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ALGORITHMS FOR RECOGNIZING BIPARTITE-HELLY AND BIPARTITE-CONFORMAL HYPERGRAPHS *, **

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Abstract. A hypergraph is Helly if every family of hyperedges of it, formed by pairwise intersecting hyperedges, has a common vertex. We consider the concepts of bipartite-conformal and (colored) bipartite-Helly hypergraphs. In the same way as conformal hypergraphs and Helly hypergraphs are dual concepts, bipartite-conformal and bipartite-Helly hypergraphs are also dual. They are useful for characterizing biclique matrices and biclique graphs, that is, the incident biclique-vertex incidence matrix and the intersection graphs of the maximal bicliques of a graph, respectively. These concepts play a similar role for the bicliques of a graph, as do clique matrices and clique graphs, for the cliques of the graph. We describe polynomial time algorithms for recognizing bipartite-conformal and bipartite-Helly hypergraphs as well as biclique matrices.

Keywords. Algorithms, bipartite graphs, biclique-Helly, biclique matrices, clique matrices, Helly property, hypergraphs.

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1. Introduction

It is well known that Helly hypergraphs and conformal hypergraphs are dual concepts, in the sense that a hypergraph is Helly if and only if its dual is conformal. We consider an extension of these concepts, namely (colored) bipartite-Helly and bipartite-conformal hypergraphs. The interest on these concepts can be justified both by their own, as combinatorial structures, and by their applications. These hypergraphs were explicitly employed in the characterizations of biclique matrices and biclique graphs. A biclique matrix can be viewed as a matrix representation of the maximal bicliques of a graph, in the same way as a clique matrix represents the maximal cliques of a graph. A biclique graph is the intersection graph of the maximal bicliques of a graph, in a similar way as a clique graph is the intersection graph of the maximal cliques of a graph. In fact, this paper has been motivated by the study of bicliques of a graph. Bicliques have been considered in many different contexts, for instance, in covering problems [1, 10]. Moreover, bicliques have already been studied in relation to the Helly property, as in [6, 7]. Finally, as for the matrices, we mention that clique matrices are related to interval graphs [5], Helly circular-arc graphs [4] and self-clique graphs [9]. Similarly, biclique matrices are related to the study of biclique graphs [8].

In this work, we describe polynomial-time algorithms for recognizing bipartite-Helly and bipartite-conformal hypergraphs. These algorithms can be viewed as counterparts of the known algorithms for recognizing Helly hypergraphs and conformal hypergraphs [2]. As applications of techniques described in this work, we present algorithms for recognizing biclique matrices. Furthermore, we employ the concept of bipartite-Helly hypergraph in order to prove that the problem of recognizing biclique graphs lies in $\mathcal{N}P$, a fact so far unknown.

In order to develop the ideas of bipartite-Helly and bipartite-conformal hypergraphs, we need further concepts related to bicliques and hypergraphs. For instance, to distinguish between the two parts of the bicliques of a graph, it would be natural to define the biclique matrix as being a $\{0, 1, -1\}$-matrix, instead of a $\{0, 1\}$-matrix, employed for representing the cliques. When considering hypergraphs, the bipartitions which are present throughout the work, lead to defining a bi-coloring of their vertices. We also employ the concept of a black section of a hypergraph, which plays a similar role for bipartite-conformal hypergraphs, as the known 2-section, employed for conformal hypergraphs [2]. Finally, recall that the Helly property requires the concept of pairwise-intersecting families of hypergraphs. Similarly, for the bipartite-Helly property, we need the equivalent concepts of monochromatic and bipartite-intersecting families.

The paper is divided as follows. In the next section, we present the main definitions and concepts related to the work. In Section 3, we describe the polynomial time algorithm for recognizing bipartite-Helly hypergraphs, while in Section 4, we present algorithms for recognizing bipartite-conformal hypergraphs. Two applications of these concepts are given in Section 5, namely, in the recognition of biclique matrices and in the $\mathcal{NP}$ containment proof for the biclique graph recognition problem. Some short remarks form the last section.

2. Preliminaries

Denote by $\mathcal{H}$ a hypergraph, with vertex set $V(\mathcal{H})$ and hyperedge set $E(\mathcal{H})$. Write $V(\mathcal{H}) = \{v_1, \ldots, v_n\}$ and $E(\mathcal{H}) = \{E_1, \ldots, E_m\}$. If $|E_i| = 2$, for all $1 \le i \le m$, we then say that the hypergraph is a *graph* and the hyperedges are *edges*. Usually, we denote a graph by G. For a graph G, write $e_k = v_i v_j$, with the meaning of $E_k = \{v_i, v_j\}$ for some k, and say that vertices v_i, v_j are *adjacent*. The *2-section* of a hypergraph $\mathcal{H}$ is a graph G_2, where $V(G_2) = V(\mathcal{H})$ and such that there is an edge $e_k = v_i v_j \in E(G_2)$ precisely when there exists some hyperedge $E_k \supseteq \{v_i, v_j\}$, for all $1 \le i \neq j \le n$.

For a graph G, say that $V' \subseteq V(G)$ is a *complete set* if v_i, v_j are adjacent, for all $v_i, v_j \in V'$. A *complete bipartite set* is a subset $B \subseteq V(G)$, which admits a bipartition $V_1 \cup V_2 = B$, where $v_i, v_j \in B$ are adjacent exactly when v_i, v_j belong to distinct parts of the bipartition. We restrict to *proper* bipartitions, that is, $V_1, V_2 \neq \emptyset$. A *clique* is a maximal complete set, while a *biclique* is a maximal complete bipartite set. The *neighborhood* of a vertex v of a graph is the subset of vertices adjacent to v. Denote by P_k a path formed by k vertices.

If G has c cliques $\{C_1, \ldots, C_c\}$, the *clique matrix* of G is the matrix $A \in \{0,1\}^{c \times n}$, defined as $a_{ki} = 1$ if and only if $v_i \in C_k$. Finally, if G has d bicliques $B_1, \ldots, B_d \subseteq V(G)$, the *biclique matrix* of G is the matrix $A \in \{0, 1, -1\}^{d \times n}$, where $a_{ki} = -a_{kj} \neq 0$, precisely when $v_i, v_j \in B_k$ and v_i, v_j are adjacent, for all $1 \le k \le n$ and $1 \le i \neq j \le n$.

Say that a hypergraph $\mathcal{H}$ is *conformal* if each clique of its 2-section is contained in some hyperedge of $\mathcal{H}$. Furthermore, say that $\mathcal{H}$ is *Helly* if every subfamily of pairwise intersecting hyperedges contains a common vertex.

A colored hypergraph $\mathcal{H}$ is a hypergraph in which there is a coloring $\mathcal{C}$ of the occurrences of each vertex in the hyperedges of $\mathcal{H}$, using the colors *white* and *black*. That is, if vertex v belongs to hyperedges $E_1, \ldots, E_k$, then v is assigned a color either white or black, in each of these hyperedges, and these colors are independent. Define a coloring of the edges of the 2-section G_2 of $\mathcal{H}$ as follows. Each $v_i v_j \in E(G_2)$ is *black* if there exists some edge $E_k \supseteq \{v_i, v_j\}$, where v_i and v_j have different colors in E_k; otherwise $v_i v_j$ is *white*. Define the *black section* of $\mathcal{H}$, as the subgraph G_b of G_2, containing exactly the black edges of G_2. Say that $\mathcal{H}$ is *bipartite-conformal*, relative to $\mathcal{C}$, when each biclique B of G_b is contained in some hyperedge of $\mathcal{H}$. That is, there is a hyperedge E_k such that $v_i v_j$ is an edge of B precisely when v_i, v_j have different colors in E_k. When every two vertices contained in a hyperedge of $\mathcal{H}$ with the same color are not adjacent in G_b , we say that $\mathcal{C}$ is a *compatible coloring* and that $\mathcal{H}$ is a *compatibly* colored hypergraph.

Given a $\{0, 1, -1\}^{m \times n}$-matrix A, the *associated hypergraph* $\mathcal{H}$ of A is the hypergraph having one vertex v_i for each column i and one hyperedge E_k for each row k of A, such that $v_i \in E_k$ precisely when $a_{ki} \neq 0$. Define a special coloring of the

$$M_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \; M_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

FIGURE 1. Row-similar matrices.

- V(H) = {v,t,s,r}, colors = {w,b}
- E(H) = $\{E_1, E_2, E_3, E_4, E_5, E_6\}$
- Hyperedges:$E_1 = \{v_w, t_b\}, E_2 = \{v_b, t_b\}$, $E_3 = \{s_b, t_b\}$, $E_4 = \{v_w, r_b, s_w\}$, $E_5 = \{v_b, r_w, t_w\}, E_6 = \{v_w, t_w\}$

FIGURE 2. Example of a colored hypergraph.

occurrences of each vertex in the hyperedges of $\mathcal{H}$ as follows: vertex $v_i \in V(\mathcal{H})$ is *white in* E_k when $a_{ki} = 1$ and v_i is *black in* E_k when $a_{ki} = -1$. When $v_i \notin E_k$ then v_i is uncolored for E_k. Such a coloring and the coloring of edges of its 2-section, is called the *canonical coloring of* $\mathcal{H}$. We also employ special concepts related to matrices, as follows.

Let A, A' be $\{0, 1, -1\}^{m \times n}$-matrices. Denote by A_k the vector consisting of row k of A. Say that row k is *dominated* by row l, when $a_{ki} = 1$ implies $a'_{li} = 1$ and $a_{ki} = -1$ implies $a'_{li} = -1$, for all $1 \leq i \leq n$, where $A'_l = A_l$ or $A'_l = -A_l$. In general, say that A, A' are *row-similar* when $A_k = A'_k$ or $A_k = -A'_k$, for all $1 \leq k \leq m$. In Figure 1 there is an example of matrices row-similar to M_1. In general, denote by M_1^* any matrix which is row-similar to M_1.

Remark that whenever A, A' are two row-similar matrices then the 2-sections G_2, G'_2 of their corresponding associated hypergraphs are isomorphic. Moreover, if $e \in E(G_2)$ and $e' \in E(G'_2)$ are two corresponding edges in an isomorphism $G_2 \cong G'_2$ then they have identical colors in the respective canonical colorings.

Given a colored hypergraph $\mathcal{H}$ and a coloring $\mathcal{C}$ of it, say that $\mathcal{C}$ *bicovers* vertices of $\mathcal{H}$ if for each v, there are hyperedges E_i, E_j such that $v \in E_i \cap E_j$ and v has different colors in E_i and E_j. On the other hand, a subfamily of hyperedges $\mathcal{E} \subseteq E(\mathcal{H})$ is *monochromatically intersecting* if, for any two hyperedges $E_i, E_j \in \mathcal{E}$, either $E_i \cap E_j = \emptyset$ or each $v \in E_i \cap E_j$ has the same color in both E_i and E_j. Consider a bipartition $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ of $\mathcal{E}$. Say that $\mathcal{E}$ is *bipartite-intersecting* if $\mathcal{E}_1, \mathcal{E}_2$ are both monochromatically intersecting, and for every pair of hyperedges $E_1 \in \mathcal{E}_1$, $E_2 \in \mathcal{E}_2$, there exists a vertex $v \in E_1 \cap E_2$, such that v has different colors in E_1 and E_2. Finally, say that $\mathcal{H}$ is *bipartite-Helly* if $\mathcal{C}$ is compatible and every bipartite-intersecting subfamily $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \subseteq E(\mathcal{H})$ contains a common vertex.

In Figure 2, there is an example of a colored hypergraph $\mathcal{H}$, using colors *white* and *black*, where v_w and v_b mean that vertex v is colored *white* and *black*,

$$A_1 = \begin{pmatrix} v_1 & v_2 & w_1 & w_2 & w_3 & w_4 \\ 1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 & -1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} v_1 & v_2 & w_1 & w_2 & w_3 & w_4 \\ 1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 & -1 & 0 \\ 1 & 1 & 0 & -1 & -1 & 0 \end{pmatrix}$$

FIGURE 3. $\{0, 1, -1\}$ matrices.

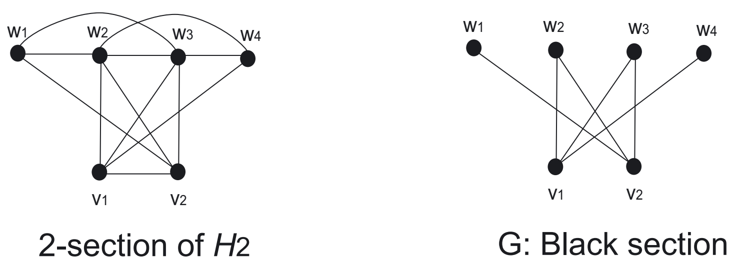

FIGURE 4. Graphs $G_2(\mathcal{H}_2)$ and $G_b(\mathcal{H}_2)$.

respectively. Observe that $E(\mathcal{H})$ bicovers $V(\mathcal{H})$. However, examining the coloring of the hyperedges E_1 and E_2, we conclude that the coloring is not compatible. On the other hand, the coloring restricted to the partial hypergraph formed by the hyperedges E_1 and E_3 is compatible. The subfamily $\{E_1, E_6\}$ is not monochromatically intersecting. On the other hand, $\{E_3\} \cup \{E_4, E_6\}$ and $\{E_5\} \cup \{E_1, E_4\}$ are examples of bipartite-intersecting subfamilies of $E(\mathcal{H})$. The latter contains a common element, while the former does not, meaning that $\mathcal{H}$ is not bipartite-Helly.

Figure 3 illustrates an example of a $\{0, 1, -1\}$ matrix with dominated rows. The last row of A_1 is dominated by the first row. The hypergraphs $\mathcal{H}_1$, $\mathcal{H}_2$, associated to the matrices A_1 and A_2, respectively, have as vertex sets $V(\mathcal{H}_1) = V(\mathcal{H}_2) = \{v_1, v_2, w_1, w_2, w_3, w_4\}$, and hyperedges $\mathcal{H}_1 = \{E_1, E_2, E_3\}$, $\mathcal{H}_2 = \{E_1, E_2, E_3'\}$, where $E_1 = \{v_1, w_2, w_3, w_4\}$, $E_2 = \{v_2, w_1, w_2, w_3\}$, $E_3 = \{v_1, w_2, w_4\}$, and $E_3' = \{v_1, v_2, w_2, w_3\}$. In Figure 4, we show the 2-section G_2 of $\mathcal{H}_2$ and the black section G_b of the hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$. Although A_1 and A_2 and their corresponding 2-sections are distinct, their black sections coincide. Observe that $\mathcal{H}_1$ is not bipartite-conformal, and that A_2 is a biclique matrix of G_b.

Notice that whenever A, A' are two row-similar matrices then the 2-sections G_2, G_2' of their corresponding associated hypergraphs are isomorphic. Moreover, if $e \in E(G_2)$ and $e' \in E(G_2')$ are two corresponding edges in the isomorphism $G_2 \cong G_2'$ then they have identical colors in the respective canonical colorings.

3. Algorithm for recognizing bipartite-Helly hypergraphs

In this section we study bipartite-Helly colored hypergraphs. We give a characterization for bipartite-Helly colored hypergraphs that leads to a polynomial time algorithm for the recognition problem.

We need the following further definitions. Let $\mathcal{H}$ be a colored hypergraph of m hyperedges, n vertices and let $\mathcal{C}$ be its coloring using colors white or black. For every subset $S' = \{v_i, v_j, v_k\}$ of three vertices of $V(\mathcal{H})$, consider every triple l_i, l_j, l_k, $1 \leq i, j, k \leq m$, where $l_i, l_k l_k$ are equal to 1 or -1, with the meaning of white or black, respectively. Let $\mathcal{E}^1_{\{l_i,l_j,l_k\}}$ be the subfamily of hyperedges of $E(\mathcal{H})$ which contains at least two vertices $v_s \in S'$, $v_r \in S'$, having colors l_s, l_r, respectively. Similarly, let $\mathcal{E}^2_{\{l_i,l_j,l_k\}}$ be the subfamily of hyperedges of $E(\mathcal{H})$ which contains at least two vertices $v_s \in S'$, $v_r \in S'$, having color $-l_s$, $-l_r$, respectively. In the example of the hypergraph of Figure 2, take $b = 1$ with $w = -1$ and consider the subset of vertices $S' = \{v, t, s\}$ together with the triples b, b, b and w, w, w. Then, $\mathcal{E}^1_{\{b,b,b\}} = \{E_2, E_3\}$, $\mathcal{E}^2_{\{b,b,b\}} = \{E_5, E_6\}$.

We start with an observation.

Observation 3.1. *Let $\mathcal{H}$= $\{E_1, E_2, \ldots, E_k\}$ be a colored hypergraph. Then, $\mathcal{H}$ is compatible if and only if every bipartite-intersecting subfamily of hyperedges $\mathcal{E}'$ =$\{E_i\} \cup \{E_j\}$ is compatible.*

As a corollary of Observation 3.1, we also derive some properties on $\{0, 1, -1\}$-matrices.

Corollary 3.2. *Let A be a $\{0, 1, -1\} -$ matrix. The columns of A form a compatible family if and only if A does not contain any matrix M_1^*.*

The following Theorem characterizes colored bipartite-Helly hypergraphs.

Theorem 3.3. *A colored hypergraph $\mathcal{H}$ is bipartite-Helly if and only if*

(1) *Every bipartite-intersecting subfamily $\mathcal{E}'$ =$\{E_i\} \cup \{E_j\}$ of $\mathcal{H}$ is compatible,*
(2) *every bipartite-intersecting subfamily $\mathcal{E}'$ =$\{E_i\} \cup \{E_j, E_k\}$ has a common element, and*
(3) *every subfamily $\mathcal{E}^1_{\{l_i,l_j,l_k\}} \cup \mathcal{E}^2_{\{l_i,l_j,l_k\}}$ has a common intersection.*

Proof. If $\mathcal{H}$ is bipartite-Helly, then the first two conditions follow directly. We prove that also the third condition holds. We need to show that $\mathcal{E}^1_{\{l_i,l_j,l_k\}} \cup \mathcal{E}^2_{\{l_i,l_j,l_k\}}$ is a bipartite intersecting subfamily, for every $\{i, j, k\}$. First, we prove that $\mathcal{E}^1_{\{l_i,l_j,l_k\}}$ and $\mathcal{E}^2_{\{l_i,l_j,l_k\}}$ are monochromatically intersecting. Let $E_r, E_s \in \mathcal{E}^1_{\{l_i,l_j,l_k\}}$. Then there is a vertex, suppose v_i, which belongs to both subsets with color l_i. If E_r, E_s intersect in a vertex having different colors in these hyperedges then $\{E_r\} \cup \{E_s\}$ is a bipartite-intersecting family which is not compatible, a contradiction. Analogously, $\mathcal{E}^2_{\{l_i,l_j,l_k\}}$ is monochromatically intersecting. Finally, it remains we prove that $\mathcal{E}^1_{\{l_i,l_j,l_k\}} \cup \mathcal{E}^2_{\{l_i,l_j,l_k\}}$ is a bipartite-intersecting family. Let $E_r \in \mathcal{E}^1_{\{l_i,l_j,l_k\}}$, and $E_s \in \mathcal{E}^2_{\{l_i,l_j,l_k\}}$. Then, there is a common vertex, suppose v_j, that belongs to E_r, E_s with different colors among them since v_j has color l_j in E_r, and $-l_j$ in E_s. Since $\mathcal{H}$ is bipartite-Helly, we conclude that $\mathcal{E}^1_{\{l_i,l_j,l_k\}} \cup \mathcal{E}^2_{\{l_i,l_j,l_k,l_k\}}$ has a common vertex.

Conversely. Let $\mathcal{E}' = \mathcal{E}_1 \cup \mathcal{E}_2$ be minimal bipartite-intersecting subfamily with no common element. Then, $|\mathcal{E}_1| + |\mathcal{E}_2| \geq 4$. Consider the case where $\mid \mathcal{E}_1 \mid = \mid \mathcal{E}_2 \mid = 2$. Let v_1 be the common vertex to $\mathcal{E}_1 \setminus \{E_{i_1}\} \cup \mathcal{E}_2$. Let l_1 be the color of v_1 in $\mathcal{E}_2$ (recall that since $\mathcal{E}_1$ is monochromatically intersecting, every vertex has the same color in $\mathcal{E}_2$). Analogously, let v_2 be the common vertex to $\mathcal{E}_1 \setminus \{E_{i_2}\} \cup \mathcal{E}_2$. Let l_2 be the color of v_2 in $\mathcal{E}_2$. Finally, let v_3 be the vertex belonging to $\mathcal{E}_1 \cup \mathcal{E}_2 \setminus \{E_{j_1}\}$ and let l_3 be the color of v_3 in $\mathcal{E}_2$. Consider $\mathcal{E}^1_{\{l_1,l_2,l_3\}} \cup \mathcal{E}^2_{\{l_1,l_2,l_3\}}$. We prove that $\mathcal{E}' = \mathcal{E}_1 \cup \mathcal{E}_2$ is included in $\mathcal{E}^1_{\{l_1,l_2,l_3\}} \cup \mathcal{E}^2_{\{l_1,l_2,l_3\}}$. Subset E_{i_1} contains v_2 and v_3. Since $E_{j_2} \in \mathcal{E}_2$ contains v_1, v_2 and v_3, the colors of v_{j_2} in E_1, E_2 and E_3 are l_1, l_2, l_3 respectively. Then, as E_{j_2} intersects E_{i_1} and both contain v_2 and v_3, their colors in E_{i_1} are $-l_2$, $-l_3$ respectively. Analogously, E_{i_2} contains v_1, v_3 with colors $-l_1$, $-l_3$, respectively. Finally, E_{j_1} contains v_1, v_2 with colors l_1, l_2 respectively. It follows that $\mathcal{E}_1 \subseteq \mathcal{E}^1_{\{l_1,l_2,l_3\}}$, $\mathcal{E}_2 \subseteq \mathcal{E}^2_{\{l_1,l_2,l_3\}}$.

The case where $\mid \mathcal{E}_1 \mid \geq 3$ is similar. We consider $\mathcal{E}_1 \setminus \{E_{i_1}\} \cup \mathcal{E}_2$, $\mathcal{E}_1 \setminus \{E_{i_2}\} \cup \mathcal{E}_2$ and $\mathcal{E}_1 \setminus \{E_{i_3}\} \cup \mathcal{E}_2$ and conclude that there are elements v_1, v_2, v_3, such that $v_j \notin E_{i_j}$. Finally, let l_1, l_2, l_3 be the colors of $v_1, v_2, v_3 \in \mathcal{E}_1$, respectively. In any case, it follows that $\mathcal{E}' = \mathcal{E}_1 \cup \mathcal{E}_2$ has a common vertex, a contradiction.

Finally, by Lemma 3.1, every bipartite-intersecting subfamily is compatible. We conclude that $\mathcal{E}$ is a bipartite-Helly hypergraph. $\triangledown$.

Theorem 3.3 leads to a polynomial time algorithm for recognizing bipartite-Helly colored hypergraphs.

The algorithm is described below. For a given colored hypergraph $\mathcal{H}$, it answers YES or NO, depending on whether $\mathcal{H}$ is bipartite-Helly. Let $\mathcal{H}$ be a colored hypergraph of m hyperedges and n vertices and let $\mathcal{C}$ be a coloring with colors white and black, represented by -1 and 1. □

Algorithm 3.4. Recognizing bipartite-Helly hypergraphs

Input: *Colored hypergraph* $\mathcal{H}$, $V(\mathcal{H}) = \{v_1, \ldots, v_n\}$ *and* $E(\mathcal{H}) = \{E_1, \ldots, E_m\}$

(1) **for** *every bipartite-intersecting subfamily* $\{E_1\} \cup \{E_j\}$ **do**
  if $\{E_1\} \cup \{E_j\}$ *is not compatible* **then return** *NO*
(2) **for** *every bipartite-intersecting subfamily* $\{E_i\} \cup \{E_j, E_k\}$ **do**
  if $E_i \cap E_j \cap E_k = \emptyset$ **then return** *NO*
(3) **for** *every* v_i, v_j, v_k, $1 \leq i, j, k \leq n$ *and every* $l = 1, -1$ **do**
  construct $\mathcal{E}^1_{\{l_i,l_j,l_k\}}$, $\mathcal{E}^2_{\{l_i,l_j,l_k\}}$
  if $\mathcal{E}^1_{\{l_i,l_j,l_k\}} \cap \mathcal{E}^2_{\{l_i,l_j,l_k\}} = \emptyset$ **then return** *NO*
return *YES*

The complexity of the above algorithm can be evaluated as follows. As a preprocessing, we compute the black section G_b of G. For this purpose, for each pair of vertices $v_a, v_b \in V(\mathcal{H})$, verify if some hyperedge E_i contains both v_a, v_b. Consequently, we can construct G_b in $O(mn^2)$ time. Next, we determine the complexity of verifying whether the coloring of a hypergraph $\mathcal{H}$ is compatible, as follows. Let E_i be a hyperedge of $\mathcal{H}$ and consider the bipartition of the vertices of E_i induced by the two colors of the hypergraph. The coloring of E_i is not compatible precisely when there is an edge of G_b formed by a pair of vertices having the same color in $\mathcal{H}$. Since there are $O(n^2)$ pairs of vertices and m hyperedges, we conclude that compatibility can be checked in $O(mn^2)$ time, for the entire hypergraph.

Now, we examine the steps of the algorithm. For Step 1, we need to consider each of the bipartite-intersecting subfamilies $E_i \cup E_j$ of $\mathcal{H}$. Therefore we compute the intersection $E_i \cap E_j$, and for each $v \in E_i \cap E_j$ verify if v has the same color in both edges. If the answer is positive or $E_i \cap E_j = \emptyset$ then $E_i \cup E_j$ is monochromatically intersecting. Consequently, Step 1 can be computed in $O(mn^2)$ time.

For Step 2, we compute first the bipartite-intersecting subfamilies of the form $\{E_1\} \cup \{E_j, E_k\}$. Using similar arguments as above, we conclude that these operations can be performed in $O(m^3 n)$ time.

Finally, for Step 3, we need to consider each triple $v_i, v_j.v_k \subseteq V(\mathcal{H})$ and each of the triples l_i, l_j, l_k, corresponding to colors white and black, respectively. Then we need to examine each hyperedege of $\mathcal{H}$ in order to construct the subfamily of hyperedges $\mathcal{E}^1_{l_i,l_j.l_k}$ and $\mathcal{E}^2_{l_i,l_j.l_k}$, employing the definitions. There are $O(n^3)$ triples and m hyperedges. Consequently, Step 3 requires $O(mn^3)$ time.

Therefore, Algorithm 3.4 requires $O(mn^3 + m^3 n)$ time.

4. Algorithms for recognizing bipartite-conformal hypergraphs

In this section we study bipartite conformal hypergrahs. The Helly property is the dual concept of conformality for hypergraphs. Similarly, we relate the bipartite-Helly property to the bipartite-conformal condition. We derive an algorithm for recognizing bipartite-conformal hypergraphs having compatible colorings. We need the following definitions.

The *dual* of a hypergraph $\mathcal{H}$ is the hypergraph $\mathcal{H}^*$, where $V(\mathcal{H}^*) = E(\mathcal{H})$, $E(\mathcal{H}^*) = V(\mathcal{H})$, and such that for $v_i^* \in V(\mathcal{H}^*)$ and $E_j^* \in E(\mathcal{H}^*)$, $v_i^* \in E_j^*$ precisely when $v_j \in E_i \in E(\mathcal{H})$. If $\mathcal{H}$ is a hypergraph with a coloring $\mathcal{C}$, then its dual hypergraph $\mathcal{H}^*$ has a coloring $\mathcal{C}^*$ defined as follows. Let $v_i \in V(\mathcal{H})$ and $E_j \in E(\mathcal{H})$, where $v_i \in E_j$. Denote by v_j^* and E_i^* the vertex and hyperedge of $\mathcal{H}^*$, corresponding to E_j and v_i, respectively. Then the color of v_j^* in E_i^* is precisely the same as the color of v_i in E_j.

Theorem 4.1. *Let $\mathcal{H}$ be a colored hypergraph, $\mathcal{C}$ its coloring and $\mathcal{H}^*$ its dual colored hypergraph. Then $\mathcal{H}$ is compatible and $\mathcal{H}$ is bipartite-conformal if and only if $\mathcal{H}^*$ is bipartite-Helly.*

Proof. Observe that $\mathcal{H}$ is compatible if and only if $\mathcal{H}^*$ is compatible. We need to prove that $\mathcal{H}$ is bipartite-conformal if and only every bipartite-intersecting family of hypedeges of $\mathcal{H}^*$ has a common vertex.

Suppose $\mathcal{H}$ is bipartite-conformal. Let G_b be its black section. Consider $\mathcal{E}_1 \cup \mathcal{E}_2$ a bipartite-intersecting family of hyperedges of $\mathcal{H}^*$, where $\mathcal{E}_1 = \{E^*{}_{i_1}, \ldots, E^*{}_{i_k}\}$, $\mathcal{E}_2 = \{E^*{}_{i_{k+1}}, \ldots, E^*{}_{i_s}\}$.

Since $\mathcal{E}_1, \mathcal{E}_2$ are monochromatically intersecting, $V_1 = \{v_{i_1}, \ldots v_{i_k}\}$, $V_2 = \{v_{i_{k+1}}, \ldots, v_{i_s}\}$ are both independent sets in G. On the other hand, since every $E_i^* \in \mathcal{E}_1$, $E_j^* \in \mathcal{E}_2$ intersect in a different color, vertices $v_i \in V_1, v_j \in V_2$ are adjacent in G. It follows that V_1, V_2 induce a complete bipartite subgraph in G. Since $\mathcal{H}$ is bipartite-conformal, there is a hyperedge E_t which contains the vertices of $V_1 \cup V_2$. It follows that E_t in $\mathcal{H}^*$ is a common vertex of $\mathcal{E}_1 \cup \mathcal{E}_2$.

Conversely, let B be a biclique of G with bipartition $V_1 = \{v_{i_1}, \ldots, v_{i_s}\}$, $V_2 = \{v_{i_{s+1}}, \ldots, v_{i_t}\}$. Consider $\mathcal{E}_1 = \{E^*{}_{i_1}, \ldots, E^*{}_{i_s}\}$, $\mathcal{E}_1 = \{E^*{}_{i_{s+1}}, \ldots, E^*{}_{i_t}\}$, hyperedges of $\mathcal{H}^*$. Since V_1, V_2 are independent sets, $\mathcal{E}_1, \mathcal{E}_2$ are monochromatically intersecting. Since every vertex of V_1 intersects every vertex of V_2, $\mathcal{E}_1 \cup \mathcal{E}_2$ is a bipartite-intersecting family in $\mathcal{H}^*$. By hypothesis there is a vertex E_t common to $\mathcal{E}_1, \mathcal{E}_2$. Then, edge E_t of $\mathcal{H}$ contains the vertices of B. $\triangledown$ $\square$

Observation 4.2. *Let A be a $\{0, 1, -1\}$-matrix which does not contain any matrix M_1^* as a submatrix. Let $\mathcal{H}$ be its associated colored hypergraph. Then $\mathcal{H}$ is bipartite-conformal if and only if the columns of A are bipartite-Helly.*

The dual relation between the bipartite-Helly and bipartite-conformal conditions, motivates the theorem below. As before, given a colored hypergraph $\mathcal{H}$, for every subset $\mathcal{E}' = \{E_i, E_j, E_k\}$ of three hyperedges of $\mathcal{H}$, $1 \le i, j, k \le m$, consider all distinct triples l_i, l_j, l_k, where each l_i, l_j or l_k is either equal to 1 or -1 (white or black, respectively). Let $\mathcal{V}^1_{\{l_i,l_j,l_k\}}$ be the subfamily of vertices of $\mathcal{H}$ which belong to at least two hyperedges $E_s \in \mathcal{E}'$, $E_r \in \mathcal{E}'$, with colors l_s, l_r, respectively. Similarly, let $\mathcal{V}^2_{\{l_i,l_j,l_k\}}$ be the subfamily of vertices of $\mathcal{H}$ which belong to at least two hyperedges $E_s \in \mathcal{E}'$, $E_r \in \mathcal{E}'$, with colors $-l_s$, $-l_r$, respectively. In the example of he hypergraph of Figure 2, consider $\mathcal{E}' = \{E_1, E_3, E_4\}$ and $l_1 = 1, l_3 = 1, l_4 = -1$. Assuming $b = 1$ and $w = -1$, then $\mathcal{V}^1_{\{l_1,l_3,l_4\}} = \{t, s\}$ and $\mathcal{V}^2_{\{l_1,l_3,l_4\}} = \emptyset$

It follows the characterization for bipartite-conformal hypergraphs, with a compatible coloring $\mathcal{C}$.

Theorem 4.3. *Let $\mathcal{H}$ be a colored hypergraph, $\mathcal{C}$ a compatible coloring of it, and G_b the black section of $\mathcal{H}$. Then $\mathcal{H}$ is bipartite-conformal if and only if every induced P_3 of G_b is contained in a hyperedge of $\mathcal{H}$ and every subfamily $\mathcal{V}^1_{\{l_i,l_j,l_k\}} \cup \mathcal{V}^2_{\{l_i,l_j,l_k\}}$ is contained in a hyperedge of $\mathcal{H}$.*

Proof. The proof is similar as that of Theorem 3.3. It is clear that every P_3 is contained in an hyperedge of $\mathcal{H}$. First, observe that $\mathcal{V}^1_{\{l_i,l_j,l_k\}}$ and $\mathcal{V}^2_{\{l_i,l_j,l_k\}}$ induce independent sets in G_b, since $\mathcal{C}$ is a compatible coloring.

Finally, let $v_r \in \mathcal{V}^1_{\{l_i,l_j,l_k\}}$, $v_s \in \mathcal{V}^2_{\{l_i,l_j,l_k\}}$. There is a hyperedge in $\mathcal{H}$ that contains v_r, v_s with different colors, meaning that in G_b they are adjacent. Then, the complete bipartite subgraph $\mathcal{V}^1_{\{l_i,l_j,l_k\}} \cup \mathcal{V}^2_{\{l_i,l_j,l_k\}}$ must be contained in a hyperedge of $\mathcal{H}$.

Conversely, let B' be a minimal bipartite subgraph of a biclique B with bipartitions $V'_1 \subseteq V_1$, $V'_2 \subseteq V_2$ ($V'_1, V'_2 \neq \emptyset$) not contained in a hyperedge. Let E_1 be the hyperedge containing $V'_1 \setminus \{v_{i_1}\} \cup V'_2$. Let l_1 be the color of vertices of V'_2 in E_1 (recall that since V'_2 is an independent set, every vertex has the same color in E_1). Analogously, let E_2 be the hyperedge containing $V'_1 \setminus \{v_{i_2}\} \cup V'_2$. Let l_2 be the color of the vertices of V'_2 in E_2. Finally, let E_3 be the hyperedge containing $V'_1 \cup V'_2 \setminus \{v_{j_1}\}$. Let l_3 be the color of vertices of V'_2 in E_3. Consider the bipartite-intersecting family $\mathcal{V}^1_{\{l_1,l_2,l_3\}} \cup \mathcal{V}^2_{\{l_1,l_2,l_3\}}$. The proof follows observing that B' is included in $\mathcal{V}^1_{\{l_1,l_2,l_3\}} \cup \mathcal{V}^2_{\{l_1,l_2,l_3\}}$, and therefore is also included in a hyperedge, a contradiction.$\triangledown$

The following algorithm for recognizing bipartite-conformal hypergraphs having a compatible coloring follows from Theorem 4.3. As before, the algorithm returns *YES* or *NO*, in case of a positive and negative recognitions, respectively. □

Algorithm 4.4. Recognizing bipartite-conformal hypergraphs

Input: *Colored hypergraph* $\mathcal{H}$, $|V(\mathcal{H})| = n$ *and* $|E(\mathcal{H})| = m$

(1) *construct the black-section* G_b *of* $\mathcal{H}$
(2) **for** *every triple of vertices* t*, forming an induced* P_3 *of* G_b **do**
    if t *is not contained in a hyperedge of* $\mathcal{H}$ **then return** *NO*
(3) **for** *every* l_i, l_j, l_k, $1 \leq i,j,k \leq m$, $l = 1, -1$ **do**
    construct sets $\mathcal{V}^1_{\{l_i,l_j,l_k\}}$, $\mathcal{V}^2_{\{l_i,l_j,l_k\}}$
    if $\mathcal{V}^1_{\{l_i,l_j,l_k\}} \cup \mathcal{V}^2_{\{l_i,l_j,l_k\}}$ *is not contained in a hyperedge of* $\mathcal{H}$
        then return *NO*
return *YES*

We evaluate the complexity of the algorithm. Step 1 requires $O(mn^2)$ time, as described in Section 3. In Step 2, we generate each triple of vertices, verify if it forms an induced P_3 and if there is any hyperedge containing it. Clearly, these operations can be done in $O(mn^3)$ time. Finally, for Step 3, we generate all triples l_i, l_j, l_k, for $1 \leq i,j \leq m$ and $l = 1, -1$, construct the subset of vertices $\mathcal{V}^1_{l_1,l_2,l_3}$, $\mathcal{V}^2_{l_1,l_2,l_3}$, and check if some hyperedge E_t contains $\mathcal{V}^1_{l_1,l_2,l_3} \cup \mathcal{V}^2_{l_1,l_2,l_3}$. There are $O(m^3)$ triples $l_i, l_j.l_k$, for each triple, we perform intersections of the hyperedges E_i, E_j, E_k, requiring $O(n)$ time, in order to construct $\mathcal{V}^1_{l_1,l_2,l_3}$ and $\mathcal{V}^2_{l_1,l_2,l_3}$. Additionally, verify, in $O(nm)$ time, if some hyperedge E_t contains $\mathcal{V}^1_{l_1,l_2,l_3} \cup \mathcal{V}^2_{l_1,l_2,l_3}$, That is, Step 3 requires $O(m^4n)$ time. Consequently, the complexity of the algorithm is $O(mn^3 + m^4n)$.

Say that a $\{0, 1, -1\}$-matrix A is *bipartite* when it admits a row-similar matrix A', such that no column of A' has both entries 1 and -1. Say that a hypergraph $\mathcal{H}$ is *bipartite* if $\mathcal{H}$ is the hypergraph associated to some bipartite matrix.

Next, we describe an algorithm for recognizing if a bipartite hypergraph $\mathcal{H}$ is bipartite-conformal. The algorithm is conceptually simple and employs the relationship between conformal and bipartite-conformal hypergraphs. The input of the algorithm is the bipartite matrix A to which $\mathcal{H}$ is associated. We transform A into a convenient matrix A', whose associated hypergraph is conformal if and only if $\mathcal{H}$ is bipartite-conformal. The algorithm answers YES or NO, respectively, to each of these alternatives.

Algorithm 4.5. Recognizing bipartite-conformal hypergraphs

Input: $\{0, 1, -1\}^{m\times n}$- *matrix* A

(1) *Partition the set of columns of* A *into two subsets* V_1, V_{-1}*, corresponding to the* $\{0, 1\}$*-columns and* $\{0, -1\}$*-columns, respectively*
(2) *Let* A' *be the matrix obtained from* A *by adding two extras rows, one containing 1's in all columns of* V_1 *and the other containing* -1*'s in all columns of* V_{-1}*, and having O's in the remaining columns*
(3) *Construct the associated hypergraph* $\mathcal{H}'$ *of* A'
(4) **if** $\mathcal{H}'$ *is conformal* **then return** *YES* **else return** *NO*

It is straightforward to conclude that the dominating operation of the above algorithm is its last step. So, the complexity of the algorithm is that of recognizing if $\mathcal{H}'$ is conformal. The latter is equivalent to recognizing if its dual is Helly, which can be done in $O(m^4n)$ time [2].

5. Applications

In this section, we describe applications of the concepts of bipartite-Helly and bipartite-conformal hypergraphs. Two kinds of applications are given. In the first, bipartite-conformal hypergraphs are employed in order to recognize biclique matrices. On the other hand, we use bipartite-Helly hypergraphs to prove a result on biclique graphs, that is, the intersection graphs of the bicliques of a graph. We show that deciding whether a given graph is a biclique graph is in the class $\mathcal{N}P$.

First, we consider the recognition of biclique matrices. These matrices have been employed in the characterization of biclique graphs [8]. Besides, they might be useful in approaching covering problems of bicliques, through matrices. Such covering problems have been considered, for example in [1,10]. A characterization of these matrices is given in terms similar to those used in the characterization of clique matrices, as below.

Theorem 5.1. [8] *Let* $A \in \{0, 1, -1\}^{n\times m}$*-matrix, and* $\mathcal{H}$ *its associated hypergraph. Then* A *is a biclique matrix of some graph if and only if*

(i) *Each row of* A *has at least one 1 and at least one* -1*,*
(ii) A *has no dominated rows,*
(iii) A *does not contain a* M_1^* *as a submatrix, and*
(iv) $\mathcal{H}$ *is bipartite-conformal, relative to its canonical coloring.*

The following property is a consequence of Theorem 5.1

Corollary 5.2. [8] *A matrix is the biclique matrix of some graph if and only if it is the biclique matrix of the 2-section of its associated hypergraph.*

For recognizing biclique matrices, we describe two algorithms. The first is based on Theorem 5.1. The second follows from Corollary 5.2 and employs an algorithm for generating the bicliques of a graph.

The first algorithm for recognizing biclique matrices follows directly from Theorem 4.3 and Theorem 5.1, by checking conditions (i),(ii), (iii) and (iv), for the associated hypergraph of A. We recall that a $\{0, 1, -1\}$-matrix A does not contain M_1^* as a submatrix if and only if the canonical coloring of its associated hypergraph $\mathcal{H}$ is compatible.

Algorithm 5.3. Recognizing biclique matrices

Input*:* $\{0, 1, -1\}^{m\times n}$*- matrix* A

(1) **if** *any row has no 1's or no 0's* **then return** *NO*
(2) **if** *A has a dominated row* **then return** *NO*
(3) *Construct the associated hypergraph $\mathcal{H}$ of A and its canonical coloring*
(4) **if** *the canonical coloring of $\mathcal{H}$ is not compatible* **then return** *NO*
(5) **if** *$\mathcal{H}$ is not bipartite-conformal, relative to its canonical coloring*
then return *NO*
(6) **return** *YES.*

We determine the complexity of the algorithm. For Step 1, clearly, $O(mn)$ steps are needed. For Step 2, for each row of A, examine all the rows, which means $O(m^2n)$ time, overall. The associated hypergraph $\mathcal{H}$ and its canonical coloring can clearly be constructed in time proportional to the size of $\mathcal{H}$, *i.e.* $O(mn)$. Checking if a coloring is compatible can be done in $O(mn^2)$ time, as described in Section 3. Finally, by Algorithm 4.4, to verify if a hypergraph is bipartite-conformal requires $O(mn^3 + m^4n)$ time, which is therefore the complexity of the present algorithm.

Alternatively, we can recognize if A is a biclique matrix by employing Corollary 2. The idea is to construct the biclique matrix of the black section G_b of the associated hypergraph of A, and verify if these two matrices are row-similar. In order to perform this operation, iteratively generate each biclique of G_b, construct its row entry B' in the biclique matrix of G_b, and search matrix A looking for a row similar to B'. If no such row exists then A is not a biclique matrix. Otherwise, remove from A the row similar to B'. If A becomes empty exactly after the generation of the last biclique of G_b then A is a biclique matrix, otherwise it is not.

The formulation below describes the process.

Algorithm 5.4. Recognizing biclique matrices
Input: $\{0, 1, -1\}^{m\times n}$ *matrix* A

(1) *construct the associated hypergraph* $\mathcal{H}$ *of* A*, its canonical coloring and black section* G_b
(2) **for** *each biclique* B *of* G_b **do**
    if $A = \emptyset$ **then return** *NO*
    construct a $\{0, 1, -1\}$*-vector* B' *corresponding to an entry of* B *in a biclique matrix of* G_b
    If A *contains a row* A_i *which is row-similar to* B'
        then *remove* A_i *from* A **else return** *NO*
(3) *if* $A = \emptyset$ **then return** *YES* **else return** *NO*

The complexity of the above algorithm can be verified as follows. Step 1 requires $O(mn^2)$ *time, as already known. In Step 2, we need to generate the bicliques of* G_b*. This can be done in* $O(n^3)$ *time per biclique [3]. To check if the entry* B' *of the biclique matrix of* G_b *is row similar to some row* A_i *of* A *requires* $O(mn)$ *time. Consequently, Step 2 requires* $O(mn^3)$ *time. Therefore the complexity of the algorithm is* $O(mn^3 + m^2n)$*.*

Finally, consider the second application, on biclique graphs. A characterization of these graphs has been described as below.

Theorem 5.5. [8] *Let* G *be a graph with no isolated vertices. Then* G *is a biclique graph if and only if* G *contains a family* $\mathcal{F}$ *of not necessarily distinct complete subgraphs covering the edges of* G*, whose associated hypergraph* $\mathcal{H}_{\mathcal{F}}$ *admits a coloring* $\mathcal{C}$ *such that*

(1) $\mathcal{H}_{\mathcal{F}}$ *bicovers* $V(G)$*.*
(2) $\mathcal{H}^*_{\mathcal{F}}$ *has no dominated hyperedeges*
(3) $\mathcal{F}$ *is a compatible coloring.*
(4) $\mathcal{H}_{\mathcal{F}}$ *is bipartite-Helly, relative to* $\mathcal{C}$*.*

The complexity of recognizing biclique graphs is unknown. However, we prove that the problem belongs to $\mathcal{NP}$.

Theorem 5.6. *Let* G *be a graph with* n *vertices. The problem of determining if* G *is a biclique graph is contained in* $\mathcal{NP}$*.*

Proof. A certificate for G being a biclique graph is a family $\mathcal{F}$ of complete subgraphs of G, satisfying the conditions of Theorem 5.5. First, we show that we can restrict to families $\mathcal{F}$ of size $O(n + m)$, where $V(G) = n$. For every vertex v_i, choose subsets $F^{i_W} \in \mathcal{F}$ and $F^{i_B} \in \mathcal{F}$ containing vertex v_i with the color white and black, respectively ($2n$ subsets). For every edge v_iv_j, consider a subset $F^{ij} \in \mathcal{F}$ that contains v_iv_j (m subsets). Finally, for every pair of adjacent vertices v_i, v_j, consider two subsets $F^{i,j} \in \mathcal{F}$ and $F^{j,i} \in \mathcal{F}$, such that $v_i \in F^{i,j}$, $v_j \in F^{j,i}$ and $v_j \notin F^{i,j}$ and $v_i \notin F^{j,i}$ ($2m$ subsets).

The subfamily $\mathcal{F}' = \{F^{i_W}, F^{i_B}, F^{i,j}, F^{i,j}, F^{i,j}\}_{i,j=1,\ldots,n}$ verifies conditions (1) – (4) of Theorem 5.5 and contains $O(n+m)$ subsets. By employing Theorem 5.1 the proof is completed. $\triangledown$ □

6. Conclusions

We have considered bipartite-Helly and bipartite-conformal hypergraphs with compatible colorings. For both types of hypergraphs, we have described characterizations and recognition algorithms. The proposed algorithms run in polynomial time in the size of the hypergraphs. As applications, we have formulated polynomial time algorithms for recognizing biclique matrices. Finally, employing the concept of bipartite-Helly hypergraphs, we have proved that the recognition problem for biclique graphs lies in $\mathcal{NP}$.

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