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An Integer Linear Programming Formulation and Branch-and-Cut Algorithm for the Capacitated m-Ring-Star Problem

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Abstract

We study the capacitated m-ring-star problem (*CmRSP*) that faces the design of minimum cost network structure that connects customers with m rings using a set of ring connections that share a distinguished node (*depot*), and optionally star connections that connect customers to ring nodes. Ring and star connections have some associated costs. Also, rings can include transit nodes, named *Steiner* nodes, to reduce the total network cost if possible. The number of customers in each ring-star (ring's customers and customer connected to it through star connections) have an upper bound (*capacity*).

These kind of networks are appropriate in optical fiber urban environments. *CmRSP* is know to be NP-Hard. In this paper we propose an integer linear programming formulation and a branch-and-cut algorithm.

Keywords: network design, m-ring-star, branch-and-cut, integer programming.

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1 Introduction

The Capacitated m-Ring-Star Problem ($CmRSP$) consists in design of a minimum cost network to connect a set of customers using: a set of m rings (cycles) containing a distinguished node (depot) shared by all of them and connections between a customer that does not belong to any ring and a ring through star connections. Rings may include transit nodes (Steiner nodes) so that star connections can be established between a customer and a ring through a transit node. Each ring and its star connections (ring-star) can contain at most Q customers. In optical fiber networks each customer in a ring receives a pair of fibers to send/receive data in a clockwise and counter-clockwise way, so if a ring connection fails, the network does not lose connectivity. Star connections are used between nearby nodes, so the repairing cost time is reasonable with respect to the ring connections' problems. $CmRSP$ was introduced by Baldacci, Dell'Amico and Salazar González in [1] where they propose exact resolution approachs based on integer linear models and branch-and-cut algorithms. In [3] is presented a resolution based on a metaheuristic approach, and [2] propose an exact resolution approach based on integer linear model and column generation algorithms.

2 Model

We consider customers denoted by $U = \{u_1, ..., u_{|U|}\}$, $W = \{w_1, ..., w_{|W|}\}$ the transit nodes, and the depot is represented by d_0 and d_1 (copy of d_0). The rings are represented with directed paths between d_0 and d_1. Ring and star connection costs are denoted by $c_{vv'}$ for $v \in V \cup \{d_0\}$, $v' \in V \cup \{d_1\}$ and d_{uv} for $u \in U$, $v \in V$ respectively with $V = U \cup W$. Variables defined for the model are: $\forall v \in V \cup \{d_0\} \quad \forall v' \in V \cup \{d_1\}$: $x_{vv'} \in \{0, 1\}$, $x_{vv'} = 1$ if v and v' are directly connected through a ring connection, otherwise $x_{vv'} = 0$. We also define $\forall u \in U, \forall v \in V$: $y_{uv} = 1$ when u is directly connected to v through a star connection, in other case $y_{uv} = 0$, and we use $\forall v \in V$: f_v for the number of customers visited in a directed path from d_0 to v in a ring-star (including star connections), $f_v \in Z$. Next we show a new formulation for the $CmRSP$.

$$\min \sum_{v \in V \cup \{d_0\}} \sum_{v' \in V \cup \{d_1\}, v' \neq v} c_{vv'} x_{vv'} + \sum_{u \in U} \sum_{v \in V, v \neq u} d_{uv} y_{uv}$$

subject to

(i) $\sum_{v \in V} x_{d_0 v} = m$

(ii) $\sum_{v \in V} x_{vd_1} = m$

(iii) $\sum_{v \in V \cup \{d_0\}, v \neq u} x_{vu} + \sum_{v \in V, v \neq u} y_{uv} = 1 \quad \forall u \in U$

(iv) $\sum_{v \in V \cup \{d_0\}, v \neq w} x_{vw} \leq 1 \quad \forall w \in W$

(v) $\sum_{v' \in V \cup \{d_0\}, v' \neq v} x_{v'v} = \sum_{v' \in V \cup \{d_1\}, v' \neq v} x_{vv'} \quad \forall v \in V$

(vi) $y_{uv} \leq \sum_{v' \in V \cup \{d_0\}, v' \neq v} x_{v'v} \quad \forall u \in U \quad \forall v \in V \quad u \neq v$

(vii) $f_u \geq 1 + \sum_{u' \in U, u' \neq u} y_{u'u} \quad \forall u \in U$

(viii) $f_w \geq \sum_{u \in U} y_{uw} \quad \forall w \in W$

(ix) $f_u \geq (f_v + 1 + \sum_{u' \in U, u' \neq u} y_{u'u}) - |U|(1 - x_{vu}) \quad \forall u \in U \quad \forall v \in V \quad u \neq v$

(x) $f_w \geq (f_v + \sum_{u \in U} y_{uw}) - |U|(1 - x_{vw}) \quad \forall w \in W \quad \forall v \in V \quad w \neq v$

(xi) $1 \leq f_v \leq Q \quad \forall v \in V$

Constraints (i) and (ii) ensure that m arcs have an origin in d_0 and m arcs arrive to d_1 respectively, and (iii) assert that every customer must belong either to some (unique) ring or be connected to one of them using a star connection. Constraints (iv) allow Steiner nodes to be part of a ring, and (v) ensure that every node has an input arc in a ring if and only if it also has an output arc. With (vi) constraints we ensure that a star connection between a customer u and node v is possible if v belongs to a ring. Constraints (vii), (viii), (ix) and (x) limit the number of customers to Q in each ring. Constraints (vii) and (viii) bound every node beyond the position in the ring-star. These are necessary for single node rings, while (ix) and (x) increase the number of customers visited from d_0 considering the current node (if it is a customer), the number of customers visited in the previous node and the star connections to u or w respectively.

Constraints (vii), (viii), (ix) and (x) also ensure subtour elimination that includes a customer. Subtours composed by only steiner nodes can be part of feasible solutions. Since by eliminating Steiner nodes subtours we can get a feasible solution with lower objective value, such feasible solutions are not minimum. However including the following constraints using $z \in Z^{|U|+|W|}$ variables we can exclude every subtour improving in some cases the computational performance.

(i) $z_{v'} \geq (z_v + 1) - |V|(1 - x_{vv'}) \quad \forall v, v' \in V \quad v \neq v'$

(ii) $1 \leq z_v \leq |V| \quad \forall v \in V$

3 Polyhedral results

In order to develop a branch-and-cut algorithm, we propose specific families of valid inequalities for the formulation given in Section 2. The Propositions 3.1, 3.2, 3.3 and 3.4 inequalities are based on a similar idea presented in [4] and are also considered in [1], adapted to the formulation presented in the Model 2. Due to space limitations, proofs are not presented. We define $U(S) = U \cap S$.

Proposition 3.1 *Capacity inequalities*

$$\forall S \subseteq V\colon \sum_{v \in S} \sum_{v' \in V \cup \{d_1\} \setminus S} x_{vv'} + \sum_{u \in U(S)} \sum_{v \in V \setminus S} y_{uv} \geq \frac{1}{Q} \Big(\sum_{v \in S} \sum_{u \in U(S)} x_{vu} + \sum_{u \in U} \sum_{v \in S} y_{uv} \Big)$$

are valid inequalities.

Proposition 3.2 *Connectivity inequalities*

$$\forall S \subseteq V, \forall u \in U(S), |U(S)| \geq 1\colon \sum_{v \in S} \sum_{v' \in V \cup \{d_1\} \setminus S} x_{vv'} \geq \sum_{v \in V \cup \{d_0\}} x_{vu} + \sum_{v \in S} y_{uv}$$

are valid inequalities.

Proposition 3.3 *Multistar inequalities*

$$\forall S \subseteq V\colon \sum_{v \in S} \sum_{v' \in V \cup \{d_1\} \setminus S} x_{vv'} + \sum_{u \in U(S)} \sum_{v \in V \setminus S} y_{uv} \geq \frac{1}{Q} \Big(\sum_{v \in S} \sum_{u \in U(S)} x_{vu} + \sum_{u \in U} \sum_{v \in S} y_{uv} + \sum_{u \in U \cup \{d_0\} \setminus S} \sum_{v \in S} x_{uv} \Big)$$

are valid inequalities.

Proposition 3.4 *Capacity bounded by constant*

$$\forall S \subseteq V, K = \Big\lceil \frac{|U(S)|}{Q} \Big\rceil\colon \sum_{v \in S} \sum_{v' \in V \cup \{d_1\} \setminus S} x_{vv'} + \sum_{u \in U(S)} \sum_{v \in V \setminus S} y_{uv} \geq K$$

are valid inequalities.

Proposition 3.5 *Customers' sum inequalities*

$$\forall u, u' \in U \quad u \neq u'\colon x_{u'u} + y_{u'u} \leq x_{ud_1} + \sum_{v' \in V, v' \neq u, v' \neq u'} x_{uv'} + y_{uv'}$$

are valid inequalities.

4 Solution methodology

We develop a branch-and-cut algorithm based on LP relaxations. To reduce the number of nodes explored, lower bounds are strengthend with valid inequalities (cutting planes) using separation procedures. Different strategies to explore the tree and variable selection criteria were analyzed. We present here different aspects of our branch-and-cut algorithm:

- *Initial heuristic:* we consider a two-phase method. In the first phase, an heuristic chooses m intial customers. The second phase is a greedy algorithm that builds the m rings (starting with the m initial customers) introducing a new customer iteratively considering some ring and star connections.
- *Separation routines:* For the family of valid inequalities in Proposition 3.5 the corresponding separation problem is tackled through direct enumeration. For family in Propositions 3.1, 3.2, 3.3, 3.4 we use a greedy heuristic as the separation procedure.
- *Branching strategies:* we tested various branching strategies that the CPLEX package offers. The best configuration for the instances tested in our formulation that we have found is using best-bound and reduced pseudo-costs for node and selection criteria respectively.

5 Computational results

Our computational experiments were conducted on a SUN UltraSparc III (CPU of 1Ghz, RAM of 2GB with SunOS 5.9). The algorithms were coded in C++, and compiled using *GNU gcc* version 3.4.6 compiler. The code was linked to CPLEX 10.1 optimization routines. We consider some instances evaluated in [1], [2] and [3]. Those instances are based on TSPLIB with 26 and 51 nodes, taking $\lfloor \alpha(n-1) \rfloor$ nodes as customers and the remaining as steiners for $\alpha \in \{0.25, 0.50, 0.75, 1.0\}$ for $m \in \{3, 4, 5\}$. We can see in Table 1 that BC outperforms CPLEX considerably in almost all instances considered. Each line shows instances parameters, the number of nodes explored nod_{CPX} and nod_{BC}, the computational time t_{CPX} and t_{BC} (* for unsolved instances within 1800 seconds) and the %gap gap_{CPX} and gap_{BC} for CPX (using best-bound and reduced pseudo-costs for node and variable selection criteria respectively, and the default cuts provided by CPLEX) and BC algorithm respectively.

6 Conclusions

In this paper we introduce a new formulation for the Capacitated m-Ring-Star Problem. We analyze computational results with different strategies related to branch-and-bound tree construction and exploration and propose valid inequalities and separation procedures to develop a branch-and-cut algorithm in order to evaluate them. Finally we evaluate CPLEX and our algorithm on different known instances and we get favorable results for our BC algorithm. As future research, it would be interesting to investigate about a primal heuristic, separation procedures alternatives for the exponential size families and analyze different variable and node selection criteria specific for this formulation.

n	$\|U\|$	$\|W\|$	m	Q	nod_{CPX}	t_{CPX}	$\%gap_{CPX}$	nod_{BC}	t_{BC}	$\%gap_{BC}$
26	12	13	3	5	26600	143	0.0	954	14	0.0
26	12	13	4	4	25000	126	0.0	192	4	0.0
26	12	13	5	3	61500	356	0.0	230	4	0.0
26	18	7	3	7	112900	*	14.37	3660	77	0.0
26	18	7	4	5	99800	*	19.32	5341	111	0.0
26	18	7	5	4	107600	*	20.0	15357	361	0.0
26	25	0	3	10	111100	*	6.71	1373	26	0.0
26	25	0	4	7	91100	*	12.89	2731	61	0.0
26	25	0	5	6	87600	*	12.19	2351	51	0.0
51	12	38	3	5	112100	*	12.26	14936	1290	0.0
51	12	38	4	4	113100	*	12.98	5180	542	0.0
51	12	38	5	3	112100	*	14.70	6150	776	0.0
51	25	25	3	10	110800	*	15.48	14287	*	2.73
51	25	25	4	7	109200	*	15.76	11348	*	3.06
51	25	25	5	6	111900	*	15.92	10842	*	8.95

Table 1: Computational times (in seconds) with $n \in \{26, 51\}$ ($n = |U| + |W| + 1$)

References

[1] R. Baldacci, M. Dell'Amico, J. Salazar González. *The Capacitated m-Ring-Star Problem*, Operations Research Vol. 55, No. 6, November-December 2007, pp. 1147-1162, INFORMS.

[2] E. Hoshino, C. Souza. *Column Generation Algorithms for the Capacitated m-Ring-Star Problem*, Lecture Notes in Computer Science, Springer Berlin / Heidelberg, Volume 5092/2008, June 2008.

[3] A. Mauttone, S. Nesmachnow, A. Olivera, F. Robledo. *A hybrid metaheuristic algorithm to solve the Capacitated m-Ring Star Problem*, 2007, International Network Optimization Conference.

[4] P. Toth, D. Vigo. *The Vehicle Routing Problem*, SIAM Monographs on Discrete Mathematics and Applications, Philadelphia, 2002.