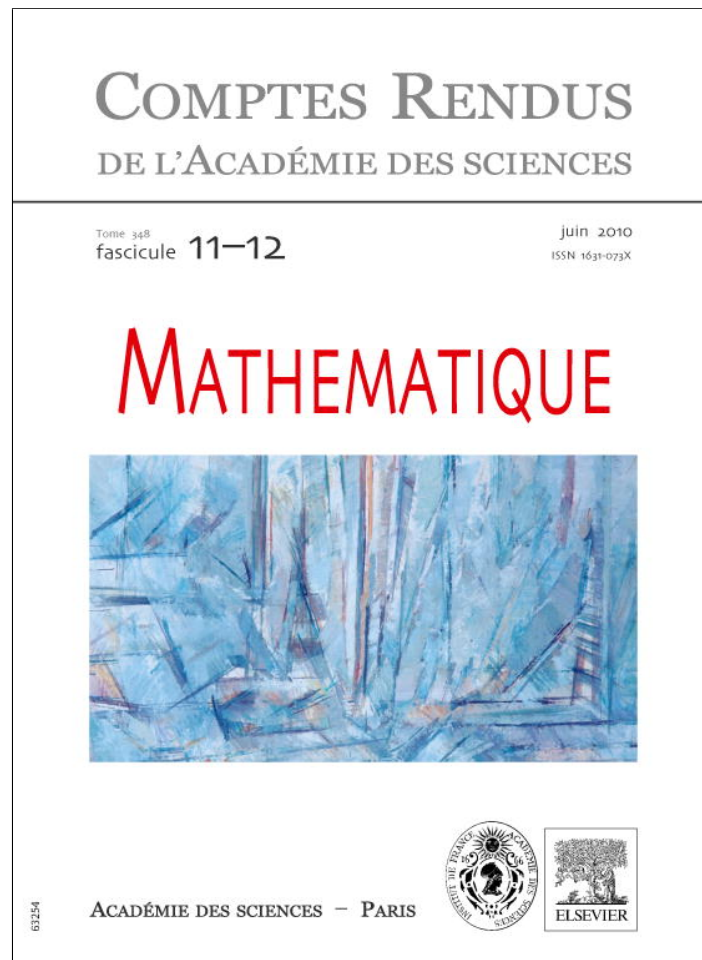


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Algebra/Group Theory

Pointed Hopf algebras over some sporadic simple groups <sup>☆</sup>*Algèbres de Hopf pointées sur quelques groupes simples sporadiques*N. Andruskiewitsch <sup>a</sup>, F. Fantino <sup>a</sup>, M. Graña <sup>b</sup>, L. Vendramin <sup>b,c</sup><sup>a</sup> Facultad de Matemática, Astronomía y Física, Universidad Nacional de Córdoba. CIEM – CONICET. Medina Allende s/n (5000) Ciudad Universitaria, Córdoba, Argentina<sup>b</sup> Departamento de Matemática – FCEyN, Universidad de Buenos Aires, Pab. I – Ciudad Universitaria (1428) Buenos Aires, Argentina<sup>c</sup> Instituto de Ciencias, Universidad de Gral. Sarmiento, J.M. Gutierrez 1150, Los Polvorines (1653), Buenos Aires, Argentina

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## ABSTRACT

Any finite-dimensional complex pointed Hopf algebra with group of group-likes isomorphic to a sporadic group, with the possible exception of the Fischer group  $Fi_{22}$ , the Baby Monster  $B$  and the Monster  $M$ , is a group algebra.

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## R É S U M É

Soit  $G$  un groupe sporadique différent du groupe de Fischer  $Fi_{22}$ , du bébé monstre  $B$  et du monstre  $M$ . Soit  $H$  une algèbre de Hopf complexe pointée de dimension finie dont le groupe des éléments dont le co-produit est égal au carré tensoriel est isomorphe à  $G$ , alors  $H$  est isomorphe à l'algèbre de groupe de  $G$ .

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## 1. Introduction

Let  $\mathbb{k}$  be an algebraically closed field of characteristic 0. In this Note, we announce a new contribution to the classification of finite-dimensional Hopf algebras over  $\mathbb{k}$ . As is known, different classes of finite-dimensional Hopf algebras have to be studied separately because the pertaining methods are radically different. There is a method for pointed Hopf algebras (those whose coradical is a group algebra  $\mathbb{k}G$ ) that has been applied with satisfactory results when  $G$  is Abelian [8]; an exposition of the method can be found in [7]. Recently, it appeared that many finite simple (or almost simple) groups  $G$  admit very few finite-dimensional, pointed Hopf algebras with coradical isomorphic to  $\mathbb{k}G$ :

- Any finite-dimensional complex pointed Hopf algebra with group of group-likes isomorphic to  $\mathbb{A}_m$ ,  $m \geq 5$ , is a group algebra [2].
- Same for the groups  $SL(2, 2^n)$ ,  $n > 1$  [10] and  $M_{20}$ ,  $M_{21} = PSL(3, 4)$  [11].
- Most of the pointed Hopf algebras over the symmetric groups have infinite dimension, with the exception of a short list of open possibilities, see [2,4] and references therein. More precisely, most of the irreducible Yetter–Drinfeld modules have infinite-dimensional Nichols algebras (see below).

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This is a report on finite-dimensional pointed Hopf algebras over sporadic simple groups. As part of our results, we have the following:

**Theorem 1.** *Let  $G$  be any sporadic simple group, different from the Fischer group  $Fi_{22}$ , the Baby Monster  $B$  and the Monster  $M$ . If  $H$  is a finite-dimensional pointed Hopf algebra with  $G(H) \simeq G$ , then  $H \simeq \mathbb{k}G$ .*

The Theorem holds more generally over any field of characteristic 0, since the property of being pointed is stable under extension of scalars.

### 1.1. Glossary

For the reader's convenience, we recall a few definitions that are central to our work. More information can be found in [5,7]. Let  $H$  be a Hopf algebra with comultiplication  $\Delta$  and bijective antipode  $S$ .

- An element  $g \neq 0$  in  $H$  is a *grouplike* if  $\Delta(g) = g \otimes g$ ; the set of all grouplikes is a group  $G(H)$  with multiplication given by the product of  $H$ .
- A *Yetter–Drinfeld module* over  $H$  is a left  $H$ -module  $M$  that bears also a structure  $\lambda : M \rightarrow H \otimes M$  of  $H$ -comodule, compatible with the action in an appropriate sense. If  $H$  is finite-dimensional, then a Yetter–Drinfeld module is the same as a module over the Drinfeld double of  $H$ . For instance, if  $H = \mathbb{k}G$  is the group algebra of a finite group  $G$ , then a Yetter–Drinfeld module over  $H$  is a left  $G$ -module  $M$  that bears also a  $G$ -graduation  $M = \bigoplus_{g \in G} M_g$ , compatibility meaning that  $h \cdot M_g = M_{hgh^{-1}}$  for all  $h, g \in G$ .
- A *rack* is a pair  $(X, \triangleright)$  where  $X$  is a non-empty set and  $\triangleright : X \times X \rightarrow X$  is an operation such that the map  $\varphi_x = x \triangleright \_$  is bijective for any  $x \in X$ , and  $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$  for all  $x, y, z \in X$ . A map  $q : X \times X \rightarrow GL(n, \mathbb{k})$  is a *2-cocycle* of degree  $n$  if

$$q_{x, y \triangleright z} q_{y, z} = q_{x \triangleright y, x \triangleright z} q_{x, z}, \quad \text{for all } x, y, z \in X.$$

- A *braided vector space* is a pair  $(V, c)$  where  $V$  is a vector space and  $c \in GL(V \otimes V)$  fulfills the braid equation:  $(c \otimes \text{id})(\text{id} \otimes c)(c \otimes \text{id}) = (\text{id} \otimes c)(c \otimes \text{id})(\text{id} \otimes c)$ . Examples:
  - (i) Any Yetter–Drinfeld module is a braided vector space in a natural way.
  - (ii) Let  $X$  be a finite rack,  $q$  a 2-cocycle of degree  $n$ ,  $V = \mathbb{k}X \otimes \mathbb{k}^n$ , where  $\mathbb{k}X$  is the vector space with basis  $e_x$ ,  $x \in X$ . We denote  $e_x v := e_x \otimes v$ . Consider the linear isomorphism  $c^q : V \otimes V \rightarrow V \otimes V$ ,  $c^q(e_x v \otimes e_y w) = e_{x \triangleright y} q_{x, y}(w) \otimes e_x v$ ,  $x, y \in X$ ,  $v, w \in \mathbb{k}^n$ . Then  $(V, c^q)$  is a braided vector space.

The braided vector spaces arising as Yetter–Drinfeld modules over group algebras of finite groups can be presented in terms of racks and cocycles, see a bit more of information below.

- We assume the reader familiar with the important notion of the *Nichols algebra* of a braided vector space, discussed at length in [7]. In short, one of the possible definitions of the Nichols algebra  $\mathfrak{B}(V)$  of a braided vector space  $(V, c)$  is as follows. Since  $c$  satisfies the braid equation, it induces a representation of the braid group  $\mathbb{B}_n$ ,  $\rho_n : \mathbb{B}_n \rightarrow GL(V^{\otimes n})$ , for each  $n \geq 2$ . Let  $Q_n = \sum_{\sigma \in \mathbb{S}_n} \rho_n(M(\sigma)) \in \text{End}(V^{\otimes n})$ , where  $M : \mathbb{S}_n \rightarrow \mathbb{B}_n$  is the so-called Matsumoto section (not a morphism of groups, but preserves product when length is preserved). Then  $\mathfrak{B}(V)$  is the quotient of the tensor algebra  $T(V)$  by  $\bigoplus_{n \geq 2} \ker Q_n$ , in fact a 2-sided ideal of  $T(V)$ . If  $c$  is the usual switch, then  $\mathfrak{B}(V)$  is just the symmetric algebra of  $V$ ; but in general the determination of a Nichols algebra is quite a difficult task.

## 2. Outline of the proof

A complete proof of Theorem 1 for the groups  $M_{22}$  and  $M_{24}$  is contained in [9]; the proof for the other groups is included in [3].

We sketch now the proof in two main reductions. The first one has been explained in several places, with detail in [7], but we include a brief summary for completeness. We remind that if  $U$  is a braided vector subspace of  $V$ , then  $\mathfrak{B}(U) \hookrightarrow \mathfrak{B}(V)$ .

### 2.1. A general reduction

Let  $G$  be a finite group,  $H$  a pointed Hopf algebra with  $G(H) \simeq G$ . Then there are two basic invariants of  $H$ , a Yetter–Drinfeld module  $V$  over  $\mathbb{k}G$  (called the infinitesimal braiding of  $H$ ) and its Nichols algebra  $\mathfrak{B}(V)$ . We have  $|G| \dim \mathfrak{B}(V) \leq \dim H$ . Therefore, the following statements are equivalent:

- (1) If  $H$  is a finite-dimensional pointed Hopf algebra with  $G(H) \simeq G$ , then  $H \simeq \mathbb{k}G$ .
- (2) If  $V \neq 0$  is a Yetter–Drinfeld module over  $\mathbb{k}G$ , then  $\dim \mathfrak{B}(V) = \infty$ .
- (3) If  $V$  is an *irreducible* Yetter–Drinfeld module over  $\mathbb{k}G$ , then  $\dim \mathfrak{B}(V) = \infty$ .

## 2.2. Looking at subracks

We focus on (3) above. The second reduction has been the basis of our recent papers. It starts from the well-known classification of irreducible Yetter–Drinfeld modules over  $\mathbb{k}G$  by pairs  $(\mathcal{O}, \rho)$ , where  $\mathcal{O}$  is a conjugacy class in  $G$  and  $\rho$  is an irreducible representation of the stabilizer  $G^s$  of a fixed point  $s \in \mathcal{O}$ . Now, the definition of the Nichols algebra  $\mathfrak{B}(\mathcal{O}, \rho)$  of the corresponding Yetter–Drinfeld module  $M(\mathcal{O}, \rho)$  just depends on the braiding. If  $\dim \rho = 1$ , then this braiding depends only on the rack  $\mathcal{O}$  and a 2-cocycle  $q : \mathcal{O} \times \mathcal{O} \rightarrow \mathbb{k}^\times$  [5]. Namely,  $\mathcal{O}$  is a rack with the product  $x \triangleright y := xyx^{-1}$ ,  $M(\mathcal{O}, \rho)$  has a natural basis  $(e_x)_{x \in \mathcal{O}}$  and the braiding is given by  $c(e_x \otimes e_y) = q_{xy} e_{x \triangleright y} \otimes e_x$ . If there exists a subrack  $X$  of  $\mathcal{O}$  such that the Nichols algebra of the braided vector space defined by  $X$  and the restriction of  $q$  is infinite dimensional, then  $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$ .

We recall some examples of racks which are relevant in this work.

- (i) Abelian racks: those racks  $X$  such that  $x \triangleright y = y$  for all  $x, y \in X$ .
- (ii)  $\mathcal{D}_p$ : the class of involutions in the dihedral group  $\mathbb{D}_p$  (of order  $2p$ ),  $p$  a prime.
- (iii)  $\mathcal{D}$ : the class of 4-cycles in  $\mathbb{S}_4$ .
- (iv) Doubles of racks: if  $X$  is a rack, then  $X^{(2)}$  denotes the disjoint union of two copies of  $X$  each acting on the other by left multiplication.

We are interested in finding subracks which are Abelian, or isomorphic to  $\mathcal{D}_p^{(2)}$  or to  $\mathcal{D}^{(2)}$ , by the following reasons:

- (A) If  $X$  is Abelian, then the corresponding braided vector space is of diagonal type. Braided vector spaces of diagonal type with finite-dimensional Nichols algebra where classified in [13]; thus, we just need to check if the matrix  $(q_{xy})$  belongs or not to the list in [13].
- (B) If  $X$  is isomorphic either to  $\mathcal{D}_p^{(2)}$  or to  $\mathcal{D}^{(2)}$ , then for some specific cocycles, the related Nichols algebras have infinite dimension [6, Ths. 4.7, 4.8].

Variations:

- (a) If  $\dim \rho > 1$ , similar arguments apply.
- (b) Sometimes the rack  $X$  is not Abelian, but the braided vector space produced by  $X$  and the 2-cocycle can be realized with an Abelian rack, by a suitable change of basis.
- (c) Let  $F < G$  be a subgroup,  $s \in F$ ,  $\mathcal{O}^F$ , resp.  $\mathcal{O}^G$  the conjugacy class of  $s$  in  $F$ , resp. in  $G$ . If  $\dim \mathfrak{B}(\mathcal{O}^F, \tau) = \infty$  for any irreducible representation  $\tau$  of  $F^s$ , then  $\dim \mathfrak{B}(\mathcal{O}^G, \rho) = \infty$  for any irreducible representation  $\rho$  of  $G^s$ .
- (d) A conjugacy class  $\mathcal{O}$  is real if  $\mathcal{O} = \mathcal{O}^{-1}$ . It is quasireal if  $\mathcal{O} = \mathcal{O}^m$  for some integer  $m$ ,  $1 < m < N$ , where  $N$  is the order of the elements in  $\mathcal{O}$ . The search of subracks isomorphic to  $\mathcal{D}_p^{(2)}$  or to  $\mathcal{D}^{(2)}$ , as well as the verification that the restriction of the cocycle  $q$  is as needed in (2.2), is greatly simplified in a real (quasireal) conjugacy class [1].
- (e) We say that a rack  $X$  is of type D if there exists a decomposable subrack  $Y = R \amalg S$  of  $X$  such that  $r \triangleright (s \triangleright (r \triangleright s)) \neq s$ , for some  $r \in R$ ,  $s \in S$ . If a conjugacy class  $\mathcal{O}$  is a rack of type D, then  $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$  for any  $\rho$  (see [2] and Theorem 8.6 of [14]).

## 2.3. Computations

We now fix a sporadic group  $G$  as in Theorem 1. We extracted relevant information from the ATLAS [15] with the `AtlasRep` package [16]. Then, we checked when a conjugacy class is real or quasireal or of type D. We used GAP [12] for the computations.

These tools allow us to apply the techniques sketched above to all pairs  $(\mathcal{O}, \rho)$  and establish the validity of (2.1).

## 2.4.

Some of these results were announced in several meetings:

- Hopf Algebras and Related Topics, A conference in honor of Professor Susan Montgomery. University of Southern California, Los Angeles, USA. February 2009.
- IV Encuentro Nacional de Álgebra, Córdoba, Argentina. August, 2008.
- First De Brún Workshop on Computational Algebra, National University of Ireland, Galway, Ireland. August, 2008
- Groupes quantiques dynamiques et catégories de fusion. CIRM, Luminy, France. April 2008.

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