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On the facets of the lift-and-project relaxations of graph subdivisions

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Abstract

We study the behavior of lift-and-project procedures for solving combinatorial optimization problems as described by Lovász and Schrijver (1991), in the context of the stable set problem on graphs. Following the work of Wolsey (1976), we investigate how to generate facets of the relaxations obtained by these procedures from facets of the relaxations of the original graph, after applying fundamental graph operations. We show our findings for the odd subdivision of an edge and its generalization, the stretching of a vertex operation.

Keywords: stable set problem, lift-and-project, graph subdivision

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1 Introduction

In a seminal paper, Lovász and Schrijver [5] introduced two lift-and-project operators, N_0 and N, which—starting from a polytope $P \subset [0,1]^n$—construct a sequence of polytopes yielding in at most n steps the convex hull of the integer points in P, $P_I = \text{conv}(P \cap \{0,1\}^n)$. (A third operator, N_+, generally does not yield a polyhedron and our results may not apply.)

A particularly interesting case is when $P_I = \text{STAB}(G)$, the stable set polytope of a simple graph $G = (V, E)$, and $P = \text{FRAC}(G)$, the fractional stable set polytope defined by the edge inequalities,

$$\text{FRAC}(G) = \{x \in [0,1]^V : x_u + x_v \leq 1, uv \in E\}.$$

(Edges will be denoted by $\{u, v\}$ or simply by uv when no confusion arises.)

In what follows, when there is no need to distinguish between N and N_0, we simply denote them by $N_\sharp$. We define $N_\sharp^0(P) = P$ and $N_\sharp^k(P) = N_\sharp(N_\sharp^{k-1}(P))$ for every integer $k \geq 1$, and for simplicity we write $N_\sharp^k(G) = N_\sharp^k(\text{FRAC}(G))$.

Lovász and Schrijver pointed out that $\text{STAB}(G) = \text{FRAC}(G)$ if and only if G is bipartite, whereas after one iteration we have $N(G) = N_0(G)$, and these are defined by the trivial, edge and odd cycle inequalities.

This brings up the idea of the $N_\sharp$-*rank* or *index* of the graph G, $r_\sharp(G)$, defined here as the smallest k for which $N_\sharp^k(G) = \text{STAB}(G)$. Thus, for bipartite graphs we have $r_\sharp(G) = 0$, for t-perfect graphs (which are not bipartite) we have $r_\sharp(G) = 1$, and in general, $r_\sharp(G) \leq |V| - 2$, with equality attained if $G = K_n$, the complete graph on n vertices. Many other properties were shown in [5].

Two questions naturally arise: are there simple characterizations of $N_0^k(G)$ or $N^k(G)$ for $k \geq 2$?, and, is it always the case that $N_0^k(G) = N^k(G)$ or even $r_0(G) = r(G)$?

The second question was raised by Lipták and Tunçel [4], who were among the first to study the ranks of $N_\sharp(G)$, and was partially answered by Au and Tunçel [1] who showed examples of graphs where $N_0^2(G) \neq N^2(G)$. But in general these questions remain unanswered.

With a view to understanding these problems, here we extend the work of Lipták and Tunçel on the $N_\sharp$-ranks of $\text{FRAC}(G)$ by studying the relationship between the facets of $N_\sharp^k(G)$ and those of its induced subgraphs. Following Wolsey [7], we focus on the *odd subdivision of an edge*, and its generalization, the *stretching of a node*.

2 Odd subdivision of an edge

Wolsey [7] introduced the *odd subdivision of an edge* defining it as follows: given the simple graph $G = (V, E)$, with $|V| = n$, construct the graph G' obtained from G by deleting the edge $v_1 v_2$, adding two new nodes v_{n+1} and v_{n+2}, and adding the edges $v_1 v_{n+1}$, $v_{n+1} v_{n+2}$, and $v_{n+2} v_2$.

Denoting by G_0 the graph G with the edge $v_1 v_2$ removed, Wolsey showed:

Proposition 2.1 ([7, Prop. 2]) *If $a^{\mathrm{T}} x \leq b$, with $a \geq \mathbf{0}$, defines a facet of $\mathrm{STAB}(G)$ different from that defined by $x_1 + x_2 \leq 1$, and $b' = \max\{a^{\mathrm{T}} x - b : x \in \mathrm{STAB}(G_0)\}$ is such that $b' > 0$, then $a^{\mathrm{T}} x + b'(x_{n+1} + x_{n+2}) \leq b + b'$ defines a facet of $\mathrm{STAB}(G')$.*

It is our purpose to generalize this result to the $N_\sharp^k(G)$ context, that is, given a valid inequality π of $N_\sharp^k(G)$ of the form

$$\pi : a^{\mathrm{T}} x \leq b, \tag{1}$$

with $a \geq \mathbf{0}$, $a \neq \mathbf{0}$, $a \neq \mathbf{e}_1 + \mathbf{e}_2$, and $b > 0$, we look for a valid inequality of $N_\sharp^k(G')$ of the form

$$\bar{\pi} : \quad a^{\mathrm{T}} x + b'(x_{n+1} + x_{n+2}) \leq b + b', \tag{2}$$

with $b' > 0$.

Notice that the inequality $x_1 + x_2 \leq 1$ in $\mathrm{FRAC}(G)$ is replaced by the three inequalities

$$x_1 + x_{n+1} \leq 1, \quad x_{n+1} + x_{n+2} \leq 1, \quad x_{n+2} + x_2 \leq 1, \tag{3}$$

in $\mathrm{FRAC}(G')$, and that these define facets of $N_\sharp^k(G')$ for all k.

Lipták and Tunçel [4] studied the $N_\sharp$-ranks of G and G', using the following:

Lemma 2.2 ([4, Lemma 17]) *Given $x \in \mathbb{R}^n$, let $\bar{x} = (x, 1 - x_1, x_1) \in \mathbb{R}^{n+2}$. Then:*

(i) *If $x \notin \mathrm{STAB}(G)$ then $\bar{x} \notin \mathrm{STAB}(G')$.*

(ii) *If $x \in N_\sharp^k(G)$ then $\bar{x} \in N_\sharp^k(G')$.*

For $\bar{x} \in \mathbb{R}^{n+2}$ let us write $\bar{x} = (x, x_{n+1}, x_{n+2})$ with $x \in \mathbb{R}^n$, and set

$$H = \{\bar{x} \in \mathbb{R}^{n+2} : x_{n+1} + x_{n+2} = 1\}.$$

The next result establishes a partial converse and a more precise version of Lemma 2.2:

Lemma 2.3 *Given $x \in \mathbb{R}^n$, let $\bar{x}^1 = (x, 1 - x_1, x_1)$ and $\bar{x}^2 = (x, x_2, 1 - x_2)$.*

(i) *If x is an extreme point of $N_\sharp^k(G)$, then $\bar{x}^1$ and $\bar{x}^2$ are extreme points of $N_\sharp^k(G')$.*

(ii) *If $\bar{x} = (x, x_{n+1}, x_{n+2}) \in N_\sharp^k(G') \cap H$, then $x \in N_\sharp^k(G)$ and $\bar{x}$ is a convex combination of $\bar{x}^1$ and $\bar{x}^2$.*

In particular, $x_1 + x_2 \leq 1$, and if $\bar{x}$ is an extreme point of $N_\sharp^k(G')$ then $\bar{x} = \bar{x}^1$ or $\bar{x} = \bar{x}^2$.

It is easy to see that if π in (1) is valid for $N_\sharp^k(G)$, then $\bar{\pi}$ in (2) is valid for $N_\sharp^k(G') \cap H$ for every b'. Thus, we need to fix b' using points outside H. To do so, let W be the set of extreme points of $N_\sharp^k(G')$ not in H, and for $\bar{x} = (x, x_{n+1}, x_{n+2}) \in W$ let us consider

$$\beta(\bar{x}) = \min\{\gamma \geq 0 : \gamma\,(1 - x_{n+1} - x_{n+2}) \geq a^{\mathrm{T}}x - b\}.$$

For every $\bar{x} \in W$ we will have

$$b' \geq \beta(\bar{x}) \quad \Rightarrow \quad a^{\mathrm{T}}x + b'\,(x_{n+1} + x_{n+2}) \leq b + b',$$

and it is natural to define

$$b' = \max\{\beta(\bar{x}) : \bar{x} \in W\}. \tag{4}$$

As a side remark, notice that the definition of b' in (4) may be viewed as a generalization of the *strength of an edge* in [3].

We have:

Theorem 2.4 *If b' is given by (4), then $\bar{\pi}$ defined in (2) is a valid inequality for $N_\sharp^k(G')$.*

Theorem 2.5 *If π in (1) defines a facet of $N_\sharp^k(G)$ different from that defined by $x_1 + x_2 \leq 1$, and b' given in (4) is positive (in particular if π is not a valid inequality for $N_\sharp^k(G_0)$), then $\bar{\pi}$ given in (2) defines a facet of $N_\sharp^k(G')$.*

By defining the $N_\sharp$-rank of a valid inequality of $\mathrm{STAB}(G)$ as the minimum k for which it is valid for $N_\sharp^k(G)$, there holds:

Corollary 2.6 *If π and $\bar{\pi}$ are defined as in Theorem 2.5, then both of them have the same $N_\sharp$-rank.*

We conclude this section with a few comments.

Proposition 2.1 is complemented by a nice structural result by Mahjoub [6]:

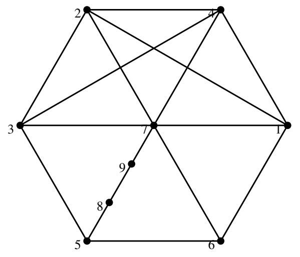

Fig. 1.

Lemma 2.7 ([6, Lemma 1]) *Let $\bar{a}^{\mathrm{T}}\bar{x} \leq b$ define a facet of* $\mathrm{STAB}(G')$ *different from those in* (3). *If both a_{n+1} and a_{n+2} are positive, then $a_{n+1} = a_{n+2}$.*

However, the converse of Proposition 2.1 is not true: some of the facets in $\mathrm{STAB}(G')$ are not obtained by this method (see [2,3]).

Moreover, Lemma 2.7 is no longer true when we consider $N_\sharp^k(G')$ instead of $\mathrm{STAB}(G')$. A counterexample is the graph G' shown in Figure 1, where the edge $\{5, 7\}$ has been subdivided and the inequality

$$x_1 + x_2 + 2x_3 + x_4 + 3x_5 + x_6 + 3x_7 + 3x_8 + 2x_9 \leq 6,$$

defines a facet of $N_0^2(G')$ (here $r_0(G) = r_0(G') = 3$).

Mahjoub presents a simpler result than that of Lemma 2.7:

Lemma 2.8 ([6, Lemma 3]) *If $\bar{a}^{\mathrm{T}}\bar{x} \leq b$ defines a facet of* $\mathrm{STAB}(G')$ *different from those in* (3), *then we cannot have $a_{n+1} > 0$ and $a_{n+2} = 0$ (and vice versa).*

By using a result by Lipták and Tunçel [4, Theorem 6], we can show:

Lemma 2.9 *Lemma 2.8 remains valid if $N_\sharp^k(G')$ is substituted for* $\mathrm{STAB}(G')$.

3 Stretching of a node

Wolsey [7] presented a generalization of the odd subdivision of an edge, called *stretching of a node*: given the graph $G = (V, E)$ and a selected node v_n, we obtain G' by separating the adjacent nodes of v_n into two non-empty subsets V_1 and V_2, introducing two new nodes v_{n+1} and v_{n+2} so that each vertex of V_ℓ is joined to $v_{n+\ell}$, $\ell = 1, 2$, and finally joining v_n to v_{n+1} and v_{n+2} only.

Since the results are analogous to those of the previous section, we briefly mention the main points.

Proposition 3.1 ([7, Prop. 3]) *If $a^{\mathrm{T}}x \leq b$ ($a \geq \mathbf{0}$) defines a facet of* $\mathrm{STAB}(G)$

such that $\max\left\{a^{\mathrm{T}}x : x \in A, x_j = 0, j \in V_\ell \cup \{n\}\right\} = b$*, for* $\ell = 1, 2$*, then* $\bar{\pi}$ *given by*

$$\bar{\pi} : \quad a^{\mathrm{T}}x + a_n\,(x_{n+1} + x_{n+2}) \leq b + a_n, \tag{5}$$

defines a facet of $\mathrm{STAB}(G')$.

Lemma 3.2 ([4, Lemma 26]) *Given* $x = (\hat{x}, x_n) \in \mathbb{R}^n$*, let* $\bar{x} = (\hat{x}, 1 - x_n, x_n, x_n) \in \mathbb{R}^{n+2}$*. Then,*

(i) *If* $x \notin \mathrm{STAB}(G)$ *then* $\bar{x} \notin \mathrm{STAB}(G')$*.*

(ii) *If* $x \in N_\sharp^k(G)$ *then* $\bar{x} \in N_\sharp^k(G')$*.*

We have:

Lemma 3.3 *If* $H_\ell = \{\bar{x} \in \mathbb{R}^{n+2} : x_n + x_{n+\ell} = 1\}$*,* $\ell = 1, 2$*, and* $\bar{x} = (\hat{x}, x_n, x_{n+1}, x_{n+2}) \in N_\sharp^k(G') \cap H_1 \cap H_2$*, then* $(\hat{x}, x_{n+1}) \in N_\sharp^k(G)$*.*

Theorem 3.4 *If* $a^{\mathrm{T}}x \leq b$ *defines a facet of* $N_\sharp^k(G)$ *and* $\bar{\pi}$ *given in (5) is such that*

(i) $\max\{\hat{a}^{\mathrm{T}}\hat{x} + a_n x_{n+2} : H_1 \setminus H_2\} = \max\{\hat{a}^{\mathrm{T}}\hat{x} + a_n x_{n+1} : \bar{x} \in H_2 \setminus H_1\} = b.$

(ii) $\bar{\pi}$ *is valid for* $N_\sharp^k(G')$*.*

Then $\bar{\pi}$ *defines a facet of* $N_\sharp^k(G')$*.*

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