}},\dots,\{Y_i^{(k)}\}_{i\in\mathbb{N}}$ are sequences of random variables with the same distribution as $\{X_i\}_{i\in\mathbb{N}}$, which are jointly independent, is not equivalent to the almost sure convergence of the polynomial (1) (see, for example, [23] for the bilinear/quadratic case). Conditions on the coefficients $a_{i_1,\dots,i_k}$ with repeated subindexes must be considered and also, in many cases, one has to impose all of them to be null in order to relate the convergence of the polynomial to that of the multilinear form. With the same spirit, for real random variables (or real function spaces), there are multilinear Khintchine inequalities but no polynomial ones, and coefficients with repeated indexes are again the problem: for Rademacher or gaussian variables, the L_p-convergence of $\sum_i a_{i,i}X_i^2$ (the diagonal quadratic form) is not related to the square summability of the coefficients $a_{i,i}$, but to the convergence of $\sum_i |a_{i,i}|$.

We show in Theorem 2 that for rotation-invariant complex random variables, coefficients with repeated indexes are not a problem, and almost sure convergence of the polynomial (1) is equivalent to that of the multilinear form (2) and, also, to the square summability of the coefficients $\{a_{i_1,\dots,i_k}\}_{i_1,\dots,i_k\in\mathbb{N}}$. Moreover, for complex gaussian and Steinhaus random variables, we present a polynomial Khintchine inequality (which has no analogue for real random variables), that allows us to relate the square summability of the coefficients also to the L_p-convergence of the polynomial and the multilinear form (Theorems 3 and 6). Another consequence of our polynomial Khintchine inequalities is a particular case of decoupling inequality, which again holds without conditions on the coefficients with repeated indexes.

In [23], Sjögren considered the convergence of gaussian quadratic and bilinear forms on full subspaces (see the definitions in Section 2). He shows that this convergence is equivalent to the coefficients defining a nuclear operator on ℓ_2, but that this is no longer true for degree three. In order to study higher degrees we introduce the standard full subspaces and show, for example, that convergence on these subspaces is equivalent to the coefficients defining a 2-dominated polynomial (or multilinear form) on ℓ_2 (see Theorem 10). We also consider non-gaussian random variables. Finally, we use our polynomial Khintchine inequality to extend in Theorem 15 a result on dominated polynomials due to Meléndez and Tonge [17].

Some of our results are proved using recent techniques on integral representation of holomorphic functions on Banach spaces introduced in [20,22]. We devote Section 3 to summarize some aspects of this theory, as well as to prove some new results needed in this work. The proofs of most of the results of the first two sections are then postponed to Section 4.

1. Polynomial Khintchine type inequalities and almost sure convergence

Let us fix some terminology. The (multi-indexed) sequence of complex numbers $\{a_{j_1,\dots,j_k}\}_{j_1,\dots,j_k\in\mathbb{N}}$ is said to be *symmetric* if

$$a_{j_1,\dots,j_k}=a_{l_1,\dots,l_k}$$

whenever $\{l_1,\dots,l_k\}$ is a permutation of $\{j_1,\dots,j_k\}$.

A complex random variable $X:(\Omega,\mathcal{A},P)\to\mathbb{C}$ is said to be *rotation-invariant* if X and $e^{i\theta}X$ have the same distribution law for all $\theta\in[0,2\pi]$. Note that for such a random variable we must have $\mathbb{E}(X)=0$, since in particular

$$\mathbb{E}(X)=\mathbb{E}\big(e^{i\pi}X\big)=e^{i\pi}\mathbb{E}(X)=-\mathbb{E}(X).$$

In the sequel, given $k\in\mathbb{N}$, we will need to work with sequences of independent complex random variables $\{X_j\}_{j\in\mathbb{N}}$, which satisfy the following hypothesis:

$$(\star)\qquad \inf_{j\in\mathbb{N}}\mathbb{E}\big(|X_j|\big)>0\quad\text{and}\quad\sup_{j\in\mathbb{N}}\mathbb{E}\big(|X_j|^{2k}\big)<\infty.$$

We will call it the $(\star)$-condition (for k). Note that, for a sequence of identically distributed non-zero random variables, this condition is merely to have a finite $2k$-th moment.

The following result makes apparent the difference between real and complex variables in terms of polynomial convergence.

Proposition 1. *Let $k\in\mathbb{N}$ and $\{X_j\}_{j\in\mathbb{N}}$ be a sequence of independent and rotation-invariant complex random variables satisfying the $(\star)$-condition. Then, there exist positive constants A_k and B_k such that for any symmetric sequence of complex numbers $\{a_{j_1,\dots,j_k}\}_{j_1,\dots,j_k\in\mathbb{N}}$ and any $n\in\mathbb{N}$, we have*

$$A_k^{-1}\left[\sum_{j_1,\dots,j_k=1}^{n}|a_{j_1,\dots,j_k}|^2\right]^{\frac12}\leqslant\big[\mathbb{E}\big(|F_n|^2\big)\big]^{\frac12}\leqslant B_k\left[\sum_{j_1,\dots,j_k=1}^{n}|a_{j_1,\dots,j_k}|^2\right]^{\frac12},$$

where $F_n=\sum_{j_1,\dots,j_k=1}^{n}a_{j_1,\dots,j_k}X_{j_1}\cdots X_{j_k}$.

If the $\{X_j\}_{j\in\mathbb{N}}$ are independent standard complex gaussian variables, we actually have

$$\big[\mathbb{E}\big(|F_n|^2\big)\big]^{1/2}=\sqrt{k!}\left(\sum_{j_1,\dots,j_k=1}^{n}|a_{j_1,\dots,j_k}|^2\right)^{1/2}.$$

As we can see in the proof of the previous proposition (in Section 4), the set of random monomials $\{X_{j_1}\cdots X_{j_k}\}_{j_1\leqslant\cdots\leqslant j_k}$ is an orthogonal system. Note that we are including monomials with repeated indexes. The implication (i) $\Rightarrow$ (ii) in the next theorem will be a consequence of this proposition, together with the martingale properties of the sequence $\sum_{j_1,\ldots,j_k=1}^{n} a_{j_1,\ldots,j_k} X_{j_1}\cdots X_{j_k}$, to be shown in Section 4. Let us note that in [23], convergence of bilinear and quadratic forms in real-valued random variables are studied. Under certain assumptions, it is shown that the almost sure convergence of the bilinear form $\sum_{i,j=1}^{\infty} a_{i,j} X_i Y_j$ is equivalent to the coefficients $\{a_{i,j}\}_{i,j\in\mathbb{N}}$ being square summable. For quadratic forms, extra conditions on the diagonal $\{a_{i,i}\}_{i\in\mathbb{N}}$ are necessary for the equivalence. We see that for complex-valued random variables the situation is different.

Throughout the article, by the convergence (in some sense) of the k-homogeneous polynomial $\sum_{j_1,\ldots,j_k} a_{j_1,\ldots,j_k} X_{j_1}\cdots X_{j_k}$ we understand the existence (in the same sense) of the limit

$$\lim_{n\to\infty} \sum_{j_1,\ldots,j_k=1}^{n} a_{j_1,\ldots,j_k} X_{j_1}\cdots X_{j_k}.$$

This convergence can be almost sure, in L_p, in full subspaces (see Section 2), etc. For multilinear forms in random variables our notion of convergence is analogous.

Theorem 2. *Given $k\in\mathbb{N}$ and a symmetric sequence $\{a_{j_1,\ldots,j_k}\}_{j_1,\ldots,j_k\in\mathbb{N}}$ of complex numbers, the following are equivalent*:

(i) $\sum_{j_1,\ldots,j_k\geqslant 1} |a_{j_1,\ldots,j_k}|^2 < \infty$.

(ii) *For every sequence $\{X_j\}_{j\in\mathbb{N}}$ of independent and rotation-invariant complex random variables which satisfies the* ($\star$)*-condition* (*for k*)*, the random series*

$$\sum_{j_1,\ldots,j_k=1}^{n} a_{j_1,\ldots,j_k} X_{j_1}\cdots X_{j_k}$$

converges almost surely.

(iii) *For every choice of k sequences $\{Y_j^{(1)}\}_{j\in\mathbb{N}},\ldots,\{Y_j^{(k)}\}_{j\in\mathbb{N}}$ of rotation-invariant complex random variables which are jointly independent and satisfy the* ($\star$)*-condition* (*for k*)*, the random series*

$$\sum_{j_1,\ldots,j_k=1}^{n} a_{j_1,\ldots,j_k} Y_{j_1}^{(1)}\cdots Y_{j_k}^{(k)}$$

converges almost surely.

If the sequence $\{a_{j_1,\ldots,j_k}\}_{j_1,\ldots,j_k\in\mathbb{N}}$ is not symmetric, the equivalence between (i) *and* (iii) *remains true.*

Now we restrict ourselves to complex gaussian variables. As we have mentioned, the proof of Proposition 1 shows the orthogonality of the whole family of functions $X_{j_1}\cdots X_{j_k}$ in L_2, including those with repeated indexes. For the other L_p's, we have the following polynomial Khintchine inequality:

Theorem 3. *If $\{X_j\}_{j\in\mathbb{N}}$ is a sequence of independent standard complex gaussian variables, then for $1 \leqslant p < \infty$ there are positive constants $A_{k,p}$ and $B_{k,p}$ such that for every symmetric sequence of complex numbers $\{a_{j_1,\dots,j_k}\}_{j_1,\dots,j_k\in\mathbb{N}}$, we have:*

$$A_{k,p}^{-1}\left[\sum_{j_1,\dots,j_k=1}^{n} |a_{j_1,\dots,j_k}|^2\right]^{\frac{1}{2}} \leqslant \left[\mathbb{E}\left(|F_n|^p\right)\right]^{\frac{1}{p}} \leqslant B_{k,p}\left[\sum_{j_1,\dots,j_k=1}^{n} |a_{j_1,\dots,j_k}|^2\right]^{\frac{1}{2}},$$

for all $n \in \mathbb{N}$, where $F_n = \sum_{j_1,\dots,j_k=1}^{n} a_{j_1,\dots,j_k} X_{j_1}\cdots X_{j_k}$.

Although this is probably known, we can derive the multilinear Khintchine inequality for complex gaussian variables from the polynomial one to obtain:

Corollary 4. *Let $\{Z_j^{(1)}\}_{j\in\mathbb{N}},\dots,\{Z_j^{(k)}\}_{j\in\mathbb{N}}$ be a finite set of sequences of standard complex gaussian variables which are jointly independent, then for $1 \leqslant p < \infty$ there are positive constants $\widetilde{A}_{k,p}$ and $\widetilde{B}_{k,p}$ such that for every sequence of complex numbers $\{b_{j_1,\dots,j_k}\}_{j_1,\dots,j_k\in\mathbb{N}}$, we have:*

$$\widetilde{A}_{k,p}^{-1}\left[\sum_{j_1,\dots,j_k=1}^{n} |b_{j_1,\dots,j_k}|^2\right]^{\frac{1}{2}} \leqslant \left[\mathbb{E}\left(|G_n|^p\right)\right]^{\frac{1}{p}} \leqslant \widetilde{B}_{k,p}\left[\sum_{j_1,\dots,j_k=1}^{n} |b_{j_1,\dots,j_k}|^2\right]^{\frac{1}{2}},$$

for all $n \in \mathbb{N}$, where $G_n = \sum_{j_1,\dots,j_k=1}^{n} b_{j_1,\dots,j_k} Z_{j_1}^{(1)}\cdots Z_{j_k}^{(k)}$.

Decoupling inequalities have evolved as a subject of great interest since their introduction by McConnell and Taqqu [15,16]. Their motivation was the study of multiple stochastic integrals (see the expository article [3] and the references therein, and also [2,4,11] for results and applications of decoupling inequalities). In these works, the polynomials and multilinear forms involved are generally required to be "tetrahedral", i.e., that the coefficients $a_{j_1,\dots,j_k}$ are zero if $j_1,\dots,j_k$ are not all different. For complex gaussian variables, as an immediate consequence of our polynomial Khintchine inequality and its multilinear analogue, we have the following particular case of decoupling inequality, without the tetrahedral assumption.

Corollary 5. *With the notation of Theorem 3 and Corollary 4 and for $1 \leqslant p, q < \infty$ we have:*

$$A_{k,p}\widetilde{B}_{k,q}\mathbb{E}\left(|G_n|^q\right)^{1/q} \leqslant \mathbb{E}\left(|F_n|^p\right)^{1/p} \leqslant \widetilde{A}_{k,q}B_{k,p}\mathbb{E}\left(|G_n|^q\right)^{1/q},$$

for all $n \in \mathbb{N}$.

Now we turn our attention to Steinhaus random variables. Recall that for a uniform random variable ϕ on the interval $[0, 2\pi]$, the (complex) random variable $e^{i\phi}$ is uniformly distributed on the complex circumference S^1, and it is called a Steinhaus random variable. For these variables we have the following.

Theorem 6. *If $\{\varphi_j\}_{j\in\mathbb{N}}$ is a sequence of independent Steinhaus random variables, then for $1 \leqslant p < \infty$ there are positive constants $\widetilde{A}_{k,p}$ and $\widetilde{B}_{k,p}$ such that for every symmetric sequence of complex numbers $\{a_{j_1,\dots,j_k}\}_{j_1,\dots,j_k\in\mathbb{N}}$, we have:*

$$\widetilde{A}_{k,p}^{-1}\left[\sum_{j_1,\dots,j_k=1}^{n}|a_{j_1,\dots,j_k}|^2\right]^{\frac{1}{2}} \leqslant \left[\mathbb{E}\left(|F_n|^p\right)\right]^{\frac{1}{p}} \leqslant \widetilde{B}_{k,p}\left[\sum_{j_1,\dots,j_k=1}^{n}|a_{j_1,\dots,j_k}|^2\right]^{\frac{1}{2}},$$

for all $n \in \mathbb{N}$, where $F_n = \sum_{j_1,\dots,j_k=1}^{n} a_{j_1,\dots,j_k}\varphi_{j_1}\cdots\varphi_{j_k}$.

Mimicking the proof of Corollary 4, we can obtain the following corollary for the multilinear situation:

Corollary 7. *Let $\{Z_j^{(1)}\}_{j\in\mathbb{N}},\dots,\{Z_j^{(k)}\}_{j\in\mathbb{N}}$ be a finite set of sequences of Steinhaus random variables which are jointly independent, then for $1 \leqslant p < \infty$ there are positive constants $\widetilde{A}_{k,p}$ and $\widetilde{B}_{k,p}$ such that for every sequence of complex numbers $\{b_{j_1,\dots,j_k}\}_{j_1,\dots,j_k\in\mathbb{N}}$, we have*:

$$\widetilde{A}_{k,p}^{-1}\left[\sum_{j_1,\dots,j_k=1}^{n}|b_{j_1,\dots,j_k}|^2\right]^{\frac{1}{2}} \leqslant \left[\mathbb{E}\left(|G_n|^p\right)\right]^{\frac{1}{p}} \leqslant \widetilde{B}_{k,p}\left[\sum_{j_1,\dots,j_k=1}^{n}|b_{j_1,\dots,j_k}|^2\right]^{\frac{1}{2}},$$

for all $n \in \mathbb{N}$, where $G_n = \sum_{j_1,\dots,j_k=1}^{n} b_{j_1,\dots,j_k}Z_{j_1}^{(1)}\cdots Z_{j_k}^{(k)}$.

It is clear that Theorem 6 and its Corollary 7 together give a decoupling inequality for Steinhaus random variables just as we did in Corollary 5 for gaussian variables.

A combination of Theorem 3, Corollary 4, Theorem 6 and Corollary 7 gives the following:

Theorem 8. *Let $\{a_{j_1,\dots,j_k}\}_{j_1,\dots,j_k\in\mathbb{N}}$ be a symmetric sequence of complex numbers. The following are equivalent*:

(i) $\sum_{j_1,\dots,j_k\geqslant 1}|a_{j_1,\dots,j_k}|^2 < \infty$ (*or any of the equivalent conditions in Theorem* 2).

(ii) *For every sequence* (*or for some sequence*) $\{X_j\}_{j\in\mathbb{N}}$ *of independent standard complex gaussian variables* (*Steinhaus random variables*), *and for every $1 \leqslant p < \infty$, the random series $\sum_{j_1,\dots,j_k=1}^{n} a_{j_1,\dots,j_k}X_{j_1}\cdots X_{j_k}$ is convergent in L_p.*

(iii) *For every sequences* (*or for some sequences*) $\{Y_j^{(1)}\}_{i\in\mathbb{N}},\dots,\{Y_j^{(k)}\}_{j\in\mathbb{N}}$ *of standard complex gaussian variables* (*Steinhaus random variables*) *which are jointly independent and for every $1 \leqslant p < \infty$, the random series $\sum_{j_1,\dots,j_k=1}^{n} a_{j_1,\dots,j_k}Y_{j_1}^{(1)}\cdots Y_{j_k}^{(k)}$ is convergent in L_p.*

If the sequence $\{a_{j_1,\dots,j_k}\}_{j_1,\dots,j_k\in\mathbb{N}}$ is not symmetric, the equivalence between (i) *and* (iii) *remains true.*

Note that as a consequence of Theorems 2 and 8, almost sure and L_p-convergence for either homogeneous polynomials or the associated multilinear form on gaussian or Steinhaus variables are all equivalent (and equivalent to square summability of the coefficients).

2. Convergence on standard full subspaces

In this section we consider polynomials and multilinear forms whose sets of convergence enjoy some linearity property. In opposition to the previous section, all the results in this one hold for both complex and real variables.

Sjögren [23] studied the convergence of bilinear and quadratic forms of standard gaussian real random variables on what he calls "full subspaces" of $\mathbb{R}^{\mathbb{N}}$, motivated by the study of convergence of some stochastic integrals. He looks at the sequence $\{X_j\}_{j\in\mathbb{N}}$ as an element of $\mathbb{R}^{\mathbb{N}}$ and, in $\mathbb{R}^{\mathbb{N}}$, considers a gaussian product measure. A "full subspace" of $\mathbb{R}^{\mathbb{N}}$ is then a linear subspace with gaussian measure 1. He shows that the convergence on a full subspace (for the gaussian measure) is equivalent to the bilinear form being nuclear on ℓ_2. He also presents a counterexample showing that for trilinear forms the convergence on full subspaces does not imply nuclearity on ℓ_2.

In order to study the same problem for n-linear forms or n-homogeneous polynomials on $\mathbb{K}^{\mathbb{N}}$, $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, and for more general random variables, we need to restrict somehow the full subspaces considered. We thus define the concept of "standard full subspace" in the construction that follows.

Let X_0 be the set of finite sequences of scalar numbers. Given a Hilbert–Schmidt injective operator $T:\ell_2\to\ell_2$, we define a norm $|||\cdot|||$ on X_0 by:

$$|||x||| = \|Tx\|_{\ell_2}.$$

We denote by X_T the completion of X_0 with respect to the norm $|||\cdot|||$. We can identify X_T with a linear subspace of $\mathbb{K}^{\mathbb{N}}$ whose gaussian measure is 1 [10, p. 59]. Therefore, X_T is a full subspace in the sense of Sjögren. We call these spaces "standard full subspaces".

It is clear that we can continuously extend the operator T to X_T. We denote by $\widetilde{T}:X_T\to\ell_2$ this extension and we have $|||x|||=\|\widetilde{T}x\|_{\ell_2}$ for all $x\in X_T$. Also, it is straightforward that $\ell_2\subset X_T$ and the inclusion $i:\ell_2\to X_T$ has the same norm as T.

Note that the standard full subspaces include the following examples: given any sequence $(\lambda_n)_{n\in\mathbb{N}}\in\ell_2$ with $\lambda_n>0$ for all n, and denoting by $(e_n)_n$ the canonical basis of ℓ_2, the mapping $T(x)=\sum_n\lambda_n x_n e_n$ defines an injective Hilbert–Schmidt operator in ℓ_2. The corresponding subspace X_T is:

$$X_T=\left\{(x_n)_n\in\mathbb{K}^{\mathbb{N}}\colon\ |||x|||^2=\sum_n\lambda_n^2|x_n|^2<\infty\right\}.$$

Now we see that standard full subspaces have measure 1 for a great variety of product probabilities. Suppose we are given a probability measure μ_1 defined on the Borel subsets of $\mathbb{K}$, such that $\int_{\mathbb{K}}|z|^2\,d\mu_1(z)=\sigma^2<\infty$, and let μ be the induced product measure on $\mathbb{K}^{\mathbb{N}}$. Then we have the following.

Theorem 9. *Take an injective Hilbert–Schmidt operator $T:\ell_2\to\ell_2$ and let X_T be the standard full subspace associated with T. If μ is defined as above, then $\mu(X_T)=1$.*

Our next objective is to relate the convergence of a random polynomial (or multilinear form) on a standard full subspace to properties of the polynomial (or multilinear form) defined by the same coefficients. First we need some definitions. Being E a Banach space, we say that a mapping $P:E\to\mathbb{K}$ is a k-homogeneous polynomial if there exists a k-linear form $\Phi:E\times\cdots\times E\to\mathbb{K}$ such that $P(x)=\Phi(x,\dots,x)$ for all $x\in E$. The space of all continuous k-homogeneous polynomials is denoted by $\mathcal{P}({}^kE)$.

For $x_1, \dots, x_m \in E$, the weak-ℓ_r norm of $(x_i)_{i=1}^m$ is defined as

$$w_r\big((x_i)_{i=1}^m\big) = \sup_{x' \in B_{E'}} \left(\sum_i \big|\langle x', x_i\rangle\big|^r\right)^{1/r}.$$

A polynomial $P \in \mathcal{P}^k(E)$ is *r-dominated* if there exists $C > 0$ such that for every finite sequence $(x_i)_{i=1}^m \subset E$ the following holds

$$\left(\sum_{i=1}^m \big|P(x_i)\big|^{\frac{r}{k}}\right)^{\frac{k}{r}} \leqslant C w_r\big((x_i)_{i=1}^m\big)^k.$$

The least of such constants C is called the *r-dominated quasi-norm* of P and will be denoted by $\|P\|_{r\text{-}dom}$. The definition for multilinear forms is analogous.

Dominated polynomials satisfy the following domination property [17]: there exists a probability measure ν on $B_{E'}$ such that for each $x \in E$ we have:

$$\big|P(x)\big| \leqslant \|P\|_{r\text{-}dom}\left(\int_{B_{E'}} \big|\langle x', x\rangle\big|^r \, d\nu\right)^{k/r}. \tag{3}$$

It is not hard to see that the convergence of a k-linear form on the product of k standard full subspaces is equivalent to the convergence on $X \times \cdots \times X$ for some standard full subspace X. Therefore, assertion (iii) in the next theorem can be also stated as convergence on the product of k standard full subspaces. We choose the following formulation for simplicity.

Theorem 10. *Let $\{a_{j_1,\dots,j_k}\}_{j_1,\dots,j_k \in \mathbb{N}}$ be a symmetric sequence of complex numbers. The following are equivalent*:

- (i) *The random series $\sum_{j_1 \leqslant N_1,\dots,j_k \leqslant N_k} a_{j_1,\dots,j_k} X_{j_1} \cdots X_{j_k}$ converges in a standard full subspace as $N_1, \dots, N_k \to \infty$.*
- (ii) *$P(x) = \sum_{j_1,\dots,j_k=1}^\infty a_{j_1,\dots,j_k} x_{j_1} \cdots x_{j_k}$ defines a 2-dominated k-homogeneous polynomial on ℓ_2.*
- (iii) *The random series $\sum_{j_1 \leqslant N_1,\dots,j_k \leqslant N_k} a_{j_1,\dots,j_k} X_{j_1}^{(1)} \cdots X_{j_k}^{(k)}$ converges in a standard full subspace as $N_1, \dots, N_k \to \infty$.*
- (iv) *$A(x) = \sum_{j_1,\dots,j_k=1}^\infty a_{j_1,\dots,j_k} x_{j_1}^1 \cdots x_{j_k}^k$ defines a 2-dominated k-linear form on ℓ_2.*

If the sequence $\{a_{j_1,\dots,j_k}\}_{j_1,\dots,j_k \in \mathbb{N}}$ is not symmetric, the equivalence between (iii) *and* (iv) *remains true.*

A combination of Theorems 9 and 10 gives the following.

Corollary 11. *Let $\{a_{j_1,\dots,j_k}\}_{j_1,\dots,j_k \in \mathbb{N}}$ be a sequence of complex numbers that define a 2-dominated k-linear form on ℓ_2. Then*

$$\sum_{j_1 \leqslant N_1,\dots,j_k \leqslant N_k} a_{j_1,\dots,j_k} Y_{j_1}^{(1)} \cdots Y_{j_k}^{(k)}$$

converges almost surely as $N_1, \dots, N_k \to \infty$ *for any sequences of* $\{Y_i^{(1)}\}_{i\in\mathbb{N}}, \dots, \{Y_i^{(k)}\}_{i\in\mathbb{N}}$ *jointly independent and identically distributed random variables with finite variance. If the coefficients are symmetric, the analogous polynomial result holds.*

It is a known fact that for degree two, nuclear and dominated polynomials (and multilinear forms) on ℓ_2 coincide (see for example [5, Section 26.4]). So we can combine Theorem 10 with Sjögren's result [23, Theorem 3] to see that the convergence of a gaussian 2-homogeneous polynomial or bilinear form on some full subspace implies the convergence on some standard full subspace. However, as we will see in Example 13 below, this is not true for degree greater than 2.

Theorem 12. *The following are equivalent*:

(i) *For every sequence* $\{X_i\}_{i\in\mathbb{N}}$ *of independent standard complex gaussian variables, the random series* $\sum_{i,j} a_{i,j} X_i X_j$ *converges in a full subspace.*
(ii) *The series* $\sum_{i,j} a_{i,j} X_i X_j$ *converges in a standard full subspace.*
(iii) *For every sequence* $\{X_i\}_{i\in\mathbb{N}}$ *and* $\{Y_j\}_{j\in\mathbb{N}}$ *of independent standard complex gaussian variables, the series* $\sum_{i,j} a_{i,j} X_i Y_j$ *converges in a full subspace.*
(iv) *The series* $\sum_{i,j} a_{i,j} X_i Y_j$ *converges in a standard full subspace.*

As a consequence, the convergence of the gaussian 2-homogeneous polynomial on a full subspace implies the almost sure convergence of the random polynomial, for any sequence of independent and identically distributed random variables $\{X_i\}_{i\in\mathbb{N}}$ with finite variance. This follows from the fact that standard full subspaces have measure 1 for the product measure on $\mathbb{K}^{\mathbb{N}}$ induced by these kind of random variables.

Since for degree two, convergence of a gaussian polynomial in a full subspace implies its convergence on some standard full subspace, one may ask if every full subspace contains a standard full subspace. The answer is negative. It can be deduced from the existence of measurable norms for which some orthonormal bases are not square summable (see [8] and [10, Chapter 1]). It also follows easily from the following example. Sjögren presented an example of a non-nuclear trilinear form that converges on a full subspace. We see that this trilinear form can be chosen so that it is not 2-dominated. Therefore, it does not converge on any standard full subspace. This shows that the previous theorem is not true for higher degrees and that there are full subspaces that contain no standard full subspaces.

Example 13. A trilinear form that converges on a full subspace but not on any standard full subspace. A full subspace not containing any standard full subspace.

Let $\rho_1 = 0$ and $\rho_{n+1} = \rho_n + n^2 + n + 1$, for all $n \in \mathbb{N}$. Sjögren's example is the following trilinear form:

$$T(X, Y, Z) = \sum_{n=1}^{\infty} a_n \left(\sum_{1 \leqslant i, j \leqslant n} X_{\rho_n+i} Y_{\rho_n+j} Z_{\rho_n+ni+j} \right),$$

with $a_n > 0$, for all n, satisfying

$$\sum_{n=1}^{\infty} a_n n^{\frac{7}{4}} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} a_n n^2 = \infty.$$

We consider the sequence $\{a_n\}_n = \{n^{-\frac{23}{8}}\}_n$, which satisfies the conditions. We want to prove that in this case the trilinear $T : \ell_2 \times \ell_2 \times \ell_2 \to \mathbb{C}$ is not 2-dominated. Let us define the sequences $\{x^l\}_l$, $\{y^l\}_l$ and $\{z^l\}_l$ to be the formed by the standard unit vectors of ℓ_2 in "the same order" as the coordinates of X, Y and Z appeared in the definition of T. That is:

$$\{x^l\}_l = \{e_{\alpha(l)}\}_l, \qquad \{y^l\}_l = \{e_{\beta(l)}\}_l, \qquad \{z^l\}_l = \{e_{\gamma(l)}\}_l,$$

where

$$\alpha = \{\rho_1 + 1, \rho_2 + 1, \rho_2 + 1, \rho_2 + 2, \rho_2 + 2, \rho_3 + 1, \rho_3 + 1, \rho_3 + 1, \rho_3 + 2, \rho_3 + 2, \ldots\},$$
$$\beta = \{\rho_1 + 1, \rho_2 + 1, \rho_2 + 2, \rho_2 + 1, \rho_2 + 2, \rho_3 + 1, \rho_3 + 2, \rho_3 + 3, \rho_3 + 1, \rho_3 + 2, \ldots\},$$
$$\gamma = \{\rho_1 + 1 \cdot 1 + 1, \rho_2 + 2 \cdot 1 + 1, \rho_2 + 2 \cdot 1 + 2, \rho_2 + 2 \cdot 2 + 1, \rho_2 + 2 \cdot 2 + 2, \ldots\}.$$

(Observe that there are lots of repetitions in the x^l's and in the y^l's.)

Let $\theta_n = \frac{n(n+1)(2n+1)}{6}$. Then

$$\sum_{l=1}^{\theta_n} \left|T\left(x^l, y^l, z^l\right)\right|^{\frac{2}{3}} = \sum_{l=1}^{n} l^2 a_l^{\frac{2}{3}} = \sum_{l=1}^{n} l^{\frac{1}{12}} \geqslant \int_0^{n-1} x^{\frac{1}{12}}\, dx = \frac{12}{13}(n-1)^{\frac{13}{12}}.$$

It is easy to see that

$$w_2\left(\{x^l\}_{l=1}^{\theta_n}\right) = w_2\left(\{y^l\}_{l=1}^{\theta_n}\right) = \sqrt{n} \quad \text{and} \quad w_2\left(\{z^l\}_{l=1}^{\theta_n}\right) = 1.$$

If T is 2-dominated, we would have

$$\left(\sum_{l=1}^{\theta_n} \left|T\left(x^l, y^l, z^l\right)\right|^{\frac{2}{3}}\right)^{\frac{3}{2}} \leqslant \|T\|_{2\text{-}dom} w_2\left(\{x^l\}_{l=1}^{\theta_n}\right) \cdot w_2\left(\{y^l\}_{l=1}^{\theta_n}\right) \cdot w_2\left(\{z^l\}_{l=1}^{\theta_n}\right),$$

and this would imply

$$(n-1)^{\frac{13}{8}} \leqslant Cn,$$

for some constant C. Since this is false, T cannot be 2-dominated.

Now, the full subspace where the random bilinear form associated to T converges, cannot contain any standard full subspace.

Let us provide a stronger version of Sjögren counterexample, namely, a non-nuclear multilinear form that converges on a standard full subspace. To this end, we denote by $\mathcal{P}_N({}^k E)$ the space of all nuclear k-homogeneous polynomials on the Banach space E, and endow it with the nuclear norm (see [6] and [18] for details).

Example 14. There are non-nuclear multilinear forms that converge on standard full subspaces.

Let us see that, for any natural number $k \geqslant 3$, there exists $Q \in \mathcal{P}(^k\ell_2)$ which is 2-dominated but it is not nuclear. For $n \geqslant 1$, in [1] the authors show that there exist polynomials $P_n \in \mathcal{P}(^k\ell_2^n)$ such that

$$\|P_n\|_{\mathcal{P}(^k\ell_2^n)} \leqslant C_k n^{1/2}$$

and

$$\|P_n\|_{\mathcal{P}_N(^k\ell_2^n)} \geqslant D_k n^{k-1/2}$$

for suitable constants C_k and D_k, which are independent of n.

Consider the following commutative diagram:

$$\begin{tikzcd} \ell_2^n \arrow[r, "P_n"] \arrow[d, "id"'] & \mathbb{C} \\ \ell_\infty^n \arrow[r, "id"'] & \ell_2^n \arrow[u, "P_n"'] \end{tikzcd}$$

Since $\|id : \ell_\infty^n \to \ell_2^n\| = \sqrt{n}$, applying the *little Grothendieck theorem* [5, p. 139] and the factorization property of dominated polynomials [17, Theorem 10] we obtain an upper bound for $\|P_n\|_{2\text{-}dom}$, namely

$$\begin{aligned} \|P_n\|_{2\text{-}dom} &\leqslant \|P_n\|_{\mathcal{P}(^k\ell_2^n)} \|id : \ell_\infty^n \to \ell_2^n\|_{2\text{-}sum}^k \\ &\leqslant \frac{2}{\sqrt{\pi}} C_k n^{1/2} \sqrt{n}^k = \widetilde{C}_k n^{1/2+k/2}, \end{aligned}$$

where $\|\cdot\|_{2\text{-}sum}$ denotes the 2-summing norm of an operator. For $2^{(k-1)/2} < d < 2^{(2k-3)/2}$ we define $Q_m = (2d)^{-m} P_{2^m} \in \mathcal{P}(^k\ell_2^{2^m})$, and obtain lower bounds for their nuclear norms:

$$\begin{aligned} \|Q_m\|_{\mathcal{P}_N(^k\ell_2^{2^m})} &\geqslant D_k (2d)^{-m} 2^{m(k-1/2)} > D_k 2^{-m} 2^{-m(k-1)/2} 2^{mk} 2^{-m/2} \\ &= D_k 2^{m(k/2-1)} \xrightarrow[m\to\infty]{} \infty. \end{aligned}$$

Since $\ell_2 = \ell_2(\ell_2^{2^m} : m \geqslant 1)$, the polynomial $Q = \bigoplus_m Q_m : \ell_2 \to \mathbb{C}$ can be defined. This polynomial cannot be nuclear because the nuclear norms of $Q_m = Q \circ \imath_m$, where $\imath_m$ is the canonical injection $\ell_2^{2^m} \hookrightarrow \ell_2$, tend to infinity. On the other hand, Q is a 2-dominated polynomial, since

$$\begin{aligned} \sum_m \|Q_m\|_{2\text{-}dom} &\leqslant \widetilde{C}_k \sum_m (2d)^{-m} 2^{m(1/2+k/2)} \\ &< \widetilde{C}_k \sum_m 2^{-m} 2^{-m(2k-3)/2} 2^{m/2} 2^{mk/2} \end{aligned}$$

$$= \widetilde{C_k} \sum_m 2^{-m(k/2-1)} < \infty,$$

and the space of 2-dominated polynomials is complete in the 2-dominated quasi-norm.

We end this section with an application of the polynomial Khintchine inequality to extend a result in [17] on dominated polynomials. Note that throughout this section, dominated polynomials were used to characterize some particular kind of convergence of random multi-indexed series. Now we take the opposite direction: we will use our results on L_p-convergence of polynomials in random variables to obtain properties of dominated polynomials. Theorem 3 in [17] states that for $2 \leqslant p < \infty$ and $1 \leqslant r \leqslant p$, if the polynomial

$$P(x) = \sum_{i_1,\dots,i_k=1}^{\infty} a_{i_1,\dots,i_k} x_{i_1} \cdots x_{i_k}$$

is r-dominated on ℓ_p, then we have $\sum_{i_1,\dots,i_k=1}^{\infty} |a_{i_1,\dots,i_k}|^2 < \infty$. We extend this result to any r and to any Banach sequence space containing ℓ_2. By a Banach sequence space we understand a Banach lattice over the natural numbers. If a Banach sequence space contains ℓ_2 (in the sense that each element of ℓ_2 is a sequence belonging to E), then a closed graph argument shows that the formal inclusion $i : \ell_2 \to E$ is continuous.

Theorem 15. *Let E be a Banach sequence space that contains ℓ_2. If the polynomial*

$$P(x) = \sum_{i_1,\dots,i_k=1}^{\infty} a_{i_1,\dots,i_k} x_{i_1} \cdots x_{i_k}$$

on E is r-dominated for some $1 \leqslant r < \infty$, then $\sum_{i_1,\dots,i_k=1}^{\infty} |a_{i_1,\dots,i_k}|^2 < \infty$.

Note that hypotheses of the previous theorem hold for Lorentz sequence spaces $d(w, p)$ with $p \geqslant 2$ and for any 2-convex Banach sequence space (see [12,13]).

3. Integral representation of holomorphic functions

In [22], the authors presented two integral representation formulas: for entire functions and for holomorphic functions on the unit ball of a Banach space. For this, they considered gaussian measures on Banach spaces and the theory of abstract Wiener spaces [9]. In [19] and [21] it is shown that many of the results stated for real separable Banach spaces in [10] remain valid in the complex setting, which is crucial to our purposes. The papers [20] and [19] were concerned with the study of the classes of holomorphic functions which can be represented using those formulas. We will make use of only a few aspects of the theory. For the sake of completeness we outline the main facts, state the known results we need and prove some new ones. We refer to [6] for the theory of polynomials and analytic functions on infinite dimensional spaces.

Given a separable Hilbert space H, if P is a finite-rank orthogonal projector in H, a cylinder set in H is a set of the form

$$C = \{x \in H\colon\ Px \in \Delta\}$$

where Δ is a Borel subset of PH. We will denote by Γ the gaussian cylinder measure defined on cylinder sets:

$$\Gamma(C) = \frac{1}{\pi^n} \int_{\Delta} e^{-|w|^2}\, dw,$$

where n is the complex dimension of PH, and the integral is with respect to the Lebesgue measure. This cylinder measure is not σ-additive, however, integrals of cylinder functions $F : H \to \mathbb{C}$ of the form $F = h \circ P$ may be defined by setting

$$\int_C F\, d\Gamma = \int_{\Delta} h\, d\Gamma_n,$$

where Γ_n is standard n-dimensional gaussian measure. A norm $\|.\|$ on H with the property that for any $\varepsilon > 0$ there is a finite-rank orthogonal projector P_ε such that for all $P \perp P_\varepsilon$,

$$\Gamma\big\{x \in H\colon\ \|Px\| > \varepsilon\big\} < \varepsilon,$$

is called *measurable* [9]. Examples of measurable norms can be constructed by considering Hilbert–Schmidt operators on H. If $S : H \to H$ is an injective Hilbert–Schmidt operator, then $\|x\|_S = \|Sx\|^{1/2}$ is a measurable norm. Upon completing $(H, \|.\|)$ one obtains a Banach space X. The natural inclusion $\iota : H \hookrightarrow X$ is continuous and dense, and (ι, H, X) is called an abstract Wiener space. A cylinder set C_X in X is one which can be described as

$$C_X = \big\{\gamma \in X\colon\ \big(\varphi_1(\gamma), \ldots, \varphi_n(\gamma)\big) \in \Delta\big\}$$

where $n \in \mathbb{N}$, $\{\varphi_k\}_{k=1}^n \subset X'$ and Δ is a Borel set in $\mathbb{C}^n$. For these sets one considers $C_H = C_X \cap H$, and defines

$$\widetilde{\Gamma}(C_X) := \Gamma(C_H).$$

The set function $\widetilde{\Gamma}$ extends to a measure W (called Wiener measure) on the Borel σ-algebra $\mathcal{B}$ of X.

Since $\iota' : X' \to H'$ has dense range, we can choose $\{z_n\}_{n\in\mathbb{N}} \subset X'$ such that the sequence $\iota'(z_n) = e_n'$ defines an orthonormal basis $\{e_n'\}_{n\in\mathbb{N}}$ of H' dual to some basis $\{e_n\}_{n\in\mathbb{N}} \subset H$. The following proposition is an analogue of [10, Corollary 4.1], where the real case is studied. Since there are not significative changes on the techniques involved for proving it, we omit the proof.

Proposition 16. *With the previous notations, $\{z_k\}_{k=1}^\infty$ is a sequence of independent and identically distributed complex gaussian random variables with mean* 0 *and variance* 1*. Moreover, given $\varphi \in X'$, then φ is a complex gaussian variable with mean* 0 *and variance $\|\iota'\varphi\|_{H'}^2$.*

As usual, we can identify H' with H via $I : H' \to H$, where for $x \in H$ and $\phi \in H'$, $\phi(x) = \langle x, I(\phi)\rangle$. Since I is conjugate linear, in order to preserve analyticity it is necessary to define involutions in H and H'. If $x = \sum x_n e_n$ is an element of H, we let $x^* = \sum \overline{x_n} e_n$. Similarly, if $\phi \in H'$, define ϕ^* so that $I(\phi^*) = I(\phi)^*$. Note that $\langle x^*, y\rangle = \overline{\langle x, y^*\rangle}$ and $\phi(x^*) = \overline{\phi^*(x)}$.

The following diagram will be useful for fixing ideas:

$$\begin{tikzcd} H \arrow[rr, "\imath"] & & X \\ H' \arrow[u, "I"] & H' \arrow[l, "*"'] & X' \arrow[l, "\imath'"'] \arrow[u, "A=\imath\circ I\circ *\circ\imath'"'] \arrow[ll, bend left, "\imath^*"] \end{tikzcd}$$

This general construction applies to the particular case when $H=\ell_2$ and, fixing a sequence of positive real numbers $(\lambda_n)_{n\in\mathbb{N}}\in\ell_2$, the measurable norm is given by the Hilbert–Schmidt operator

$$S:\ell_2\to\ell_2$$
$$S\big((x_n)_n\big)=(\lambda_n x_n)_n.$$

In this way we obtain $\ell_2\hookrightarrow\overline{(\ell_2,\|.\|_S)}=B_0\subset\mathbb{C}^{\mathbb{N}}$ and, since the finite dimensional projectors induce the gaussian measures μ_n on the Borel sets of $\mathbb{C}^n$, we conclude that $\widetilde{\Gamma}$ extends to a measure W, which is the same measure whose existence is ensured by Kolmogorov's existence theorem. Also, the sequence $\{z_n\}_{n\in\mathbb{N}}\subset X'$ is explicitly determined by the set of linear functionals in B_0', represented via the Riesz theorem by $\{\frac{1}{\lambda_n^2}e_n\}_{n\in\mathbb{N}}\subset B_0$. With this choice, it holds that $\{\imath' z_n\}_{n\in\mathbb{N}}=\{e_n'\}_{n\in\mathbb{N}}$, where $\{e_n'\}_{n\in\mathbb{N}}$ is the dual basis of the standard orthonormal basis for ℓ_2.

We do not need the integral formula in its general version, so we just state the following theorem.

Theorem 17. *(See [21, Teorema 3.2.7].) If $\{\varphi_j\}_{j=1}^k\subset B_0'$, then*

$$\int_{B_0} e^{z(\gamma)}\prod_{j=1}^k\overline{\varphi_j(\gamma)}\,dW(\gamma)=\prod_{j=1}^k\overline{\varphi_j\big(\imath\circ I\circ\imath'(z)\big)}\quad\textit{for all } z\in B_0'.$$

Recall that a k-homogeneous polynomial p defined on B_0 is of finite type if there exists $\{\varphi_j\}_{j=1}^N\subset B_0'$ such that $p(\gamma)=\sum_{j=1}^N\varphi_j^k(\gamma)$. The space of finite type polynomials is denoted by $\mathcal{P}_f({}^kB_0)$. From the polarization formula, it can be seen that the product of k different linear functionals is also of finite type.

It is possible to define on $\mathcal{P}_f({}^kB_0)$ the operator

$$\mathcal{T}:\mathcal{P}_f\big({}^kB_0\big)\to\mathcal{P}_f\big({}^kB_0'\big)$$
$$\big[\mathcal{T}(p)\big](z)=\int_{B_0}e^{z(\gamma)}\overline{p(\gamma)}\,dW(\gamma).$$

Since $\imath':B_0'\to\ell_2'$ has dense range, according to Theorem 17, there exists a unique $P\in\mathcal{P}({}^k\ell_2)$ such that the following diagram commutes:

$$
\begin{tikzcd}
\ell_2 \arrow[rrd, dashed, "P"] & & \\
\ell_2' \arrow[u, "I"] & B_0' \arrow[l, "*\circ\iota'"] \arrow[r, "\mathcal{T}(p)"'] & \mathbb{C}
\end{tikzcd}
$$

Namely, since

$$[\mathcal{T}(p)](z) = \overline{p\big(\iota \circ I \circ \iota'(z)\big)} = \overline{p \circ \iota\big[\big(I \circ \iota^*(z)\big)^*\big]},$$

we have

$$P(x) = \overline{p\big[\iota\big(x^*\big)\big]}.$$

From this “extension” property, we can define:

$$
\begin{gathered}
T : \mathcal{P}_f\big({}^k B_0\big) \to \mathcal{P}\big({}^k \ell_2\big) \\
\big[T(p)\big](x) = \overline{p\big[\iota\big(x^*\big)\big]}.
\end{gathered}
$$

Note that we could have defined the operator T independently of the integral representation formula. However, we will see below that this representation approach has some advantages.

We are ready to show the link between random variables and polynomials which will allow us to obtain the polynomial Khintchine inequalities. We will employ some usual notations. For $\alpha = (\alpha_1, \ldots, \alpha_n)$ a multi-index, we define $\ell(\alpha) = n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$. We recall that the sequence $\{z_n\}_{n\in\mathbb{N}} \subset B_0'$ was chosen satisfying $\iota' z_n = e_n'$, and that these vectors form an orthonormal basis for ℓ_2'.

Given a multi-index $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$, set

$$z^\alpha(\gamma) = \prod_{j=1}^{\ell(\alpha)} \big[z_j(\gamma)\big]^{\alpha_j} \quad \text{and} \quad \big(\iota^* z\big)^\alpha(x) = \prod_{j=1}^{\ell(\alpha)} \big[e_j'(x)\big]^{\alpha_j}.$$

Working on a separable Hilbert space, Dwyer [7] defined Hilbert–Schmidt k-functionals and O. Lopushansky and A. Zagorodnyuk study, in [14], the Hilbert space of k-homogeneous polynomials $\mathcal{P}_h({}^k H)$ over H, which is intimately related to the operator T. We need the following results from [14]:

Proposition 18. *The inclusion $\mathcal{P}_h({}^k H) \hookrightarrow \mathcal{P}({}^k H)$ is continuous and $\|P\| \leqslant \|P\|_h$ for all $P \in \mathcal{P}_h({}^k H)$.*

Proposition 19. *Given $Q_n \in \mathcal{P}_h({}^n H)$ and $Q_m \in \mathcal{P}_h({}^m H)$, then we have $Q_n Q_m \in \mathcal{P}_h({}^{n+m} H)$ and $\|Q_n Q_m\|_h \leqslant \|Q_n\|_h \|Q_m\|_h$.*

Proposition 20. *The set of polynomials $\{\sqrt{\frac{k!}{\alpha!}}(\iota^* z)^\alpha\}_{|\alpha|=k}$ forms an orthonormal basis of $\mathcal{P}_h({}^k H)$.*

For simplicity of notation the closure of the span of $\{z^\alpha\}_{|\alpha|=k}$ in $L^2(W)$ will be denoted by $L^2_k(W)$. Remark 1 in [20] states that it is possible to define an isomorphism $\widetilde{T}: L^2_k(W) \to \mathcal{P}_h({}^k\ell_2)$, because $\{\frac{z^\alpha}{\sqrt{\alpha!}}\}_{|\alpha|=k}$ and $\{\frac{\sqrt{|\alpha|!}}{\sqrt{\alpha!}}(i^*z)^\alpha\}_{|\alpha|=k}$ are orthonormal bases of $L^2_k(W)$ and $\mathcal{P}_h({}^k\ell_2)$ respectively, and we have $T(z^\alpha) = (i^*z)^\alpha$. Moreover, we have

$$\|g_k\|_2 = \sqrt{k!}\,\big\|\widetilde{T}(g_k)\big\|_h \tag{4}$$

for any $g_k \in L^2_k(W)$.

Note that if we are given any linear combination of products of k linear functionals, we can compute its L^2-norm in terms of the Hilbertian norm of the polynomial associated via $\widetilde{T}$. So, if we think of linear functionals as complex gaussian variables, we are able to compute L^2-norms of these linear combinations as Hilbertian norms of the associated polynomials.

4. The proofs

In this section we present the proofs of the results stated in Sections 1 and 2. In order to prove Proposition 1 it is convenient to state the following simple result:

Lemma 21. *Let X be a rotation-invariant complex random variable. If for some $k \in \mathbb{N}_0$ we have $\mathbb{E}(|X|^{2k}) < \infty$, then*

$$\mathbb{E}\big(X^m\overline{X}^n\big) = \delta_{m,n}\mathbb{E}\big(|X|^{2m}\big) \quad \textit{for all } m, n \leqslant k.$$

In particular, $\mathbb{E}(X^m) = 0$ for $1 \leqslant m \leqslant k$.

Proof. Let $\theta \in (0, 2\pi)$, since $e^{i\theta}X$ has the same distribution law than X, it is a matter of fact that

$$\mathbb{E}\big(X^m\overline{X}^n\big) = \mathbb{E}\big(\big[e^{i\theta}X\big]^m\overline{\big[e^{i\theta}X\big]}^n\big) = e^{i(m-n)\theta}\mathbb{E}\big(X^m\overline{X}^n\big).$$

If $m, n \leqslant k$, then $|\mathbb{E}(X^m\overline{X}^n)| \leqslant [\mathbb{E}(|X|^{2k})]^{(m+n)/2k} < \infty$. So $\mathbb{E}(X^m\overline{X}^n)$ must be 0 for $m \neq n$, and we obtain the stated result.

Proof of Proposition 1. We need to compute

$$\mathbb{E}\big(|F_n|^2\big) = \sum_{\substack{l_1,\ldots,l_k=1\\ j_1,\ldots,j_k=1}}^{n} a_{l_1,\ldots,l_k}\overline{a_{j_1,\ldots,j_k}}\mathbb{E}(X_{l_1}\cdots X_{l_k}\overline{X_{j_1}}\cdots\overline{X_{j_k}}).$$

Given $J = (l_1, l_2, \ldots, l_k) \in \{1, 2, \ldots, n\}^k$, we define $\mathcal{R}(J) = (\mathcal{R}(J)_m)_{1\leqslant m\leqslant n}$ by $\mathcal{R}(J)_m = \sum_{r=1}^k \delta_{m,l_r}$. This new multi-index counts how many times each number is repeated in J.

The symmetry of the sequence $\{a_{j_1,\ldots,j_k}\}$ allows us to denote $a_{\mathcal{R}(j_1,\ldots,j_k)} = a_{j_1,\ldots,j_k}$. Calling $\alpha = \mathcal{R}(l_1, \ldots, l_k)$, $\beta = \mathcal{R}(j_1, \ldots, j_k)$, we have

$$\mathbb{E}\big(|F_n|^2\big) = \sum_{|\alpha|=k}\sum_{|\beta|=k}\binom{k}{\alpha}\binom{k}{\beta}a_\alpha\overline{a_\beta}\mathbb{E}\big(X^\alpha\overline{X^\beta}\big),$$

where X^α stands for $X_1^{\alpha_1}\cdots X_k^{\alpha_k}$. We can use the independence of $\{X_j\}_{j\in\mathbb{N}}$ and Lemma 21 to obtain:

$$\mathbb{E}\big(|F_n|^2\big)=\sum_{|\alpha|=k}\sum_{|\beta|=k}\binom{k}{\alpha}\binom{k}{\beta}a_\alpha\overline{a_\beta}\prod_{s=1}^n\mathbb{E}\big(X_s^{\alpha_s}\overline{X_s^{\beta_s}}\big)$$
$$=\sum_{|\alpha|=k}\binom{k}{\alpha}^2|a_\alpha|^2\prod_{s=1}^n\mathbb{E}\big(|X_s|^{2\alpha_s}\big).$$

For the left side of the inequality, observe that

$$\prod_{s=1}^n\mathbb{E}\big(|X_s|^{2\alpha_s}\big)\geqslant\prod_{s=1}^n\mathbb{E}\big(|X_s|\big)^{2\alpha_s}\geqslant\Big[\inf_{j\in\mathbb{N}}\mathbb{E}\big(|X_j|\big)\Big]^{2k}.$$

The obvious estimation $1\leqslant\binom{k}{\alpha}\leqslant k!$ gives the following:

$$\mathbb{E}\big(|F_n|^2\big)\geqslant\Big[\inf_{j\in\mathbb{N}}\mathbb{E}\big(|X_j|\big)\Big]^{2k}\sum_{|\alpha|=k}\binom{k}{\alpha}|a_\alpha|^2$$
$$=\Big[\inf_{j\in\mathbb{N}}\mathbb{E}\big(|X_j|\big)\Big]^{2k}\left(\sum_{j_1,\dots,j_k=1}^n|a_{j_1,\dots,j_k}|^2\right).$$

On the other hand, for any multi-index α, since $|\alpha|=k$, at most k numbers of the set $\{\alpha_1,\dots,\alpha_n\}$ are different from 0. Moreover, none of them can be greater than k, so we have:

$$\mathbb{E}\big(|F_n|^2\big)\leqslant\sum_{|\alpha|=k}k!\binom{k}{\alpha}|a_\alpha|^2\Big[\sup_{j\in\mathbb{N}}\mathbb{E}\big(|X_j|^{2k}\big)\Big]$$
$$=k!\Big[\sup_{j\in\mathbb{N}}\mathbb{E}\big(|X_j|^{2k}\big)\Big]\left(\sum_{j_1,\dots,j_k=1}^n|a_{j_1,\dots,j_k}|^2\right).$$

From both inequalities, we can take

$$A_k^{-1}=\Big[\inf_{j\in\mathbb{N}}\mathbb{E}\big(|X_j|\big)\Big]^k\quad\text{and}\quad B_k=\sqrt{k!}\Big[\sup_{j\in\mathbb{N}}\mathbb{E}\big(|X_j|^{2k}\big)\Big]^{1/2}.$$

If the variables are gaussian, since

$$\mathbb{E}\big(|F_n|^2\big)=\sum_{|\alpha|=k}\binom{k}{\alpha}^2|a_\alpha|^2\prod_{s=1}^n\int_\Omega\big|X_s(\omega)\big|^{2\alpha_s}\,dP(\omega),$$

we must compute

$$\prod_{s=1}^{n}\int_{\Omega}\left|X_s(\omega)\right|^{2\alpha_s}\,dP(\omega)=\prod_{s=1}^{n}\int_{\mathbb{C}}|w|^{2\alpha_s}e^{-|w|^2}\,\frac{dw}{\pi}$$
$$=\prod_{s=1}^{n}\int_{0}^{2\pi}\int_{0}^{+\infty}\rho^{2\alpha_s+1}e^{-\rho^2}\,\frac{d\rho\,d\theta}{\pi}$$
$$=\alpha!.$$

Then,

$$\mathbb{E}\big(|F_n|^2\big)=\sum_{|\alpha|=k}k!\binom{k}{\alpha}|a_\alpha|^2=k!\left(\sum_{j_1,\dots,j_k=1}^{n}|a_{j_1,\dots,j_k}|^2\right). \qquad \square$$

In order to prove Theorem 2, we need the following lemma, for which we adapt some ideas from [23].

Lemma 22. *Suppose that $\{X_j\}_{j\in\mathbb{N}}$ is a sequence of independent and rotation-invariant complex random variables which, for some $k>1$, satisfies the $(\star)$-condition. Then, there exists $\varepsilon>0$ such that for any sequence of complex numbers $\{a_j\}_{j\in\mathbb{N}}$, we have*

$$P\left(\left|\sum_{j=1}^{n}a_jX_j\right|^2\geqslant\varepsilon^2\sum_{j=1}^{n}|a_j|^2\right)\geqslant\varepsilon. \tag{5}$$

Proof. By homogeneity, it is sufficient to prove that the inequality holds assuming that $\sum_{j=1}^{n}|a_j|^2=1$. Also, since the variables X_j are independent and rotation-invariant, the distribution laws of $\sum_{j=1}^{n}a_jX_j$ and $\sum_{j=1}^{n}|a_j|X_j$ coincide. Thus, we can assume that $\{a_j\}_{j\in\mathbb{N}}$ is a sequence of non-negative real numbers.

Since

$$\left|\sum_{j=1}^{n}a_jX_j\right|^2=\left(\mathfrak{Re}\sum_{j=1}^{n}a_jX_j\right)^2+\left(\mathfrak{Im}\sum_{j=1}^{n}a_jX_j\right)^2$$
$$=\left(\sum_{j=1}^{n}a_j\,\mathfrak{Re}\,X_j\right)^2+\left(\sum_{j=1}^{n}a_j\,\mathfrak{Im}\,X_j\right)^2,$$

it is enough to prove that

$$P\left(\left(\sum_{j=1}^{n}a_j\mathfrak{Re}X_j\right)^2\geqslant\varepsilon^2\right)\geqslant\varepsilon. \tag{6}$$

Since $\mathfrak{Re}\,X_j=\mathfrak{Im}(e^{i\frac{\pi}{2}}X_j)$ and X_j is rotation-invariant, $\mathfrak{Re}\,X_j$ and $\mathfrak{Im}\,X_j$ are identically distributed. In particular, $\mathbb{E}(|X_j|)\leqslant\mathbb{E}(|\mathfrak{Re}\,X_j|)+\mathbb{E}(|\mathfrak{Im}\,X_j|)=2\mathbb{E}(|\mathfrak{Re}\,X_j|)$, and then we have $0<\inf_{j\in\mathbb{N}}\mathbb{E}(|\mathfrak{Re}\,X_j|)$. Also, since $|\mathfrak{Re}\,X_j|\leqslant|X_j|$, it follows that $\sup_{j\in\mathbb{N}}\mathbb{E}(|\mathfrak{Re}\,X_j|^{2k})<\infty$.

We assume first that $\mathbb{E}(|\mathfrak{Re}\, X_j|^2) = 1$, for all $j \in \mathbb{N}$. If we show that there exists $\delta > 0$ such that $P(|\mathfrak{Re}\, X_j| \geqslant \delta) \geqslant \delta$ for all $j \in \mathbb{N}$, we can apply [23, Lemma 1] to obtain (6) for some $\varepsilon > 0$. Suppose the contrary: that for any $\delta > 0$, the inequality $P(|\mathfrak{Re}\, X_j| \geqslant \delta) \geqslant \delta$ does not hold at least for some $j \in \mathbb{N}$. In particular, choosing a sequence $\{\delta_s\}_{s\in\mathbb{N}}$ such that $\delta_s \to 0$, let X_{j_s} satisfy $P(|\mathfrak{Re}\, X_{j_s}| \geqslant \delta_s) < \delta_s$.

For each δ_s, and for suitable $R_s > \delta_s$, we can write

$$\begin{aligned}
\mathbb{E}\big(|\mathfrak{Re}\, X_{j_s}|\big) &= \int_0^{+\infty} P\big(|\mathfrak{Re}\, X_{j_s}| > t\big)\, dt \\
&= \int_0^{\delta_s} P\big(|\mathfrak{Re}\, X_{j_s}| > t\big)\, dt + \int_{\delta_s}^{R_s} P\big(|\mathfrak{Re}\, X_{j_s}| > t\big)\, dt + \int_{R_s}^{+\infty} P\big(|\mathfrak{Re}\, X_{j_s}| > t\big)\, dt.
\end{aligned}$$

Obviously,

$$\int_0^{\delta_s} P\big(|\mathfrak{Re}\, X_{j_s}| > t\big)\, dt \leqslant \delta_s.$$

By Chebyshev inequality, we have

$$P\big(|\mathfrak{Re}\, X_{j_s}| > t\big) \leqslant \frac{\mathbb{E}(|\mathfrak{Re}\, X_{j_s}|^2)}{t^2} = \frac{1}{t^2},$$

so we can choose $R_s = \frac{1}{\delta_s^{1/2}}$ to obtain

$$\int_{R_s}^{+\infty} P\big(|\mathfrak{Re}\, X_{j_s}| > t\big)\, dt \leqslant \int_{R_s}^{+\infty} \frac{1}{t^2}\, dt = \frac{1}{R_s} = \delta_s^{1/2}.$$

Finally, since $t \mapsto P(|\mathfrak{Re}\, X_{j_s}| > t)$ is a decreasing function, we have the following bound for the remaining integral:

$$\int_{\delta_s}^{R_s} P\big(|\mathfrak{Re}\, X_{j_s}| > t\big)\, dt \leqslant P\big(|\mathfrak{Re}\, X_{j_s}| > \delta_s\big)(R_s - \delta_s) < \delta_s(R_s - \delta_s).$$

Combining these inequalities, we get

$$\mathbb{E}\big(|\mathfrak{Re}\, X_{j_s}|\big) \leqslant \delta_s + \delta_s^{1/2} + \delta_s(R_s - \delta_s) = \delta_s - \delta_s^2 + 2\delta_s^{1/2} \xrightarrow[s\to\infty]{} 0,$$

which is a contradiction, because we know that $\inf_{j\in\mathbb{N}} \mathbb{E}(|\mathfrak{Re}\, X_j|) > 0$.

So (5) holds whenever $\mathbb{E}(|\mathfrak{Re}\, X_j|^2) = 1$.

In the general case, the ($\star$)-condition implies

$$0 < \Big[\inf_{j\in\mathbb{N}} \mathbb{E}\big(|\mathfrak{Re}\, X_j|\big)\Big]^2 \leqslant \mathbb{E}\big(|\mathfrak{Re}\, X_j|^2\big) \leqslant \sup_{j\in\mathbb{N}} \mathbb{E}\big(|\mathfrak{Re}\, X_j|^2\big) < \infty.$$

So, the result is valid for the variables $Y_j = \frac{\mathfrak{Re}\, X_j}{\sqrt{\mathbb{E}(|\mathfrak{Re}\, X_j|^2)}}$, which means that there exists $\varepsilon > 0$ such that for any sequence of real numbers $\{b_j\}_{j\in\mathbb{N}}$,

$$P\left(\left|\sum_{j=1}^n b_j Y_j\right| \geqslant \varepsilon\left(\sum_{j=1}^n b_j^2\right)^{1/2}\right) \geqslant \varepsilon.$$

Thus, for any sequence of real numbers $\{a_j\}_{j\in\mathbb{N}}$,

$$P\left(\left|\sum_{j=1}^n a_j\, \mathfrak{Re}\, X_j\right| \geqslant \varepsilon\left(\sum_{j=1}^n a_j^2 \mathbb{E}\big(|\mathfrak{Re}\, X_j|^2\big)\right)^{1/2}\right) \geqslant \varepsilon.$$

Standard calculations give us the desired inequality. $\square$

We also need the following result proved in [23, Lemma 2]:

Lemma 23. *Let $\{W_j\}_{j\in\mathbb{N}}$ be a sequence of random variables which satisfies that there is a constant $\delta > 0$ such that*

$$P\big(|W_j| \geqslant \delta\big) \geqslant \delta,$$

for all $j \in \mathbb{N}$. Then, there exists $\eta > 0$ such that

$$P\left(\sum_{j=1}^n c_j |W_j|^2 \geqslant \eta \sum_{j=1}^n c_j\right) \geqslant \eta,$$

for every $c_j \geqslant 0$ and every $n \in \mathbb{N}$.

Proof of Theorem 2. Let us see that (i) implies (ii). For each n, consider the random variable

$$Z_n = \sum_{j_1,\dots,j_k=1}^n a_{j_1,\dots,j_k} X_{j_1} \cdots X_{j_k},$$

and the σ-algebra $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. We can write

$$Z_{n+1} = Z_n + X_{n+1}A_1 + X_{n+1}^2 A_2 + \cdots + X_{n+1}^{k-1} A_{k-1} + X_{n+1}^k a_{n+1,\dots,n+1},$$

where $Z_n, A_1, \dots, A_{k-1}$ are $\mathcal{F}_n$-measurable random variables. Then,

$$\mathbb{E}(Z_{n+1}|\mathcal{F}_n) = Z_n + A_1\mathbb{E}(X_{n+1}|\mathcal{F}_n) + A_2\mathbb{E}\big(X_{n+1}^2\big|\mathcal{F}_n\big) + \cdots + A_{k-1}\mathbb{E}\big(X_{n+1}^{k-1}\big|\mathcal{F}_n\big) + a_{n+1,\dots,n+1}\mathbb{E}\big(X_{n+1}^k\big|\mathcal{F}_n\big).$$

Since $\mathbb{E}(X_{n+1}^j|\mathcal{F}_n) = \mathbb{E}(X_{n+1}^j)$, and from Lemma 21, $\mathbb{E}(X_{n+1}^j) = 0$, for all $1 \leqslant j \leqslant k$, it follows that $\{Z_n\}_n$ is a martingale relative to $\{\mathcal{F}_n\}_n$.

By (i) and Proposition 1, the martingale $\{Z_n\}_n$ is bounded in L_2, hence it is bounded in L_1, and so it converges almost surely.

To prove that (ii) implies (iii), from the sequences $\{Y_j^{(1)}\}_{i\in\mathbb{N}}, \dots, \{Y_j^{(k)}\}_{j\in\mathbb{N}}$ we can construct a sequence $\{X_j\}_{j\in\mathbb{N}}$ of independent and rotation-invariant complex random variables, verifying the $(\star)$-condition. Indeed, we can identify

$$\big\{Y_j^{(r)}\big\}_{j\in\mathbb{N}} \rightsquigarrow \{X_{(j-1)k+r}\}_{j\in\mathbb{N}} \quad \text{for } r = 1, 2, \dots, k.$$

Given k natural numbers $\{l_1, l_2, \dots, l_k\}$, let

$$r_1 \equiv l_1 \mod (k), \qquad r_2 \equiv l_2 \mod (k), \qquad \dots, \qquad r_k \equiv l_k \mod (k)$$

for $r_1, r_2, \dots, r_k \in \{1, 2, \dots, k\}$. Setting

- $b_{l_1,\dots,l_k} = 0$ if $\{r_1, r_2, \dots, r_k\}$ is not a complete residue system modulo k, or else,
- $b_{l_1,\dots,l_k} = \frac{a_{\frac{l_{\tau(1)}-r_{\tau(1)}+k}{k},\dots,\frac{l_{\tau(k)}-r_{\tau(k)}+k}{k}}}{k!}$, where τ is a permutation of $\{1, \dots, k\}$ satisfying $1 = r_{\tau(1)} < r_{\tau(2)} < \cdots < r_\tau(k) = k$.

Then, the multilinear random mapping $\sum_{j_1,\dots,j_k=1}^n a_{j_1,\dots,j_k} Y_{j_1}^{(1)} \cdots Y_{j_k}^{(k)}$ can be viewed as the random polynomial

$$\sum_{l_1,\dots,l_k=1}^{nk} b_{l_1,\dots,l_k} X_{l_1} \cdots X_{l_k}.$$

Note that $\{b_{l_1,\dots,l_k}\}_{l_s\geqslant 1}$ is a symmetric sequence and, since

$$\sum_{l_1,\dots,l_k=1}^{nk} |b_{l_1,\dots,l_k}|^2 = \frac{1}{k!} \sum_{j_1,\dots,j_k=1}^{n} |a_{j_1,\dots,j_k}|^2,$$

we have $\sum_{l_1,\dots,l_k\geqslant 1} |b_{l_1,\dots,l_k}|^2 < \infty$. Now, applying (i) $\Rightarrow$ (ii), we deduce that the random polynomial $\sum_{l_1,\dots,l_k=1}^{nk} b_{l_1,\dots,l_k} X_{l_1} \cdots X_{l_k}$ is almost surely convergent and consequently the random series $\sum_{j_1,\dots,j_k=1}^n a_{j_1,\dots,j_k} Y_{j_1}^1 \cdots Y_{j_k}^k$ is almost surely convergent too.

It only remains to prove that (iii) implies (i). We will prove by induction on k that, if $\{a_{j_1,\dots,j_k}\}_{j_1,\dots,j_k\in\mathbb{N}}$ is a sequence of complex numbers, and $\{Y_j^{(1)}\}_{j\in\mathbb{N}}, \dots, \{Y_j^{(k)}\}_{j\in\mathbb{N}}$ are independent and rotation-invariant complex random variables satisfying the $(\star)$-condition for k, then there exists $\delta_k > 0$ such that, for all $n \in \mathbb{N}$,

$$P\left(\left|\sum_{j_1,\dots,j_k=1}^{n} a_{j_1,\dots,j_k} Y_{j_1}^{(1)} \cdots Y_{j_k}^{(k)}\right|^2 \geqslant \delta_k^2 \sum_{j_1,\dots,j_k=1}^{n} |a_{j_1,\dots,j_k}|^2\right) \geqslant \delta_k.$$

It is clear that from this inequality the result follows. For $k = 1$, if $\{Y_j\}_{j\in\mathbb{N}}$ is a sequence satisfying the hypothesis, then from Lemma 22, there exists $\delta_1 > 0$ such that

$$P\left(\left|\sum_{i=1}^{n} a_i Y_i\right|^2 \geqslant \delta_1^2 \sum_{i=1}^{n} |a_i|^2\right) \geqslant \delta_1. \tag{7}$$

Suppose that the result is valid for $k-1$ and let $\{Y_j^{(1)}\}_{j\in\mathbb{N}}, \dots, \{Y_j^{(k)}\}_{j\in\mathbb{N}}$ and $\{a_{j_1,\dots,j_k}\}_{j_1,\dots,j_k\in\mathbb{N}}$ be as in the statement. We have

$$\sum_{j_1,\dots,j_k=1}^{n} a_{j_1,\dots,j_k} Y_{j_1}^{(1)} \cdots Y_{j_k}^{(k)} = \sum_{j_1=1}^{n}\left(\sum_{j_2,\dots,j_k=1}^{n} a_{j_1,\dots,j_k} Y_{j_2}^{(2)} \cdots Y_{j_k}^{(k)}\right) Y_{j_1}^{(1)}.$$

Conditional on the values of $Y_{j_2}^{(2)}, \dots, Y_{j_k}^{(k)}$ $(1 \leqslant j_2, \dots, j_k \leqslant n)$ we set $c_{j_1} = \sum_{j_2,\dots,j_k=1}^{n} a_{j_1,\dots,j_k} Y_{j_2}^{(2)} \cdots Y_{j_k}^{(k)}$. Then we can find, as in (7), some $\delta_1 > 0$ such that

$$P\left(\left|\sum_{j_1=1}^{n} c_{j_1} Y_{j_1}^{(1)}\right|^2 \geqslant \delta_1^2 \sum_{j_1=1}^{n} |c_{j_1}|^2\right) \geqslant \delta_1. \tag{8}$$

Let us define $W_{j_1} = \sum_{j_2,\dots,j_k=1}^{n} \widetilde{a}_{j_1,\dots,j_k} Y_{j_2}^{(2)} \cdots Y_{j_k}^{(k)}$, where

$$\widetilde{a}_{j_1,\dots,j_k} = \frac{a_{j_1,\dots,j_k}}{\sum_{j_2,\dots,j_k=1}^{n} |a_{j_1,\dots,j_k}|^2}.$$

By the inductive hypothesis, the sequence $\{W_{j_1}\}_{j_1}$ satisfies the hypothesis of Lemma 23 with some δ_{k-1} (note that the $(\star)$-condition for k implies the $(\star)$-condition for $k-1$). We can write

$$\sum_{j_1=1}^{n}\left|\sum_{j_2,\dots,j_k=1}^{n} a_{j_1,\dots,j_k} Y_{j_2}^{(2)} \cdots Y_{j_k}^{(k)}\right|^2 = \sum_{j_1=1}^{n}\left(\sum_{j_2,\dots,j_k=1}^{n} |a_{j_1,\dots,j_k}|^2\right) |W_{j_1}|^2.$$

By Lemma 23, there exists some $\eta > 0$ such that

$$P\left(\sum_{j_1=1}^{n}\left(\sum_{j_2,\dots,j_k=1}^{n} |a_{j_1,\dots,j_k}|^2\right)|W_{j_1}|^2 \geqslant \eta\left(\sum_{j_1,\dots,j_k=1}^{n} |a_{j_1,\dots,j_k}|^2\right)\right) \geqslant \eta. \tag{9}$$

The result follows from (8) and (9) with $\delta_k = \delta_1 \eta$. $\square$

Proof of Theorem 3. Proposition 1 shows that $\mathbb{E}(|F_n|^2)^{1/2} = \sqrt{k!}(\sum_{i_1,\dots,i_k=1}^{n} |a_{i_1,\dots,i_k}|^2)^{1/2}$. Since for any $q > 2$, we have the inclusions

$$L^q(\Omega_1,\mathfrak{A}_1,P_1)\subset L^2(\Omega_1,\mathfrak{A}_1,P_1)\subset L^1(\Omega_1,\mathfrak{A}_1,P_1).$$

It is sufficient to prove the left inequality for $p=1$ and the right one for infinitely many $p>2$.

Let us first show that for any even integer $p>2$ and $n\in\mathbb{N}$, we have:

$$\mathbb{E}\big(|F_n|^p\big)^{1/p}\leqslant B_{k,p}\left(\sum_{i_1,\dots,i_k=1}^{n}|a_{i_1,\dots,i_k}|^2\right)^{1/2}.$$

We define the function

$$\Phi: B_0\to\mathbb{C}$$
$$\Phi(\gamma)=\sum_{i_1,\dots,i_k=1}^{n}a_{i_1,\dots,i_k}z_{i_1}(\gamma)\cdots z_{i_k}(\gamma),$$

where B_0 and the z_j's are defined in Section 3. By Proposition 16, F_n and Φ are identically distributed, so it is enough to check that

$$\|\Phi\|_p\leqslant B_{k,p}\left(\sum_{i_1,\dots,i_k=1}^{n}|a_{i_1,\dots,i_k}|^2\right)^{1/2}.$$

Write $p=2r$ for r a natural number greater than 1. Now we can use (4) and Theorem 17 to get

$$\|\Phi\|_p=\left[\int\limits_{B_0}\big|\Phi(\gamma)\big|^{2r}\,dW(\gamma)\right]^{1/2r}=\big\|\Phi^r\big\|_2^{2/p}=\big[(kr)!\big\|\widetilde{T}\big(\Phi^r\big)\big\|_h^2\big]^{1/p}=\big[(kr)!\big\|\widetilde{T}(\Phi)^r\big\|_h^2\big]^{1/p}.$$

Now we use Proposition 19 and (4) to obtain

$$\big[(kr)!\big\|\widetilde{T}(\Phi)^r\big\|_h^2\big]^{1/p}\leqslant\big[(kr)!\big\|\widetilde{T}(\Phi)\big\|_h^{2r}\big]^{1/p}=\sqrt[p]{(kr)!}\big\|\widetilde{T}(\Phi)\big\|_h=\frac{\sqrt[p]{(kr)!}}{\sqrt{k!}}\|\Phi\|_2.$$

Since $\|\Phi\|_2=\sqrt{k!}(\sum_{i_1,\dots,i_k=1}^{n}|a_{i_1,\dots,i_k}|^2)^{1/2}$ as in Proposition 1, we conclude that:

$$\mathbb{E}\big(|F_n|^p\big)^{1/p}=\|\Phi\|_p\leqslant\sqrt[p]{(kr)!}\left(\sum_{i_1,\dots,i_k=1}^{n}|a_{i_1,\dots,i_k}|^2\right)^{1/2}.$$

It remains to prove the left inequality for $p=1$. This is a consequence of Hölder's inequality:

$$\begin{aligned}\mathbb{E}\big(|F_n|^2\big)&=\int\limits_{\Omega_1}\big|F_n(\omega)\big|^2\,dP_1(\omega)=\int\limits_{\Omega_1}\big|F_n(\omega)\big|^{2/3}\big|F_n(\omega)\big|^{4/3}\,dP_1(\omega)\\&\leqslant\left(\int\limits_{\Omega_1}\big|F_n(\omega)\big|\,dP_1(\omega)\right)^{2/3}\left(\int\limits_{\Omega_1}\big|F_n(\omega)\big|^4\,dP_1(\omega)\right)^{1/3}.\end{aligned}$$

Hence, we have

$$\mathbb{E}\big(|F_n|^2\big)^{3/2} = \bigg(\int_{\Omega_1} \big|F_n(\omega)\big|^2 \, dP_1(\omega)\bigg)^{3/2} \leqslant \mathbb{E}\big(|F_n|\big)\mathbb{E}\big(|F_n|^4\big)^{1/2} \leqslant \frac{B_{k,4}^2}{k!}\mathbb{E}\big(|F_n|\big)\mathbb{E}\big(|F_n|^2\big),$$

which is $\mathbb{E}(|F_n|^2)^{1/2} \leqslant \frac{B_{k,4}^2}{k!}\mathbb{E}(|F_n|) \leqslant \frac{B_{k,4}^2}{k!}\mathbb{E}(|F_n|^p)^{1/p}$ for $p \geqslant 1$. □

Proof of Corollary 4. We can construct a sequence $\{X_i\}_{i\in\mathbb{N}}$ of independent standard complex gaussian variables, just as we did in the proof of Theorem 2, to obtain the identification $\{Z_i^{(r)}\}_{i\in\mathbb{N}} \rightsquigarrow \{X_{(i-1)k+r}\}_{i\in\mathbb{N}}$ for $r = 1, 2, \ldots, k$. Then, the multilinear random mapping G_n can be thought of, for suitable $\{a_{j_1,\ldots,j_k}\}_{j_s\geqslant 1}$, as the random polynomial

$$F_n = \sum_{j_1,\ldots,j_k=1}^{nk} a_{j_1,\ldots,j_k} X_{i_1} \cdots X_{i_k}.$$

Therefore, $\mathbb{E}(|F_n|^p)^{1/p} = \mathbb{E}(|G_n|^p)^{1/p}$ for any $1 \leqslant p < \infty$. Since $\{a_{j_1,\ldots,j_k}\}_{j_1,\ldots,j_k\in\mathbb{N}}$ is a symmetric sequence, we can apply Theorem 3 to conclude that $\mathbb{E}(|F_n|^p)^{1/p}$ is asymptotically equivalent to

$$\bigg(\sum_{j_1,\ldots,j_k=1}^{nk} |a_{j_1,\ldots,j_k}|^2\bigg)^{1/2} = \frac{1}{\sqrt{k!}}\bigg(\sum_{j_1,\ldots,j_k=1}^{n} |b_{j_1,\ldots,j_k}|^2\bigg)^{1/2},$$

and the result follows. □

In the proof of Theorem 6 we will use the following result, the proof of which is a simple exercise:

Lemma 24. *Given two independent random variables φ and X, φ a Steinhaus variable and X a standard complex gaussian variable, then $Y = \varphi|X|$ is a standard complex gaussian variable.*

Proof of Theorem 6. Following the ideas of Theorem 3, it is sufficient to prove that for $p \geqslant 2$, we have:

$$\bigg[\mathbb{E}\bigg(\bigg|\sum_{j_1,\ldots,j_k=1}^{n} a_{j_1,\ldots,i_k}\varphi_{j_1}\cdots\varphi_{j_k}\bigg|^p\bigg)\bigg]^{1/p} \leqslant \widetilde{B}_{k,p}\bigg(\sum_{j_1,\ldots,j_k=1}^{n} |a_{j_1,\ldots,j_k}|^2\bigg)^{1/2}.$$

From Theorem 3, we know that for every sequence $\{Z_i\}_{i\in\mathbb{N}}$ of independent standard complex gaussian variables,

$$\bigg[\mathbb{E}\bigg(\bigg|\sum_{j_1,\ldots,j_k=1}^{n} b_{j_1,\ldots,i_k} Z_{j_1}\cdots Z_{j_k}\bigg|^p\bigg)\bigg]^{1/p} \leqslant B_{k,p}\bigg(\sum_{j_1,\ldots,j_k=1}^{n} |b_{j_1,\ldots,i_k}|^2\bigg)^{1/2}.$$

Take $\{X_i\}_{i\in\mathbb{N}}$ a sequence of independent standard complex gaussian variables, which is independent from the sequence $\{\varphi_i\}_{i\in\mathbb{N}}$ of Steinhaus random variables. We will consider a new symmetric sequence defined by

$$b_{j_1,\dots,j_k} = \frac{a_{j_1,\dots,j_k}}{\mathbb{E}(|X_{j_1}|\cdots|X_{j_k}|)}.$$

From Lemma 24, we have $\varphi_i|X_i| \sim Z_i$ for all $i \geqslant 1$. So, letting $Q_n = \sum_{j_1,\dots,j_k=1}^n b_{j_1,\dots,j_k} \times Z_{j_1}\cdots Z_{j_k}$, we can compute

$$\begin{aligned}
\mathbb{E}\big(|Q_n|^p\big) &= \int_{\Omega_1}\int_{\Omega_2}\Bigg|\sum_{j_1,\dots,j_k=1}^n b_{j_1,\dots,j_k}\prod_{j=1}^k \varphi_{j_j}(w_1)\big|X_{j_j}(w_2)\big|\Bigg|^p d\mu_2(w_2)\,d\mu_1(w_1)\\
&\geqslant \int_{\Omega_1}\Bigg(\int_{\Omega_2}\Bigg|\sum_{j_1,\dots,j_k=1}^n b_{j_1,\dots,j_k}\prod_{j=1}^k \varphi_{j_j}(w_1)\big|X_{j_j}(w_2)\big|\Bigg|\,d\mu_2(w_2)\Bigg)^p d\mu_1(w_1)\\
&\geqslant \int_{\Omega_1}\Bigg|\int_{\Omega_2}\sum_{j_1,\dots,j_k=1}^n b_{j_1,\dots,j_k}\prod_{j=1}^k \varphi_{j_j}(w_1)\big|X_{j_j}(w_2)\big|\,d\mu_2(w_2)\Bigg|^p d\mu_1(w_1)\\
&= \int_{\Omega_1}\Bigg|\sum_{j_1,\dots,j_k=1}^n b_{j_1,\dots,j_k}\mathbb{E}\big(|X_{j_1}|\cdots|X_{j_k}|\big)\varphi_{j_1}(w_1)\cdots\varphi_{j_k}(w_1)\Bigg|^p d\mu_1(w_1)\\
&= \mathbb{E}\Bigg(\Bigg|\sum_{j_1,\dots,j_k=1}^n a_{j_1,\dots,j_k}\varphi_{j_1}\cdots\varphi_{j_k}\Bigg|^p\Bigg).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\Bigg(\Bigg|\sum_{j_1,\dots,j_k=1}^n a_{j_1,\dots,j_k}\varphi_{j_1}\cdots\varphi_{j_k}\Bigg|^p\Bigg)^{1/p}\\
&\quad\leqslant B_{k,p}\Bigg(\sum_{j_1,\dots,j_k=1}^n |b_{j_1,\dots,j_k}|^2\Bigg)^{1/2}\\
&\quad= B_{k,p}\Bigg(\sum_{j_1,\dots,j_k=1}^n \frac{|a_{j_1,\dots,j_k}|^2}{\mathbb{E}^2(|X_{j_1}|\cdots|X_{j_k}|)}\Bigg)^{1/2}\\
&\quad\leqslant \frac{B_{k,p}}{\min_{1\leqslant j_1,\dots,j_k\leqslant n}\mathbb{E}(|X_{j_1}|\cdots|X_{j_k}|)}\Bigg(\sum_{j_1,\dots,j_k=1}^n |a_{j_1,\dots,j_k}|^2\Bigg)^{1/2}. \qquad \square
\end{aligned}$$

Proof of Theorem 9. Let us write T in its spectral decomposition: $T(x) = \sum_j \lambda_j\langle x, e_j\rangle f_j$, for suitable orthonormal bases $(e_j)_j$ and $(f_j)_j$.

From the following iterated limit:

$$\lim_{N\to\infty}\Big(\lim_{k\to\infty} e^{-\frac{1}{N}\sum_{j=1}^{k}\lambda_j^2|x_j|^2}\Big) = \begin{cases} \lim_{N\to\infty} e^{-\frac{|||x|||^2}{N}} = 1, & x \in X_T, \\ \lim_{N\to\infty} 0 = 0, & x \notin X_T, \end{cases}$$

we have:

$$\lim_{N\to\infty}\Big(\lim_{k\to\infty} e^{-\frac{1}{N}\sum_{j=1}^{k}\lambda_j^2|x_j|^2}\Big) = \chi_{X_T}(x).$$

Fix $N \in \mathbb{N}$, the sequence of functions

$$g_{N,k} : \mathbb{K}^{\mathbb{N}} \to \mathbb{R}$$
$$g_{N,k}(x) = e^{-\frac{1}{N}\sum_{j=1}^{k}\lambda_j^2|x_j|^2}$$

converges to the function

$$g_N(x) = \begin{cases} e^{-\frac{|||x|||^2}{N}}, & x \in X_T, \\ 0, & x \notin X_T. \end{cases}$$

Since we have the bound $|e^{-\frac{1}{N}\sum_{j=1}^{k}\lambda_j^2|x_j|^2}| \leqslant 1$ for all $k \in \mathbb{N}$, applying the Lebesgue dominated convergence theorem, we conclude that:

$$\int_{\mathbb{K}^{\mathbb{N}}} g_N(x)\, d\mu(x) = \lim_{k\to\infty} \int_{\mathbb{K}^{\mathbb{N}}} g_{N,k}(x)\, d\mu(x).$$

Also, $\{g_N(x)\}_{N\in\mathbb{N}}$ is an increasing sequence of non-negative functions converging to $\chi_{X_T}(x)$. From the monotone convergence theorem we obtain:

$$\int_{\mathbb{K}^{\mathbb{N}}} \chi_{X_T}(x)\, d\mu(x) = \lim_{N\to\infty} \int_{\mathbb{K}^{\mathbb{N}}} g_N(x)\, d\mu(x) = \lim_{N\to\infty}\Big(\lim_{k\to\infty} \int_{\mathbb{K}^{\mathbb{N}}} g_{N,k}(x)\, d\mu(x)\Big).$$

Since $g_{N,k}$ are cylinder functions, these integrals are computed as

$$\lim_{N\to\infty}\Big(\lim_{k\to\infty} \int_{\mathbb{K}^k} g_{N,k}(x)\, d\mu_k(x)\Big) = \lim_{N\to\infty}\Big(\lim_{k\to\infty} \prod_{j=1}^{k} \int_{\mathbb{K}} e^{-\lambda_j^2|x_j|^2/N}\, d\mu_1(x_j)\Big).$$

Since $t \mapsto e^{-\lambda_j^2 t/N}$ is a convex function, from Jensen inequality we have

$$\int_{\mathbb{K}} e^{-\lambda_j^2|x_j|^2/N}\, d\mu_1(x_j) \geqslant e^{-\lambda_j^2 \int_{\mathbb{K}} |x_j|^2\, d\mu_1(x_j)/N} = e^{-\lambda_j^2\sigma^2/N}.$$

Therefore,

$$\lim_{N\to\infty}\left(\lim_{k\to\infty}\int\limits_{\mathbb{K}^k} g_{N,k}(x)\,d\mu_k(x)\right) \geqslant \lim_{N\to\infty}\left(\lim_{k\to\infty}\prod_{j=1}^{k} e^{-\lambda_j^2\sigma^2/N}\right) = 1.$$

Consequently, $\mu(X_T) = 1 = \mu(\mathbb{K}^{\mathbb{N}})$. $\square$

Proof of Theorem 10. First, we prove the equivalence between (i) and (ii). Let X_T be a standard full subspace such that $\sum_{i_1\leqslant N_1,\ldots,i_k\leqslant N_k} a_{i_1,\ldots,i_k} X_{i_1}\cdots X_{i_k}$ converges in X_T to a polynomial $\widetilde{P}$ as $N_1,\ldots,N_k \to \infty$. By the polynomial Banach–Steinhaus theorem, $\widetilde{P}$ is a continuous k-homogeneous polynomial on X_T.

Let us define on $\widetilde{T}(X_T) \subset \ell_2$ the following polynomial $g(\widetilde{T}x) = \widetilde{P}(x)$. Since

$$\bigl|g(y)\bigr| = \bigl|g(\widetilde{T}x)\bigr| = \bigl|\widetilde{P}(x)\bigr| \leqslant \|\widetilde{P}\| \|\!|x|\!\|^k = \|\widetilde{P}\| \|\widetilde{T}x\|_{\ell_2}^k = \|\widetilde{P}\| \|y\|_{\ell_2}^k,$$

we can continuously extend g to ℓ_2, preserving its norm.

Then, $\widetilde{P} = g \circ \widetilde{T}$, and we can define $P : \ell_2 \to \mathbb{C}$ by $P = g \circ \widetilde{T} \circ i$. Since $\widetilde{T} \circ i = T$ and the Hilbert–Schmidt operators on ℓ_2 are absolutely 2-summing, it follows that $P = g \circ T$ is 2-dominated.

Conversely, being $P : \ell_2 \to \mathbb{C}$ 2-dominated, by [17, Theorem 14], there exist a regular Borel probability measure μ on (B_{ℓ_2}, w^*) and a k-homogeneous polynomial $Q : L_2(\mu) \to \mathbb{C}$ such that the following diagram commutes:

$$\begin{tikzcd}
\ell_2 \arrow[r, "P"] \arrow[d, "i_X"'] & \mathbb{C} \\
C_{(B_{\ell_2}, w^*)} \arrow[r, "j_2"] & L_2(\mu) \arrow[u, "Q"']
\end{tikzcd}$$

The operator j_2 is absolutely 2-summing and so is $j_2 \circ i_X$. We can identify the image $j_2 \circ i_X(\ell_2) \subset L_2(\mu)$ with ℓ_2 and thus we have an injective Hilbert–Schmidt operator $T : \ell_2 \to \ell_2$ that verifies $P = Q|_{\ell_2} \circ T$. To see that the series converges in X_T we have to show that there exists a continuous k-homogeneous polynomial $\widetilde{P} : X_T \to \mathbb{C}$ that coincides with P in X_0. Let $\widetilde{P} = Q|_{\ell_2} \circ \widetilde{T}$. From the following inequality

$$\bigl|\widetilde{P}(x)\bigr| = \bigl|Q(\widetilde{T}x)\bigr| \leqslant \|Q\| \|\widetilde{T}x\|^k \leqslant \|Q\| \|\!|x|\!\|^k,$$

the result follows.

Using polarization formula it is clear that (i) implies (iii). That (iii) implies (iv) follows just as the implication from (i) to (ii). By [17, Theorem 6], (ii) and (iv) are equivalent, and this ends the proof. $\square$

Proof of Theorem 15. Since any r-dominated polynomials is also r'-dominated for any $r' > r$, we can assume $r \geqslant n$. Take $\{X_i\}_{i\in\mathbb{N}}$ a sequence of independent standard complex gaussian variables and put $Z^n = (X_1, \ldots, X_n, 0, \ldots)$. For each $n \in \mathbb{N}$ we can use Khintchine's inequality with $p = \frac{r}{k} \geqslant 1$ to obtain:

$$
\begin{aligned}
\left[\sum_{i_1,\dots,i_k=1}^{n} |a_{i_1,\dots,i_k}|^2\right]^{\frac{1}{2}} &\leqslant A_{k,p}\big[\mathbb{E}\big(\big|P\big(Z^n\big)\big|^p\big)\big]^{\frac{1}{p}} \\
&\leqslant A_{k,p}\|P\|_{r\text{-}dom}\left[\mathbb{E}\left[\left[\int\limits_{B_{E'}} \big|\langle x', Z^n\rangle\big|^r \,d\mu\left(x'\right)\right]^{\frac{kp}{r}}\right]\right]^{\frac{1}{p}} \\
&= A_{k,p}\|P\|_{r\text{-}dom}\left[\mathbb{E}\left(\int\limits_{B_{E'}} \big|\langle x', Z^n\rangle\big|^r \,d\mu\left(x'\right)\right)\right]^{\frac{1}{p}} \\
&= A_{k,p}\|P\|_{r\text{-}dom}\left[\int\limits_{B_{E'}} \mathbb{E}\big(\big|\langle x', Z^n\rangle\big|^r\big) \,d\mu\left(x'\right)\right]^{\frac{1}{p}} \\
&= A_{k,p}\|P\|_{r\text{-}dom}\mathbb{E}\big(|X_1|^r\big)^{\frac{1}{p}}\left[\int\limits_{B_{E'}} \big\|x'\big\|_{\ell_2}^r \,d\mu\left(x'\right)\right]^{\frac{1}{p}} \\
&\leqslant A_{k,p}\|P\|_{r\text{-}dom}\mathbb{E}\big(|X_1|^r\big)^{\frac{1}{p}}\big\|i' : E' \to \ell_2\big\|^{r/p}.
\end{aligned}
$$

Since the last bound is independent of n, the result follows. $\square$

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References

[1] Daniel Carando, Verónica Dimant, Extension of polynomials and John's theorem for symmetric tensor products, Proc. Amer. Math. Soc. 135 (6) (2007) 1769–1773 (electronic).

[2] Alejandro de Acosta, A decoupling inequality for multilinear functions of stable vectors, Probab. Math. Statist. 8 (1987) 71–76.

[3] Victor H. de la Peña, Decoupling inequalities: a second generation of martingale inequalities, in: Probability Towards 2000, New York, 1995, in: Lecture Notes in Statist., vol. 128, Springer, New York, 1998, pp. 151–164.

[4] Victor H. de la Peña, Stephen J. Montgomery-Smith, Jerzy Szulga, Contraction and decoupling inequalities for multilinear forms and U-statistics, Ann. Probab. 22 (4) (1994) 1745–1765.

[5] Andreas Defant, Klaus Floret, Tensor Norms and Operator Ideals, North-Holland Math. Stud., vol. 176, North-Holland, Amsterdam, 1993, xi, 566 pp.

[6] Seán Dineen, Complex Analysis on Infinite Dimensional Spaces, Springer Monogr. Math., Springer, London, 1999, xv, 543 pp., DM 179.00.

[7] Thomas A.W. Dwyer, Partial differential equations in generalized Fischer spaces for Hilbert–Schmidt holomorphy type, PhD thesis, University of Maryland, 1971.

[8] Victor Goodman, A divergence theorem for Hilbert space, Trans. Amer. Math. Soc. 164 (1972) 411–426.

[9] Leonard Gross, Abstract Wiener spaces, in: Proc. Fifth Berkeley Sympos. Math. Statist. and Probability, vol. II: Contributions to Probability Theory, Berkeley, CA, 1965/1966, Univ. California Press, Berkeley, CA, 1967, pp. 31–42 (Part 1).

[10] Hui Hsiung Kuo, Gaussian Measures in Banach Spaces, Lecture Notes in Math., vol. 463, Springer-Verlag, Berlin, 1975.

[11] Stanisław Kwapień, Decoupling inequalities for polynomial chaos, Ann. Probab. 15 (3) (1987) 1062–1071.
[12] Joram Lindenstrauss, Lior Tzafriri, Classical Banach Spaces I. Sequence Spaces, Ergeb. Math. Grenzgeb. (Results in Mathematics and Related Areas), vol. 92, Springer-Verlag, Berlin–Heidelberg–New York, 1977.
[13] Joram Lindenstrauss, Lior Tzafriri, Classical Banach Spaces II. Function Spaces, Ergeb. Math. Grenzgeb. (Results in Mathematics and Related Areas), vol. 97, Springer-Verlag, Berlin, 1979.
[14] Oleh V. Lopushansky, Andriy V. Zagorodnyuk, Hilbert spaces of analytic functions of infinitely many variables, Ann. Polon. Math. 81 (2) (2003) 111–122.
[15] Terry R. McConnell, Murad S. Taqqu, Double integration with respect to symmetric stable processes, Technical Report 618, Cornell University, 1984.
[16] Terry R. McConnell, Murad S. Taqqu, Decoupling inequalities for multilinear forms in independent symmetric random variables, Ann. Probab. 14 (3) (1986) 943–954.
[17] Yolanda Meléndez, Andrew Tonge, Polynomials and the Pietsch domination theorem, Math. Proc. R. Ir. Acad. Ser. A 99 (2) (1999) 195–212.
[18] Jorge Mujica, Complex Analysis in Banach Spaces. Holomorphic Functions and Domains of Holomorphy in Finite and Infinite Dimensions, North-Holland Math. Stud., vol. 120, Notas Mat., vol. 107, North-Holland, Amsterdam/New York/Oxford, 1986, xi, 434 pp.
[19] Damián Pinasco, Representable functions on the unit ball of a Banach space, Mediterr. J. Math., doi:10.1007/s00009-010-0093-5.
[20] Damián Pinasco, L^p-representable functions on Banach spaces, J. Math. Anal. Appl. 335 (1) (2007) 480–497.
[21] Damián Pinasco, Representación integral de funciones holomorfas en espacios de Banach, Tesis Doctoral, Universidad de Buenos Aires, Argentina, 2008.
[22] Damián Pinasco, Ignacio Zalduendo, Integral representation of holomorphic functions on Banach spaces, J. Math. Anal. Appl. 308 (1) (2005) 159–174.
[23] Peter Sjögren, On the convergence of bilinear and quadratic forms in independent random variables, Studia Math. 71 (3) (1981/1982) 285–296.