Math. Log. Quart., 1 – 18 (2012) / **DOI** 10.1002/malq.201100095

Quantifier elimination for elementary geometry and elementary affine geometry

Rafael Grimson[1,*], **Bart Kuijpers**[2], and **Walied Othman**[3]

[1] Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires Pabellón I, Ciudad Universitaria (1428), Buenos Aires, Argentina
[2] Theoretical Computer Science Group, Hasselt University & Transnationale Universiteit Limburg, Agoralaan, Gebouw D, 3590 Diepenbeek, Belgium
[3] Geographisches Institut, Universität Zürich, Winterthurerstrasse 190, 8057 Zürich, Switzerland

Received 9 November 2011, revised 21 March 2012, accepted 22 March 2012
Published online XXXX

Key words Quantifier elimination, euclidean geometry, affine geometry, geometric constructions, semi-algebraic geometry.

MSC (2010) 03C10, 51M05, 03B30

We introduce new first-order languages for the elementary n-dimensional geometry and elementary n-dimensional affine geometry ($n \geq 2$), based on extending $\mathsf{FO}(\beta, \equiv)$ and $\mathsf{FO}(\beta)$, respectively, with new function symbols. Here, β stands for the betweenness relation and $\equiv$ for the congruence relation. We show that the associated theories admit effective quantifier elimination.

1 Introduction

1.1 Origins of the problem

For any fixed natural number n, the elementary n-dimensional Euclidean geometry, $\mathcal{E}_n$, is a theory dealing with the elementary properties of the n-dimensional Euclidean space. In this context, *elementary* means the portion of geometry that can be developed within first-order logic without the help of set-theoretic notions. Tarski's axiom system for this theory, already presented by him in his course given at the Warsaw University in 1926–27 and finally published in [16] and [22], is based on two primitive notions: *betweenness* and *equidistance*. The theory $\mathcal{E}_n$ is complete but not categorical: its models are, up to isomorphisms, the n-dimensional Cartesian spaces over some real closed fields [22]. The first axiom system based on these primitive notions was proposed by Veblen [24].

The elementary theory of n-dimensional affine geometry, $\mathcal{A}_n$, is a complete theory as well. Its only primitive notion is the *betweenness* relation[1]. The interested reader can consult [23] and [1, Chapter 7] for more references and historical remarks on the development of these theories.

For every fixed natural number n, we introduce two new first-order theories, $\mathcal{E}'_n$ and $\mathcal{A}'_n$. These new theories are extensions by definitions of $\mathcal{E}_n$ and $\mathcal{A}_n$, respectively, and admit effective quantifier elimination.

There are classical examples of this technique, based on extending the signature of a theory with finitely many new symbols—expressing properties already definable by quantified formulas in the original language—to obtain a new theory that admits quantifier elimination and has the same expressive power as the original language. For instance, by adding the binary relation symbol "$<$" to the signature $\langle +, \times, 0, 1\rangle$, Tarski [21] obtained a theory, $\mathcal{R}$, for real closed fields that admits quantifier elimination . Another classic example is that of the congruence relations in Presburger arithmetic (cf. [5]).

* Corresponding author: e-mail: rgrimson@dm.uba.ar

[1] In her monograph [19], Szmielew showed that this last primitive notion can be replaced by parallelity, leading to a more abstract development of affine geometry, including representation theorems for subsystems of the axiom system of affine geometry.

Like in Szczebra and Tarski [18], the detailed discussion will be restricted to the case $n = 2$, i.e., to the geometry of the plane. We denote by $\mathcal{E}$ and $\mathcal{A}$ the theories $\mathcal{E}_2$ and $\mathcal{A}_2$, respectively. In Section 7.3, we indicate how our results can be extended to higher dimensions.

Languages that admit the elimination of quantifiers for elementary algebra and fragments of geometry have been the subjects of several investigations, but as far as we know, no language for elementary geometry which allows quantifier elimination has been proposed. In a way, this omission is surprising, because quantifier elimination is a natural requirement of expressibility for a language.

Quantifier elimination methods have been mainly used to obtain decision procedures. Recently, within the theory of constraint databases [9], quantifier-elimination techniques have also been used to evaluate queries. In particular, within the context of spatial databases, the languages $\mathsf{FO}(\beta, \equiv)$ and $\mathsf{FO}(\beta)$ have been proposed [6] as query languages for geometric databases. The results we present here lead to a query evaluation procedure for these query languages.

We remark that, sharing some primitive notions, the languages that we obtain are related to the languages used in constructive geometry [11–13]. One difference is the absence of constant symbols in our language. In the presence of constant symbols, formulas can define relations that are not invariant under similarity transformations of the plane. Our languages preserve this basic characteristic of Euclidean geometry.

1.2 Outline and Summary

The paper is organized as follows. In Section 2, we introduce the concepts of affine-invariant and similarity-invariant relation. We also introduce the theories of real closed fields, $\mathcal{R}$, elementary affine geometry, $\mathcal{A}$, and elementary Euclidean geometry, $\mathcal{E}$, with their associated languages $\mathsf{FO}(+, \times, <, 0, 1)$, $\mathsf{FO}(\beta)$ and $\mathsf{FO}(\beta, \equiv)$, respectively. Being all three complete theories, we fix a standard model for each and use the fact that a formula holds in this model if and only if it is true in the corresponding theory.

We stress the difference between geometric variables and algebraic variables, and introduce the concept of *translation*. In particular, we recall the existence of a translation from $\mathsf{FO}(\beta, \equiv)$ (and hence, also from $\mathsf{FO}(\beta)$) to $\mathsf{FO}(+, \times, <, 0, 1)$. This translation is based on the fact that the Euclidean plane can be embedded in the Cartesian plane by taking coordinates in a fixed coordinate system.

We recall that the theory of real closed fields, $\mathcal{R}$, admits quantifier elimination, and we denote by $\mathfrak{E}_{\mathcal{R}}$ a quantifier-elimination function for this theory. We prove that no finite predicative extension of $\mathsf{FO}(\beta)$ or $\mathsf{FO}(\beta, \equiv)$ admits quantifier elimination.

In Section 3, we define the *basic segment-arithmetic functions*, $\oplus$ and $\otimes$, the *affine projection function*, π, and for the two *basic metric functions*, $\pi^{\perp}$ and κ (corresponding to the orthogonal projection and the segment construction function), and expand the signatures of $\mathsf{FO}(\beta)$ and $\mathsf{FO}(\beta, \equiv)$ adding new function symbols for some of these basic functions, and the 0-ary relation symbol $\top$. The interpretation of the new symbols in the resulting languages, $\mathsf{FO}(\beta, \top, \oplus, \otimes, \pi)$ and $\mathsf{FO}(\beta, \equiv, \top, \oplus, \otimes, \pi^{\perp}, \kappa)$, are given by $\mathsf{FO}(\beta)$-formulas and $\mathsf{FO}(\beta, \equiv)$-formulas respectively. In this way, the resulting theories, $\mathcal{A}'$ and $\mathcal{E}'$ are *extensions by definitions* of $\mathcal{A}$ and $\mathcal{E}$ respectively. This ensures that the new languages have the same expressive power as the original ones and also the existence of translations $\mathcal{B}$ from $\mathsf{FO}(\beta, \top, \oplus, \otimes, \pi)$ to $\mathsf{FO}(\beta)$ and $\mathcal{M}$ from $\mathsf{FO}(\beta, \equiv, \top, \oplus, \otimes, \pi^{\perp}, \kappa)$ to $\mathsf{FO}(\beta, \equiv)$.

In Section 4, we define a translation $\mathcal{S} : \mathsf{FO}(+, \times, <, 0, 1)_{\mathrm{QF,AI}} \to \mathsf{FO}(\beta, \top, \oplus, \otimes, \pi)_{\mathrm{QF}}$, translating any formula in the affine-invariant quantifier-free fragment of $\mathsf{FO}(+, \times, <, 0, 1)$ into the quantifier-free fragment of $\mathsf{FO}(\beta, \top, \oplus, \otimes, \pi)$ in such a way that, for any affine-invariant quantifier-free $\mathsf{FO}(+, \times, <, 0, 1)$-formula φ, $\mathcal{S}(\varphi)$ and φ *define the same relation*. The technical difficulty in the construction of this translation is due to the absence of constant symbols in $\mathsf{FO}(\beta, \top, \oplus, \otimes, \pi)$ to use as coordinate system and the subsequent need to use some of the variables already involved in the formula as a reference system.

Analogously, in Section 5, we define a translation $\mathcal{T} : \mathsf{FO}(+, \times, <, 0, 1)_{\mathrm{QF,SI}} \to \mathsf{FO}(\beta, \equiv, \top, \oplus, \otimes, \pi^{\perp}, \kappa)_{\mathrm{QF}}$, translating any formula in the similarity-invariant quantifier-free fragment of $\mathsf{FO}(+, \times, <, 0, 1)$ into the quantifier-free fragment of $\mathsf{FO}(\beta, \equiv, \top, \oplus, \otimes, \pi^{\perp}, \kappa)$.

In Section 6, we define $\mathfrak{E}_{\mathcal{A}'} := \mathcal{S} \circ \mathfrak{E}_{\mathcal{R}} \circ \mathcal{C} \circ \mathcal{B} : \mathsf{FO}(\beta, \top, \oplus, \otimes, \pi) \to \mathsf{FO}(\beta, \top, \oplus, \otimes, \pi)_{\mathrm{QF}}$ as the composition of the translations $\mathcal{C}$, $\mathcal{B}$ and $\mathcal{S}$ with the quantifier-elimination function $\mathfrak{E}_{\mathcal{R}}$. The map $\mathfrak{E}_{\mathcal{A}'}$ results to be a quantifier-elimination function for the theory $\mathcal{A}'$. In this sense, we prove that $\mathcal{A}'$ is a *conservative extension* of $\mathcal{A}$ that admits quantifier elimination. Analogously, we prove that the map $\mathfrak{E}_{\mathcal{E}'} := \mathcal{T} \circ \mathfrak{E}_{\mathcal{R}} \circ \mathcal{C} \circ \mathcal{M} : \mathsf{FO}(\beta, \top, \oplus, \otimes, \pi) \to \mathsf{FO}(\beta, \top, \oplus, \otimes, \pi)_{\mathrm{QF}}$ is a quantifier-elimination function for the theory $\mathcal{E}'$.

In the last section we discuss the problem of finding *minimal* languages for elementary geometry and elementary affine geometry that admit the elimination of quantifier; this problem is interesting from a metamathematical viewpoint. Finally, we briefly indicate how, performing only minor changes in the argumentation, analogous constructions could be carried on for higher-dimensional theories.

For the fluidity of the exposition, we do not prove every geometrical statement in our argumentation. The missing arguments may be filled in using basic tools from analytic geometry.

2 Preliminaries and definitions

In Tarski's formalization of elementary geometry [22], points are treated as individuals and represented by first-order variables. Its only primitive notions, in terms of which all geometrical notions turn to be definable, are the *betweenness* and the *equidistance* relations.

We recall that, being a first-order theory, this formalization does not provide variables to denote geometrical figures (point sets) nor classes of geometrical figures. However, it is possible to express in the resulting formalism all the results that form the subject matter of geometry courses as taught in secondary schools and which are formulated in terms of some special classes of geometrical figures such as straight lines, circles, segments, triangles and, in general, polygons with any fixed number of vertices, as well as certain relations between geometrical figures in these classes such as congruence and similarity. This possibility is mainly a consequence of the fact that, in each of these classes, every geometrical figure is determined by a fixed finite number of points.

The representation theorem for elementary geometry [22] states that a necessary and sufficient condition for a structure to be a model of this theory is that it is isomorphic with the Cartesian space over some real closed field. In addition, this theory is shown to be complete and decidable.

2.1 Semi-algebraic and geometric relations

Let $\mathbb{R}$ be the set of real numbers and let $\mathbb{E}$ the universe of a model of Tarski's elementary plane geometry isomorphic to $\mathbb{R}^2$. We call $\mathbb{E}$ the *Euclidean plane* and we refer to $\mathbb{R}^2$ as the *Cartesian plane*. We fix an Euclidean coordinate system in $\mathbb{E}$, that is, we fix an origin O and two points E_1 and E_2 such that the segments $\overline{OE_1}$ and $\overline{OE_2}$ are orthogonal and congruent. We observe that, not being collinear, the points O, E_1, E_2 define, in particular, an affine coordinate system. We define C_{O,E_1,E_2} as the function from $\mathbb{E}$ to $\mathbb{R}^2$ that maps points to their coordinates with respect to the coordinate system O, E_1, E_2.

We shall deal with the following two different kinds of relations.

Definition 2.1 A k-ary *semi-algebraic relation* $(k \geq 1)$ is a subset of $\mathbb{R}^k$ that can be described as a boolean combination of sets of the form

$$\{(x_1, \dots, x_k) \in \mathbb{R}^k \mid p(x_1, \dots, x_k) > 0\},$$

where $p \in \mathbb{Z}[X_1, \dots, X_k]$ is a polynomial with integer coefficients in the variables $X_1, \dots, X_k$.

A k-ary *geometric relation* $(k \geq 1)$ is a subset of $\mathbb{E}^k$ such that its image under C^k_{O,E_1,E_2} is a semi-algebraic relation of $\mathbb{R}^{2k}$.

We have allowed only integer coefficients in the definition of semi-algebraic relation for simplicity: as we shall see, in this way semi-algebraic relations correspond exactly to the relations definable in the language $\mathsf{FO}(+, \times, <, 0, 1)$.

We shall refer to variables ranging over $\mathbb{E}$ as *geometric variables*, whereas variables ranging over $\mathbb{R}$ will be called *algebraic variables*. Also, for ease of reading, we shall consistently use the letters $o, p, q, r, s, u, v, e_1, e_2, p_1, p_2, \dots$, to represent geometric variables, and $a, b, x, y, t, x_1, y_1, x_2, y_2, \dots$, for algebraic variables. Variables ranging over the natural numbers $\mathbb{N}$ will be denoted by $i, j, k, l, m\, n$. Finally, we differentiate geometric variables from points in $\mathbb{E}$ writing p_i and $\underline{p_i}$ respectively. In this way, p_i is a geometric variable while $\underline{p_i}$ represents some fixed point in $\mathbb{E}$. Analogously, we write x_i for algebraic variables and $\underline{x_i}$ for fixed elements of $\mathbb{R}$.

2.2 Affine and similarity transformations of the plane

Definition 2.2 An *affine transformation* of $\mathbb{R}^2$ is a bijective function $f : \mathbb{R}^2 \to \mathbb{R}^2$, for which there exist $\underline{a_{11}}, \underline{a_{12}}, \underline{a_{21}}, \underline{a_{22}}, \underline{b_1}, \underline{b_2} \in \mathbb{R}$, with

$$f(x,y)^T = \begin{pmatrix} \underline{a_{11}} & \underline{a_{12}} \\ \underline{a_{21}} & \underline{a_{22}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \underline{b_1} \\ \underline{b_2} \end{pmatrix}.$$

An *affine transformation* of $\mathbb{E}$ is a bijective function $f : \mathbb{E} \to \mathbb{E}$, such that $C^{-1}_{O,E_1,E_2} \circ f \circ C_{O,E_1,E_2}$ is an affine transformation of $\mathbb{R}^2$.

Since any two affine coordinate systems are equal up to an affine transformation of the plane, the notion of affine transformation of $\mathbb{E}$ is independent of the chosen affine coordinate system O, E_1, E_2.

In particular, translation, rotation, scaling, and reflection over an axis are affine transformations. We remark that our definition of affine transformation coincides with what are sometimes called *non-degenerate* affine transformations.

We denote by $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$ the norm of points in the Cartesian plane, $\|(x,y)\| = \sqrt{x^2+y^2}$.

Definition 2.3 A *similarity transformation* of $\mathbb{R}^2$ is a bijective function $f : \mathbb{R}^2 \to \mathbb{R}^2$, for which there exist $\underline{r} \in \mathbb{R}$, $\underline{r} > 0$ such that for all pairs, $(\underline{x_1}, \underline{y_1})$ and $(\underline{x_2}, \underline{y_2})$, of points in $\mathbb{R}^2$, the following holds:

$$\|f(\underline{x_1}, \underline{y_1}) - f(\underline{x_2}, \underline{y_2})\| = r \cdot \|(\underline{x_1}, \underline{y_1}) - (\underline{x_2}, \underline{y_2})\|.$$

A *similarity transformation* of $\mathbb{E}$ is a bijective function $f : \mathbb{E} \to \mathbb{E}$, such that $C^{-1}_{O,E_1,E_2} \circ f \circ C_{O,E_1,E_2}$ is a similarity transformation of $\mathbb{R}^2$.

Since any two Euclidean coordinate systems are equal up to a similarity transformation of the plane, the notion of similarity transformation of $\mathbb{E}$ is independent of the chosen Euclidean coordinate system O, E_1, E_2.

In particular, translation, rotation, dilatations, and reflection over an axis are similarity transformations. Clearly, any similarity transformation is an affine transformation but the converse does not hold.

2.3 Affine-invariant and similarity-invariant relations

Now, we define the concept of an affine-invariant relation.

Definition 2.4 A k-ary geometric relation P is called *affine invariant* if for any tuple $(\underline{p_1}, \ldots, \underline{p_k})$ in $\mathbb{E}^k$ and any affine transformation f of $\mathbb{E}$, we have that $(\underline{p_1}, \ldots, \underline{p_k}) \in P$ implies $(f(\underline{p_1}), \ldots, f(\underline{p_k})) \in P$.

A $2k$-ary semi-algebraic relation Q is called *affine invariant* if for any tuple $(\underline{x_1}, \underline{y_1}, \ldots, \underline{x_k}, \underline{y_k})$ in $\mathbb{R}^{2k}$ and any affine transformation f of $\mathbb{R}^2$, we have that $(\underline{x_1}, \underline{y_1}, \ldots, \underline{x_k}, \underline{y_k})$ implies $(f(\underline{x_1}, \underline{y_1}), \ldots, f(\underline{x_k}, \underline{y_k})) \in Q$.

We remark that affine-invariant semi-algebraic relations range over pairs of real numbers while affine-invariant geometric relations range over points in the plane $\mathbb{E}$. As an example for previous definition, we consider the geometric relation $\mathsf{L} \subset \mathbb{E}^3$ consisting of triples $(p,q,r) \in \mathbb{E}^3$ that are collinear. Since any affine transformation preserves collinearity, this relation is affine invariant. A finer relation that will play an important role is β, which consists of all triples $(p,q,r) \in \mathbb{E}^3$ for which q belongs to the closed line segment between p and r. Clearly, β is also affine invariant. Their semi-algebraic counterparts are subsets of $\mathbb{R}^6$ and can be defined algebraically, as will be shown later.

Certainly, not all geometric relations are affine invariant. For instance, the unary relation $\{O\}$, containing the origin of the coordinate system O, E_1, E_2, is not affine invariant.

Definition 2.5 A k-ary geometric relation P is called *similarity invariant* if for any tuple $(\underline{p_1}, \ldots, \underline{p_k})$ in $\mathbb{E}^k$ and any similarity transformation f of $\mathbb{E}$, we have that $(\underline{p_1}, \ldots, \underline{p_k}) \in P$ implies $(f(\underline{p_1}), \ldots, f(\underline{p_k})) \in P$.

A $2k$-ary semi-algebraic relation Q is called *similarity invariant* if for any tuple $(\underline{x_1}, \underline{y_1} ..., \underline{x_k}, \underline{y_k})$ in $\mathbb{R}^{2k}$ and any similarity transformation f of $\mathbb{R}^2$, we have that $(\underline{x_1}, \underline{y_1} ..., \underline{x_k}, \underline{y_k})$ implies $(f(\underline{x_1}, \underline{y_1}), \ldots, f(\underline{x_k}, \underline{y_k})) \in Q$.

Since similarity transformations are affine transformations, affine-invariant relations are similarity invariant.

In Euclidean geometry there is no notion of unit length. Hence, no intrinsic metric can be defined in the Euclidean plane. Although, the relation $\equiv$, consisting of all quadruples $(\underline{p}, \underline{r}, \underline{q}, \underline{s}) \in \mathbb{E}^4$ such that the segments $\overline{pr}$ and $\overline{qs}$ are congruent (i.e., for which the distance between $\underline{p}$ and $\underline{r}$ is equal to the distance between $\underline{q}$ and $\underline{s}$), is a similarity-invariant relation. It gives an example of a similarity-invariant relations that is not affine invariant.

Further examples of affine-invariant (and thus, similarity-invariant) geometric relations concern parallelism and equal ratio. Indeed, if four points define two parallel lines, then the results of any affine transformation applied to them, also define two parallel lines. Also, the ratio of a triple $(\underline{p}, \underline{q}, \underline{r})$ of collinear points, defined (when $\underline{p} \neq \underline{r}$) as $\frac{\|C_{O,E_1,E_2}(\underline{q})-C_{O,E_1,E_2}(\underline{p})\|}{\|C_{O,E_1,E_2}(\underline{r})-C_{O,E_1,E_2}(\underline{p})\|}$ and denoted $(\underline{p} : \underline{q} : \underline{r})$, is independent of the affine coordinate system O, E_1, E_2 of $\mathbb{E}$. Therefore, the 6-ary geometric relation *equal ratio* $(\underline{p} : \underline{q} : \underline{r}) = (\underline{p'} : \underline{q'} : \underline{r'})$ is affine invariant.

2.4 The theories $\mathcal{R}$, $\mathcal{E}$, $\mathcal{A}$ and their expressive power

We work in first-order logic with equality and suppose that first-order formulas are built using the connectives $\neg$ and $\wedge$ and the existential quantifier $\exists$. The symbols $\vee$, $\rightarrow$, $\forall$ and $\neq$ stand for their usual abbreviations.

We introduce now the first-order languages $\mathsf{FO}(+,\times,<,0,1)$, $\mathsf{FO}(\beta,\equiv)$ and $\mathsf{FO}(\beta)$ together with their standard interpretations and characterize their expressive power.

Definition 2.6 Suppose that σ is a first-order signature, S is a σ-structure, and ψ a $\mathsf{FO}(\sigma)$-formula with m free variables. The *relation defined* by ψ in S is the set of m-tuples in $|S|^m$ that satisfy ψ.

If $k < m$ and $(\underline{s_1}, \dots, \underline{s_k})$ is a k-tuple of elements in $|S|^k$, we define the *relation defined* by $\psi[\underline{s_1}, \dots, \underline{s_k}]$ in S as the set of $(m-k)$-tuples $(\underline{s_{k+1}}, \dots, \underline{s_m})$ of elements in $|S|^{m-k}$ such that $(\underline{s_1}, \dots, \underline{s_m})$ satisfy ψ.

Since we consider only one interpretation of each language, we shall use the same symbol for relation/ functional symbols and their interpretations, not to overload the notation. We also refer to *the relation defined by a formula* without reference to the structure considered. As we shall see, the theories $\mathcal{R}$, $\mathcal{E}$ and $\mathcal{A}$ define precisely the semi-algebraic, similarity-invariant and affine-invariant geometric relations, respectively.

The language $\mathsf{FO}(\beta)$ is the first-order language with a signature consisting only of the ternary relation symbol β. As the standard interpretation for this language, we consider the structure $\langle \mathbb{E}, \beta \rangle$, where variables are assumed to range over the Euclidean plane $\mathbb{E}$ and where $(\underline{p}, \underline{q}, \underline{r}) \in \beta$ if and only if $\underline{p}$, $\underline{q}$ and $\underline{r}$ are collinear points and $\underline{q}$ belongs to the closed line segment between $\underline{p}$ and $\underline{r}$. In particular, $(\underline{p}, \underline{p}, \underline{q}) \in \beta$ for any $\underline{p}, \underline{q} \in \mathbb{E}$. We denote by $\mathcal{A}$ the first-order theory resulting from this standard interpretation. The next proposition follows immediately from [6, Proposition 5.4].

Proposition 2.7 *The relations definable in $\mathcal{A}$, correspond exactly to the affine-invariant geometric relations.*

The language $\mathsf{FO}(\beta,\equiv)$ is the first-order language with a signature consisting only of the ternary relation symbol β and the quaternary relation symbol $\equiv$. As the standard interpretation for this language, we consider the structure $\langle \mathbb{E}, \beta, \equiv \rangle$, where variables are assumed to range over the Euclidean plane $\mathbb{E}$, β is defined as before and $(\underline{p}, \underline{r}, \underline{q}, \underline{s}) \in \equiv$ if and only if the segments $\overline{\underline{p}\underline{r}}$ and $\overline{\underline{q}\underline{s}}$ are congruent. We denote by $\mathcal{E}$ the first-order theory resulting from this standard interpretation. For the sake of readability and following the tradition, we denote $\equiv (p_i, p_j, p_k, p_l)$ by $p_i p_j \equiv p_k p_l$. The next proposition follows immediately from [6, Proposition 5.5].

Proposition 2.8 *The relations definable in $\mathcal{E}$, correspond exactly to the similarity-invariant geometric relations.*

Finally, $\mathsf{FO}(+,\times,<,0,1)$ is a first-order language with a signature consisting of the binary function symbols $+$ and $\times$; the binary relation symbol $<$; and the constant symbols 0 and 1. We call this language *the language of real closed fields*. As its standard interpretation, we consider the structure $\langle \mathbb{R}, +, \times, <, 0, 1 \rangle$, that is, the reals with the well-known functions, relation and constants. We denote by $\mathcal{R}$ the theory resulting from this interpretation, usually called *the theory of the real closed fields*. The following proposition is an immediate consequence of [3, Theorem 2.74].

Proposition 2.9 *The relations definable in $\mathcal{R}$, correspond exactly to the semi-algebraic relations.*

Clearly, not any $\mathsf{FO}(+,\times,<,0,1)$-formula defines a similarity-invariant relation. The formula $x_1 = 0 \wedge y_1 = 0$ exemplifies this. We shall denote by $\mathsf{FO}(+,\times,<,0,1)_{\mathrm{SI}}$ the similarity-invariant fragment of $\mathsf{FO}(+,\times,<,0,1)$, i.e., the set of $\mathsf{FO}(+,\times,<,0,1)$-formulas defining similarity-invariant semi-algebraic relations. Analogously, we denote by $\mathsf{FO}(+,\times,<,0,1)_{\mathrm{AI}}$ the affine-invariant fragment of $\mathsf{FO}(+,\times,<,0,1)$.

2.5 Translations

In order to compare relations defined on the Euclidean plane with relations defined on the Cartesian plane, we introduce the following definitions.

Let us call the languages with geometric variables (whose standard interpretation is given over $\mathbb{E}$) *geometric languages*; $\mathsf{FO}(\beta)$ and $\mathsf{FO}(\beta,\equiv)$, as well as the new languages we are going to introduce, are examples of geometric languages.

Definition 2.10 Let φ be a formula in a geometric language defining the m-ary geometric relation G_φ $(m \geq 0)$ and let ψ be a $\mathsf{FO}(+,\times,<,0,1)$-formula defining the $2m$-ary semi-algebraic relation A_ψ. If, for any points $\underline{p_1},\ldots,\underline{p_m}$ in $\mathbb{E}$, with coordinates $(\underline{x_1},\underline{y_1}),\ldots,(\underline{x_m},\underline{y_m})$ with respect to the coordinate system O, E_1, E_2,

$$G_\varphi(\underline{p_1},\ldots,\underline{p_m}) \quad \text{holds if and only if} \quad A_\psi(\underline{x_1},\underline{y_1},\ldots,\underline{x_m},\underline{y_m}) \text{ holds},$$

then φ and ψ are said to *define the same relation*.

We remark that, since $\mathsf{FO}(\beta,\equiv)$-formulas define similarity-invariant relations, in the case $\varphi \in \mathsf{FO}(\beta,\equiv)$, the previous definition is independent of the Euclidean coordinate system O, E_1, E_2. Analogously, for $\varphi \in \mathsf{FO}(\beta)$ the definition remains invariant if we change O, E_1, E_2 to any other affine coordinate system.

The following two fundamental examples are basic results in analytic geometry.

Example 2.11 The $\mathsf{FO}(+,\times,<,0,1)$-formula

$$\mathsf{equidistance}_{\mathsf{coord}}(x_1,y_1,x_2,y_2,x_3,y_3,x_4,y_4) := (x_1-x_2)^2+(y_1-y_2)^2 = (x_3-x_4)^2+(y_3-y_4)^2$$

and the $\mathsf{FO}(\beta,\equiv)$-formula $p_1p_2 \equiv p_3p_4$ define the same relation.

Example 2.12 Another important example is given by the $\mathsf{FO}(+,\times,<,0,1)$-formula

$$\begin{aligned}\beta_{\mathsf{coord}}(x_i,y_i,x_j,y_j,x_k,y_k) \;:=\; & [(x_i-x_j)(y_k-y_j) = (x_k-x_j)(y_i-y_j)]\wedge\\ & [((x_k-x_j)(x_j-x_i)>0) \vee ((x_k-x_j)(x_j-x_i)=0)]\wedge\\ & [((y_k-y_j)(y_j-y_i)>0) \vee ((y_k-y_j)(y_j-y_i)=0)]\end{aligned}$$

and the $\mathsf{FO}(\beta)$-formula $\beta(p_i,p_j,p_k)$. They both define the same relation.

Definition 2.13 Given two syntactic fragments $\mathcal{L}_1$ and $\mathcal{L}_2$, of two first-order languages with a fixed interpretation, a recursive function $\mathcal{M}$ that maps any $\mathcal{L}_1$-formula, φ, to a $\mathcal{L}_2$-formula, $\mathcal{M}(\varphi)$, defining the same relation as φ will be called a *translation* between these fragments.

Based on the Examples 2.11 and 2.12, we define a translation $\mathcal{C}$ from $\mathsf{FO}(\beta,\equiv)$ to $\mathsf{FO}(+,\times,<,0,1)_{\mathrm{SI}}$. For any $i \in \mathbb{N}$ and any point $\underline{p_i}$ in $\mathbb{E}$, we denote by $\underline{x_i}$ and $\underline{y_i}$ the first and the second coordinates of $\underline{p_i}$ with respect to our fixed coordinate system O, E_1, E_2. Since for all $\underline{p_1},\underline{p_2},\underline{p_3},\underline{p_4} \in \mathbb{E}$, $\mathcal{E} \models \beta[\underline{p_1},\underline{p_2},\underline{p_3}]$ if and only if $\mathcal{R} \models \beta_{\mathsf{coord}}[\underline{x_1},\underline{y_1},\underline{x_2},\underline{y_2},\underline{x_3},\underline{y_3}]$ and $\mathcal{E} \models [\underline{p_1},\underline{p_2}] \equiv [\underline{p_3},\underline{p_4}]$ if and only if $\mathcal{R} \models \mathsf{equidistance}_{\mathsf{coord}}[\underline{x_1},\underline{y_1},\underline{x_2},\underline{y_2},\underline{x_3},\underline{y_3},\underline{x_4},\underline{y_4}]$, we immediately obtain a translation, $\mathcal{C}$, defined on the quantifier-free fragment of $\mathsf{FO}(\beta,\equiv)$.

Indeed, $\mathcal{C}$ is obtained by translating $p_i = p_j$ by $x_i = x_j \wedge y_i = y_j$ and by defining $\mathcal{C}(\beta(p_i,p_j,p_k))$ as $\beta_{\mathsf{coord}}(x_i,y_i,x_j,y_j,x_k,y_k)$, $\mathcal{C}(p_ip_j \equiv p_kp_l)$ as $\mathsf{equidistance}_{\mathsf{coord}}(x_i,y_i,x_j,y_j,x_k,y_k,x_l,y_l)$, and further $\mathcal{C}(\varphi \wedge \psi)$ as $\mathcal{C}(\varphi)\wedge\mathcal{C}(\psi)$ and $\mathcal{C}(\neg\varphi)$ as $\neg\mathcal{C}(\varphi)$. We extend $\mathcal{C}$, by recursion on the quantifier-depth, to the whole language $\mathsf{FO}(\beta,\equiv)$ defining $\mathcal{C}(\exists p_i \varphi)$ as $\exists x_i \exists y_i \mathcal{C}(\varphi)$.

A direct induction on the structure of the formulas shows that for any $\mathsf{FO}(\beta,\equiv)$-formula φ with m free variables and for any points $\underline{p_1},\ldots,\underline{p_m}$ in $\mathbb{E}$, with coordinates $(\underline{x_1},\underline{y_1}),\ldots,(\underline{x_m},\underline{y_m})$ the following holds:

$$\mathcal{E} \models \varphi[\underline{p_1},\ldots,\underline{p_m}] \quad \text{if and only if} \quad \mathcal{R} \models \mathcal{C}(\varphi)[\underline{x_1},\underline{y_1},\ldots,\underline{x_m},\underline{y_m}].$$

We summarize these result in the following proposition.

Proposition 2.14 *The function $\mathcal{C}$ is a translation from $\mathsf{FO}(\beta,\equiv)$ to $\mathsf{FO}(+,\times,<,0,1)_{\mathrm{SI}}$.*

In particular, we obtain the following corollary.

Corollary 2.15 *The function $\mathcal{C}|_{\mathsf{FO}(\beta)}$ is a translation from $\mathsf{FO}(\beta)$ to $\mathsf{FO}(+,\times,<,0,1)_{\mathrm{AI}}$.*

2.6 Quantifier elimination for $\mathcal{R}$, $\mathcal{E}$ and $\mathcal{A}$

Definition 2.16 Let $\mathcal{S}$ be a first-order theory over a signature σ. We say that a theory $\mathcal{S}$ has *quantifier elimination* if for every formula $\varphi \in \mathsf{FO}(\sigma)$ there is a quantifier-free formula with the same free-variables $\psi \in \mathsf{FO}(\sigma)$ such that $\mathcal{S} \models \varphi \leftrightarrow \psi$.

We remark that if the signature σ does not have constant symbols then it has no quantifier-free sentences. Some authors admit (cf. [10]) that the formula ψ in the Definition 2.16 may have more free variables than the original formula φ as long as these two formulas are equivalent; in this way, the quantifier-free formula equivalent to a true sentence may be $p = p$. Others (cf. [14]) say that a theory $\mathcal{S}$ has quantifier elimination if every formula *with at least one free variable* is equivalent to a quantifier-free formula with the same free variables. We prefer to use the notion given by Definition 2.16 and to solve this inconvenience adding to σ a new constant predicate symbol $\top$ that holds in the structure $\mathcal{S}$ and furnishes a quantifier-free true sentence.

For any first-order signature σ, we denote by $\mathsf{FO}(\sigma)_{\mathrm{QF}}$ the quantifier-free fragment of $\mathsf{FO}(\sigma)$.

A recursive function $\mathfrak{E}_{\mathcal{S}} : \mathsf{FO}(\sigma) \to \mathsf{FO}(\sigma)_{\mathrm{QF}}$ is called a *quantifier-elimination function* if for any $\mathsf{FO}(\sigma)$-formula φ, $\mathfrak{E}_{\mathcal{S}}(\varphi)$ is a quantifier-free $\mathsf{FO}(\sigma)$-formula, equivalent to and with the same free-variables as φ. If such a function exists, the theory is said to admit *effective* quantifier elimination.

In the 1930s, Tarski showed that the theory of real closed fields, $\mathcal{R}$, admits effective quantifier elimination (cf. [21], or [2] for a modern account). In the same article, Tarski used this result and an interpretation of the Euclidean plane in the Cartesian plane, to give a decision procedure for elementary geometry (cf. [15]). We denote by $\mathfrak{E}_{\mathcal{R}}$ a quantifier-elimination function for the theory of real closed fields.

Since the theories $\mathcal{E}$ and $\mathcal{A}$ do not have constant symbols, they do not admit quantifier elimination. We prove the following stronger result: it is not possible to obtain a theory that admits quantifier elimination by extending $\mathsf{FO}(\beta)$ (nor $\mathsf{FO}(\beta, \equiv)$) with finitely many relation symbols.

For every $k \in \mathbb{N}$, consider the semi-algebraic affine-invariant relation P^k consisting of the triplets of aligned points $(\underline{o}, \underline{p}, \underline{s})$ such that the segment $\overline{\underline{os}}$ is equal to k times the segment $\overline{\underline{op}}$. Clearly, if $k \neq j$ then the relations P^k and P^j are different. This implies that there are countably infinite different ternary affine-invariant relations. By Proposition 2.7, all these ternary relations are definable in $\mathsf{FO}(\beta)$. We denote by ψ_k a $\mathsf{FO}(\beta)$-formula defining the relation P^k.

Proposition 2.17 *Any extension of* $\mathsf{FO}(\beta)$ *with a finite number of new relation symbols does not admit quantifier elimination.*

P r o o f. We suppose than an extension of $\mathsf{FO}(\beta)$ with a finite number of new relation symbols is given. If this extension admitted quantifier elimination, all the different ternary relations $P^i(o, p, q)$, $i \in \mathbb{N}$, would be definable in this language by quantifier-free formulas. Since there are no constant nor function symbols in the new language, the only terms that can be built in the extended language using the variables o, p and q are the atomic terms o, p and q themselves. Thus, the number of different atomic formulas that can be built using only the given variables is finite. Hence, the number of non-equivalent quantifier-free formulas in this language is finite. Therefore, the extended language cannot define, without quantifiers, all the infinite different relations defined by the quantified $\mathsf{FO}(\beta)$-formulas ψ^k, $k \in \mathbb{N}$. This concludes the proof. □

The previous proof yields immediately the following corollary.

Corollary 2.18 *Any extension of* $\mathsf{FO}(\beta, \equiv)$ *with a finite number of new relation symbols does not admit quantifier elimination.*

3 The new languages

In this section, we extend by definitions the languages of elementary geometry and elementary affine geometry obtaining the new languages $\mathsf{FO}(\beta, \equiv, \top, \oplus, \otimes, \pi^{\perp}, \kappa)$ and $\mathsf{FO}(\beta, \top, \oplus, \otimes, \pi)$, respectively. Their associated theories $\mathcal{E}'$ and $\mathcal{A}'$ will be shown to admit quantifier elimination in the following section.

The new symbols introduced are the two *basic segment arithmetic functions*, $\oplus$ and $\otimes$, the *affine projection function* π, and the two *basic metric functions*, $\pi^{\perp}$ and κ.

First, we show how to define some affine-invariant relations in the language $\mathsf{FO}(\beta)$, that we need later on to define these functions.

Collinear: The formula:

$$\mathsf{L}(p, q, r) := \beta(p, q, r) \vee \beta(p, r, q) \vee \beta(q, p, r)$$

expresses that the points p, q and r are collinear. We remark that this is a quantifier-free expression.

Parallel: The formula:

$$\mathsf{P}(p,q,r,s) := (\mathsf{L}(p,q,r) \wedge \mathsf{L}(p,q,s)) \vee r = s \vee \forall u(\neg \mathsf{L}(p,q,u) \vee \neg \mathsf{L}(r,s,u))$$

expresses that the segments $\overline{pq}$ and $\overline{rs}$ are parallel. We remark that the first line expresses that the four points are aligned or that a segment is just a point, in both cases $\overline{pq}$ and $\overline{rs}$ are considered parallel. The second line expresses that no point is collinear with p and q and with r and s, at the same time.

We shall also need the following similarity-invariant relation.

Right angle: The $\mathsf{FO}(\beta, \equiv)$-formula:

$$\mathsf{R}(p,q,r) := \neg \mathsf{L}(p,q,r) \wedge \exists o(\beta(o,p,q) \wedge or \equiv rq \wedge op \equiv pq)$$

expresses that the points p, q and r form a non-degenerate triangle with a straight angle at p.

Finally, the following formula is used to define the new symbol $\top$.

(1) $$\forall p(p = p)$$

3.1 The two basic segment-arithmetic functions

Now, we present the formulas that implicitly define the *basic segment-arithmetic functions*.

Sum: The relation "the vector $\overrightarrow{os}$ is the result of the vector sum of $\overrightarrow{op}$ and $\overrightarrow{oq}$" is, certainly, an affine-invariant geometric relation. Thus, by Proposition 2.7, there exists a $\mathsf{FO}(\beta)$-formula $\mathsf{Sum}(o,p,q,s)$ defining it.
Let $\underline{x_1}, \underline{x_2}, \underline{x_3}$ be three real numbers. Using the coordinates in the fixed coordinate system O, E_1, E_2 to define points in $\mathbb{E}$, we consider $\underline{o} = (0,0), \underline{p_1} = (\underline{x_1}, 0), \underline{p_2} = (\underline{x_2}, 0)$ and $\underline{p_3} = (\underline{x_3}, 0)$. Then, the relation $\mathsf{Sum}(\underline{o}, \underline{p_1}, \underline{p_2}, \underline{p_3})$ holds if and only if $\underline{x_1} + \underline{x_2} = \underline{x_3}$ as real numbers. This allows us to translate the semi-algebraic addition into the geometric context.

Equal Ratio: We consider the 5-ary relation: "o, p and q are collinear, $o \neq p$, $o \neq r$ and s is the unique point, collinear with o and r, that satisfies $(o : p : q) = (o : r : s)$". This is an affine-invariant geometric relation and since $\mathsf{FO}(\beta)$ is a complete language for these relations, there exists an $\mathsf{FO}(\beta)$-formula $\mathsf{EqualRatio}(o,p,q,r,s)$ defining it.
Let $\underline{x_1}, \underline{x_2}, \underline{x_3}$ be three real numbers. Using the coordinates in the fixed coordinate system O, E_1, E_2 to define points in $\mathbb{E}$, we consider $\underline{o} = (0,0), \underline{e_1} = (1,0), \underline{p_1} = (\underline{x_1}, 0), \underline{p_2} = (\underline{x_2}, 0)$ and $\underline{p_3} = (\underline{x_3}, 0)$. We have that, for $\underline{x_2} \neq 0$, $\mathsf{EqualRatio}(\underline{o}, \underline{e_1}, \underline{p_1}, \underline{p_2}, \underline{p_3})$ holds if and only if $\underline{x_1} \cdot \underline{x_2} = \underline{x_3}$ as real numbers. This allow us to translate the semi-algebraic product into the geometric context.

Constructions, similar to Sum and $\mathsf{EqualRatio}$, to deal with segment arithmetic can be found already in Descartes [4], in Hilbert's book [8] and also in [16] (cf. also [7] for a contemporary account).

We remark that for every $\underline{o}, \underline{p}, \underline{q} \in \mathbb{E}$ there exists a unique $\underline{s}$ satisfying $\mathsf{Sum}(\underline{o}, \underline{p}, \underline{q}, \underline{s})$. On the other hand, for every $\underline{o}, \underline{p}, \underline{q}, \underline{r}$ there exists at most one $\underline{s}$ satisfying $\mathsf{EqualRatio}(\underline{o}, \underline{p}, \underline{q}, \underline{r}, \underline{s})$.

We conclude that the following two $\mathsf{FO}(\beta)$-formulas define functional relations with respect to their last variable:

(2) $$\mathsf{Sum}(o,p,q,s);$$

(3) $$\mathsf{EqualRatio}(o,p,q,r,s) \vee [(\neg \mathsf{L}(o,p,q) \vee \neg \mathsf{L}(o,r,s) \vee o = p \vee o = r) \wedge s = o].$$

3.2 The affine projection function

We present the formula that defines the *affine projection function*.

Affine Projection: We want to define the following relation: "the points o, p and q form an affine coordinate system and s is the projection, parallel to $\overline{oq}$, of r on the line $\overline{op}$, or o, p and q are aligned and

$s = o$". Being an affine-invariant geometric relation, we know that the relation is definable in $\mathsf{FO}(\beta)$. Explicitly, we can define it as:

(4) $$\neg\mathsf{L}(o,p,q) \wedge [(\mathsf{L}(r,o,p) \wedge s = r) \vee (\neg\mathsf{L}(r,o,p) \wedge \mathsf{L}(s,o,p) \wedge \mathsf{P}(r,s,o,q))] \vee (\mathsf{L}(o,p,q) \wedge s = o).$$

We remark that the formula defines a functional relation in s. We call this function the *affine projection function* and denote it by $\pi : \mathbb{E}^4 \to \mathbb{E}$.

3.3 The two basic metric functions

Now, we present the two formulas that implicitly define the *basic metric functions*.

Orthogonal Projection: We consider the 4-ary relation defined by

(5) $$(\mathsf{L}(o,p,q) \wedge s = q) \vee (\neg\mathsf{L}(o,p,q) \wedge \mathsf{L}(o,p,s) \wedge (\mathsf{R}(s,q,o) \vee \mathsf{R}(s,q,p))).$$

When $o \neq p$, the last formula defines that s is the orthogonal projection of q over the line passing through o and p. We remark that this formula defines also a functional relation in s.

Segment Construction: The axiom of segment construction states that $\exists s(\beta(p,o,s) \wedge os \equiv qr)$. This axiom appears in Tarski's axiomatization of elementary geometry [22] (cf. the first congruence axiom in Hilbert's text [8]). We introduce the following $\mathsf{FO}(\beta,\equiv)$-formula closely related to it:

(6) $$(o = p \wedge s = o) \vee (o \neq p \wedge \beta(p,o,s) \wedge os \equiv qr).$$

We remark that this relation defined by this formula is functional in s. If $\underline{o} \neq \underline{p}$, the unique $\underline{s}$ satisfying it is the point in the ray opposite to $\overrightarrow{\underline{op}}$ such that the segments $\overline{\underline{qr}}$ and $\overline{\underline{os}}$ are congruent.

The functions implicitly defined by these $\mathsf{FO}(\beta,\equiv)$-formulas with respect to their last variable are called the *basic metric functions* and are denoted by κ and $\pi^{\perp}$, respectively.

3.4 The language $\mathsf{FO}(\beta,\top,\oplus,\otimes,\pi)$ and the theory $\mathcal{A}'$

We extend the theory $\mathcal{A}$ by definitions (cf., e.g., [17, Section 4.6]) using the formula (1) to define the 0-ary relation symbol $\top$, the formula (2) to define the function symbol $\oplus$, the formula (3) to define the function symbol $\otimes$ and the formula (4) to define the function symbol π. In this way, we obtain the theory $\mathcal{A}'$ in the extended language $\mathsf{FO}(\beta,\top,\oplus,\otimes,\pi)$.

Being an extension by definitions, the expressive power of the expanded language is the same as that of the original one and there exists (cf. [17]) a translation $\mathcal{B} : \mathsf{FO}(\beta,\top,\oplus,\otimes,\pi) \to \mathsf{FO}(\beta)$ in the sense of Definition 2.13. If $\varphi \in \mathsf{FO}(\beta,\top,\oplus,\otimes,\pi)$ happens to be a $\mathsf{FO}(\beta)$-formula, then $\mathcal{B}(\varphi)$ is just φ. Essentially, via this map, a formula in $\mathsf{FO}(\beta,\top,\oplus,\otimes,\pi)$ is translated to an $\mathsf{FO}(\beta)$-formula replacing each occurrence of a new symbol, by its defining formula in $\mathsf{FO}(\beta)$. This is summarized in the following proposition.

Proposition 3.1 *The map*

$$\mathcal{B} : \mathsf{FO}(\beta,\top,\oplus,\otimes,\pi) \to \mathsf{FO}(\beta)$$

is a translation.

For the sake of legibility, we shall use the following suggestive notation for terms in the language $\mathsf{FO}(\beta,\top,\oplus,\otimes,\pi)$. We write $p \oplus_o q$ for $\oplus(o,p,q)$; $q \otimes_{o,p} r$ for $\otimes(o,p,q,r)$; and $\pi_{opq}(r)$ for $\pi(o,p,q,r)$.

The next lemma follows directly from the definitions.

Lemma 3.2 *We consider $\underline{x_1}, \underline{x_2} \in \mathbb{R}$ and three affine-independent points $\underline{o}, \underline{e_1}, \underline{e_2} \in \mathbb{E}$. We further denote $\underline{p_1} = (\underline{x_1}, 0)$ and $\underline{p_2} = (\underline{x_2}, 0)$, where the coordinates are taken with respect to the affine coordinate system $\underline{o}, \underline{e_1}, \underline{e_2}$. Then, the standard interpretation of the term $\underline{p_1} \oplus_{\underline{o}} \underline{p_2}$ is the point with coordinates $(\underline{x_1} + \underline{x_2}, 0)$, and the standard interpretation of $\underline{p_1} \otimes_{\underline{o},\underline{e_1}} \underline{p_2}$ has coordinates $(\underline{x_1} \cdot \underline{x_2}, 0)$.*

We further define the following abbreviations:

$$\mathsf{AffCoord}^1_{o,e_1,e_2}(p) := \pi_{o,e_1,e_2}(p); \text{ and}$$
$$\mathsf{AffCoord}^2_{o,e_1,e_2}(p) := \pi_{o,e_2,e_1}(p) \otimes_{o,e_2} e_1.$$

When the points $\underline{o}, \underline{e_1}, \underline{e_2}$ form an affine coordinate system, it follows immediately that the term $\mathsf{AffCoord}^1_{\underline{o},\underline{e_1},\underline{e_2}}(\underline{p})$ can be interpreted geometrically as the projection, in the direction of $\overline{\underline{oe_2}}$, of the point $\underline{p}$ over the line $\overline{\underline{oe_1}}$. Under the same hypothesis and denoting by $\underline{p'}$ the projection parallel to $\overline{\underline{oe_1}}$ of the point $\underline{p}$ over the line $\overline{\underline{oe_2}}$, the term $\mathsf{AffCoord}^2_{\underline{o},\underline{e_1},\underline{e_2}}(\underline{p})$ represents the unique point $\underline{q}$ on the line $\overline{\underline{oe_1}}$ that satisfies $(\underline{o} : \underline{e_1} : \underline{q}) = (\underline{o} : \underline{e_2} : \underline{p'})$.

We state this result for further reference.

Lemma 3.3 *We suppose that the points $\underline{o}, \underline{e_1}, \underline{e_2}$ form an affine coordinate system and that the point $\underline{p}$ has coordinates $(\underline{x}, \underline{y})$ in this coordinate system. Then, the term $\mathsf{AffCoord}^1_{\underline{o},\underline{e_1},\underline{e_2}}(\underline{p})$ is naturally interpreted as the point with coordinates $(\underline{x}, 0)$ and the term $\mathsf{AffCoord}^2_{\underline{o},\underline{e_1},\underline{e_2}}(\underline{p})$ as the point with coordinates $(\underline{y}, 0)$, always with respect to the same coordinate system.*

3.5 The language $\mathsf{FO}(\beta, \equiv, \top, \oplus, \otimes, \pi^\perp, \kappa)$ and the theory $\mathcal{E}'$

As we did with the theory $\mathcal{A}$, we extend the theory $\mathcal{E}$ by definitions using the formula (1) to define the 0-ary relation symbol $\top$, the formula (2) to define the function symbol $\oplus$, the formula (3) to define the function symbol $\otimes$, the formula (5) to define the function symbol $\pi^\perp$ and the formula (6) to define the function symbol κ. In this way, we obtain the theory $\mathcal{E}'$ in the extended language $\mathsf{FO}(\top, \oplus, \otimes, \pi^\perp, \kappa)$.

Being an extension by definitions, the expressive power of the expanded language is the same as that of the original one and there exists a translation $\mathcal{M} : \mathsf{FO}(\beta, \equiv, \top, \oplus, \otimes, \pi^\perp, \kappa) \to \mathsf{FO}(\beta, \equiv)$. Essentially, via this map, a formula in $\mathsf{FO}(\top, \oplus, \otimes, \pi^\perp, \kappa)$ is translated to an $\mathsf{FO}(\beta, \equiv)$-formula replacing each occurrence of a new symbol, by its defining formula in $\mathsf{FO}(\beta, \equiv)$.

Proposition 3.4 *The map*

$$\mathcal{M} : \mathsf{FO}(\beta, \top, \oplus, \otimes, \pi) \to \mathsf{FO}(\beta)$$

is a translation.

We shall use the notation previously introduce for the symbols $\oplus$ and $\otimes$ and we denote by $\pi^\perp_{op}(q)$ the $\mathsf{FO}(\beta, \equiv, \top, \oplus, \otimes, \pi^\perp, \kappa)$-term $\pi^\perp(o, p, q)$.

We further define the following abbreviations:

$$\mathsf{EuCoord}^1_{o,e_1,e_2}(p) := \pi^\perp_{o,e_1}(p);$$
$$\mathsf{EuCoord}^2_{o,e_1,e_2}(p) := \pi^\perp_{o,e_2}(p) \otimes_{o,e_2} e_1; \text{ and}$$
$$\varepsilon(o, p, q) := \kappa(o, o \oplus_{-\pi^\perp_{op}(q)} q, o, p).$$

The following result follows immediately from the definitions.

Lemma 3.5 *We suppose that the points $\underline{o}, \underline{e_1}, \underline{e_2}$ form an Euclidean coordinate system and that the point $\underline{p}$ has coordinates $(\underline{x}, \underline{y})$ in this coordinate system. Then, the term $\mathsf{EuCoord}^1_{\underline{o},\underline{e_1},\underline{e_2}}(\underline{p})$ is naturally interpreted as the point with coordinates $(\underline{x}, 0)$ and the term $\mathsf{EuCoord}^2_{\underline{o},\underline{e_1},\underline{e_2}}(\underline{p})$ as the point with coordinates $(\underline{y}, 0)$, always with respect to the same coordinate system.*

Lemma 3.6 *If the points $\underline{o}, \underline{e_1}, \underline{e_2}$ form an affine coordinate system, then the points $\underline{o}, \underline{e_1}, \varepsilon(\underline{o}, \underline{e_1}, \underline{e_2})$ form an Euclidean coordinate system.*

Proof. Let us suppose that the three points are affine independent. The segments $\overline{\underline{oe_1}}$ and $\overline{\underline{o}\varepsilon(\underline{o}, \underline{e_1}, \underline{e_2})}$ and congruent by construction (cf. the definition of κ). Since the point $\varepsilon(\underline{o}, \underline{e_1}, \underline{e_2})$ belongs to the line $\overline{\underline{o}(\underline{q} - \pi^\perp_{\underline{op}}(\underline{q}))}$ that is perpendicular to the line $\overline{\underline{op}}$, the three points form an Euclidean coordinate system. □

4 The translation $\mathcal{S}$ of $\mathsf{FO}(+, \times, <, 0, 1)_{\mathrm{QF,AI}}$-formulas to $\mathsf{FO}(\beta, \top, \oplus, \otimes, \pi)_{\mathrm{QF}}$-formulas

In the present section, we define a translation from the quantifier-free affine-invariant fragment of $\mathsf{FO}(+, \times, <, 0, 1)$ into the quantifier-free fragment of $\mathsf{FO}(\beta, \top, \oplus, \otimes, \pi)$.

The main result of the present section is the following theorem.

Theorem 4.1 *There exists a translation*

$$\mathcal{S} : \mathsf{FO}(+, \times, <, 0, 1)_{\mathrm{QF,AI}} \to \mathsf{FO}(\beta, \top, \oplus, \otimes, \pi)_{\mathrm{QF}}.$$

4.1 A translation given an affine coordinate system for $\mathbb{E}$

We assume that the variables used in $\mathsf{FO}(+, \times, <, 0, 1)$-formulas are $x_1, y_1, x_2, y_2, \ldots$ and we define a map (not a translation)

$$\mathcal{S}_{o,e_1,e_2} : \mathsf{FO}(+, \times, <, 0, 1)_{\mathrm{QF,AI}} \to \mathsf{FO}(\beta, \top, \oplus, \otimes, \pi)_{\mathrm{QF}}.$$

The image, $\mathcal{S}_{o,e_1,e_2}(\varphi)$, of an $\mathsf{FO}(+, \times, <, 0, 1)$-formula φ in the variables $x_1, y_1, \ldots, x_m, y_m$, involves the variables $o, e_1, e_2, p_1, p_2, \ldots, p_m$.

First, we define it for $\mathsf{FO}(+, \times, <, 0, 1)$-terms, by induction on structure of the term, as follows:

$$\begin{aligned}
\mathcal{S}_{o,e_1,e_2}(0) &:= o,\\
\mathcal{S}_{o,e_1,e_2}(1) &:= e_1,\\
\mathcal{S}_{o,e_1,e_2}(x_i) &:= \mathsf{AffCoord}^1_{o,e_1,e_2}(p_i),\\
\mathcal{S}_{o,e_1,e_2}(y_i) &:= \mathsf{AffCoord}^2_{o,e_1,e_2}(p_i),\\
\mathcal{S}_{o,e_1,e_2}(t_1 + t_2) &:= \mathcal{S}_{o,e_1,e_2}(t_1) \oplus_o \mathcal{S}_{o,e_1,e_2}(t_2) \text{ and}\\
\mathcal{S}_{o,e_1,e_2}(t_1 \times t_2) &:= \mathcal{S}_{o,e_1,e_2}(t_1) \otimes_{o,e_1} \mathcal{S}_{o,e_1,e_2}(t_2) \text{ where } t_1 \text{ and } t_2 \text{ are } \mathsf{FO}(+, \times, <, 0, 1)\text{-terms.}
\end{aligned}$$

We remark that the image of an $\mathsf{FO}(+, \times, <, 0, 1)$-term involving the variables $x_1, y_1, \ldots, x_m, y_m$, through the map $\mathcal{S}_{o,e_1,e_2}$, is an $\mathsf{FO}(\beta, \top, \oplus, \otimes, \pi)$-term in the variables o, e_1, e_2 and $p_1, \ldots, p_m$. The map $\mathcal{S}_{o,e_1,e_2}$ allows us to translate the two basic semi-algebraic operations ($+$ and $\times$) to the geometric setting, as is proved in the next proposition.

Proposition 4.2 *Let us assume that $\underline{o}, \underline{e_1}, \underline{e_2}$ are three affine-independent points. Let t be a $\mathsf{FO}(+, \times, <, 0, 1)$-term in the variables $x_1, y_1, \ldots, x_m, y_m$ and consider points $\underline{p_1}, \ldots, \underline{p_m}$ in $\mathbb{E}$, with coordinates $(\underline{x_1}, \underline{y_1}), \ldots, (\underline{x_m}, \underline{y_m})$ with respect to the coordinate system $\underline{o}, \underline{e_1}, \underline{e_2}$.*

Then, $\mathcal{S}_{o,e_1,e_2}(t)[\underline{o}, \underline{e_1}, \underline{e_2}, \underline{p_1}, \ldots, \underline{p_m}]$ has coordinates $(t[\underline{x_1}, \underline{y_1}, \ldots, \underline{x_m}, \underline{y_m}], 0)$ in the affine coordinate system $\underline{o}, \underline{e_1}, \underline{e_2}$.

P r o o f. We prove the proposition by induction in length of the term t. If it is an atomic term, the conclusion follows directly from Lemma 3.3. It remains to prove the cases $t = r + s$ and $t = r \times s$, where r and s are shorter $\mathsf{FO}(+, \times, <, 0, 1)$-terms. But these cases are direct consequence of Lemma 3.2. □

Now, we define the translation of atomic formulas. The case of the relation symbol "$<$" is based in a case analysis. We define $\mathcal{S}_{o,e_1,e_2}(t_1 = t_2)$ as $\mathcal{S}_{o,e_1,e_2}(t_1) = \mathcal{S}_{o,e_1,e_2}(t_2)$; and $\mathcal{S}_{o,e_1,e_2}(t_1 < t_2)$ as $(\mathcal{S}_{o,e_1,e_2}(t_1) \neq \mathcal{S}_{o,e_1,e_2}(t_2)) \wedge (\varphi_1 \vee \varphi_2 \vee \varphi_3)$, where

$$\begin{aligned}
\varphi_1 &:= \beta(\mathcal{S}_{o,e_1,e_2}(t_2), o, e_1) \wedge \beta(\mathcal{S}_{o,e_1,e_2}(t_1), \mathcal{S}_{o,e_1,e_2}(t_2), e_1);\\
\varphi_2 &:= \beta(o, e_1, \mathcal{S}_{o,e_1,e_2}(t_2)) \wedge (\beta(\mathcal{S}_{o,e_1,e_2}(t_1), o, e_1) \vee \beta(o, \mathcal{S}_{o,e_1,e_2}(t_1), \mathcal{S}_{o,e_1,e_2}(t_2))); \text{ and}\\
\varphi_3 &:= \beta(o, \mathcal{S}_{o,e_1,e_2}(t_2), e_1) \wedge (\beta(\mathcal{S}_{o,e_1,e_2}(t_1), o, e_1) \vee \beta(o, \mathcal{S}_{o,e_1,e_2}(t_1), \mathcal{S}_{o,e_1,e_2}(t_2))).
\end{aligned}$$

Finally, we extend the map $\mathcal{S}_{o,e_1,e_2}$ to the whole quantifier-free fragment of $\mathsf{FO}(+, \times, <, 0, 1)$ in the natural way, simply translating the conjunctions as conjunctions and negations as negations. The resulting formula always has o, e_1, e_2 as extra free variables and one *geometric variable* for each couple of *coordinate-variables* in the original formula.

To lighten the notation, we write $\mathcal{S}_{\underline{o},\underline{e_1},\underline{e_2}}(\varphi)$ for $\mathcal{S}_{o,e_1,e_2}(\varphi)[\underline{o},\underline{e_1},\underline{e_2}]$.

Proposition 4.3 *Let us assume that $\underline{o},\underline{e_1},\underline{e_2} \in \mathbb{E}$ form an affine coordinate system, and that φ is a quantifier-free $\mathsf{FO}(+,\times,<,0,1)$-formula in the variables $x_1, y_1, \ldots, x_m, y_m$. Consider points $\underline{p_1},\ldots,\underline{p_m}$ in $\mathbb{E}$, with coordinates $(\underline{x_1},\underline{y_1}),\ldots,(\underline{x_m},\underline{y_m})$ with respect to the coordinate system $\underline{o},\underline{e_1},\underline{e_2}$. Then, $\mathcal{A}' \models \mathcal{S}_{\underline{o},\underline{e_1},\underline{e_2}}(\varphi)[\underline{p_1},\ldots,\underline{p_m}]$ if and only if $\mathcal{R} \models \varphi[\underline{x_1},\underline{y_1},\ldots,\underline{x_m},\underline{y_m}]$.*

P r o o f. It is sufficient to prove the proposition for atomic formulas. The case of a formula of the form $t_1 = t_2$ is a consequence of Proposition 4.2. Let us assume that φ is of the form $t_1 < t_2$.

Let us denote by $\underline{t_1}$ the real number $t_1[\underline{x_1},\underline{y_1},\ldots,\underline{x_m},\underline{y_m}]$, and by $\underline{t_2}$ the real number $t_2[\underline{x_1},\underline{y_1},\ldots,\underline{x_m},\underline{y_m}]$.

Then, the points $\mathcal{S}_{\underline{o},\underline{e_1},\underline{e_2}}(t_1)[\underline{p_1},\ldots,\underline{p_m}]$ and $\mathcal{S}_{\underline{o},\underline{e_1},\underline{e_2}}(t_2)[\underline{p_1},\ldots,\underline{p_m}]$ have, respectively, by Proposition 4.2, coordinates $(\underline{t_1},0)$ and $(\underline{t_2},0)$ in the coordinate system $\underline{o},\underline{e_1},\underline{e_2}$.

We can assume, with out loss of generality, that $\underline{t_1} \neq \underline{t_2}$. Now, we claim that $\underline{t_1} < \underline{t_2}$ holds if and only if $(\varphi_1 \vee \varphi_2 \vee \varphi_3)[\underline{p_1},\ldots,\underline{p_m}]$ holds. Let us assume that $0 < \underline{t_2} < 1$, the remaining cases ($\underline{t_2} = 0, \underline{t_2} = 1, \underline{t_2} < 0$ and $\underline{t_2} > 1$) can be handled analogously.

Clearly, since $0 < \underline{t_2} < 1$, $(\varphi_1 \vee \varphi_2)[\underline{p_1},\ldots,\underline{p_m}]$ is false. Since, under the above hypothesis, $\underline{t_1}$ is less than $\underline{t_2}$ if and only if "0 is between $\underline{t_1}$ and 1, or $\underline{t_1}$ is between 0 and $\underline{t_2}$", $\varphi_3[\underline{p_1},\ldots,\underline{p_m}]$ holds if and only if $\underline{t_1} < \underline{t_2}$ holds. Hence, we have proved the claim and completed the proof of the proposition. □

4.2 Finding a basis

The map $\mathcal{S}_{o,e_1,e_2}$ is not a translation because it adds the three new free variables o, e_1 and e_2. We show how to use the variables $p_1,\ldots,p_m$ already involved in the formula, considering three different situations:

(1) when all the variables represent the same point;

(2) when all the variables represent points that are aligned and two are different; and

(3) when there are three variables representing affine-independent points.

To distinguish these cases, we define the three $\mathsf{FO}(\beta,\top,\oplus,\otimes,\pi)$-formulas $\mathsf{AffBasis}$, $\mathsf{Aligned}^m$ and Equal^m, and their $\mathsf{FO}(+,\times,<,0,1)$-counterparts. First, we define

$$\mathsf{AffBasis}(p_1,p_2,p_3) := \neg\mathsf{L}(p_1,p_2,p_3) \text{ and}$$
$$\mathsf{AffBasis}_{\mathsf{coord}}(x_1,y_1,x_2,y_2,x_3,y_3) := (x_2 - x_1)\times(y_3 - y_1) - (y_2 - y_1)\times(x_3 - x_1) \neq 0.$$

We remark that these are quantifier-free formulas. Both formulas define the same affine-invariant relation: that the three points form an affine coordinate system. That the second formula defines this relation is a consequence of the fact that the oriented area of the parallelogram with vertices at $(0,0), (a,b), (a+c,b+d)$, and (c,d), is given by the determinant of the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Further on, we consider, for $m \in \mathbb{N}$, $m \geq 3$, the formulas

$$\mathsf{Aligned}^m_{\mathsf{coord}}(x_1,y_1,\ldots,x_m,y_m) := \bigwedge_{1\leq i<j<k\leq m} \neg\mathsf{AffBasis}_{\mathsf{coord}}(x_i,y_i,x_j,y_j,x_k,y_k);$$
$$\mathsf{Aligned}^m(p_1,\ldots,p_m) := \bigwedge_{1\leq i<j<k\leq m} \neg\mathsf{AffBasis}(p_i,p_j,p_k);$$
$$\mathsf{Equal}^m_{\mathsf{coord}}(x_1,y_1,\ldots,x_m,y_m) := \bigwedge_{2\leq i\leq m} (x_1 = x_i)\wedge(y_1 = y_i), \text{ and}$$
$$\mathsf{Equal}^m(p_1,\ldots,p_m) := \bigwedge_{2\leq i\leq m} (p_1 = p_i).$$

We remark that these four formulas define affine-invariant relations. The first two define the same relation, namely, that the points are aligned. The last two also define the same relation, namely, that all the points are the same.

For the remainder of this section, let us assume that $\varphi(x_1, y_1, \ldots, x_m, y_m)$ is a quantifier-free $\mathsf{FO}(+, \times, <, 0, 1)$-formula defining an affine-invariant relation. For $i, j, k \in \mathbb{N}$ such that $1 \leq i < j < k \leq m$, let us denote by $\varphi^{\langle i,j,k \rangle}$ the formula

$$\mathsf{AffBasis}_{\mathsf{coord}}(x_i, y_i, x_j, y_j, x_k, y_k) \wedge \varphi(x_1, y_1, \ldots, x_m, y_m).$$

We remark that $\varphi^{\langle i,j,k \rangle}$ defines an affine-invariant relation.

Lemma 4.4 *The formula* $\varphi^{\langle i,j,k \rangle}$ *and* $\mathsf{AffBasis}(p_i, p_j, p_k) \wedge \mathcal{S}_{p_i,p_j,p_k}(\varphi)$ *define the same relation.*

Proof. For any $\underline{p_1}, \ldots, \underline{p_m} \in \mathbb{E}$ we consider $(\underline{x_1}, \underline{y_1}), \ldots, (\underline{x_m}, \underline{y_m})$ to be their coordinates in some fixed affine coordinate system. We prove that

$$\mathcal{R} \models \mathsf{AffBasis}_{\mathsf{coord}}[\underline{x_i}, \underline{y_i}, \underline{x_j}, \underline{y_j}, \underline{x_k}, \underline{y_k}] \wedge \varphi[\underline{x_1}, \underline{y_1}, \ldots, \underline{x_m}, \underline{y_m}].$$

if and only if

$$\mathcal{A}' \models \mathsf{AffBasis}[\underline{p_i}, \underline{p_j}, \underline{p_k}] \wedge \mathcal{S}_{\underline{p_i},\underline{p_j},\underline{p_k}}(\varphi)[\underline{p_1}, \ldots, \underline{p_m}].$$

On the one hand, if $\underline{p_i}, \underline{p_j}, \underline{p_k}$ are affine dependent, then both formulas are clearly false. On the other hand, if $\underline{p_i}, \underline{p_j}, \underline{p_k}$ are affine independent, since φ is affine invariant, Proposition 4.3 implies that $\varphi[\underline{x_1}, \underline{y_1}, \ldots, \underline{x_m}, \underline{y_m}]$ holds if and only if $\mathcal{S}_{\underline{p_i},\underline{p_j},\underline{p_k}}(\varphi)[\underline{p_1}, \ldots, \underline{p_m}]$ holds.

Hence, $\varphi^{\langle i,j,k \rangle}$ and $\mathsf{AffBasis}(p_i, p_j, p_k) \wedge \mathcal{S}_{p_i,p_j,p_k}(\varphi)$ define the same relation. □

Let us denote by $\varphi^{\langle * \rangle}$ the $\mathsf{FO}(+, \times, <, 0, 1)$-formula

$$\mathsf{Equal}^m_{\mathsf{coord}}(x_1, y_1, \ldots, x_m, y_m) \wedge \varphi(x_1, y_1, \ldots, x_m, y_m).$$

Lemma 4.5 *The formula* $\varphi^{\langle * \rangle}$ *defines the same relation as* $\mathsf{Equal}^m(p_1, \ldots, p_m)$ *or as* $\neg\top$*, and which of the two is the case is decidable.*

Proof. Since the theory of real closed fields is recursively decidable (cf., e.g., [3]), it is, in particular, effectively decidable whether $\varphi^{\langle * \rangle}$ is satisfiable or not. If it is unsatisfiable, it defines the same relation as $\neg\top$.

Suppose, on the other hand, that it its satisfiable. We prove that, under this assumption, $\varphi^{\langle * \rangle}$ defines the same relation as Equal^m. Clearly, if a tuple of pairs of coordinates satisfies $\varphi^{\langle * \rangle}$, then all the pairs are equal. But since $\varphi^{\langle * \rangle}$ defines an affine-invariant relation, its truth value is invariant under translations. Hence, it is satisfied by all m-tuples of equal pairs of coordinates. Whence, $\varphi^{\langle * \rangle}$ defines the same relation as Equal^m, and the proof is complete. □

To take care of those cases where all the points are aligned and two are different, we define a new map $\mathcal{S}_{o,e_1}$, differing from $\mathcal{S}_{o,e_1,e_2}$ only in the third and fourth rules in the definition of the term-translation. We remark that the these rules are the only ones where the map $\mathcal{S}_{o,e_1,e_2}$ involves e_2. So, we have

$$\begin{aligned}
\mathcal{S}_{o,e_1}(0) &:= o; \\
\mathcal{S}_{o,e_1}(1) &:= e_1; \\
\mathcal{S}_{o,e_1}(x_i) &:= p_i; \\
\mathcal{S}_{o,e_1}(y_i) &:= o; \\
\mathcal{S}_{o,e_1}(t_1 + t_2) &:= \mathcal{S}_{o,e_1}(t_1) \oplus_o \mathcal{S}_{o,e_1}(t_2); \text{ and} \\
\mathcal{S}_{o,e_1}(t_1 \times t_2) &:= \mathcal{S}_{o,e_1}(t_1) \otimes_{o,e_1} \mathcal{S}_{o,e_1}(t_2).
\end{aligned}$$

For $i \leq m$, let us denote by $\varphi^{\langle i \rangle}$ the formula

$$\mathsf{Aligned}^m_{\mathsf{coord}}(x_1, y_1, \ldots, x_m, y_m) \wedge ((x_1 \neq x_i) \vee (y_1 \neq y_i)) \wedge \varphi(x_1, y_1, \ldots, x_m, y_m).$$

Arguing as in the proofs of Proposition 4.2 and Lemma 4.4, we obtain the following result.

Lemma 4.6 *The formulas,* $\varphi^{\langle i \rangle}$ *and*

$$\mathsf{Aligned}^m(p_1, \ldots, p_m) \wedge (p_1 \neq p_i) \wedge S_{p_1,p_i}(\varphi)(p_1, \ldots, p_m)$$

define the same relation.

The three previous lemmas motivate the following definitions. Consider the formulas

$$\alpha_m := \bigwedge_{2\le i\le m} ((x_1 = x_i) \wedge (y_1 = y_i)) \vee \neg((x_1 = x_i) \wedge (y_1 = y_i));$$

$$\beta_m := \bigwedge_{1\le i<j<k\le m} (\mathsf{AffBasis}_{\mathsf{coord}}(x_i, y_i, x_j, y_j, x_k, y_k) \vee \neg\mathsf{AffBasis}_{\mathsf{coord}}(x_i, y_i, x_j, y_j, x_k, y_k)).$$

Clearly, α_m and β_m are logically valid.

Proof. (Proof of Theorem 4.1.) Given $\varphi(x_1, y_1, \ldots, x_m, y_m)$, a quantifier-free $\mathsf{FO}(+, \times, <, 0, 1)$-formula defining an affine-invariant relation, we define $\widetilde{\varphi}$ as the result of a first distribution of the conjunctions over the disjunctions in $\varphi \wedge \alpha_m \wedge \beta_m$. We remark that, since α_m and β_m are logically valid, $\widetilde{\varphi}$ is equivalent to φ. It is also quantifier-free and affine invariant. To clarify the meaning of previous distribution, we remark that any disjunct in $\widetilde{\varphi}$ contains, for any $1 \le i < j < k \le m$, $\mathsf{AffBasis}_{\mathsf{coord}}(x_i, y_i, x_j, y_j, x_k, y_k)$ or $\neg\mathsf{AffBasis}_{\mathsf{coord}}(x_i, y_i, x_j, y_j, x_k, y_k)$ as a conjunct and for any $1 < i \le m$, it also contains $((x_1 = x_i) \wedge (y_1 = y_i))$ or $((x_1 \neq x_i) \vee (y_1 \neq y_i))$ as a conjunct.

We define the translation $\mathcal{S}(\varphi)$ as the disjunction of the translation of each disjunct γ in $\widetilde{\varphi}$. Each disjunct is translated using the Lemmas 4.4, 4.5 and 4.6.

First, we consider the case where, for some $i, j, k \in \mathbb{N}$, γ contains the formula $\mathsf{AffBasis}_{\mathsf{coord}}(x_i, y_i, x_j, y_j, x_k, y_k)$ as a conjunct. Let us assume that γ is of the form $\mathsf{AffBasis}_{\mathsf{coord}}(x_i, y_i, x_j, y_j, x_k, y_k) \wedge \delta$, where (i, j, k) is the first triple, in the lexicographical order, such that $\mathsf{AffBasis}_{\mathsf{coord}}(x_i, y_i, x_j, y_j, x_k, y_k)$ is a conjunct of γ. Then, we define

$$\mathcal{S}(\gamma) := \mathsf{AffBasis}(p_i, p_j, p_k) \wedge \mathcal{S}_{p_i,p_j,p_k}(\delta)(p_1, \ldots, p_m).$$

By Lemma 4.4, $\mathcal{S}(\gamma)$ defines the same relation as γ.

Now, we assume now that γ contains no conjunct of the form $\mathsf{AffBasis}_{\mathsf{coord}}\,(x_i, y_i, x_j, y_j, x_k, y_k)$. Hence, γ contains $\mathsf{Aligned}^m_{\mathsf{coord}}(x_1, y_1, \ldots, x_m, y_m)$ as a conjunct. If it contains, for some $1 < i \le m$, $\neg((x_1 = x_i) \wedge (y_1 = y_i))$ as a conjunct, let us write $\gamma = \mathsf{Aligned}^m_{\mathsf{coord}}(x_1, y_1, \ldots, x_m, y_m) \wedge \neg((x_1 = x_i) \wedge (y_1 = y_i) \wedge \delta$ for the first i with this property, and define

$$\mathcal{S}(\gamma) := \mathsf{Aligned}(p_1, \ldots, p_m) \wedge (p_1 \neq p_i) \wedge S_{p_1,p_i}(\delta)(p_1, \ldots, p_m).$$

By Lemma 4.6, $\mathcal{S}(\gamma)$ defines the same relation as γ.

Finally, we assume that γ contains no conjunct of the form $\neg((x_1 = x_i) \wedge (y_1 = y_i))$. Then, it contains $\mathsf{Equal}^m_{\mathsf{coord}}(x_1, y_1, \ldots, x_m, y_m)$ as a conjunct. We define $\mathcal{S}(\gamma) := \neg\top$ or $\mathcal{S}(\gamma) := \mathsf{Equal}^m(p_1, \ldots, p_m)$, in order to obtain a $\mathsf{FO}(\beta, \top, \oplus, \otimes, \pi)$-formula defining the same relation, what is possible by Lemma 4.5.

We finally define

$$\mathcal{S}(\varphi) = \bigvee_{\gamma \text{ disjunct in } \widetilde{\varphi}} \mathcal{S}(\gamma)$$

Clearly, $\mathcal{S}(\varphi)$ is quantifier free and defines the same relation as φ. Whence, $\mathcal{S} : \mathsf{FO}(+, \times, <, 0, 1)_{\mathrm{QF,AI}} \to \mathsf{FO}(\beta, \top, \oplus, \otimes, \pi)_{\mathrm{QF}}$ is a translation, and the proof is completed. □

5 The translation $\mathcal{T}$ of $\mathsf{FO}(+, \times, <, 0, 1)_{\mathrm{QF,SI}}$-formulas to $\mathsf{FO}(\beta, \equiv, \top, \oplus, \otimes, \pi^\perp, \kappa)_{\mathrm{QF}}$-formulas

We define a translation from the quantifier-free similarity-invariant fragment of $\mathsf{FO}(+, \times, <, 0, 1)$ into the quantifier-free fragment of $\mathsf{FO}(\beta, \equiv, \top, \oplus, \otimes, \pi^\perp, \kappa)$.

The main result of the present section is the following theorem.

Theorem 5.1 *There exists a translation*

$$\mathcal{T} : \mathsf{FO}(+, \times, <, 0, 1)_{\mathrm{QF,SI}} \to \mathsf{FO}(\beta, \equiv, \top, \oplus, \otimes, \pi^\perp, \kappa)_{\mathrm{QF}}.$$

As in the last section, we assume that the variables used in $\mathsf{FO}(+,\times,<,0,1)$-formulas are $x_1, y_1, x_2, y_2, \dots$ and we define a map (not a translation)

$$\mathcal{T}_{o,e_1,e_2} : \mathsf{FO}(+,\times,<,0,1)_{\mathrm{QF,AI}} \to \mathsf{FO}(\beta,\top,\oplus,\otimes,\pi)_{\mathrm{QF}}.$$

The image, $\mathcal{T}_{o,e_1,e_2}(\varphi)$, of an $\mathsf{FO}(+,\times,<,0,1)$-formula φ in the variables $x_1, y_1, \dots, x_m, y_m$, involves the variables $o, e_1, e_2, p_1, p_2, \dots, p_m$.

First, we define it for $\mathsf{FO}(+,\times,<,0,1)$-terms, by induction in structure of the term, as follows:

$$\begin{aligned}
\mathcal{T}_{o,e_1,e_2}(0) &:= o,\\
\mathcal{T}_{o,e_1,e_2}(1) &:= e_1,\\
\mathcal{T}_{o,e_1,e_2}(x_i) &:= \mathsf{EuCoord}^1_{o,e_1,e_2}(p_i),\\
\mathcal{T}_{o,e_1,e_2}(y_i) &:= \mathsf{EuCoord}^2_{o,e_1,e_2}(p_i),\\
\mathcal{T}_{o,e_1,e_2}(t_1+t_2) &:= \mathcal{T}_{o,e_1,e_2}(t_1) \oplus_o \mathcal{T}_{o,e_1,e_2}(t_2), \text{ and}\\
\mathcal{T}_{o,e_1,e_2}(t_1\times t_2) &:= \mathcal{T}_{o,e_1,e_2}(t_1) \otimes_{o,e_1} \mathcal{T}_{o,e_1,e_2}(t_2), \text{ where } t_1 \text{ and } t_2 \text{ are } \mathsf{FO}(+,\times,<,0,1)\text{-terms.}
\end{aligned}$$

The next proposition is the Euclidean analogous to Proposition 4.2. Its proof is completely analogous to that of Proposition 4.2, using Lemma 3.5 instead of Lemma 3.3.

Proposition 5.2 *Let us assume that $\underline{o}, \underline{e_1}, \underline{e_2}$ form an Euclidean coordinate system. Let t be a $\mathsf{FO}(+,\times,<,0,1)$-term in the variables $x_1, y_1, \dots, x_m, y_m$ and consider points $\underline{p_1}, \dots, \underline{p_m}$ in $\mathbb{E}$, with coordinates $(\underline{x_1}, \underline{y_1}), \dots, (\underline{x_m}, \underline{y_m})$ with respect to the coordinate system $\underline{o}, \underline{e_1}, \underline{e_2}$.*

Then, $\mathcal{T}_{o,e_1,e_2}(t)[\underline{o}, \underline{e_1}, \underline{e_2}, \underline{p_1}, \dots, \underline{p_m}]$ has coordinates $(t[\underline{x_1}, \underline{y_1}, \dots, \underline{x_m}, \underline{y_m}], 0)$ in the Euclidean coordinate system $\underline{o}, \underline{e_1}, \underline{e_2}$.

The map $\mathcal{T}_{o,e_1,e_2}$ is defined on atomic formulas and extended to the whole quantifier-free fragment of $\mathsf{FO}(+,\times,<,0,1)$ in an analogous way as $\mathcal{S}_{o,e_1,e_2}$ was defined. Also, we write $\mathcal{T}_{\underline{o},\underline{e_1},\underline{e_2}}(\varphi)$ for $\mathcal{T}_{o,e_1,e_2}(\varphi)[\underline{o}, \underline{e_1}, \underline{e_2}]$.

The next proposition and its proof are the Euclidean analogous to Proposition 4.3.

Proposition 5.3 *Let us assume that $\underline{o}, \underline{e_1}, \underline{e_2} \in \mathbb{E}$ form an Euclidean coordinate system, and that φ is a quantifier-free $\mathsf{FO}(+,\times,<,0,1)$-formula in the variables $x_1, y_1, \dots, x_m, y_m$. Consider points $\underline{p_1}, \dots, \underline{p_m}$ in $\mathbb{E}$, with coordinates $(\underline{x_1}, \underline{y_1}), \dots, (\underline{x_m}, \underline{y_m})$ with respect to the coordinate system $\underline{o}, \underline{e_1}, \underline{e_2}$. Then, $\mathcal{E}' \models \mathcal{T}_{\underline{o},\underline{e_1},\underline{e_2}}(\varphi)[\underline{p_1}, \dots, \underline{p_m}]$ if and only if $\mathcal{R} \models \varphi[\underline{x_1}, \underline{y_1}, \dots, \underline{x_m}, \underline{y_m}]$.*

The map $\mathcal{T}_{o,e_1,e_2}$ is not a translation because it adds the three new free variables o, e_1 and e_2. We use the same strategy as in the case of $\mathcal{S}_{o,e_1,e_2}$ to use the variables $p_1, \dots, p_m$ already involved in the formula. That is, we considering the three different situations:

(1) when all the variables represent the same point;

(2) when all the variables represent points that are aligned and two are different; and

(3) when there are three variables representing affine-independent points.

Since the Euclidean relations among aligned points coincide with the affine relation among these points, Cases (1) and (2) are translated exactly as in the affine case. The next lemma show how to manage the third case.

Let us assume that $\varphi(x_1, y_1, \dots, x_m, y_m)$ is a quantifier-free $\mathsf{FO}(+,\times,<,0,1)$-formula defining a similarity-invariant relation. We recall that for $i, j, k \in \mathbb{N}$ such that $1 \le i < j < k \le m$, we denote by $\varphi^{\langle i,j,k\rangle}$ the formula

$$\mathsf{AffBasis}_{\mathsf{coord}}(x_i, y_i, x_j, y_j, x_k, y_k) \wedge \varphi(x_1, y_1, \dots, x_m, y_m).$$

We remark that $\varphi^{\langle i,j,k\rangle}$ defines a similarity-invariant relation.

Lemma 5.4 *The formulas $\varphi^{\langle i,j,k\rangle}$ and $\mathsf{AffBasis}(p_i, p_j, p_k) \wedge \mathcal{T}_{p_i,p_j,\varepsilon(p_i,p_j,p_k)}(\varphi)$ define the same relation.*

P r o o f. For any $\underline{p_1}, \dots, \underline{p_m} \in \mathbb{E}$ we consider $(\underline{x_1}, \underline{y_1}), \dots, (\underline{x_m}, \underline{y_m})$ to be their coordinates in some fixed Euclidean coordinate system. We prove that

$$\mathcal{R} \models \mathsf{AffBasis}_{\mathsf{coord}}[\underline{x_i}, \underline{y_i}, \underline{x_j}, \underline{y_j}, \underline{x_k}, \underline{y_k}] \wedge \varphi[\underline{x_1}, \underline{y_1}, \dots, \underline{x_m}, \underline{y_m}].$$

if and only if

$$\mathcal{E}' \models \mathsf{AffBasis}[\underline{p_i}, \underline{p_j}, \underline{p_k}] \wedge \mathcal{T}_{\underline{p_i},\underline{p_j},\varepsilon(\underline{p_i},\underline{p_j},\underline{p_k})}(\varphi)[\underline{p_1}, \ldots, \underline{p_m}]$$

On the one hand, if $\underline{p_i}, \underline{p_j}, \underline{p_k}$ are affine dependent, then both formulas are clearly false. On the other hand, if $\underline{p_i}, \underline{p_j}, \underline{p_k}$ are affine independent, Lemma 3.6 implies that $\underline{p_i}, \underline{p_j}, \varepsilon(\underline{p_i}, \underline{p_j}, \underline{p_k})$ form an Euclidean coordinate system.

Thus, since φ is similarity invariant, Proposition 5.3 implies that the sentence $\varphi[\underline{x_1}, \underline{y_1}, \ldots, \underline{x_m}, \underline{y_m}]$ holds if and only if $\mathcal{T}_{\underline{p_i},\underline{p_j},\varepsilon(\underline{p_i},\underline{p_j},\underline{p_k})}(\varphi)[\underline{p_1}, \ldots, \underline{p_m}]$ holds.

Hence, $\varphi^{\langle i,j,k\rangle}$ and $\mathsf{AffBasis}(p_i, p_j, p_k) \wedge \mathcal{T}_{p_i,p_j,p_k}(\varphi)$ define the same relation, what completes the proof. □

Proof. (Proof of Theorem 5.1.) Given $\varphi(x_1, y_1, \ldots, x_m, y_m)$, a quantifier-free $\mathsf{FO}(+, \times, <, 0, 1)$-formula defining a similarity-invariant relation, we define $\widetilde{\varphi}$ as in the proof of Theorem 4.1. We define the translation $\mathcal{T}(\varphi)$ as the disjunction of the translation of each disjunct γ in $\widetilde{\varphi}$. Each disjunct is translated using the Lemmas 5.4, 4.5 and 4.6.

Let γ be a disjunct in the disjunction $\widetilde{\varphi}$ of the form $\mathsf{AffBasis}_{\mathsf{coord}}(x_i, y_i, x_j, y_j, x_k, y_k) \wedge \delta$, where (i, j, k) is the first triple, in the lexicographical order, such that $\mathsf{AffBasis}_{\mathsf{coord}}(x_i, y_i, x_j, y_j, x_k, y_k)$ is a conjunct of γ. Then, we define

$$\mathcal{T}(\gamma) := \mathsf{AffBasis}(p_i, p_j, p_k) \wedge \mathcal{T}_{p_i,p_j,\varepsilon(p_i,p_j,p_k)}(\delta)(p_1, \ldots, p_m).$$

By Lemma 5.4, $\mathcal{T}(\gamma)$ defines the same relation as γ.

The other two cases (γ contains $\mathsf{Aligned}^m_{\mathsf{coord}} \wedge (p_1 \neq p_i)$ for some $i \in \mathbb{N}$ or γ contains $\mathsf{Equal}^m_{\mathsf{coord}}$ as conjuncts) are treated in a way completely analogous to the affine case. Since affine and Euclidean relation among aligned points coincide, the map

$$\mathcal{T}(\varphi) = \bigvee_{\gamma \text{ disjunct in } \widetilde{\varphi}} \mathcal{T}(\gamma)$$

obtained in this way $\mathcal{T} : \mathsf{FO}(+, \times, <, 0, 1)_{\mathrm{QF,SI}} \to \mathsf{FO}(\beta, \equiv, \top, \oplus, \otimes, \pi^{\perp}, \kappa)_{\mathrm{QF}}$ is a translation, and the proof is completed. □

6 Quantifier elimination for the theories $\mathcal{A}'$ and $\mathcal{E}'$

Theorem 6.1 *The theory $\mathcal{A}'$ defines exactly the affine-invariant geometric relations and admits effective quantifier elimination.*

Proof. We prove that

$$\mathcal{S} \circ \mathfrak{E}_{\mathcal{R}} \circ \mathcal{C} \circ \mathcal{B} : \mathsf{FO}(\beta, \top, \oplus, \otimes, \pi) \to \mathsf{FO}(\beta, \top, \oplus, \otimes, \pi)_{\mathrm{QF}}$$

is an effective quantifier-elimination function.

Since $\mathcal{S}$, $\mathfrak{E}_{\mathcal{R}}$, $\mathcal{C}$ and $\mathcal{B}$ are recursive functions, their composition is recursive.

Let φ be a $\mathsf{FO}(\beta, \top, \oplus, \otimes, \pi)$-formula. By Corollary 2.15 and Proposition 3.1, the $\mathsf{FO}(+, \times, <, 0, 1)$-formula $\psi := \mathcal{C}(\mathcal{B}(\varphi))$ defines the same relation as φ. In particular, it defines an affine-invariant relation. We recall from Section 2.6, that $\mathfrak{E}_{\mathcal{R}}(\psi)$ is a quantifier-free $\mathsf{FO}(+, \times, <, 0, 1)$-formula, equivalent to ψ. In particular, it defines the same relation as φ. Thus, by Theorem 4.1, $\mathcal{S}(\mathfrak{E}_{\mathcal{R}}(\psi))$ is a quantifier-free $\mathsf{FO}(\beta, \top, \oplus, \otimes, \pi)$-formula defining the same relation as φ.

Being an extension by definitions of a complete theory, $\mathcal{A}'$ is complete. Thus, two $\mathsf{FO}(\beta, \top, \oplus, \otimes, \pi)$-formulas define the same relation (under the standard interpretation) if and only if they are equivalent in $\mathcal{A}'$.

Hence, for any $\mathsf{FO}(\beta, \top, \oplus, \otimes, \pi)$-formula φ, $\mathcal{S}(\mathfrak{E}_{\mathcal{R}}(\mathcal{C}(\mathcal{B}(\varphi)))) \in \mathsf{FO}(\beta, \top, \oplus, \otimes, \pi)$ is quantifier free and equivalent to φ.

Whence, $\mathcal{A}'$ admits effective quantifier elimination. □

In an analogous way, we obtain the following result.

Theorem 6.2 *The theory $\mathcal{E}'$ defines exactly the similarity-invariant geometric relations and admits effective quantifier elimination.*

Proof. Arguing as in the previous proof, using Theorem 5.1 instead of Theorem 4.1, and Theorem 3.4 instead of 3.1 we conclude that

$$\mathcal{T} \circ \mathfrak{E}_{\mathcal{R}} \circ \mathcal{C} \circ \mathcal{M} : \mathsf{FO}\big(\beta, \equiv, \top, \oplus, \otimes, \pi^{\perp}, \kappa\big) \to \mathsf{FO}\big(\beta, \equiv, \top, \oplus, \otimes, \pi^{\perp}, \kappa\big)_{\mathrm{QF}}$$

is an effective quantifier-elimination function for $\mathcal{E}'$. □

7 Final remarks

7.1 Discussion on the primitive notions

Elementary Euclidean and affine geometry do not admit quantifier elimination in their respective original languages. We have added new symbols to the underlying signature to allow quantifier-elimination. We now discuss the minimality of the resulting signatures.

Fixed a language $\mathcal{L}$ and an interpretation, we shall say that a symbol in the underlying signature is *dispensable* if any property definable in $\mathcal{L}$ can be also defined by a quantifier-free formula not involving that symbol. We remark that, since we require the formula to be quantifier-free, this notion is more subtle than what is usually understood by *independence* of the primitive notions.

We briefly argue that $\top$ and $\otimes$ are indispensable in both extended signatures.

The case of $\top$ is immediate. Since $\mathsf{FO}(\beta, \top, \oplus, \otimes, \pi)$ and $\mathsf{FO}\big(\beta, \equiv, \top, \oplus, \otimes, \pi^{\perp}, \kappa\big)$ have no constant symbols, no quantifier free sentence can be constructed with out it. Hence, it is indispensable.

To prove that $\otimes$ is indispensable in $\mathsf{FO}(\beta, \top, \oplus, \otimes, \pi)$, consider the formula $z \neq o \wedge o \oplus_z o = r \otimes_{z,o} r$, defining that the three points are collinear and that the ratio $(z : o : r)$ is equal to $\pm\sqrt{2}$. [20, Theorem 2] implies that this cannot be defined only with β and $\oplus$; since the function π does not add expressive power on collinear points, we conclude that $\otimes$ is indispensable in $\mathsf{FO}(\beta, \top, \oplus, \otimes, \pi)$. The proof that $\otimes$ is indispensable in $\mathsf{FO}(\beta, \equiv, \top, \oplus, \otimes, \pi^{\perp}, \kappa)$ is completely analogous. The dispensability of the symbols β, π and $\oplus$ in $\mathsf{FO}(\beta, \top, \oplus, \otimes, \pi)$ remains an open problem.

Consider the two $\mathsf{FO}\big(\beta, \equiv, \top, \oplus, \otimes, \pi^{\perp}, \kappa\big)$-formulas

$$\equiv (o, p, q, r) \leftrightarrow o \oplus_p o = \kappa(o, p, q, r) \quad \text{and} \quad \beta(p, q, r) \leftrightarrow (p = \kappa(q, r, q, p) \vee q = r).$$

The truth of both formulas in $\mathcal{E}'$ is easy to verify. Hence, the symbols β and $\equiv$ can be replaced in any $\mathsf{FO}(\beta, \equiv, \top, \oplus, \otimes, \pi^{\perp}, \kappa)$-formula by the right side of these formulas[2]. Thus, β and $\equiv$ are dispensable in the language $\mathsf{FO}(\beta, \equiv, \top, \oplus, \otimes, \pi^{\perp}, \kappa)$. We conclude that the language $\mathsf{FO}(\top, \oplus, \otimes, \pi^{\perp}, \kappa)$ defines exactly the similarity-invariant properties of the Euclidean plane and admits the elimination of quantifiers. The dispensability of the symbols $\oplus, \pi^{\perp}$ and κ in $\mathsf{FO}(\top, \oplus, \otimes, \pi^{\perp}, \kappa)$ remains an open problem.

7.2 Axiom systems for the new languages $\mathsf{FO}(\beta, \top, \oplus, \otimes, \pi)$ and $\mathsf{FO}\big(\top, \oplus, \otimes, \pi^{\perp}, \kappa\big)$

Tarski's complete axiom system for elementary Euclidean geometry can be transformed to a complete axiom system for the theory $\mathcal{E}'$ in the language $\mathsf{FO}\big(\beta, \equiv, \top, \oplus, \otimes, \pi^{\perp}, \kappa\big)$ adjoining the axiom $\top$ and the implicit definitions of the new function symbols (replacing in formulas given in Section 3 the variable s by the corresponding instantiated function symbol). Finally, replacing in the resulting axiom system, each occurrence of β and $\equiv$ by the equivalent $\mathsf{FO}\big(\top, \oplus, \otimes, \pi^{\perp}, \kappa\big)$-formulas recently introduced, we obtain an axiom system in the language $\mathsf{FO}\big(\top, \oplus, \otimes, \pi^{\perp}, \kappa\big)$ for the corresponding theory. We remark that an analogous procedure can be followed to axiomatize the affine case. The resulting axioms are all universal (also called, quantifier-free) with the exception of the lower-dimensional axiom and the continuity axiom-schema. A natural question remains open: Is it possible to extend our signature with finitely many new functions to obtains a purely universal axiomatization in the line of constructive analysis?

[2] The second formula is analogous to the abbreviation (3) in [12].

7.3 Extension to higher dimensions

Our results can easily be extended to n-dimensional spaces for $n > 2$. We briefly indicate how.

In the affine case, we replace the projection function symbol π by π^n whose interpretation is defined as follows.

$$\pi^n(p, o, e_1, e_2, \ldots, e_n)$$

is the projection, parallel to the affine hull of $o, e_2, \ldots e_n$, of p over $\overline{oe_1}$, if p belongs to the affine hull of $o, e_1, e_2, \ldots e_n$ and o otherwise. A direct generalization of our proofs (using π^n to coordinate the space) shows that the language $\mathsf{FO}(\beta, \top, \oplus, \otimes, \pi^n)$, interpreted over the n-dimensional Euclidean space, defines exactly the affine-invariant relations on the n-dimensional Euclidean space and admits quantifier elimination.

On the other hand, a straightforward generalization to dimension n of our proofs shows that the language $\mathsf{FO}\big(\top, \oplus, \otimes, \pi^{\perp}, \kappa\big)$, interpreted over the n-dimensional Euclidean space $\big($where $\pi^{\perp}$ is, as before, interpreted as the orthogonal projection over a line$\big)$, defines exactly the similarity-invariant relations and admits the elimination of quantifiers.

Acknowledgements This work was partially supported by the grants UBACYT 20020100300067 (Argentina) and FWO G.0344.05 (Flanders).

References

[1] P. Balbiani, V. Goranko, R. Kellerman, and D. Vakarelov, Logical theories for fragments of elementary geometry, in: Handbook of Spatial Logics, edited by M. Aiello, I. E. Pratt-Hartmann, and J. F. A. K. van Benthem (Springer-Verlag, Secaucus, NJ, 2007), pp. 343–428.
[2] S. Basu, R. Pollack, and M. F. Roy, Algorithms in Real Algebraic Geometry, Algorithms and Computation in Mathematics Vol. 10 (Springer-Verlag, Secaucus, NJ, 2006).
[3] J. Bochnak, M. Coste, and M. F. Roy, Real Algebraic Geometry, Ergebnisse der Mathematik und ihrer Grenzgebiete (Springer-Verlag, 1998).
[4] R. Descartes, La Géométrie (Ian Maire, Leiden, 1637).
[5] H. Enderton, A mathematical introduction to logic (Harcourt/Academic Press, 2000).
[6] M. Gyssens, J. Van den Bussche, and D. Van Gucht, Complete geometric query languages, J. Comput. Syst. Sci. **58**(3), 483–511 (1999).
[7] R. Hartshorne, Geometry: Euclid and Beyond (Springer-Verlag, 2000).
[8] D. Hilbert, Grundlagen der Geometrie (Teubner-Verlag, Leipzig, 1899).
[9] G. Kuper, L. Libkin, and J. Paredaens, editors, Constraint Databases (Springer-Verlag, 2000).
[10] D. Marker, Model theory: an introduction, Graduate texts in mathematics Vol. 217 (Springer-Verlag, 2002).
[11] N. Moler and P. Suppes, Quantifier-free axioms for constructive plane geometry, Compos. Math. **20**, 143–152 (1968).
[12] V. Pambuccian, Constructive axiomatizations of plane absolute, euclidean and hyperbolic geometry, Math. Log. Q. **47**(1), 129–136 (2001).
[13] V. Pambuccian, Axiomatizing geometric constructions, J. Appl. Log. **6**(1), 24–46 (2008).
[14] B. Poizat, A course in model theory: an introduction to contemporary mathematical logic (Universitext, Springer-Verlag, 2000).
[15] W. Schwabhäuser, Über die Vollständigkeit der elementaren euklidischen Geometrie, Math. Log. Q. **2**(10–15), 137–165 (1956).
[16] W. Schwabhäuser, W. Szmielew, and A. Tarski, Metamathematische Methoden in der Geometrie, Hochschultext (Springer-Verlag, 1983).
[17] J. Shoenfield, Mathematical logic (Addison-Wesley, 1967).
[18] L. Szczebra and A. Tarski, Metamathematical discussion of some affine geometries, Fundam. Math. **104**, 155–192 (1979).
[19] W. Szmielew, From Affine to Euclidean Geometry, An Axiomatic Approach (Kluwer Academic Publishers, The Netherlands, 1983).
[20] A. Tarski, Sur les ensembles d?finissables de nombres r?els, Fundam. Math. **17**, 210–239 (1931).
[21] A. Tarski, A decision method for elementary algebra and geometry (University of California Press, Berkeley, CA, second edition, 1951).
[22] A. Tarski, What is elementary geometry?, in: The Axiomatic Method, With Special Reference to Geometry and Physics. Proceedings of an International Symposium held at the Univ. of Calif., Berkeley, Dec. 26, 1957-Jan. 4, 1958, edited by L. Henkin, P. Suppes and A. Tarski, Studies in Logic and the Foundations of Mathematics (North-Holland Publishing Co., Amsterdam, 1959), pp. 16–29.
[23] A. Tarski and S. Givant, Tarski's system of geometry, Bull. Symb. Log. **5**, 175–214 (1999).
[24] O. Veblen, A system of axioms for geometry, Trans. Am. Math. Soc. **5**, 343–384 (1904).

MATHEMATICAL LOGIC QUARTERLY

2011

Professor
Dr. Benedikt Löwe
MLQ Editorial Office
Institute for Logic, Language and Computation
Universiteit van Amsterdam
Postbus 94242
1090 GE Amsterdam
The Netherlands
E-MAIL: mlq-science@uva.nl

WILEY Publishers Since 1807

Manuscript No.

Author/Title/Issue No.

Required Fields may be filled in using Acrobat Reader

Please correct your galley proofs and return them within 14 days together with the completed reprint order form. The editors reserve the right to publish your article with editors' corrections if your proofs do not arrive in time.

After having received your corrections, your paper will be published online up to several weeks ahead of the print edition in the EarlyView service of Wiley Online Library (wileyonlinelibrary.com).
Please keep in mind that reading proofs is your responsibility. Corrections should therefore be clear. The use of standard proof correction marks is recommended. Corrections listed in an electronic file should be sorted by line numbers. Please do not incorporate your corrections into the electronic PDF file.
LaTeX and Word files are sometimes slightly modified by the production department to follow general presentation rules of the journal.

Note that the quality of the halftone figures is not as high as the final version that will appear in the issue.

Check the enclosed galley proofs very carefully, paying particular attention to the formulas (including line breakings introduced in production), figures, numerical values, tabulated data and layout of the pages.

A black box ( ]) or a question at the end of the paper (after the references) signals unclear or missing information that specifically requires **your attention.** Note that the author is liable for damages arising from incorrect statements, including misprints.

The main aim of proofreading is to correct errors that may have occurred during the production process, **and not to modify the content of the paper**. Corrections that may lead to a change in the page layout should be avoided.

Note that sending back a corrected **manuscript** file **is of no use.**

If your paper contains **color figures**, please fill in the Color Print Authorization and note the further information given on the following pages. Clearly mark desired **color print figures** in your proof corrections.

Return the corrected proofs preferably by e-mail or by courier or post.

Please do not send your corrections to the printing house but to the Editorial Office:

E-MAIL mlq-science@uva.nl

Please limit corrections to errors in the text; cost incurred for any further changes or additions will be charged to the author, unless such changes have been agreed upon by the editor.

Full color reprints, PDF files, Issues, Color Print, and Cover Posters may be ordered by filling out the accompanying form.

Order Form

MATHEMATICAL LOGIC QUARTERLY

2011

Professor
Dr. Benedikt Löwe
MLQ Editorial Office
Institute for Logic, Language and Computation
Universiteit van Amsterdam
Postbus 94242
1090 GE Amsterdam
The Netherlands
E-MAIL: mlq-science@uva.nl

WILEY Publishers Since 1807

Please complete this form and return it by FAX or e-mail.

Manuscript No.

Author/Title/Issue No.

Required Fields may be filled in using Acrobat Reader

Reprints/Issues/PDF Files/Posters

Whole issues, reprints and PDF files (300 dpi) for an unlimited number of printouts are available at the rates given on the third page. Reprints and PDF files can be ordered before *and after* publication of an article. All reprints will be delivered in full color, regardless of black/white printing in the journal.

Reprints

Please send me and bill me for

☐ full color reprints with color cover

☐ full color reprints with color cover ***and*** customized color sheet

Issues

Please send me and bill me for

☐ entire issues

PDF

Please send me and bill me for

❐ a PDF file (300 dpi) for an unlimited number of printouts **with customized cover sheet.**

The PDF file will be sent to your e-mail address.

Send PDF file to:

Please note that posting of the final published version on the open internet is not permitted. For author rights and re-use options, see the Copyright Transfer Agreement at http://www.wiley.com/go/ctavchglobal.

Cover Posters

Posters are available of all the published covers in two sizes (see attached price list). **Please send me and bill me for**

☐ A2 (42 × 60 cm/17 × 24in) posters

☐ A1 (60 × 84 cm/24 × 33in) posters

Mail reprints and/or issues and/or posters to (no P.O. Boxes):

Color print authorization

Please bill me for

☐ color print figures (total number of color figures)

❐ YES, please print Figs. No. ____ in color.

❐ NO, please print all color figures in black/white.

VAT number:

Tax-free charging can only be processed with the **VAT** number of the institute/company. To prevent delays, please provide us with the VAT number with this order.

Purchase Order No.:

Terms of payment:

☐ Please send an invoice ☐ Cheque is enclosed

Please charge my credit card

VISA ☐ MasterCard ☐ AMERICAN EXPRESS ☐

Expiry date

Card no.	
Card Verification Code	

Date, Signature ____________________

Send bill to:

Signature ____________________

Date ____________________

Subscriptions

For ordering information, claims and any enquiry concerning your journal subscription please go to www.wileycustomerhelp.com or contact your nearest office.

Americas: e-mail: cs-journals@wiley.com; Tel: +1 781 388 8598 or +1 800 835 6770 (toll free in the USA & Canada)

Europe, Middle East and Africa: e-mail: cs-journals@wiley.com; Tel: +44 (0) 1865 778315.

Asia Pacific: e-mail: cs-journals@wiley.com; Tel: +65 6511 8000.

Japan: For Japanese speaking support, e-mail: cs-japan@wiley.com; Tel: +65 6511 8010 or Tel (toll-free): 005 316 50 480.

Price List for Reprints 2011 – Mathematical Logic Quarterly

The prices listed below are valid only for orders received in the course of 2011. Minimum order is 50 copies.
Reprints can be ordered before *and after* publication of an article. All reprints are delivered with color cover and color figures. If more than 500 copies are ordered, special prices are available upon request.
Single issues are available to authors at a reduced price.
The prices include mailing and handling charges. All prices are subject to local VAT/sales tax.

Reprints with color cover Size (pages)	Price for orders of (in Euro) 50 copies	100 copies	150 copies	200 copies	300 copies	500 copies*
1–4	330,–	385,–	425,–	445,–	548,–	752,–
5–8	470,–	556,–	608,–	636,–	784,–	1077,–
9–12	610,–	717,–	786,–	824,–	1016,–	1396,–
13–16	744,–	874,–	958,–	1004,–	1237,–	1701,–
17–20	885,–	1040,–	1138,–	1196,–	1489,–	2022,–
for every additional 4 pages	140,–	164,–	175,–	188,–	231,–	315,–
for additional customized colored cover sheet	**190,—**	**340,—**	**440,—**	**650,—**	**840,—**	**990,—**

PDF file (300 dpi, unlimited number of printouts, customized cover sheet)	€ 330.00
Issues	€ 38.00 per copy for up to 10 copies.*
Cover Posters	• A2 (42 × 60 cm/17 × 24in) € 49.00 • A1 (60 × 84 cm/24 × 33in) € 69.00

*Prices for more copies available on request.

Special offer: If you order 100 or more reprints you will receive a pdf file (300 dpi, unlimited number of printouts, color figures) and an issue for free.

Color figures

If your paper contains **color figures**, please notice that, generally, these figures will appear in color in the online PDF version and all reprints of your article at no cost. This will be indicated by a note „(online color at: www.mlq-journal.org)“ in the caption. The print version of the figures in the journal hardcopy will be black/white unless the author explicitely requests a color print publication and contributes to the additional printing costs.

	Approximate color print figure charges
First figure	€ 495.00
Each additional figure	€ 395.00 Special prices for more color print figures on request

If you wish color figures in print, please answer the **color print authorization** questions on the second page of our Order Form and clearly mark the desired color print figures in your proof corrections.

Information regarding VAT:

Please note that for German sales tax purposes the charge for *color print* is considered a service and therefore is subject to German sales tax. For **institutional** customers in other countries the tax will be waived, i.e. the Recipient of Service is liable for VAT. Members of the EU will have to present a VAT identification number. Customers in other countries may also be asked to provide according tax identification information.