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The *logbook* of Pointed Hopf algebras over the sporadic simple groups ☆

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ABSTRACT

In this paper we give details of the proofs performed with GAP of the theorems of our paper [N. Andruskiewitsch, F. Fantino, M. Graña, L. Vendramin, Pointed Hopf algebras over the sporadic simple groups, J. Algebra 325 (1) (2011) 305–320].

Contents

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Introduction

In these notes, we give details on the proofs performed with [GAP,B] of the theorems of our paper [AFGV2]. Here we discuss the algorithms implemented for studying Nichols algebras over non-abelian groups. For each group we refer to files where the results of our computations are shown. These logs files can be downloaded from our webpages:

http://www.mate.uncor.edu/~fantino/afgv-sporadic
http://mate.dm.uba.ar/~lvendram/afgv-sporadic

We believe, and hope, that the details in this work are enough to guide the reader to repeat and corroborate our calculations.

Throughout the paper, we follow the notations and conventions of [AFGV2]. We write ATLAS for any of the references [CC+,WPN+,WWT+].

1. Algorithms

In this section, we explain our algorithms to implement the techniques presented in [AFGV2, Section 2.2].

1.1. Algorithms for type D

Algorithm 1 checks if a given conjugacy class $\mathcal{O}_r^G$ of a finite group G is of type D.

Notice that in Algorithm 1 we need to run over all the conjugacy class $\mathcal{O}_r^G$ to look for the element s such that the conditions of [AFGV1, Prop. 3.4] are satisfied. This is not always an easy task. To avoid this problem, we have *the random variation* of Algorithm 1, that is our Algorithm 2. The key is to pick randomly an element x in the group G and to check if r and $s = xrx^{-1}$ satisfy the conditions of

Algorithm 1: Type D

```
for s ∈ O_r^G do
    if (rs)^2 ≠ (sr)^2 then
        Compute the group H = ⟨r, s⟩
        if O_r^H ∩ O_s^H = ∅ then
            return true                        /* the class is of type D */
        end
    end
end
return false                                   /* the class is not of type D */
```

Algorithm 2: Random variation of Algorithm 1

```
forall i: 1 ⩽ i ⩽ N do                         /* the number of iterations */
    x ∈ G;                                     /* randomly chosen */
    s = xrx^-1
    if (rs)^2 ≠ (sr)^2 then
        Compute the group H = ⟨r, s⟩
        if O_r^H ∩ O_s^H = ∅ then
            return true                        /* the class is of type D */
        end
    end
end
return false                                   /* the class is not of type D */
```

[AFGV1, Prop. 3.4]; if not we repeat the process N times, where N is fixed. This naive variation of Algorithm 1 turns out to be very powerful and allows us to study big sporadic groups such as the Janko group J_4 or the Fischer group Fi'_{24}.

For large groups, it is more economical to implement Algorithms 1 and 2 in a recursive way. Let G be a finite group represented faithfully, for example, as a permutation group or inside a matrix group over a finite field. We compute $\mathcal{O}^G_{g_1}, \dots, \mathcal{O}^G_{g_n}$, the set of conjugacy classes of G. To decide if these conjugacy classes are of type D, we restrict the computations to be done inside a nice subgroup of G.

Assume that the list of all maximal subgroups of G, up to conjugacy, is known, namely $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_k$, with non-decreasing order. Also assume that it is possible to restrict our good representation of G to every maximal subgroup $\mathcal{M}_i$. Here we say that a representation is *good* if it allows us to perform our computations in a reasonable time.

Fix $i \in \{1, \dots, k\}$. Let $h \in M_i$ and let $\mathcal{O}^{\mathcal{M}_i}_h$ be the conjugacy class of h in $\mathcal{M}_i$. Since $\mathcal{M}_i$ is a subgroup of G, the element h belongs to a conjugacy class of G, say $\mathcal{O}^G_h$. So, if the class $\mathcal{O}^{\mathcal{M}_i}_h$ is of type D, then the class $\mathcal{O}^G_h$ is of type D too.

Notice that to implement Algorithm 3 we need to have not only a good representation for the group G, but we need to know how to restrict the good representation of G to all its maximal subgroups. This information appears in the ATLAS for many of the sporadic simple groups. So, using the GAP interface to the ATLAS we could implement Algorithm 3 for the sporadic simple groups.

1.2. Structure constants

Some conjugacy classes of involutions are studied with [AFGV2, Prop. 1.7, Eq. (1.4)]. For example, in the proof of Theorem 2.2 we claim that the conjugacy class 2A of $L_5(2)$ gives only infinite-dimensional Nichols algebras. This follows from [AFGV2, Prop. 1.8] because $S(\text{2A}, \text{3A}, \text{3A}) = 42$.

```
gap> ct := CharacterTable("L5(2)");;
gap> ClassNames(ct);;
gap> ClassMultiplicationCoefficient(ct, ct.2a, ct.3a, ct.3a);
42
```

Algorithm 3: Type D: Using maximal subgroups

$\mathcal{O}^G_{g_1}, \dots, \mathcal{O}^G_{g_n}$ is the set of conjugacy classes of G
$S = \{1, 2, \dots, n\}$

foreach *maximal subgroup* $\mathcal{M}$ **do**
  Compute $\mathcal{O}^{\mathcal{M}}_{h_1}, \dots, \mathcal{O}^{\mathcal{M}}_{h_m}$, the set of conjugacy classes of $\mathcal{M}$
  foreach i: $1 \leqslant i \leqslant m$ **do**
    Identify $\mathcal{O}^{\mathcal{M}}_{h_i}$ with a conjugacy class in G: $h_i \in \mathcal{O}^G_{g_{\sigma(i)}}$
    if $\sigma(i) \in S$ **then**
      if $\mathcal{O}^{\mathcal{M}}_{h_i}$ *is of type D* **then**
        Remove $\sigma(i)$ from S
        if $S = \emptyset$ **then**
          return true /* the group is of type D */
        end
      end
    end
  end
end
foreach $j \in S$ **do**
  if $\mathcal{O}^G_{g_j}$ *is of type D* **then**
    Remove s from S
    if $S = \emptyset$ **then**
      return true /* the group is of type D */
    end
  end
end
return S /* conjugacy classes not of type D */

1.3. *Bases for permutation groups*

Let G be a group acting on a set X. A subset B of X is called a *base* for G if the identity is the only element of G which fixes every element in B, see [DM, Section 3.3]. In other words,

$$\{g \in G \mid g \cdot b = b, \text{ for all } b \in B\} = 1.$$

Lemma 1.1. *Let G be a group acting on a set X. Let B be a subset of X. The following are equivalent*:

(1) *B is a base for G.*
(2) *For all $g, h \in G$ we have*: *$g \cdot b = h \cdot b$ for all $b \in B$ implies $g = h$.*

Proof. If B is a base, then $g \cdot b = h \cdot b \Rightarrow (h^{-1}g) \cdot b = b \Rightarrow h^{-1}g = 1 \Rightarrow h = g$. The converse is trivial. □

Let G be a permutation group. With the GAP function `BaseOfGroup` we compute a base for G. We use `OnTuples` to encode a permutation and `RepresentativeAction` to decode the information. This enables us to reduce the size of our log files.

1.4. *An algorithm for involutions*

We describe here an algorithm used to discard some conjugacy classes of involutions. In this work, we use this algorithm for the classes called 2A in $O_7(3)$, $S_6(2)$, $S_8(2)$, Co_2, Fi_{22}, Fi_{23} and B.

Let $\mathcal{O}$ be one of the classes 2A in $O_7(3)$, $S_6(2)$, $S_8(2)$ or Co_2, and $g \in \mathcal{O}$. It is enough to consider the irreducible representations ρ of the corresponding centralizer such that $\rho(g) = -1$, see [AFGV2, (1.3)]. For these conjugacy classes, the remaining representations ρ satisfy $\deg \rho > 4$, as can be seen from the character tables. By [AFGV2, Lemma 1.3], we are reduced to find an involution x such that $gh = hg$, for $h = xgx^{-1}$, and to compute the multiplicities of the eigenvalues of $\rho(h)$. For this last task,

Table 1
Classes not of type D in some alternating and symmetric groups.

Group	Not of type D	Log file
$\mathbb{A}_9$	(1 2 3)	`A9/A9.log`
$\mathbb{A}_{11}$	(1 2 3) (1 2 3 4 5 6 7 8 9 10 11) (1 2 3 4 5 6 7 8 9 11 10)	`A11/A11.log`
$\mathbb{A}_{12}$	(1 2 3) (1 2 3 4 5 6 7 8 9 10 11) (1 2 3 4 5 6 7 8 9 10 12)	`A12/A12.log`
$\mathbb{S}_{12}$	(1 2) (1 2 3)	`S12/S12.log`

we use [AFGV2, Remark 1.4]. See the proofs of Theorems 2.4, 2.13 and Lemma 2.15, for the classes 2A of $O_7(3)$, 2A of $S_6(2)$ and 2A of $S_8(2)$, respectively, and the file `Co2/2A.log` for 2A of Co_2.

Let $\mathcal{O}$ be the class 2A of Fi_{22}; it has 3510 elements. Let ρ be an irreducible representation of the corresponding centralizer. The group $O_7(3)$ is a maximal subgroup of Fi_{22} and the class 2A of $O_7(3)$ is contained in the class 2A of Fi_{22} – see the file `Fi22/2A.log`. By [AFGV2, Lemma 1.5], $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.

Let $\mathcal{O}$ be the class 2A of Fi_{23}. Let ρ be an irreducible representation of the corresponding centralizer. The group $S_8(2)$ is a maximal subgroup of Fi_{23} and the class 2A of $S_8(2)$ is contained in the class 2A of Fi_{23}. By [AFGV2, Lemma 1.5], $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.

Let $\mathcal{O}$ be the class 2A of B. Let ρ be an irreducible representation of the corresponding centralizer. The group Fi_{23} is a maximal subgroup of B and the class 2A of Fi_{23} is contained in the class 2A of B – see the file `B/step2.log`. By [AFGV2, Lemma 1.5], $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.

2. Some auxiliary groups

In this section we study some groups that appear as subquotients of some sporadic simple groups.

2.1. The groups $\mathbb{A}_9$, $\mathbb{A}_{11}$, $\mathbb{A}_{12}$, $\mathbb{S}_{12}$

In [AFGV1], we classify the conjugacy classes of type D in alternating and symmetric groups. In Table 1, we list all non-trivial permutations such that their conjugacy classes are not of type D for the groups $\mathbb{A}_9$, $\mathbb{A}_{11}$, $\mathbb{A}_{12}$, $\mathbb{S}_{12}$.

2.2. The group $L_5(2)$

This group has order 9 999 360. It has 27 conjugacy classes. To study Nichols algebras over this group we use the representation inside $\mathbb{S}_{31}$ given in the ATLAS.

Lemma 2.1. *Every non-trivial conjugacy class of* $L_5(2)$*, except* 2A *and those with representatives of order* 31*, is of type D.*

Proof. We perform Algorithm 1, see the file `L5(2)/L5(2).log` for details. □

Theorem 2.2. *The group* $L_5(2)$ *collapses.*

Proof. Let $\mathcal{O}$ be the class 2A; then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$ for any irreducible representation ρ of the corresponding centralizer since $S(2A, 3A, 3A) = 42$ and [AFGV2, Prop. 1.8] applies. Now the result follows from Lemma 2.1 and from [AFGV2, Lemma 1.2] for the conjugacy classes with representatives of order 31 since these are quasi-real of type $j = 2$ and $g^{j^2} \neq g$. □

Table 2
Involutions in $O_8^+(2)$.

Class	Size	
2A	1575	$S(2A, 3E, 3E) = 81$
2B	3780	$S(2B, 3E, 3E) = 108$
2C	3780	$S(2C, 3E, 3E) = 108$
2D	3780	$S(2D, 3E, 3E) = 108$
2E	56 700	$S(2E, 3E, 3E) = 486$

2.3. *The group $O_7(3)$*

This group has order 4 585 351 680. It has 58 conjugacy classes. For computations we use a representation inside $\mathbb{S}_{351}$ given in the ATLAS.

Lemma 2.3. *Every non-trivial conjugacy class of $O_7(3)$, except* 2A, *is of type D.*

Proof. We perform Algorithm 2, see the file `O7(3)/O7(3).log` for details. □

Theorem 2.4. *The orthogonal group $O_7(3)$ collapses.*

Proof. By Lemma 2.3 it remains to study the conjugacy class 2A. For this conjugacy class we use [AFGV2, Lemma 1.3]. See the file `O7(3)/2A.log` for details. □

2.4. *The group $O_8^+(2)$*

This group has order 174 182 400. It has 53 conjugacy classes. For the computations we construct a permutation representation.

Lemma 2.5. *Every non-trivial conjugacy class with representative of order distinct from* 2, 3 *is of type D.*

Proof. We perform Algorithm 2, see the file `O8+(2)/O8+2(2).log` for details. □

Theorem 2.6. *The group $O_8^+(2)$ collapses.*

Proof. By Lemma 2.5 it remains to consider the conjugacy classes with representative of order 2 or 3. For the five conjugacy classes of involutions in $O_8^+(2)$ we use [AFGV2, Prop. 1.8]. See Table 2 for details.

On the other hand, the conjugacy classes 3A, 3B, 3C, 3D, 3E of $O_8^+(2)$ are real, so [AZ, Lemma 2.2], cf. [AFGV2, Lemma 1.2], applies, and the result follows. □

2.5. *The group $O_{10}^-(2)$*

This group has order 25 015 379 558 400. It has 115 conjugacy classes. For the computations we use the representation inside $\mathbb{S}_{495}$ given in the ATLAS.

Lemma 2.7. *Every non-trivial conjugacy class, except the conjugacy classes* 2A, 3A, 11A, 11B, 33A, 33B, 33C, 33D, *is of type D.*

Proof. We perform Algorithm 2, see the file `O10-(2)/O10-(2).log` for details. □

Theorem 2.8. *The group $O_{10}^-(2)$ collapses.*

Table 3
Some Chevalley groups of type D.

Group	Order	Conjugacy classes	Log file
$G_2(3)$	4 245 696	23	`G2(3)/G2(3).log`
$G_2(5)$	5 859 000 000	44	`G2(5)/G2(5).log`

Proof. By Lemma 2.7 it remains to study the conjugacy classes 2A, 3A, 11A, 11B, 33A, 33B, 33C, 33D. Let $\mathcal{O}$ be the class 2A; then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$ for any irreducible representation ρ of the corresponding centralizer since $S(2A, 3F, 3F) = 243$ and [AFGV2, Prop. 1.8] applies. For the class 3A use [AZ, Lemma 2.2], since it is a real conjugacy class. And for the classes 11A, 11B, 33A, 33B, 33C, 33D use [AFGV2, Lemma 1.2], since the classes 11A, 11B (resp. 33A, 33B, 33C, 33D) are quasi-real of type $j = 3$ (resp. $j = 4$) with $g^{j^2} \neq g$. □

2.6. The exceptional group $G_2(4)$

In this section we prove that the group $G_2(4)$ collapses. This group has order 251 596 800. It has 32 conjugacy classes. In particular, the conjugacy classes with representatives of order 2 or 3 are the following:

Name	Centralizer size
2A	61 440
2B	3840
3A	60 480
3B	180

Lemma 2.9. *The conjugacy classes of $G_2(4)$ with representatives of order distinct from 2, 3 are of type D.*

Proof. We perform Algorithm 3, see the file `G2(4)/G2(4).log` for details. □

Theorem 2.10. *The group $G_2(4)$ collapses.*

Proof. By Lemma 2.9, it remains to study the conjugacy classes with representatives of order 2 or 3. For the two conjugacy classes of involutions use [AFGV2, Prop. 1.8], because $S(2A, 3B, 3B) = 171$ and $S(2B, 3A, 3A) = 126$. For the conjugacy classes with representatives of order 3 use [AFGV2, Lemma 1.2], because these conjugacy classes are real. □

2.7. The exceptional groups $G_2(3)$ and $G_2(5)$

For the orders and number of conjugacy classes of the groups $G_2(3)$ and $G_2(5)$ see Table 3. We have the following result.

Theorem 2.11. *The groups $G_2(3)$ and $G_2(5)$ are of type D. Hence, they collapse.*

Proof. We perform Algorithm 3, see Table 3 for the log files. □

2.8. The symplectic group $S_6(2)$

This group has order 1 451 520. It has 30 conjugacy classes. For the computations we use the representation of $S_6(2)$ inside $\mathbb{S}_{28}$ given in the ATLAS.

Lemma 2.12. *Every non-trivial conjugacy class of $S_6(2)$, with the possible exception of 2A, 2B, 3A, is of type D.*

Proof. We perform Algorithm 2, see the file `S6(2)/S6(2).log` for details. □

Theorem 2.13. *The group $S_6(2)$ collapses.*

Proof. By Lemma 2.12 it remains to study the classes 2A, 2B, 3A. For the conjugacy class 2A use [AFGV2, Lemma 1.3], see the file `S6(2)/2A.log`. For the conjugacy class 2B use [AFGV2, Prop. 1.8], since $S(2\text{B}, 3\text{C}, 3\text{C}) = 27$. For the conjugacy class 3A use [AZ, Lemma 2.2], since this conjugacy class is real. □

2.9. The symplectic group $S_8(2)$

This group has order 47 377 612 800. It has 81 conjugacy classes. For the computations we use the representation of $S_8(2)$ inside $\mathbb{S}_{120}$ given in the ATLAS.

Lemma 2.14. *Every non-trivial conjugacy class, except* 2A, 2B, 3A, *is of type D.*

Proof. We perform Algorithm 2, see the file `S8(2)/S8(2).log` for details. □

Lemma 2.15. *Let $\mathcal{O}$ be one of the classes* 2A, 2B, 3A. *Then* $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$ *for any irreducible representation ρ of the corresponding centralizer.*

Proof. For the real conjugacy class 3A use [AFGV2, Lemma 1.2]. For the conjugacy class 2A use [AFGV2, Lemma 1.3], see the file `S8(2)/2A.log` for details. For the conjugacy class 2B use [AFGV2, Prop. 1.8], since $S(2\text{B}, 3\text{C}, 3\text{C}) = 135$. □

2.10. The automorphism group of the Tits group

In this subsection we study Nichols algebras over the group $\text{Aut}(^2F_4(2)') \simeq \text{Aut}(^2F_4(2)) \simeq {}^2F_4(2)$, the automorphism group of the Tits group, see [GL]. This group has order 35 942 400. It has 29 conjugacy classes.

Lemma 2.16. *Every non-trivial conjugacy class of $^2F_4(2)$, with the exception of the class* 2A, *of size* 1755, *is of type D.*

Proof. We perform Algorithm 1. See the file `2F4(2)/2F4(2).log` for details. □

Theorem 2.17. *The group $^2F_4(2)$ collapses.*

Proof. By Lemma 2.16 it remains to study the conjugacy class 2A. For this conjugacy class use [AFGV2, Prop. 1.8], because $S(2\text{A}, 3\text{A}, 3\text{A}) = 27$. □

2.11. Direct products

In this subsection we study some direct products of groups that appear as subgroups or subquotients of the sporadic simple groups.

The group $\mathbb{A}_4 \times G_2(4)$. In this group every non-trivial conjugacy class with representative of order distinct from 2, 3, 6 is of type D. This follows from Lemma 2.9 and [AFGV2, Lemma 2.8].

The group $3 \times G_2(3)$. In this group every non-trivial conjugacy class with representative of order distinct from 3 is of type D since every non-trivial conjugacy class of $G_2(3)$ is of type D. This follows from Theorem 2.11 and [AFGV2, Lemma 2.8].

Table 4
Conjugacy classes not known of type D.

G	Conjugacy classes not necessarily of type D	Reference
M_{11}	8A, 8B, 11A, 11B	Algorithm 3
M_{12}	11A, 11B	Algorithm 3
M_{22}	11A, 11B	Algorithm 3
M_{23}	23A, 23B	Algorithm 3
M_{24}	23A, 23B	Algorithm 3
J_1	15A, 15B, 19A, 19B, 19C	Algorithm 3
J_2	2A, 3A	Algorithm 3
J_3	5A, 5B, 19A, 19B	Algorithm 3
Suz	3A	Algorithm 3
Ru	29A, 29B	Algorithm 3
HS	11A, 11B	Algorithm 3
He	all collapse	Algorithm 3
McL	11A, 11B	Algorithm 3
Co_3	23A, 23B	Algorithm 3
Co_2	2A, 23A, 23B	Algorithm 3
$O'N$	31A, 31B	Algorithm 3
Fi_{22}	2A, 22A, 22B	Algorithm 3
$T = {}^2F_4(2)'$	2A	Algorithm 3
Co_1	3A, 23A, 23B	Section 3.5
Fi_{23}	2A, 23A, 23B	Section 3.4
HN	all collapse	Section 3.6
Th	all collapse	Section 3.2
Ly	33A, 33B, 37A, 37B, 67A, 67B, 67C	Section 3.1
J_4	29A, 37A, 37B, 37C, 43A, 43B, 43C	Section 3.3
Fi'_{24}	23A, 23B, 27B, 27C	Section 3.7
	29A, 29B, 33A, 33B, 39C, 39D	
B	2A, 16C, 16D, 32A	Section 3.8
	32B, 32C, 32D, 34A	
	46A, 46B, 47A, 47B	
M	32A, 32B, 41A, 46A, 46B	Section 3.9
	47A, 47B, 59A	
	59B, 69A, 69B, 71A, 71B	
	87A, 87B, 92A, 92B, 94A, 94B	

The group $\mathbb{A}_9 \times \mathbb{S}_3$. In this group every non-trivial conjugacy class with representative of order distinct from 2, 3 is of type D. See the file `A9xS3/A9xS3.log` for the computations.

The group $\mathbb{S}_5 \times L_3(2)$. In this group every non-trivial conjugacy class with representative of order distinct from 2, 3, 4, 6, 7 is of type D. This was proved with Algorithm 1 – see the file `S5xL3(2)/S5xL3(2).log` for details.

The group $\mathbb{A}_6 \times U_3(3)$. In this group every conjugacy class with representative of order 28 or 35 is of type D. This was proved with Algorithm 1 – see the file `A6xU3(3)/A6xU3(3).log` for details.

The group $\mathbb{S}_5 \times \mathbb{S}_9$. In this group every non-trivial conjugacy class with representative of order distinct from 2, 3, 6 is of type D. This was proved with Algorithm 2 – see the file `S5xS9/S5xS9.log` for details.

3. Proof of Theorem II in [AFGV2]

In Table 4 we list the conjugacy classes of the sporadic simple groups that are not necessarily of type D. We use the phrase "all collapse" to indicate those groups where all non-trivial conjugacy classes are of type D. Also we give a reference about either the algorithms or else the subsections where these groups are treated.

In the next subsections, we deal with some large sporadic groups (for small groups see Table 5). In order to study these groups we study their maximal subgroups with Algorithms 1 or 2. The fusion of

Table 5
Log files for the sporadic groups studied with Algorithm 3.

G	Log file	G	Log file
M_{11}	`M11/M11.log`	Ru	`Ru/Ru.log`
M_{12}	`M12/M12.log`	HS	`HS/HS.log`
M_{22}	`M22/M22.log`	He	`He/He.log`
M_{23}	`M23/M23.log`	McL	`McL/McL.log`
M_{24}	`M24/M24.log`	Co_3	`Co3/Co3.log`
J_1	`J1/J1.log`	Co_2	`Co2/Co2.log`
J_2	`J2/J2.log`	$O'N$	`ON/ON.log`
J_3	`J3/J3.log`	Fi_{22}	`Fi22/Fi22.log`
Suz	`Suz/Suz.log`	T	`T/T.log`

the conjugacy classes of the maximal subgroups of a sporadic group is stored in the ATLAS, up to the Monster group M. For this group the fusion of the conjugacy classes is known only for some of its maximal subgroups; furthermore, the list of all maximal subgroups of M is not known. In the case of the Baby Monster group B all the maximal subgroups and the fusions of conjugacy classes are known except the fusion of the conjugacy classes of the sixth maximal subgroup.

In each case we split the proof into several steps according to the corresponding maximal subgroup. We collect in tables and logs the relevant information.

3.1. The Lyons group Ly

Step 1. *The maximal subgroup $\mathcal{M}_1 \simeq G_2(5)$.*

In $G_2(5)$ every non-trivial conjugacy class is of type D, see Theorem 2.11. Therefore, the conjugacy classes 2A, 3A, 3B, 4A, 5A, 5B, 6A, 6B, 6C, 7A, 8A, 8B, 10A, 10B, 12A, 12B, 15A, 15B, 15C, 20A, 21A, 21B, 24A, 24B, 24C, 25A, 30A, 30B, 31A, 31B, 31C, 31D, 31E of *Ly* are of type D.

Step 2. *The maximal subgroup $\mathcal{M}_4 \simeq 2.\mathbb{A}_{11}$.*

Consider the short exact sequence $1 \to 2 \to 2.\mathbb{A}_{11} \to \mathbb{A}_{11} \to 1$. From Table 1, every non-trivial conjugacy class of $\mathbb{A}_{11}$ is of type D except the class of the 3-cycles and the classes of the 11-cycles. Thus, every non-trivial conjugacy class in $2.\mathbb{A}_{11}$ with representative of order distinct from 2, 3, 6, 11, 22 is of type D, by [AFGV2, Lemma 2.7]. Hence the conjugacy classes 9A, 14A, 18A, 28A, 40A, 40B, 42A, 42B of *Ly* are of type D.

Step 3. *The maximal subgroup $\mathcal{M}_6 \simeq 3^5 : (2 \times M_{11})$.*

We construct a permutation representation of $\mathcal{M}_6$ and apply Algorithm 1 in this maximal subgroup. We check that all conjugacy classes of $\mathcal{M}_6$ with representative of order 11, 22 are of type D – see the file `Ly/step3.log`. By the fusion of the conjugacy classes, the conjugacy classes 11A, 11B, 22A, 22B of *Ly* are of type D.

Remark 3.1. Not necessarily of type D: 33A, 33B, 37A, 37B, 67A, 67B, 67C.

3.2. The Thompson group Th

See the file `Th/fusions.log` for the fusion of conjugacy classes.

Step 1. *The maximal subgroup $\mathcal{M}_3 \simeq 2^{1+8}.\mathbb{A}_9$.*

Consider the short exact sequence $1 \to 2^{1+8} \to 2^{1+8}.\mathbb{A}_9 \to \mathbb{A}_9 \to 1$. From Table 1, every non-trivial conjugacy class of $\mathbb{A}_9$ is of type D except the class of the 3-cycles. By [AFGV2, Lemma 2.7], every conjugacy class in $2^{1+8}.\mathbb{A}_9$ with representative of order 5, 7, 9, 10, 14, 15, 18, 20, 28, 30, 36 is of type D. Hence the conjugacy classes 5A, 7A, 10A, 14A, 15A, 15B, 18A, 18B, 20A, 28A, 30A, 30B, 36A, 36B, 36C of *Th* are of type D.

Step 2. *The maximal subgroup* $\mathcal{M}_2 \simeq 2^5.L_5(2)$.

Consider the short exact sequence $1 \to 2^5 \to 2^5.L_5(2) \to L_5(2) \to 1$. By Lemma 2.1, in $L_5(2)$ every non-trivial conjugacy class with representative of order distinct from 2 and 31 is of type D. Therefore, by [AFGV2, Lemma 2.7], every conjugacy class in $\mathcal{M}_2$ with representative of order 3, 6, 12, 21, 24 is of type D. Hence the classes 3A, 3C, 12D, 21A, 24A, 24B of *Th* are of type D.

On the other hand, each group $L_5(2)$ and $2^5.L_5(2)$ have six classes of elements of order 31; let $\{\mathcal{O}_1, \dots, \mathcal{O}_6\}$ be these classes in $L_5(2)$. We can check that for every i, j, with $1 \leqslant i \neq j \leqslant 6$, there exist $r \in \mathcal{O}_i$ and $s \in \mathcal{O}_j$ such that $(rs)^2 \neq (sr)^2$. Then the same occurs in the corresponding classes of $2^5.L_5(2)$. Since the fusion of the conjugacy classes from $2^5.L_5(2)$ to *Th* establish that 31a, 31c, 31d go to 31A, and 31b, 31e, 31f go to 31B, then the conjugacy classes 31A and 31B of *Th* are of type D. We note that here the classes of this maximal subgroup are named in lower case letter because they are not necessarily named as in the ATLAS.

Step 3. *The maximal subgroup* $\mathcal{M}_{12} \simeq L_2(19).2$.

In this maximal subgroup, the conjugacy classes with representatives of order 2, 3, 6, 19 are of type D – see the file `Th/step3.log`. Then, by the fusion of the conjugacy classes, the conjugacy classes 2A, 3B, 19A of *Th* are of type D.

Step 4. *The maximal subgroup* $\mathcal{M}_5 \simeq (3 \times G_2(3)) : 2$.

From Theorem 2.11, the group $G_2(3)$ is of type D. Thus, the conjugacy class 13A of *Th* is of type D. On the other hand, by Section 2.11 and [AFGV2, Lemma 2.8], the conjugacy classes 39A, 39B of *Th* are of type D. See the file `Th/step4.log` for the fusion of the conjugacy classes $3 \times G_2(3) \to (3 \times G_2(3)) : 2$.

Step 5. *The maximal subgroup* $\mathcal{M}_6 \simeq 3.3^2.3.(3 \times 3^2).3^2 : 2\mathbb{S}_4$.

By the fusion of the conjugacy classes and Algorithm 1, we see that the classes 4A, 8A, 12A, 12B, 12C of *Th* are of type D. See the file `Th/step5.log` for details.

Step 6. *The maximal subgroup* $\mathcal{M}_7 \simeq 3^2.3^3.3^2.3^2 : 2\mathbb{S}_4$.

By the fusion of the conjugacy classes and Algorithm 1, we see that the conjugacy classes 4B, 6A, 6B, 6C, 8B, 9A, 9B, 9C, 24C, 24D, 27A, 27B, 27C of *Th* are of type D. See the file `Th/step6.log` for details.

3.3. *The Janko group* J_4

See the file `J4/fusions.log` for the fusion of conjugacy classes.

Step 1. *The maximal subgroup* $\mathcal{M}_1 \simeq 2^{11} : M_{24}$.

We use the maximal subgroup $\mathcal{M}_1 \simeq 2^{11} : M_{24}$. Consider the short exact sequence $1 \to 2^{11} \to 2^{11} : M_{24} \to M_{24} \to 1$. We know that every non-trivial conjugacy class of M_{24} with representative of order distinct from 23 is of type D. By [AFGV2, Lemma 2.7], every non-trivial conjugacy class in $2^{11} : M_{24}$ with representative of order distinct from 2, 4, 8, 16, 23 is of type D. Hence the conjugacy classes 3A, 5A, 6A, 6B, 6C, 7A, 7B, 10A, 10B, 12A, 12B, 12C, 14A, 14B, 14C, 14D, 15A, 20A, 20B, 21A, 21B, 24A, 24B, 28A, 28B, 30A of J_4 are of type D. Also, this maximal subgroup has a primitive permutation representation on 2^{11} points. We construct this primitive group and use Algorithm 2 to determine that the only conjugacy class with representative of order 16 in $\mathcal{M}_1$ is of type D – see the file `J4/step1.log`. Hence, the conjugacy class 16A of J_4 is of type D.

Step 2. *The maximal subgroup* $\mathcal{M}_4 \simeq 2^{3+12}.(\mathbb{S}_5 \times L_3(2))$.

Consider the short exact sequence $1 \to 2^{3+12} \to 2^{3+12}.(\mathbb{S}_5 \times L_3(2)) \to (\mathbb{S}_5 \times L_3(2)) \to 1$. Every non-trivial conjugacy class of $\mathbb{S}_5 \times L_3(2)$ with representative of order distinct from 2, 3, 4, 6, 7 is of type D, see Section 2.11. By [AFGV2, Lemma 2.7], every conjugacy class in $2^{3+12}.(\mathbb{S}_5 \times L_3(2))$ with representative of order $35, 42$ is of type D. Hence the conjugacy classes 35A, 35B, 42A, 42B of J_4 are of type D.

Step 3. *The maximal subgroup* $\mathcal{M}_5 \simeq U_3(11).2$.

We perform Algorithm 2 – see the file `J4/step3.log`. Hence the conjugacy classes 8A, 8B, 11A, 11B of J_4 are of type D.

Step 4. *The maximal subgroup* $\mathcal{M}_6 \simeq M_{22}.2$.

We perform Algorithm 1 – see the file `M22/M22.2.log`. Hence the conjugacy classes 4B, 8C of J_4 are of type D.

Step 5. *The maximal subgroup* $\mathcal{M}_7 \simeq 11_+^{1+2} : (5 \times 2\mathbb{S}_4)$.

We perform Algorithm 1 – see the file `J4/step5.log`. Hence the conjugacy classes 4A, 22A, 22B, 40A, 40B, 44A, 66A, 66B of J_4 are of type D.

Step 6. *The maximal subgroup* $\mathcal{M}_8 \simeq L_2(32).5$.

We perform Algorithm 1 – see the file `J4/step6.log`. Hence the conjugacy classes 31A, 31B, 31C, 33A, 33B of J_4 are of type D.

Step 7. *The maximal subgroup* $\mathcal{M}_9 \simeq L_2(23).2$.

We perform Algorithm 1 – see the file `J4/step7.log`. Hence the conjugacy classes 2A, 2B, 23A of J_4 are of type D.

Step 8. *The maximal subgroup* $\mathcal{M}_{13} \simeq 37 : 12$.

We construct a permutation representation of this maximal subgroup. By the fusion of the conjugacy classes, the classes 4a and 4b of $\mathcal{M}_{13}$ go to the conjugacy class 4C of J_4. We find r in 4A, s in 4B of $\mathcal{M}_{13}$ such that $(rs)^2 \neq (sr)^2$ – see the file `J4/step8.log`. Hence the conjugacy class 4C of J_4 is of type D. We note that here the classes of this maximal subgroup are named with lower case letters because they are not necessarily named as in the ATLAS.

Remark 3.2. Not necessarily of type D: 29A, 37A, 37B, 37C, 43A, 43B, 43C.

3.4. The Fischer group Fi_{23}

See the file `Fi23/fusions.log` for the fusion of conjugacy classes.

Step 1. *The maximal subgroup* $\mathcal{M}_1 \simeq 2.Fi_{22}$.

Consider the short exact sequence $1 \to 2 \to 2.Fi_{22} \to Fi_{22} \to 1$. By Table 4, every non-trivial conjugacy class of Fi_{22} with representative of order distinct from 2, 22 is of type D. By [AFGV2, Lemma 2.7], every conjugacy class in $2.Fi_{22}$ with representative of order 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 20, 21, 26, 42 is of type D. Then the conjugacy classes 3A, 3B, 3C, 3D, 5A, 6A, 6B, 6C, 6D, 6E, 6F, 6G, 6H, 6I, 6J, 6K, 6L, 6M, 6N, 6O, 7A, 9B, 9C, 9E, 10A, 10B, 10C, 11A, 12A, 12B, 12C, 12E, 12F, 12G, 12H, 12I, 12J, 12K, 12L, 12M, 12N, 12O, 13A, 13B, 14A, 14B, 16A, 16B, 20A, 20B, 21A, 26A, 26B of Fi_{23} are of type D.

Step 2. *The maximal subgroup* $\mathcal{M}_4 \simeq S_8(2)$.

By Lemma 2.14 and the fusion of the conjugacy classes, the conjugacy classes 2B, 2C, 4A, 4B, 4C, 4D, 8C, 15A, 15B, 17A of Fi_{23} are of type D.

Step 3. *The maximal subgroup* $\mathcal{M}_5 \simeq O_7(3) \times \mathbb{S}_3$.

By Lemma 2.3 and [AFGV2, Lemma 2.8], every non-trivial conjugacy class of $O_7(3) \times \mathbb{S}_3$ with representative of order distinct from 2, 3, 6 is of type D. Thus by the fusion of the conjugacy classes, the conjugacy classes 9D, 12D, 18A, 18B, 18C, 18E, 18F, 18H, 39A, 39B of Fi_{23} are of type D.

Step 4. *The maximal subgroup* $\mathcal{M}_3 \simeq 2^2.U_6(2).2$.

We perform Algorithm 2 in this maximal subgroup to see that the conjugacy classes 22A, 22B, 22C of Fi_{23} are of type D. See the file `Fi23/step4.log` for details.

Step 5. *The maximal subgroup* $\mathcal{M}_2 \simeq O_8^+(3) : \mathbb{S}_3$.

We perform Algorithm 2 to see that the conjugacy class 27A of Fi_{23} is of type D. See the file `Fi23/step5.log` for details.

Step 6. *The maximal subgroup* $\mathcal{M}_{10} \simeq (2^2 \times 2^{1+8}).(3 \times U_4(2)).2$.

We perform Algorithm 1 in this maximal subgroup to see that the classes 9A, 18D, 18G, 24A, 24B, 24C, 36A of Fi_{23} are of type D. See `Fi23/step6.log` for details.

Step 7. *The maximal subgroup* $\mathcal{M}_{12} \simeq \mathbb{S}_4 \times S_6(2)$.

We perform Algorithm 1 in this maximal subgroup to see that the conjugacy class 36B Fi_{23} is of type D. See the file `Fi23/step7.log` for details.

Step 8. *The maximal subgroup* $\mathcal{M}_9 \simeq \mathbb{S}_{12}$.

From Table 1, every non-trivial conjugacy class of $\mathbb{S}_{12}$ with representative of order distinct of 2, 3, 11 is of type D. Thus, from the fusion of the conjugacy classes, the conjugacy classes 8A, 8B, 28A, 30A, 30B, 30C, 35A, 42A, 60A of Fi_{23} are of type D.

Remark 3.3. Not necessarily of type D: 2A, 23A, 23B.

3.5. The Conway group Co_1

See the file `Co1/fusions.log` for the fusion of conjugacy classes.

Step 1. *The maximal subgroups $\mathcal{M}_1 \simeq Co_2$ and $\mathcal{M}_4 \simeq Co_3$.*

From Table 4 and the fusion of the conjugacy classes, the conjugacy classes 3B, 3C, 4A, 4B, 4C, 4D, 4F, 5B, 5C, 6C, 6D, 6E, 6F, 6G, 6I, 7B, 8B, 8C, 8D, 8E, 9B, 9C, 10D, 10E, 10F, 11A, 14B, 15D, 15E, 16A, 16B, 20B, 20C, 21C, 22A, 28A, 30D, 30E of Co_1 are of type D.

Step 2. *The maximal subgroup $\mathcal{M}_{16} \simeq \mathbb{A}_9 \times \mathbb{S}_3$.*

We know that every non-trivial conjugacy class of $\mathbb{A}_9 \times \mathbb{S}_3$ with representative of order distinct from $2, 3$ is of type D, see Section 2.11. Then, by the fusion of conjugacy classes, the classes 4E, 5A, 6A, 6B, 6H, 9A, 9B, 10A, 10B, 12L, 12M, 15A, 15C, 30A, 30C of Co_1 are of type D.

Step 3. *The maximal subgroup $\mathcal{M}_3 \simeq 2^{11} : M_{24}$.*

Consider the short exact sequence $1 \to 2^{11} \to 2^{11} : M_{24} \to M_{24} \to 1$. By Table 4 and [AFGV2, Lemma 2.7], every non-trivial conjugacy class of $2^{11} : M_{24}$ with representative of order distinct from 23 is of type D. By the fusion of the conjugacy classes, the conjugacy class 3D of Co_1 is of type D.

Step 4. *The maximal subgroup $\mathcal{M}_7 \simeq (\mathbb{A}_4 \times G_2(4)) : 2$.*

Note that $\mathbb{A}_4 \times G_2(4)$ is a subgroup of $\mathcal{M}_7$. Now by Section 2.11 and the fusion of conjugacy classes, the conjugacy classes 14A, 26A of Co_1 are of type D.

Step 5. *The maximal subgroup $\mathcal{M}_{14} \simeq (\mathbb{A}_6 \times U_3(3)) : 2$.*

The group $\mathbb{A}_6 \times U_3(3)$ is a subgroup of Co_1. It has two conjugacy classes with representatives of order 28 and four conjugacy classes with representatives of order 35, which are all of type D – see Section 2.11. The maximal subgroup $\mathcal{M}_{14}$ has only one conjugacy class with representative of order 28, hence this conjugacy class is of type D. The fussion of conjugacy classes says that the conjugacy class 28A of $\mathcal{M}_{14}$ goes to 28B of Co_1; thus the conjugacy class 28B of Co_1 is of type D. On the other hand, the conjugacy class 35A of Co_1 is of type D since Co_1 has only one conjugacy class with representative of order 35.

Step 6. *The maximal subgroup $\mathcal{M}_2 \simeq 3.Suz.2$.*

We perform Algorithm 2 in this maximal subgroup to see that the conjugacy classes 7A, 8A, 8F, 10C, 13A, 15B, 21A, 21B, 30B, 33A, 39A, 39B, 42A of Co_1 are of type D. See the file `Suz/3.Suz.2.log` for the computations.

Step 7. *The maximal subgroup $\mathcal{M}_5 \simeq 2^{1+8}.O_8^+(2)$.*

By Lemma 2.5 and [AFGV2, Lemma 2.7] the conjugacy classes of $2^{1+8}.O_8^+(2)$ with representative of order 5, 7, 9, 10, 14, 15, 18, 20, 28, 30, 36, 40, 60 are of type D. Thus, by the fusion of the conjugacy classes, the conjugacy classes 20A, 36A, 40A, 60A of Co_1 are of type D.

On the other hand, to study other conjugacy classes of Co_1 we use a script that performs an algorithm similar to Algorithm 2 that we explain briefly here – see the file `Co1/step7.log`. First we compute all conjugacy classes of Co_1 and $\mathcal{M}_5$. Then we study the conjugacy classes of $\mathcal{M}_5$ with representative of order 12, 18, 24 and discard the corresponding conjugacy classes in Co_1 when those are of type D. At the end of the log file we see that only two conjugacy classes of Co_1, both with representatives of order 12, were not discarded. One of these classes has centralizer of order 48, the other 72. These are the conjugacy classes 12L and 12M of Co_1, and these classes were considered in Step 2. Therefore, besides the conjugacy classes of the previous paragraph, the conjugacy classes 12A, 12B, 12C, 12D, 12E, 12F, 12G, 12H, 12I, 12J, 12K, 18A, 18B, 18C, 24A, 24B, 24C, 24D, 24E, 24F of Co_1 are of type D.

Remark 3.4. Not necessarily of type D: 3A, 23A, 23B.

3.6. The Harada–Norton group HN

See the file `HN/fusions.log` for the fusion of conjugacy classes.

Step 1. *The maximal subgroup* $\mathcal{M}_1 \simeq \mathbb{A}_{12}$.

By Table 1, in this maximal subgroup every non-trivial conjugacy class with representative of order distinct from $3, 11$ is of type D. Therefore, the conjugacy classes 2A, 2B, 5A, 5E, 6A, 6B, 6C, 7A, 9A, 15A, 20C, 21A, 30A, 35A, 35B of *HN* are of type D. It remains to prove that the conjugacy class 11A of *HN* is of type D. For that purpose, let $r = (1\,2\,3\,4\,5\,6\,7\,8\,9\,10\,11)$ and $s = (1\,2\,3\,4\,5\,6\,7\,8\,9\,11\,10)$ be elements in $\mathbb{A}_{12}$. It is easy to see that $(rs)^2 \neq (sr)^2$ and that r and s belong to different conjugacy classes in the group $\langle r, s\rangle \simeq \mathbb{A}_{11}$. Then, the conjugacy class 11A of *HN* is of type D.

Step 2. *The maximal subgroup* $\mathcal{M}_{11} \simeq M_{12}.2$.

We perform Algorithm 1 and obtain that every non-trivial conjugacy class is of type D – see the file `M12/M12.2.log`. By the fusion of the conjugacy classes, the conjugacy classes 3A, 3B, 4A, 4B, 4C, 12C of *HN* are of type D.

Step 3. *The maximal subgroup* $\mathcal{M}_2 \simeq 2.HS.2$.

We perform Algorithm 2 and obtain that every conjugacy class with representative of order $5, 8, 10, 12, 14, 20, 22, 40$ is of type D – see the file `HS/2.HS.2.log`. Therefore, the conjugacy classes 5B, 8A, 8B, 10A, 10B, 10C, 10F, 10G, 10H, 12A, 12B, 14A, 20A, 20B, 22A, 40A, 40B of *HN* are of type D.

Step 4. *The maximal subgroup* $\mathcal{M}_{14} \simeq 3^{(1+4)} : 4.\mathbb{A}_5$.

We use Algorithm 1 and obtain that every conjugacy class with representative of order $5, 10, 15, 20, 30$ is of type D – see the file `HN/step4.log`. Therefore, the conjugacy classes 5C, 5D, 10D, 10E, 15B, 15C, 20D, 20E, 30B, 30C of *HN* are of type D.

Step 5. *The maximal subgroup* $\mathcal{M}_3 \simeq U_3(8).3_1$.

We perform Algorithm 2 and obtain that every conjugacy class with representative of order 19 is of type D – see the file `HN/step5.log`. Therefore, the conjugacy classes 19A, 19B of *HN* are of type D.

Step 6. *The maximal subgroup* $\mathcal{M}_{10} \simeq 5^{2+1+2}.4.\mathbb{A}_5$.

We perform Algorithm 1 and obtain that every conjugacy class with representative of order 25 is of type D – see the file `HN/step6.log`. Therefore, the conjugacy classes 25A, 25B of *HN* are of type D.

3.7. The Fischer group Fi'_{24}

The log files concerning this group are stored in the folder F3+. See the file F3+/fusions.log for the fusion of the conjugacy classes. The list of (representatives of conjugacy classes of) maximal subgroups of Fi'_{24} can be found in [LW].

Step 1. *The maximal subgroup $\mathcal{M}_1 \simeq Fi_{23}$.*

By Section 3.4, we know that every non-trivial conjugacy class of Fi_{23} with representative of order distinct from 2, 23 is of type D. Hence, the conjugacy classes 3A, 3B, 3C, 3D, 4A, 4B, 4C, 5A, 6A, 6B, 6C, 6D, 6E, 6F, 6G, 6H, 6I, 6J, 7A, 8A, 8B, 9A, 9B, 9C, 9E, 9F, 10A, 10B, 11A, 12A, 12B, 12C, 12D, 12E, 12F, 12G, 12H, 12K, 12L, 12M, 13A, 14A, 15A, 15C, 16A, 17A, 18A, 18B, 18C, 18D, 18E, 18F, 20A, 21A, 22A, 24A, 24B, 24E, 26A, 27A, 28A, 30A, 30B, 35A, 36C, 36D, 39A, 39B, 42A, 60A of Fi'_{24} are of type D.

Step 2. *The maximal subgroups $\mathcal{M}_{13} \simeq He.2$ and $\mathcal{M}_{14} \simeq He.2$.*

We perform Algorithm 3 and obtain that every non-trivial conjugacy class is of type D – see the file He/He.2.log. Then the conjugacy classes 2A, 2B, 3E, 6K, 7B, 12I, 12J, 14B, 21B, 21C, 21D, 24C, 24D, 42B, 42C of Fi'_{24} are of type D.

Step 3. *The maximal subgroup $\mathcal{M}_4 \simeq O^-_{10}(2)$.*

By Lemma 2.7 and the fusion of the conjugacy classes, the conjugacy classes 8C, 15B, 18G, 18H, 20B of Fi'_{24} are of type D.

Step 4. *The maximal subgroup $\mathcal{M}_5 \simeq 3^7.O_7(3)$.*

We consider the short exact sequence $1 \to 3^7 \to 3^7.O_7(3) \to O_7(3) \to 1$. By Lemma 2.3, every non-trivial conjugacy class of $O_7(3)$ with representative of order distinct from 2 is of type D. By [AFGV2, Lemma 2.7], every non-trivial conjugacy class of $\mathcal{M}_5$ with representative of order distinct from 24, 36, 45 is of type D. Therefore, the conjugacy classes 24F, 24G, 36A, 36B, 45A, 45B of Fi'_{24} are of type D.

Step 5. *The maximal subgroup $\mathcal{M}_{20} \simeq \mathbb{A}_6 \times L_2(8):3$.*

We perform Algorithm 1 and obtain that every conjugacy class with representative of order 9 is of type D – see the file F3+/step5.log. Therefore, the conjugacy class 9D of *HN* is of type D.

Remark 3.5. Not necessarily of type D: 23A, 23B, 27B, 27C, 29A, 29B, 33A, 33B, 39C, 39D.

3.8. The Baby Monster group B

For this group we compute the fusion of the conjugacy classes in each step.

Step 1. *The maximal subgroup $\mathcal{M}_2 \simeq 2^{1+22}.Co_2$.*

We consider the short exact sequence $1 \to 2^{1+22} \to 2^{1+22}.Co_2 \to Co_2 \to 1$. By Table 4, every non-trivial conjugacy class of Co_2 with representative of order distinct from 2, 23 is of type D. Therefore, by [AFGV2, Lemma 2.7], the conjugacy classes 5A, 5B, 6A, 6B, 6C, 6D, 6E, 6F, 6G, 6H, 6I, 6J, 6K, 7A, 9A, 9B, 10A, 10B, 10C, 10D, 10E, 10F, 11A, 12A, 12B, 12C, 12D, 12E, 12F 12G, 12H, 12I, 12J, 12K, 12L, 12M, 12N, 12O, 12P, 12Q, 12R, 12S, 14A, 14B, 14C, 14D, 14E, 15A, 15B, 18F, 20A, 20B, 20C, 20D, 20E, 20F, 20G, 20H, 20I, 20J, 24A, 24B, 24C, 24D, 24E, 24F, 24G, 24H, 24I, 24J, 24K, 24M, 28A, 28B, 28C,

28D, 28E, 30A, 30B, 30C, 30D, 30E, 30F, 30G, 30H, 36A, 40A, 40B, 40C, 40D, 44A, 48A, 48B, 56A, 56B of B are of type D. See the file `B/step1.log` for the fusion of the conjugacy classes $2^{1+22}.Co_2 \to B$.

Step 2. *The maximal subgroup* $\mathcal{M}_3 \simeq Fi_{23}$.

By Section 3.4, every non-trivial conjugacy class of Fi_{23} distinct of 2A, 23A, 23B is of type D. Therefore, the conjugacy classes 2B, 2D, 3A, 3B, 4D, 4E, 4G, 4H, 8J, 8K, 13A, 16G, 17A, 18A, 18B, 18C, 18D, 18E, 21A, 22A, 22B, 24L, 26B, 27A, 35A, 36B, 36C, 39A, 42B of B are of type D. See the file `B/step2.log` for the fusion of conjugacy classes $Fi_{23} \to B$.

Step 3. *The maximal subgroup* $\mathcal{M}_{18} \simeq \mathbb{S}_5 \times M_{22} : 2$.

We perform Algorithm 1 and obtain that every conjugacy class of $\mathcal{M}_{18}$ with representative of order 4, 8, 33, 42, 55, 66, 70 is of type D. Hence the conjugacy classes 4A, 4B, 4C, 4F, 8B, 8C, 8E, 8I, 33A, 42C, 55A, 66A, 70A of B are of type D. See the file `B/step3.log` for the fusion of conjugacy classes $\mathbb{S}_5 \times M_{22} : 2 \to B$ and the computations.

Step 4. *The maximal subgroup* $\mathcal{M}_4 \simeq 2^{9+16}.S_8(2)$.

By Lemma 2.14, every non-trivial conjugacy class, except 2A, 2B, 3A is of type D. By [AFGV2, Lemma 2.7] and the fusion of conjugacy classes – see the file `B/step4.log` – the conjugacy classes 34B, 34C, 40E, 42A, 60A, 60B, 60C of B are of type D.

Step 5. *The maximal subgroup* $\mathcal{M}_{16} \simeq \mathbb{S}_4 \times {}^2F_4(2)$.

See the file `B/step5.log` for the fusion of the conjugacy classes from this maximal subgroup into B. By Lemma 2.16 and [AFGV2, Lemma 2.8], we deduce that every non-trivial conjugacy class of $\mathcal{M}_{16}$ with representative of order distinct from 2, 3, 4 is of type D. Thus, the conjugacy classes 4I, 8A, 8D, 8F, 8G, 8H, 8L, 8N, 12T, 16A, 16B, 16E, 16F, 26A, 52A of B are of type D.

Step 6. *The maximal subgroup* $\mathcal{M}_{28} \simeq L_2(17).2$.

We perform Algorithm 1 and obtain that every conjugacy class of $\mathcal{M}_{28}$ with representative of order 8, 16 is of type D. Then, by the fusion of the conjugacy classes, the conjugacy classes 8M, 16H of B are of type D. See the file `B/step6.log` for the fusion of the conjugacy classes and the computations.

Step 7. *The maximal subgroup* $\mathcal{M}_5 \simeq Th$.

By Section 3.2, every non-trivial conjugacy class in Th is of type D. Therefore, by the fusion of the conjugacy classes – see the file `B/step7.log` – the conjugacy classes 4J, 19A, 24N, 31A, 31B of B are of type D.

Step 8. *The maximal subgroup* $\mathcal{M}_{30} \simeq 47 : 23$.

We use this maximal subgroup to show that the classes 23A, 23B of B are of type D. Indeed, the classes 23a and 23d of $\mathcal{M}_{30}$ fuse into the class 23A of B and there exist elements r and s in the classes 23a and 23d of $\mathcal{M}_{30}$, respectively, such that $(rs)^2 \neq (sr)^2$. Then the class 23A of B is of type D. On the other hand, the classes 23u and 23v of $\mathcal{M}_{30}$ fuse into the class 23B of B and there exist elements r and s in the classes 23u and 23v of $\mathcal{M}_{30}$, respectively, such that $(rs)^2 \neq (sr)^2$. Then the class 23B of B is of type D. See the file `B/step8.log` for the details. We note that here the classes of this maximal subgroup are named in lower case letter because they are not necessarily named as in the ATLAS.

Step 9. *The maximal subgroup* $\mathcal{M}_1 \simeq 2.({}^2E_6(2)) : 2$.

We know that $2 \times U_3(8)$ is a subgroup of B, because $2 \times U_3(8)$ is a subgroup of $\mathcal{M}_1$, see [W]. In $U_3(8)$ there exist non-conjugate elements r and s, both of order 19, such that $(rs)^2 \neq (sr)^2$ – see the file `B/step9.log`. Since B has only one conjugacy class of elements of order 38, the conjugacy class 38A of B is of type D.

Remark 3.6. Not necessarily of type D: 2A, 16C, 16D, 32A, 32B, 32C, 32D, 34A, 46A, 46B, 47A, 47B.

3.9. *The Monster group M*

For the known maximal subgroups of M, we use the order in which they appear listed in the ATLAS.

Step 1. *The maximal subgroup* $\mathcal{M}_1 \simeq 2.B$.

By Section 3.8 and [AFGV2, Lemma 2.7], the conjugacy classes 3A, 3B, 5A, 5B, 6A, 6B, 6C, 6D, 6E, 7A, 8A, 8B, 8C, 8D, 8E, 8F, 9A, 9B, 10A, 10B, 10C, 10D, 10E, 11A, 12A, 12B, 12C, 12E, 12F, 12G, 12H, 12I, 13A, 14A, 14B, 17A, 18A, 18B, 18C, 18D, 18E, 19A, 21A, 22A, 22B, 23A, 23B, 24A, 24B, 24C, 24D, 24F, 24G, 24H, 24I, 25A, 26A, 27A, 28A, 28B, 28C, 33B, 35A, 36A, 36B, 36C, 36D, 38A, 39A, 40A, 40B, 40C, 40D, 42A, 44A, 44B, 48A, 50A, 52A, 54A, 55A, 66A, 70A, 78A, 84A, 104A, 104B, 110A of M are of type D. See the file `M/step1.log` for the fusion of conjugacy classes $2.B \to M$.

Step 2. *The maximal subgroup* $\mathcal{M}_2 \simeq 2^{1+24}.Co_1$.

By Section 3.5 and [AFGV2, Lemma 2.7], the conjugacy classes 7B, 13B, 14C, 15A, 15B, 15C, 15D, 20A, 20B, 20C, 20D, 20E, 20F, 21B, 21D, 26B, 28D, 30A, 30B, 30C, 30D, 30E, 30F, 30G, 33A, 35B, 39C, 39D, 42B, 42C, 42D, 52B, 56A, 56B, 56C, 60A, 60B, 60C, 60D, 60E, 60F, 66B, 70B, 78B, 78C, 84B, 88A, 88B of M are of type D. See the file `M/step2.log` for the fusion of conjugacy classes $2^{1+24}.Co_1 \to M$.

Step 3. *The maximal subgroup* $\mathcal{M}_9 \simeq \mathbb{S}_3 \times Th$.

By Section 3.2 and [AFGV2, Lemma 2.8], every non-trivial conjugacy class in this maximal subgroup with representative of order distinct from 2, 3 is of type D. Moreover, in this maximal subgroup, there is only one conjugacy class with representative of order 3 that is not of type D: that corresponding to the 3-cycles in $\mathbb{S}_3$. Hence, the conjugacy classes 3C, 4A, 4D, 6F, 12D, 12J, 21C, 24E, 24J, 27B, 31A, 31B, 39B, 57A, 62A, 62B, 84C, 93A, 93B of M are of type D. See the file `M/step3.log` for the fusion of conjugacy classes $\mathbb{S}_3 \times Th \to M$.

Step 4. *The maximal subgroup* $\mathcal{M}_{40} \simeq L_2(29).2 \simeq \mathbf{PGL}(2, 29)$.

We use a representation of this maximal subgroup inside $\mathbb{S}_{30}$ given in the ATLAS, see [BW]. This maximal subgroup has only one conjugacy class of elements of order 29 which is of type D. Therefore, the conjugacy class 29A of M is of type D. See the file `M/step4.log` for details.

Step 5. *The maximal subgroup* $\mathcal{M}_{23} \simeq (L_3(2) \times S_4(4) : 2).2$.

We use a representation of this maximal subgroup inside $\mathbb{S}_{184}$ given in the ATLAS, see [BW]. In this maximal subgroup the conjugacy classes with representatives of order 16, 34, 51, 68, 119 are of type D. Therefore, the conjugacy classes 16A, 16B, 16C, 34A, 51A, 68A, 119A, 119B of M are of type D. See the file `M/step5.log` for details.

Step 6. *The maximal subgroup* $\mathcal{M}_{21} \simeq (\mathbb{A}_5 \times U_3(8):3_1):2$.

We use a representation of this maximal subgroup inside $\mathbb{S}_{518}$ given in the ATLAS, see [BW]. In this maximal subgroup, the conjugacy classes with representatives of order 95 are of type D. Therefore, the conjugacy classes 95A, 95B of M are of type D. See the file `M/step6.log` for details.

Step 7. *The maximal subgroup* $\mathcal{M}_3 \simeq 3.Fi_{24}$.

Let $H = Fi_{24}$. From the ATLAS we know that the group $K = \mathbb{S}_5 \times \mathbb{S}_9$ is a maximal subgroup of H. From 2.11 every conjugacy class of K with representative of order 4 is of type D. Also, from the fusion of conjugacy classes $K \to H$, every conjugacy class of Fi_{24} with representative of order 4 is of type D. Therefore, by [AFGV2, Lemma 2.7], the conjugacy classes 4A, 4B, 4C, 4D of M are of type D. Also, since Fi'_{24} is a maximal subgroup of Fi_{24}, the conjugacy classes of Fi_{24} with representatives of order 15, 45 are of type D – see Section 3.7. Therefore, by [AFGV2, Lemma 2.7] and the fusion of conjugacy classes $Fi'_{24} \to Fi_{24}$, the conjugacy class 45A of M is of type D. For details about these observations and the fusion of conjugacy classes $3.Fi_{24} \to M$ see the file `M/step7.log`.

Step 8. *The subgroup HN.*

Since $(\mathbb{D}_{10} \times HN).2$ is a maximal subgroup of M, HN is a subgroup of M. Therefore, by Section 3.6 and the fusion of the conjugacy classes $HN \to M$, the conjugacy classes 2A, 2B of M are of type D. For the fusion of the conjugacy classes see the file `M/step8.log`.

Step 9. *The subgroup* $2 \times 47:23$.

We use this subgroup to show that the classes 46C, 46D of M are of type D. Indeed, the classes 46A and 46B of $2 \times 47:23$ fuse into the class 46C of M and there exist elements r and s in the classes 46A and 46B of $2 \times 47:23$, respectively, such that $(rs)^2 \neq (sr)^2$. Then the class 46C of M is of type D. The conjugacy class 46D is treated analogously. See the file `M/step9.log` for the details.

Remark 3.7. Not necessarily of type D: 32A, 32B, 41A, 46A, 46B, 47A, 47B, 59A, 59B, 69A, 69B, 71A, 71B, 87A, 87B, 92A, 92B, 94A, 94B.

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Appendix A. Real and quasi-real conjugacy classes

In this appendix we list all real and quasi-real conjugacy classes of the groups studied in [AFGV2]. The information about real conjugacy classes of a given group G is easy to obtain from the character table of G using the GAP function `RealClasses`. Similarly, with the GAP function `PowerMaps` it is easy to determine the quasi-real conjugacy classes of a given group.

The function `QuasiRealConjugacyClasses` returns the list of quasi-real conjugacy classes and its type.

```
gap> QuasiRealClasses := function( ct )
>  local nc, oc, a, b, p, c, j, rc;
>
>  nc := NrConjugacyClasses(ct);
>  oc := OrdersClassRepresentatives(ct);
```

```
>  rc := RealClasses(ct);
>
>  a := [];
>  b := [];
>
>  for c in [1..nc] do
>    if not c in rc then
>      for j in [2..oc[c]-2] do
>        p := PowerMap(ct, j);
>        if p[c] = c then
>          if j-1 mod oc[c] <> 0 then
>            if not c in b then
>              Add(a, [c, j]);
>              Add(b, c);
>            fi;
>          fi;
>        fi;
>      od;
>    fi;
>  od;
>  return a;
>end;
```

The group $L_5(2)$. The conjugacy classes 7A, 7B, 15A, 15B, 21A, 21B, 31A, 31B, 31C, 31D, 31E, 31F are quasi-real of type $j = 2$, and the classes 14A, 14B are quasi-real of type $j = 9$. The remaining conjugacy classes are real.

The groups $O_8^+(2)$, $S_6(2)$ and $S_8(2)$. In these groups every conjugacy class is real.

The group $O_{10}^-(2)$. The conjugacy classes 3B, 3C, 6B, 6C, 6F, 6G, 6H, 6I, 6L, 6M, 6T, 6U are neither real nor quasi-real. The conjugacy classes 11A, 11B, 35A, 35B are quasi-real of type $j = 3$, the classes 9B, 9C, 15B, 15C, 15F, 15G, 33A, 33B, 33C, 33D are quasi-real of type $j = 4$, the classes 12B, 12C, 12E, 12F, 12I, 12J, 12N, 12O, 12R, 12S, 18A, 18B, 18C, 18D, 24C, 24D, 30B, 30C are quasi-real of type $j = 7$. The remaining conjugacy classes are real.

The group $G_2(4)$. The conjugacy classes 12B, 12C are quasi-real of type $j = 7$. The remaining conjugacy classes are real.

The Tits group. The conjugacy classes 8A, 8B are quasi-real of type $j = 5$, the conjugacy classes 16A, 16B, 16C, 16D are quasi-real of type $j = 9$. The remaining conjugacy classes are real.

The Mathieu groups. In any of the Mathieu simple groups, every conjugacy class is real or quasi-real. See Table 6 for the details concerning not real but quasi-real conjugacy classes.

The Conway groups. In the Conway groups Co_1, Co_2 and Co_3 every conjugacy class is real or quasi-real. The quasi-real not real conjugacy classes are listed in Table 7.

The Janko groups. In the Janko groups J_1 and J_2 every conjugacy class is real. In the Janko group J_3 the conjugacy classes 19A, 19B are quasi-real of type $j = 4$ and the remaining conjugacy classes are real. In the group J_4 every conjugacy class is real, with the exceptions of the following classes which are quasi-real:

(1) 7A, 7B, 21A, 21B, 35A, 35B (of type $j = 2$);
(2) 14A, 14B, 14C, 14D, 28A, 28B (of type $j = 9$);
(3) 42A, 42B (of type $j = 11$).

Table 6
Mathieu groups: quasi-real classes.

	Classes	Type
M_{11}	8A, 8B, 11A, 11B	$j = 3$
M_{12}	11A, 11B	$j = 3$
M_{22}	7A, 7B	$j = 2$
	11A, 11B	$j = 3$
M_{23}	7A, 7B, 15A, 15B, 23A, 23B	$j = 2$
	11A, 11B	$j = 3$
	14A, 14B	$j = 9$
M_{24}	7A, 7B, 15A, 15B, 21A, 21B, 23A, 23B	$j = 2$
	14A, 14B	$j = 9$

Table 7
Conway groups: quasi-real classes.

	Classes	Type
Co_1	23A, 23B, 39A, 39B	$j = 2$
Co_2	15B, 15C, 23A, 23B	$j = 2$
	14B, 14C	$j = 9$
	30B, 30C	$j = 17$
Co_3	23A, 23B	$j = 2$
	11A, 11B, 20A, 20B, 22A, 22B	$j = 3$

Table 8
Fischer groups: quasi-real classes.

	Classes	Type
Fi_{22}	11A, 11B, 16A, 16B, 22A, 22B	$j = 3$
	18A, 18B	$j = 7$
Fi_{23}	16A, 16B, 22B, 22C	$j = 3$
	23A, 23B	$j = 2$
Fi'_{24}	23A, 23B	$j = 2$
	18G, 18H	$j = 7$

The Fischer groups. In the Fischer groups Fi_{22}, Fi_{23} and Fi'_{24} every conjugacy class is real or quasi-real. The quasi-real not real conjugacy classes are listed in Table 8.

The Highman–Sims group. The conjugacy classes 11A, 11B, 20A, 20B are quasi-real of type $j = 3$. The remaining conjugacy classes are real.

The Lyons group. The conjugacy classes 11A, 11B, 22A, 22B are quasi-real of type $j = 3$, the conjugacy classes 33A, 33B are quasi-real of type $j = 4$. The remaining conjugacy classes are real.

The Harada–Norton group. Every conjugacy class is real, with the exceptions of the following classes which are quasi-real:

(1) 19A, 19B (of type $j = 4$);
(2) 35A, 35B (of type $j = 3$);
(3) 40A, 40B (of type $j = 7$).

The Held group. Every conjugacy class is real, with the exceptions of the following classes which are quasi-real:

(1) 7A, 7B, 7D, 7E, 21C, 21D (of type $j = 2$);
(2) 14A, 14B, 14C, 14D, 28A, 28B (of type $j = 9$).

The MacLaughlin group. Every conjugacy class is real, with the exceptions of the following classes which are quasi-real:

(1) 7A, 7B, 15A, 15B (of type $j = 2$);
(2) 11A, 11B (of type $j = 3$);
(3) 9A, 9B (of type $j = 4$);
(4) 14A, 14B (of type $j = 9$);
(5) 30A, 30B (of type $j = 17$).

The O'Nan group. Every conjugacy class is real, with the exceptions of the following classes which are quasi-real:

(1) 31A, 31B (of type $j = 2$);
(2) 20A, 20B (of type $j = 3$).

The Rudvalis group Ru. The conjugacy classes 16A, 16B are quasi-real of type $j = 5$. The remaining conjugacy classes are real.

The Suzuki group Suz. The conjugacy classes 6B, 6C, with centralizers of size 1296 are neither real nor quasi-real. The classes 9A, 9B are quasi-real of type $j = 4$, and the classes 18A, 18B are quasi-real of type $j = 7$. The remaining conjugacy classes are real.

The Thompson group. Every conjugacy class is real, with the exceptions of the following classes which are quasi-real:

(1) 15A, 15B, 31A, 31B, 39A, 39B (of type $j = 2$);
(2) 27B, 27C (of type $j = 4$);
(3) 24C, 24D (of type $j = 5$);
(4) 12A, 12B, 24A, 24B, 36B, 36C (of type $j = 7$);
(5) 30A, 30B (of type $j = 17$).

The Baby Monster group. The conjugacy classes 23A, 23B, 31A, 31B, 47A, 47B are quasi-real of type $j = 2$, the classes 30G, 30H are quasi-real of type $j = 17$, the classes 32C, 32D, 46A, 46B are quasi-real of type $j = 3$. The remaining conjugacy classes are real.

The Monster group. The conjugacy classes 23A, 23B, 31A, 31B, 39C, 39D, 47A, 47B, 69A, 69B, 71A, 71B, 87A, 87B, 93A, 93B, 95A, 95B, 119A, 119B are quasi-real of type $j = 2$. The classes 40C, 40D, 44A, 44B, 46A, 46B, 46C, 46D, 56B, 56C, 59A, 59B, 88A, 88B, 92A, 92B, 94A, 94B, 104A, 104B are quasi-real of type $j = 3$. The classes 62A, 62B, 78B, 78C are quasi-real of type $j = 5$. The remaining conjugacy classes are real.

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