A generalization of the membrane-plate analogy to non-homogeneous polygonal domains consisting of homogeneous subdomains

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The well-known membrane-plate analogy that relates the natural frequencies when dealing with polygonal homogeneous domains is herein extended to non-homogeneous systems comprised of homogeneous subdomains. The analogy is generalized and demonstrated and it is shown that certain restrictions among the frequency parameters of the membranes and plates arise. Several examples of membranes and plates with interfaces separating areas with different material properties are numerically solved with different approaches. The subdomains are separated by straight, curved, and closed line interfaces. It is shown that the analogy is verified provided that the restrictions are satisfied. The analogy is first demonstrated and presented as a practical methodology to find the natural frequencies of membranes knowing the corresponding ones of the plates or vice versa. Second, the plate and membrane vibration problems, governed by the bi-Laplacian and Laplacian differential operators, respectively, can be solved without distinction, though under certain conditions, i.e., solve one of them and deduce the other using the analogy. Various numerical examples validate the analogy. © *2010 Acoustical Society of America.* [DOI: 10.1121/1.3337222]

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I. INTRODUCTION

The *membrane-plate analogy* is known[1] for the problem of natural vibrations between simply supported plane plates and fixed membranes in homogeneous polygonal domains. Regarding the origin of the analogy, Timoshenko and Woinowsky-Krieger[2] referenced an earlier work of Marcus.[3] Also Conway and Farham[4] are frequently cited for his work on the vibration problem. On the other hand, the non-homogeneous membranes have received the attention of researches.[5–7] However, the above-mentioned analogy may be lost when some complexities are present, and extensions to it have been addressed by several authors, such as compression buckling, hygrothermal buckling, and vibration of sandwich plates,[8] laminated plates,[9] shallow shells,[10] intermediate partial supports,[11] or, as is the topic of this study, the consideration of non-homogeneous domains made of homogeneous subdomains.[12] Here, the original idea is extended to non-homogeneous domains (distributed in homogeneous regions). In particular, polygonal domains with various interfaces are tackled. Despite that the study deals with a theoretical demonstration, it also proposes a methodology of practical interest for non-homogeneous domains. Let us assume constant thickness h_0 (the dimension in the z direction) and accept that T_0 (force per unit of length) when used in the Helmholtz equation is uniform and the same for all the regions. That is, each region is made of a homogeneous material but, in general, with the following physical characteristics varying from region to region, i.e., mass density $\rho_j = \rho_0 r_j$, Young's modulus $E_j = E_0 e_j$, and Poisson's ratio $\nu_j = \nu_0 n_j$, where ρ_0, E_0, and ν_0 are arbitrary reference constants and ρ_j, E_j, and ν_j are constant within each region. A schematic drawing of a rectangular domain with one interface (two regions, I and II) is shown in Fig. 1, though obviously the number is arbitrary. The most general analogy that will be found in what follows includes, as a particular case, the classical one (homogeneous materials) and will yield the restrictions to be satisfied in order for the new analogy to be valid.

In short, the analogy object of the present study is within the following definition: The aim is to find a relationship among the mode shapes of the membrane v_j and plate w_j, where j denotes one of the homogeneous domains.

$$v_j = G_j(w_j), \tag{1}$$

in such way that some condition among the non-dimensional frequency parameters Ω of both systems (subscripts m and p stand for membrane and plate, respectively)

$$\Omega_p = g(\Omega_m) \tag{2}$$

exists. It can be shown below that one is led to additional restrictions that will be only verified for homogeneous domains. This approach is not shown herein for the sake of brevity. Alternatively, a more general expression will be proposed,

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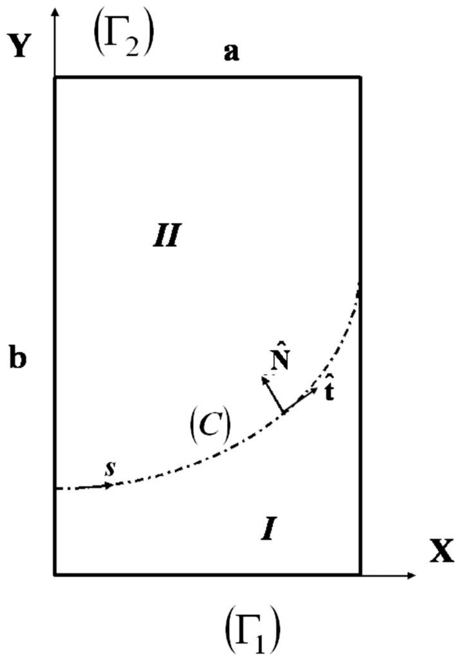

FIG. 1. Membrane configuration for the particular case of two regions.

$$v_j = \alpha_j \nabla^2 w_j + \beta_j w_j, \tag{3}$$

in which the constants α_j and β_j will be found provided that the boundary conditions (BCs) and the continuity conditions (CCs) are satisfied for membranes and find any $\Omega_p = g(\Omega_m)$. At the end, we will find the following relationship between membrane and plate non-dimensional frequency parameters:

$$\Omega_p = K\Omega_m^2, \tag{4}$$

in which K is a constant value that is found with the material properties involved in the problem.

Although not included, the analogy is also valid for the plate buckling of plates with uniform plane load.

II. GOVERNING EQUATIONS

The governing equation of the elastic, homogeneous, isotropic, vibrating plate (Germain–Lagrange)[2,13] with constant thickness h_0 in z direction is

$$\nabla^2(\nabla^2 w_j) - \Omega_p^2 f_j^2 w_j = 0, \tag{5}$$

where $w_j = w_j(X,Y)$ is the plate mode shape for each j region that should satisfy the BCs and the CCs at the interface. In what follows, and without loss of generality, it will be assumed that $j = \mathrm{I}, \mathrm{II}$. The adopted plate frequency parameter is

$$\Omega_p \equiv \sqrt{\frac{\rho_0 h_0}{D_0}}\omega_p a^2, \tag{6}$$

where ω_p are the circular natural frequencies of the whole plate, a is an arbitrary reference length, and the flexural rigidity D_0 writes

$$D_0 \equiv \frac{E_0 h_0^3}{12(1-\nu_0^2)} \tag{7}$$

and

$$f_j^2 \equiv \frac{r_j(1-\nu_0^2 n_j^2)}{e_j(1-\nu_0^2)}. \tag{8}$$

In turn, the differential equations governing the natural vibration of a homogeneous membrane of constant thickness h_0 under a constant force per unit of length T_0 (Helmholtz equation[14]) is

$$\nabla^2 v_j + \Omega_m^2 r_j v_j = 0, \tag{9}$$

where $v_j = v_j(X,Y)$ denotes the membrane mode shape of each region, which should satisfy the BCs on the boundaries and the CCs at the interface, and where the adopted membrane frequency parameter is

$$\Omega_m = \sqrt{\frac{\rho_0 h_0}{T_0}}\omega_m a, \tag{10}$$

where ω_m are the successive circular natural frequencies of the whole membrane. It is accepted, for the sake of simplicity, that both plates and membranes have the same mass distribution.

In this problem, the BCs are considered simply supported for the plate and fixed for the membrane. That is, with $j = \mathrm{I}, \mathrm{II}$ (see Fig. 1),

$$w_j(0,y) = w_j(a,y) = 0, \tag{11a}$$

$$\frac{\partial^2 w_j}{\partial x^2}(0,y) = \frac{\partial^2 w_j}{\partial x^2}(a,y) = 0, \tag{11b}$$

$$w_{\mathrm{I}}(x,0) = w_{\mathrm{II}}(x,b) = 0, \tag{11c}$$

$$\frac{\partial^2 w_{\mathrm{I}}}{\partial y^2}(x,0) = \frac{\partial^2 w_{\mathrm{II}}}{\partial y^2}(x,b) = 0. \tag{11d}$$

It is important to observe that, from BC (11), the following consequences yield

$$\nabla^2 w_j(0,y) = \nabla^2 w_j(a,y) = 0, \tag{12a}$$

$$\nabla^2 w_{\mathrm{I}}(x,0) = \nabla^2 w_{\mathrm{II}}(x,b) = 0. \tag{12b}$$

Equations (12a) and (12b) are derived from Eq. (11b). In turn, Eq. (12b) comes from Eq. (11d). Also

$$v_j(0,y) = v_j(a,y) = 0, \tag{13a}$$

$$v_{\mathrm{I}}(x,0) = v_{\mathrm{II}}(x,b) = 0. \tag{13b}$$

Now, with respect to the CCs that should hold over the interface, it is true that plates require two geometric (or essential) conditions to be fulfilled. They are related with the function w and the slope in the normal sense, $\partial w/\partial N$, and two force (or natural) conditions (bending moment M_N and the shear force V_N in the Nz plane). In the membrane case, two geometric conditions are to be imposed: function and slope in the normal direction.

A. Continuity condition for plane plates over interface $\mathcal{C}$ (Fig. 1)

The interface $\mathcal{C}$ limits the regions with different material properties. Let us recall[2,13] that a bending force M_N over an interface curve writes

$$M_N = -D\left[\nabla^2 w + (\nu - 1)\frac{\partial^2 w}{\partial t^2}\right], \tag{14}$$

and the total shear force V_N in the area element with normal N, parallel to z, is

$$V_N = -D\left[\frac{\partial}{\partial N}(\nabla^2 w) + (1-\nu)\frac{\partial^3 w}{\partial N\,\partial t^2}\right], \tag{15}$$

where

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial N^2} + \frac{\partial^2 w}{\partial t^2}, \tag{16}$$

in which $\partial(\cdot)/\partial N$ and $\partial(\cdot)/\partial t$ are directional derivatives. The bending stiffness D in Eqs. (14) and (15) is, for each region of the plate,

$$D_j = \frac{E_j h_0^3}{12(1-\nu_j^2)} = \frac{r_j}{f_j^2} D_0. \tag{17}$$

Then, the four CCs over the interface are given as follows.

- Function:

$$(w_{\mathrm{I}})_{(\mathcal{C})} = (w_{\mathrm{II}})_{(\mathcal{C})} (\equiv w^*). \tag{18a}$$

- Slope with respect to normal direction:

$$\left(\frac{\partial w_{\mathrm{I}}}{\partial N}\right)_{(\mathcal{C})} = \left(\frac{\partial w_{\mathrm{II}}}{\partial N}\right)_{(\mathcal{C})} \left(\equiv \frac{\partial w^*}{\partial N}\right). \tag{18b}$$

- Bending moment:

$$(D_{\mathrm{I}}\nabla^2 w_{\mathrm{I}})_{(\mathcal{C})} - (D_{\mathrm{II}}\nabla^2 w_{\mathrm{II}})_{(\mathcal{C})} = -[D_{\mathrm{I}}(\nu_{\mathrm{I}} - 1) - D_{\mathrm{II}}(\nu_{\mathrm{II}} - 1)]\frac{\partial^2 w^*}{\partial t^2}. \tag{18c}$$

- Shear in plane tz:

$$\left(D_{\mathrm{I}}\frac{\partial(\nabla^2 w_{\mathrm{I}})}{\partial N}\right)_{(\mathcal{C})} - \left(D_{\mathrm{II}}\frac{\partial(\nabla^2 w_{\mathrm{II}})}{\partial N}\right)_{(\mathcal{C})} = -[D_{\mathrm{I}}(1-\nu_{\mathrm{I}}) - D_{\mathrm{II}}(1-\nu_{\mathrm{II}})]\frac{\partial^3 w^*}{\partial N\,\partial t^2}. \tag{18d}$$

The definitions using $w^* = w^*(s)$ and its derivatives are just a simplifying notation.

B. Continuity condition for membranes over $\mathcal{C}$

The CCs and its consequences over the interface write the following.

- Function:

$$(v_{\mathrm{I}})_{(\mathcal{C})} = (v_{\mathrm{II}})_{(\mathcal{C})}. \tag{19a}$$

- Slope along N:

$$\left(\frac{\partial v_{\mathrm{I}}}{\partial N}\right)_{(\mathcal{C})} = \left(\frac{\partial v_{\mathrm{II}}}{\partial N}\right)_{(\mathcal{C})}. \tag{19b}$$

III. STATEMENT OF AN ANALOGY

In what follows the analogy proposed in the Introduction is stated and demonstrated. As mentioned before, the analogy that is object of the present study is within the following definition (that includes the well-known analogy valid for homogeneous polygonal domains as a particular case): Find a relationship among the membrane and plate mode shapes

$$v_j = G_j(w_j), \tag{20}$$

in such way that some condition among the frequencies of both systems

$$\Omega_p = g(\Omega_m) \tag{21}$$

exists. A second approach (inversely to the first one) would perhaps be possible; that is, the proposition of a functional relationship among frequencies and then, the derivation of the link between the mode shapes. Here the first approach is followed.

In order to start the search, let us rewrite Eq. (5) in the following format (recall that each f_j is constant):

$$\nabla^2(\nabla^2 w_j - \Omega_p f_j w_j) + \Omega_p f_j(\nabla^2 w_j - \Omega_p f_j w_j) = 0. \tag{22}$$

By simple observation $G_j(\cdot)$ [from Eq. (20)] may be selected as

$$v_j = G_j(w_j) \equiv \nabla^2 w_j - \Omega_p f_j w_j. \tag{23}$$

This is no more that the classical relationship extended to non-homogeneous domains. To start with a proposition similar to the classical analogy was the motivation of the present approach. It was be shown (not included herein) that it conduces to additional restrictions that will only be verified for homogeneous domains. Thus a more general expression will have to be used,

$$v_j = \alpha_j \nabla^2 w_j + \beta_j w_j, \tag{24}$$

in which the constants α_j and β_j will be found provided that the BCs and CCs are satisfied for membranes and find any $\Omega_p = g(\Omega_m)$. Particularly, if $\alpha_j = 1$ and $\beta_j = -\Omega_p f_j$, we obtain Eq. (23). Equation (23) would lead to the conclusion that it is only verified by homogenous domain cases. The need of a more complex relationship among the membrane and plate mode shapes [Eq. (24)] becomes apparent (Sec. III A).

A. Generalization of the analogy

A more general $G_j(w_j)$ is introduced in Eq. (24); i.e., $v_j = \alpha_j \nabla^2 w_j + \beta_j w_j$. The constants α_j and β_j will be found satisfying the BCs and CCs for membranes and with the aim of finding any $\Omega_p = g(\Omega_m)$. With relationship (24) and Eq. (5), Eq. (9) yields

$$(\beta_j + \alpha_j \Omega_m^2 r_j)\nabla^2 w_j + (\alpha_j \Omega_p^2 f_j^2 + \beta_j \Omega_m^2 r_j) w_j = 0. \tag{25}$$

These Helmholtz equations for each plate mode shape w_j are not—in general—satisfied. The only simultaneous conditions to identically verify Eq. (25) are

$$\beta_j + \alpha_j \Omega_m^2 r_j = 0, \tag{26a}$$

$$\alpha_j \Omega_p^2 f_j^2 + \beta_j \Omega_m^2 r_j = 0, \tag{26b}$$

from which and for each j the following results:

$$\beta_j = -\alpha_j \Omega_m^2 r_j, \tag{27a}$$

$$\Rightarrow \alpha_j(\Omega_p^2 f_j^2 - \Omega_m^4 r_j^2) = 0. \tag{27b}$$

In order for Eq. (27b) to be satisfied, it only remains that [if $\alpha_j=0$ due to Eq. (27a) $\Rightarrow \beta_j=0 \Rightarrow$ and due to Eq. (24) $v_j \equiv 0$]

$$\alpha_j \neq 0, \tag{28a}$$

$$\Omega_p = \Omega_m^2 \frac{r_j}{f_j}, \tag{28b}$$

since Ω_p, Ω_m, r_j, and f_j are essentially positive. As before, we find a possible function $g(\cdot)$, but a first additional restriction should be verified as

$$r_{\mathrm{I}} f_{\mathrm{II}} = r_{\mathrm{II}} f_{\mathrm{I}} \Rightarrow \frac{r}{f} = \text{const}, \tag{29}$$

in order for the frequencies to be region independent. Then, the frequencies relationship may be written as

$$\Omega_p = \Omega_m^2 \frac{r_{\mathrm{I}}}{f_{\mathrm{I}}} = \Omega_m^2 \frac{r_{\mathrm{II}}}{f_{\mathrm{II}}} = \Omega_m^2 \frac{r}{f}. \tag{30}$$

In this case the α_j's remain free and Eq. (24) with Eq. (27a) may be written as follows:

$$v_j = G_j(w_j) = \alpha_j(\nabla^2 w_j - \Omega_m^2 r_j w_j). \tag{31}$$

Let us now impose for $v_j = G_j(w_j)$ the BCs and the CCs for the membrane [taking also into account the plate BCs—recall Eqs. (11a), (11b), (12a), and (12b)]. First, we will analyze the CCs. Making use of Eq. (31) we impose Eqs. (19a) and (19b), taking into account Eq. (18a) and their consequences,

$$(\alpha_{\mathrm{I}} \nabla^2 w_{\mathrm{I}})_{(\mathcal{C})} - (\alpha_{\mathrm{II}} \nabla^2 w_{\mathrm{II}})_{(\mathcal{C})} = \Omega_m^2(\alpha_{\mathrm{I}} r_{\mathrm{I}} - \alpha_{\mathrm{II}} r_{\mathrm{II}}) w^*, \tag{32a}$$

$$\left(\alpha_{\mathrm{I}} \frac{\partial \nabla^2 w_{\mathrm{I}}}{\partial N}\right)_{(\mathcal{C})} - \left(\alpha_{\mathrm{II}} \frac{\partial \nabla^2 w_{\mathrm{II}}}{\partial N}\right)_{(\mathcal{C})} = \Omega_m^2(\alpha_{\mathrm{I}} r_{\mathrm{I}} - \alpha_{\mathrm{II}} r_{\mathrm{II}}) \frac{\partial w^*}{\partial N}. \tag{32b}$$

After comparing Eq. (32) with Eqs. (18c) and (18d), the following restrictions are mandatory. They avoid incoherences and, on the other hand, lead to the simpler and most direct conditions.

$$\alpha_{\mathrm{I}} r_{\mathrm{I}} - \alpha_{\mathrm{II}} r_{\mathrm{II}} = 0, \tag{33a}$$

$$\alpha_j = C D_j, \tag{33b}$$

$$D_{\mathrm{I}}(1-\nu_{\mathrm{I}}) - D_{\mathrm{II}}(1-\nu_{\mathrm{II}}) = 0, \tag{33c}$$

where C is an arbitrary constant. These restrictions must be added to Eq. (29). However, and due to Eq. (33b), it may be deduced that Eq. (33a) is equivalent to Eq. (29). In effect the use of Eqs. (17) and (33b) leads us to this conclusion. On the other hand after taking into account Eq. (33a), Eq. (33c) may be written as

$$r_{\mathrm{II}}(1-\nu_{\mathrm{I}}) = r_{\mathrm{I}}(1-\nu_{\mathrm{II}}) \Rightarrow \frac{r}{1-\nu} = \text{const}. \tag{34}$$

Moreover, with this relationship and Eq. (29) and recalling Eq. (8), the following result is found:

$$e_{\mathrm{II}}(1+\nu_{\mathrm{I}}) = e_{\mathrm{I}}(1+\nu_{\mathrm{II}}). \tag{35}$$

Then, if, for instance, we choose

$$r_{\mathrm{II}} = k r_{\mathrm{I}} (\Rightarrow f_{\mathrm{II}} = k f_{\mathrm{I}}), \tag{36}$$

k should be real and positive for real materials. Equation (34) automatically defines the following relationship:

$$(1-\nu_{\mathrm{II}}) = k(1-\nu_{\mathrm{I}}). \tag{37}$$

There are infinite possibilities of choosing the ν_j's (and not necessarily between 0 and 0.5, as is required in lineal elasticity). Furthermore, due to Eq. (35), a relation among the e_j is obtained.

$$e_{\mathrm{II}} = \frac{2 - k(1-\nu_{\mathrm{I}})}{(1+\nu_{\mathrm{I}})} e_{\mathrm{I}}. \tag{38}$$

Summing up, the present proposition (24) consisted in linking each membrane mode shape v_j as a linear combination of the plate mode shape and its Laplacian for each region. With this proposition, an analogy for non-homogeneous domains composed of homogeneous subdomains is found, with restrictions [Eqs. (29) and (34)]. This analogy works in the following way: *One* real or fictitious system is solved in order to find frequencies and mode shapes of *other* real system.

Obviously, the above presented analogy includes the classical one, i.e., when $r_{\mathrm{I}} = r_{\mathrm{II}}$, $f_{\mathrm{I}} = f_{\mathrm{II}}$, and $\nu_{\mathrm{I}} = \nu_{\mathrm{II}}$ (i.e., $\alpha_j = 1$).

The analogy is valid for all orders of frequencies and for an arbitrary number of regions. Successive relationships such as Eqs. (29) and (34) must be employed for each region when more than two regions are present. At present, the authors cannot assert the uniqueness of an analogy of this type; i.e., no guarantee can be provided that other analogies do not exist.

IV. EXAMPLES

In this section various numerical examples illustrate the analogy between plates and membranes with non-homogeneous domains with homogeneous subdomains. The necessary conditions have to be fulfilled in each case. Some cases are two-way statements but others are only in one way; i.e., a fictitious plate serves to find the natural frequencies of a real membrane (see example 1c below). Figure 2 shows the five different solved configurations: a rectangular/square plate/membrane with two regions separated by either a straight line parallel to one of the boundaries, an inclined straight line, a curved line, and a closed line (i.e., a closed region inside another one). Finally a non-rectangular domain was studied: a triangular plate/membrane with three regions separated by lines parallel to one side.

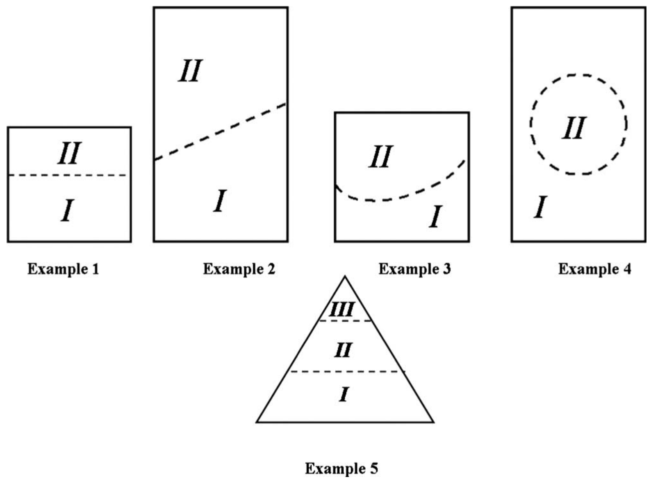

FIG. 2. Numerical examples. Schemes of the non-homogeneous domains.

The first group of numerical examples deals with a square shaped membrane/plate with two regions separated by a straight line parallel to one side. If X and Y are the axes located at the left inferior corner, the line is $Y=0.6a$, in which $0 \leq X \leq a$, $a=1$ (side of the square). Three different illustrations are presented in this case (examples 1a, 1b, and 1c). In all cases, the reference properties are $\nu_0=0.3$, E_0, and ρ_0, and the parameter values are depicted in Table I. As can be observed, conditions (29) and (34) are verified. The plate frequencies were found with the finite element software ALGOR (Ref. 15) for cases 1a and 1b. The well-known Levy solution[2] was employed to solve the vibration plate problem for case 1c and the membrane frequencies were found solving Eq. (9) with FLEXPDE.[16] The frequencies of the plates and the membranes were numerically found and compared in Table II. The error was calculated in all the examples with the following expression:

$$\epsilon\% = 100\left[\frac{r\Omega_m^2}{f\Omega_p} - 1\right]. \tag{39}$$

The following three examples deal with rectangular or square shapes with other interfaces. First a rectangular membrane/plate with two regions separated by an inclined straight line ($Y=0.3X+0.7$) is addressed (axes X and Y are located at the left inferior corner) (example 2). The sides of the rectangle are $a=1$ (X direction) and $b=1.8$ (Y direction). In this case $\nu_0=0.5$, $r_{\mathrm{I}}=1$, $e_{\mathrm{I}}=1$, $n_{\mathrm{I}}=1$, $r_{\mathrm{II}}=2$, $e_{\mathrm{II}}=2/3$, $n_{\mathrm{II}}=0$, $f_{\mathrm{I}}=1$, $f_{\mathrm{II}}=2$, and $\Omega_m^2/\Omega_p=f/r=1$. The membrane frequencies Ω_m were found using a trigonometric series approach as reported in a previous paper by the authors.[12] The plate frequencies Ω_p were found with ALGOR.[15] Next, the case of two regions separated by a curved interface $Y=0.8X^2-0.5X+0.4$ was tackled (the same data hold for the other properties) (example 3). In the membrane case, Eq. (9) was solved straightforwardly with FLEXPDE.[16] The plate frequencies were found also with FLEXPDE after rewriting Eq. (5) into two coupled membrane equations and the appropriate boundary conditions. This is necessary since this finite element software handles up to second order differential equations. A rectangular membrane/plate ($a=1$, $b=1.8$) with a region limited by a closed curve (a centered circumference of radius $R=0.4$) (example 4) was studied. Again both membrane and plates were solved with FLEXPDE. Table III contains the numerical values of the first three natural frequencies for the three cases.

TABLE I. Example 1. Data: parameters r_j, e_j, and n_j, and calculated values of f_j $(j=1,2)$.

Parameter	1a	1b	1c
r_{I}	1	0.6	1
r_{II}	2	1	2
e_{I}	1	1.3	1
e_{II}	2/3	1	0.461 5
n_{I}	5/3	1.619 03	1
n_{II}	0	0.476 2	−1.333
f_{I}	0.907 841	0.622 52	1
f_{II}	1.815 682	1.037 53	2
$\Omega_m^2/\Omega_p=f/r$	0.907 841	1.037 53	1

TABLE II. Example 1. First four natural frequencies for a square domain with a linear interface parallel to one side. Membrane equation (9) was solved with FLEXPDE (Ref. 16) and plate equation (5) was solved with ALGOR (Ref. 15) for cases 1a and 1b. Plate of case 1c was tackled with Levy solution. The error is computed with Eq. (39).

Example	Order of frequency	Membrane Ω_m^2	Plate Ω_p	Error $\epsilon\%$
1a	1	14.2537	15.7044	−0.024
	2	33.0705	36.4348	−0.019
	3	36.7375	40.4691	−0.005
	4	61.1121	67.3298	−0.020
1b	1	26.4785	25.5251	−0.017
	2	63.2931	61.0124	−0.014
	3	65.5396	63.1734	−0.007
	4	108.3759	104.4600	−0.004
1c	1	14.2537	14.2563	−0.018
	2	33.0705	33.0713	−0.002
	3	36.7375	36.7373	0.0005
	4	61.1121	61.1080	0.006

TABLE III. Examples 2–4. First three natural frequencies for the membranes (example 2—see Ref. 12—and examples 2 and 3 with FLEXPDE, see Ref. 16) and plates [example 2 with ALGOR (Ref. 15) and examples 3 and 4 with FLEXPDE]. The error was computed with Eq. (39).

Example	Order of frequency	Membrane Ω_m^2	Plate Ω_p	Error $\epsilon\%$
2	1	7.6618	7.6662	−0.06
	2	15.728	15.737	−0.06
	3	23.473	23.502	−0.12
3	1	11.133	11.141	−0.072
	2	27.649	27.660	−0.040
	3	31.751	31.787	−0.113
4	1	7.5006	7.5037	−0.040
	2	16.132	16.126	−0.037
	3	25.185	25.248	−0.249

TABLE IV. Example 5. First five natural frequencies for a triangular membrane/plate with three regions. The error is computed with Eq. (39).

Order of frequency	Membrane Ω_m^2	Membrane $\times r/f$ $\Omega_m^2 r/f$	Plate Ω_p	Error $\epsilon\%$
1	33.301 90	42.621 23	42.621 23	0
2	73.470 89	94.031 26	94.031 25	0
3	89.789 84	114.917 0	114.916 9	0
4	117.985 2	151.002 6	150.994 8	0.005
5	153.384 8	196.308 6	196.278 7	−0.015

Finally, example 5 deals with an equilateral triangular domain with three regions, as is shown in Fig. 2. The data for this case are $\rho_0=1$, $E_0=1$, $\nu_0=0.3$, $r_1=1$, $r_2=2$, $r_3=3$, $e_1=1$, $e_2=0.8$, $e_3=0.6$, $n_1=0.2$, $n_2=0.1$, and $n_3=0$. Consequently $r_j/f_j=r/f=\Omega_p/\Omega_m^2=1.2798$. The results are reported in Table IV. Both membrane and plates were solved using FLEXPDE.[16]

As can be observed, errors are negligible. They are within the numerical inevitable errors that are involved in both the finite element calculations, as well as the other analytical approaches. Recall that the analogy is theoretically correct.

V. CONCLUSIONS

An analogy between membranes and simply supported plates for non-homogeneous domains comprised of homogeneous subdomains has been presented. It is verified within polygonal domains. Concretely, the analogy is derived from Eq. (24) that relates each membrane mode shape v as a linear combination of the plate mode shape w and its Laplacian. With this proposition, some restrictions arise [Eqs. (29) and (34)]. Several examples illustrate its validity provided that the required restrictions are satisfied. Through the demonstrated analogy, the plate and membrane vibration problems governed by the bi-Laplacian and Laplacian differential operators, respectively, can be solved without distinction, under certain conditions, i.e., solve one of them and deduce the other using the relationship. Obviously, the analogy includes the classic one when the whole domain is homogeneous.

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