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Proper Hamiltonian Paths in Edge-Colored Multigraphs

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Abstract

A c-edge-colored multigraph has each edge colored with one of the c available colors and no two parallel edges have the same color. A proper hamiltonian path is a path containing all the vertices of the multigraph such that no two adjacent edges have the same color. In this work we establish sufficient conditions for a multigraph to have a proper hamiltonian path, depending on several parameters such as the number of edges, the rainbow degree, etc.

Keywords: Multigraph, Proper Hamiltonian Path, Edge-Coloring

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1 Introduction

The study of problems modeled by edge-colored graphs gave place to important developments over the last years. In particular, problems arising in molecular biology are often modeled by means of colored graphs, i.e., graphs with colored edges and/or vertices [8]. Given such a graph, original problems correspond to extracting subgraphs such as Hamiltonian and Eulerian paths or cycles colored in a specied pattern [1,2,4]. The most natural pattern in such a context is that of a proper coloring, which entails adjacent edges/vertices having different colors. Properly colored paths and cycles have applications in various other elds, as in VLSI for compacting a programmable logical array [7]. Although a large body of work has already been done[2,3], in most of that previous work the number of colors was restricted to two. For instance, while it is well known that properly edge-colored hamiltonian cycles can be found efficiently in 2-edge-colored complete graphs, it is a long standing question whether there exists a polynomial algorithm for nding such hamiltonian cycles in edge-colored complete graphs with three colors or more [4]. Notice that the hamiltonian path problem was solved recently in [5] in the case of complete graphs, with an arbitrary number of colors. In this work we give sufficient conditions involving various parameters as the number of edges, rainbow degree, etc, in order to guaranty the existence of properly edge-colored hamiltonian paths in edge-colored multigraphs. Recent work on cycles and paths involving colored degrees in edge-colored graphs are found in [1,6].

Formally, let $I_c = \{1, 2, \cdots, c\}$ be a set of $c \geq 2$ colors. Throughout this paper, G^c denotes a *c-edge-colored connected multigraph* such that, each edge is colored with one color in I_c and no two parallel edges joining the same pair of vertices have the same color. Let n (m) be the number of vertices (edges) of G^c. If H is a subgraph of G^c, then $N^i_H(x)$ denotes the set of vertices of H adjacent to x with an edge of color i. Whenever H is isomorphic to G^c, we write $N^i(x)$ instead of $N^i_{G^c}(x)$. The *colored i-degree* of a vertex x, denoted by $d^i(x)$, is the cardinality of $N^i(x)$. The *rainbow degree* of a vertex x, denoted by $rd(x)$, is the number of different colors on the edges incident to x. An edge with endpoints x and y is denoted by xy, and its color by $c(xy)$. A *rainbow complete multigraph* is the one having all possible colored edges between any pair of vertices. A subgraph of G^c is said to be *properly edge-colored*, if any two adjacent edges in this subgraph differ in color. A *hamiltonian path* (*cycle*) is a path (cycle) containing all vertices of the multigraph. We will use one family of 2-edge-colored multigraphs, denoted by $H_{k,k+3}$, without proper hamiltonian paths defined as follows. For $k \geq 1$, consider a complete red graph on k vertices and join it with red edges to an independent set on $k+3$ vertices. Finally, superpose a complete blue graph on $2k+3$ vertices.

2 Main results

The two lemmas below will be useful in view of Theorem 2.3. For all results, except Theorem 2.8, G^c will denote a 2-edge-colored multigraph on colors *red* and *blue*, denoted by r and b, respectively.

Lemma 2.1 *If G^c contains a proper cycle C of length $n-2$ and $d^b_C(x) + d^b_C(y) > |C|$, where $G^c - C = \{x, y\}$, then G^c has a proper hamiltonian path with ends x and y, and starting and ending with blue edges.*

Lemma 2.2 *If $m \geq (n-1)(n-2) + n$, then G^c has a proper hamiltonian cycle if n is even, and a proper cycle of length $n-1$ otherwise.*

Theorem 2.3 *For $n \neq 5, 7$, if $m \geq f_1(n) = n^2 - 3n + 4$, then G^c has a proper hamiltonian path.*

Proof. By induction on n. For small values of n, say $n \leq 9$, the argument can be completed by a tedious analysis, so we exclude the details. Suppose $n \geq 10$. By a Theorem of [1], if for every vertex $d^r(x) \geq \lceil \frac{n+1}{2} \rceil$ and $d^b(x) \geq \lceil \frac{n+1}{2} \rceil$, then G^c has a proper hamiltonian path. Suppose therefore that for some vertex, say x, and for some color, say red, $d^r(x) < \lceil \frac{n+1}{2} \rceil$.

Claim 1: $d^r(x) + d^b(x) \geq 3$. Otherwise, if $d^r(x) + d^b(x) \leq 2$, then $m \leq n(n-1) - 2n + 4 = f_1(n)$, a contradiction unless all inequalities become equalities, i.e., $d^r(x) + d^b(x) = 2$. In particular, $G^c - x$ is a rainbow complete multigraph. If y is a neighbor of x in $G^c - x$, then we may easily find a proper hamiltonian path in $G^c - x$ starting at y with a color different from $c(xy)$. Then, we can join x to this path in order to find a hamiltonian one in G^c.

Claim 2: Neither $d^r(x) = 0$ nor $d^b(x) = 0$. Assume by contradiction that either $d^r(x) = 0$ or $d^b(x) = 0$. In that case $G^c - x$ has at least $n^2 - 3n + 4 - (n-1) = [(n-1)-1][(n-1)-2] + (n-1)$ edges. By Lemma 2.2, $G^c - x$ has a proper cycle of length $n-1$ or $n-2$. If $G^c - x$ has a proper hamiltonian cycle then in a trivial maner we join x to the cycle in order to obtain a proper hamiltonian path. Assume therefore that $n-1$ is odd and that $G^c - x$ has a proper cycle C of length $n-2$. Let y be the vertex outside C in $G^c - x$. As in Lemma 2.1 we may show that $d^r_C(x) + d^r_C(y) \leq |C|$ and $d^b_C(x) + d^b_C(y) \leq |C|$, otherwise, we can find a proper hamiltonian path between x and y. It follows that the number of edges of G^c is at most $n(n-1) - 2(n-2) + 2 = n^2 - 3n + 4$, again a contradiction unless all inequalities become equalities. In particular, there is a red and a blue edge between x and y. In that case, take the proper path starting at, say x, containing all the vertices of C. Then join y to x using one edge xy with the appropriate color. This proper path is hamiltonian.

Let us now complete the proof. By Claims 1 and 2 there exist two distinct neighbors of x, say y and z, in G^c such that $c(xy) = r$ and $c(xz) = b$. Replace now the vertices x, y, z by a new vertex s and add colored edges between s and $G^c - \{x, y, z\}$ such that $N^b(s) = N^b_{G^c-\{x,y,z\}}(y)$ and $N^r(s) = N^r_{G^c-\{x,y,z\}}(z)$. The resulting graph, G' has $n-2$ vertices and at least $n^2 - 3n + 4 - [(n-1) + \lceil \frac{n+1}{2} \rceil + 2(n-2)] > f_1(n-2)$ edges. By the induction hypothesis, G' has a proper hamiltonian path and so does the graph G^c. □

Theorem 2.3 is the best possible for $n \neq 5, 7$. In fact, consider a rainbow complete 2-edge-colored multigraph on $n-2$ vertices for n odd. Add two new vertices x_1 and x_2. Then add the red edge x_1x_2 and all red edges between $\{x_1, x_2\}$ and the complete graph. Although the resulting graph has n^2-3n+3 edges, it has no proper hamiltonian path, since at least one of the vertices x_1 or x_2 cannot be attached to any such path. Indeed, for n odd, the two extremal edges of any proper hamiltonian path must differ in colors. If $n = 5, 7$, Theorem 2.3 does not hold for the graphs $H_{k,k+3}$, $k = 1, 2$.

In the rest of the section, we will deal with c-edge-colored multigraphs with given number of edges, such that each vertex has rainbow degree equal to c. We will start with the case $c = 2$ and later we will study the case $c \geq 3$. We establish the following preliminary lemma and definition.

Lemma 2.4 *Suppose that for every vertex x in G^c, $rd(x) = 2$ and $n \geq 14$. If $m \geq (n-3)(n-4) + 3n - 2$, then G^c has two matchings M^r and M^b on colors, say red and blue, such that $|M^r| = \lfloor \frac{n}{2} \rfloor$ and $|M^b| \geq \lceil \frac{n-2}{2} \rceil$.*

Definition 2.5 A path P is compatible with a matching M if its edges belong alternatively to M and not to M.

Theorem 2.6 *Under the conditions of Lemma 2.4, G^c has a proper hamiltonian path.*

Proof (Sketch) Suppose n even (the odd case is similar). Let us suppose that G^c has not a proper hamiltonian path. We will show that G^c has less than $(n-3)(n-4) + 3n - 2$ edges. By Lemma 2.4, G^c has two matchings M^r, M^b, such that $|M^r| = \frac{n}{2}$ and $|M^b| \geq \frac{n-2}{2}$. Take the longest proper path $P = x_1y_1x_2y_2 \ldots x_py_p$, compatible with the matching M^r. Suppose $2p < n$, otherwise, P is a proper hamiltonian path. Since $|M^r| = \frac{n}{2}$, $c(x_1y_1) = c(x_py_p) = r$. Otherwise, we can easily extend the path by adding an edge of the matching to P. So, as P is properly colored, the edges x_iy_i are red

$(i = 1, \ldots, p)$ and the edges $y_i x_{i+1}$ are blue $(i = 1, \ldots, p-1)$. Let $e_1, e_2, \ldots, e_s$ be the edges of M^r in $G^c - P$, $s = \frac{n-2p}{2}$. It can be shown that the worst scenario is when there is just one edge of M^r in $G^c - P$, so $2p = n-2$. Indeed, if there are more, then there exist more possibilites to extend P. Now, we will count the blue edges that are missing if we cannot extend the path P to a proper hamiltonian one.

- If we have blue edges between the endpoints of e_1 and x_1 or y_p, we have a hamiltonian path and we are done. So, there are 4 blue missing edges.
- If there are 3 or more blue edges between the edpoints of the edge e_1 and the endpoints of the edges $y_i x_{i+1}$ $(i = 1, \ldots, p-1)$, we can easily add e_i to the path P. Otherwise, there are $2\frac{2p-2}{2}$ blue missing edges.
- If the path P is also a proper cycle, it is easy to extend P using e_1 since, as $rd = 2$, we have two different possibilites. First, if there exists one blue edge between e_1 and P, we just take e_1, then this blue edge and finally we go through P in the appropriate direction. If that edge does not exist, then there is a blue edge, f_1, parallel to e_1. Now, as the graph is connected we have a red edge between e_1 and P, so we obtain the hamiltonian path as before but starting with f_1 and then taking the red edge that goes to P. Otherwise, if the path is not a proper cycle, we can see that there are $2p-1$ blue missing edges.

By adding up all these numbers, we conclude that there are $4p + 1 = 2n - 3$ blue missing edges. Now, considering the blue matching instead of the red one and repeating the above arguments, we conclude that there are at least $n - 2$ red missing edges. Adding all these missing edges, we obtain that G^c has at most $n(n-1) - (3n-5) < (n-3)(n-4) + 3n - 2$ edges. □

Theorem 2.6 is the best possible for $n \geq 14$. Indeed, for n odd, consider a complete blue graph, say A, on $n-3$ vertices. Add 3 new vertices v_1, v_2, v_3 and join them to a vertex v in A with blue edges. Finally, superpose the obtained graph with a complete red graph on n vertices. Although the resulting 2-edge-colored multigraph has $(n-3)(n-4) + 3n - 3$ edges, it has no proper hamiltonian path since one of the vertices v_1, v_2, v_3 cannot belong to such a path. For $n \leq 13$, the graphs $H_{k,k+3}$, $k = 2, 3, 4, 5$, have more than $(n-3)(n-4) + 3n - 2$ edges but they have not a proper hamiltonian path.

Lemma 2.7 *Let G be a connected non-colored simple graph on n vertices, $n \geq 9$. If $m \geq \frac{(n-2)(n-3)}{2} + 3$, then G has a matching M of size $|M| = \lfloor \frac{n}{2} \rfloor$.*

Theorem 2.8 *Let G^c be a c-edge-colored multigraph on n vertices, $n \geq 14$ and $c \geq 3$. Assume that for every vertex x of G^c, $rd(x) = c$. If $m \geq \frac{c(n-2)(n-3)}{2} +$*

$2c+1$, *then* G^c *has a proper hamiltonian path.*

Proof (Sketch) It is clear that there exists a color, say red, such that the spanning red subgraph of G^c has at least $\frac{(n-2)(n-3)}{2}+3$ edges. Therefore, by Lemma 2.7, there is a red matching of size $\lfloor \frac{n}{2} \rfloor$. Now, remove the red edges of G^c, color the rest of the multigraph with some new color, say black, and remove parallel edges. If the obtained graph has a matching of size $\lceil \frac{n-2}{2} \rceil$, then the red subgraph superposed with this black subgraph form a 2-edge-colored multigraph such that conditions of Theorem 2.6 are satisfied. So, the result clearly holds. Otherwise, G^c has a very particular structure and we can show how to find a proper hamiltonian path. $\square$

Theorem 2.8 is the best possible. In fact, consider a rainbow complete multigraph, say A, on $n-2$ vertices. Add 2 new vertices v_1, v_2 and then join them to a vertex v of A with all possible colors. The resulting c-edge-colored multigraph has $\frac{c(n-2)(n-3)}{2}+2c$ edges and has no proper hamiltonian path.

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