This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2017.2776598, IEEE Transactions on Automatic Control

Control Strategies for Non-zero Set-point Regulation of Linear Impulsive Systems

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***Abstract*—Despite its potential use in several meaningful problems linear impulsive systems have not been extensively studied to account for dynamics in which the equilibria are out of the origin. In this work a novel characterization of the system equilibria and invariant regions - derived from the definition of two underlying discrete-time systems - is given, and based on this characterization, an unconstrained feedback control and a zone Model Predictive Control strategies are proposed. The controllers are tested in two drug administration problems: an intravenous bolus administration of Lithium ions and a nonlinear HIV infection dynamics under Zidovudine treatment.**

***Index Terms*—Impulsive Systems, non-zero regulation, feedback control, MPC, drug administration.**

I. Introduction

IMPULSIVE control systems (ICS) have received a great attention in the last decade, specially in the field of biomedical research. One central problem has been the scheduling of drug administration in the treatment of several human diseases, as it was stated in the Bellman's seminal work [1]. The dynamics of the human immunodeficiency virus (HIV), where the intake of drugs twice or three times a day can be directly interpreted as an impulsive input, is the most studied case in the literature (*i.e.*, [3], [15], [20]). Other meaningful biomedical cases are malaria [4], influenza with co-infections [2], tumor-bearing [5] and Type I diabetes [12], [19].

Despite its potential use in these important problems, the regulation to non-zero set-points - which is the case in most applications - has received little attention in the context of ICS. For instance, in [6], the regulation problem is approached by using Lyapunov function methods and sufficient conditions are given for both, the solvability of the tracking problem and the output-tracking offset-free property. However, it is assumed that the output reference (an equilibrium) is at the origin; otherwise, the methodology does not work appropriately.

Recently, a version of model predictive control (MPC) for ICS has been developed in [21], with an application to the dosing of intravenous bolus of Lithium ions upon oral intake described in [8]. The strategy also covers the problem to steer a linear ICS (LICS) to a zone defined by a 'therapeutic window' not including the origin. Furthermore, it accounts for feasibility at both, impulsive and continuous time. However, the formulation is based on polytopic invariant target sets, whose calculation is not a trivial task and could be difficult to characterize in many applications. In the field of ICS invariant set characterization, [14] also provides different conditions for the invariance of nonlinear ICS.

The zone MPC ([9], [10]) is an MPC formulation that is less general than the one having invariants sets as target sets ([21], [11]), but more general than the typical one having equilibrium points as set-points ([17]). In this framework, the state is steered to an equilibrium set - instead to an equilibrium point - making no differences between points inside the set. Furthermore, this control strategy is formulated in a tracking scenario where the equilibrium set can be far from the origin. By means of the use of artificial/intermediary variables that are only forced to lie in the equilibrium space, these kind of controllers ensure feasibility for any change of the target set. It also provides an enlarged domain of attraction, given by the controllable set to the entire equilibrium space, instead of the controllable set to a given point or invariant terminal set (as it is described in [17]).

The contributions of this paper - which is an extension of [18] that includes stability results - are in three-fold: first, a novel dynamic characterization of LICS is developed. Non-zero equilibria are analyzed and described by means of two underlying linear discrete-time systems which naturally arise when the time instants before and after the impulsive time are considered. Second, sufficient conditions to stabilize the system at the defined equilibrium regions are presented. Finally, an efficient unconstrained feedback control and a zone MPC control - which guarantees feasibility and convergence (attractivity) to the state window target - are proposed. The performance of both strategies is illustrated by two biomedical applications, lying in the central problem of scheduling of medicaments.

II. Notation

$\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the sets of non-negative integers, reals, column vectors of length n and n by m matrices, respectively. Given a matrix M, $\rho(M)$ denotes its spectral radius (see [16]), and $\mathcal{R}(M)$ denotes its range (column space). Given a function $f:\mathbb{R}\to\mathbb{R}^n$, $f(a^+)\triangleq\lim_{t\to a^+}f(t)$, *i.e.*, $f(a^+)$ is the limit of $f(t)$ when t approaches a from the right. The convex hull of a collection of sets $\mathcal{V}_i$, $i=1,2,\cdots,k$ (*i.e.*, the smallest convex set containing all the sets) is denoted as $\mathrm{ch}\{\mathcal{V}_1,\mathcal{V}_2,\cdots,\mathcal{V}_k\}$. Given a nonempty closed set $\mathcal{V}$, the distance from a point x to $\mathcal{V}$ is denoted by $\mathrm{dist}_{\mathcal{V}}(x)\triangleq\min_{y\in\mathcal{V}}\|y-x\|$, where $\|\cdot\|$ is the Euclidean norm. The angle between subspaces $\mathcal{X}_1$ and $\mathcal{X}_2$, $\theta_{\mathcal{X}_1,\mathcal{X}_2}$, is defined as the largest principal angle. If the dimension of $\mathcal{X}_1$ and $\mathcal{X}_2$ is 1, then $\theta_{\mathcal{X}_1,\mathcal{X}_2}\triangleq\min\{\mathrm{acos}\left(\left|x_1^\top x_2\right|/\|x_1\|\,\|x_2\|\right):x_1\in\mathcal{X}_1,x_2\in\mathcal{X}_2\}$.

III. Preliminaries

The class of dynamic systems of interest in this paper consists in a set of linear impulsive first-order differential equations of the form

$$\dot{x}(t) = A_c x(t),\quad x(0)=x_0,\quad t\neq\tau_k, \tag{1a}$$
$$x(\tau_k T+) = A_d x(\tau_k)+Bu(\tau_k),\quad k\in\mathbb{N}, \tag{1b}$$

where $t\in\mathbb{R}$ denotes the time, τ_k, $k\in\mathbb{N}$, denotes the impulse time instants, $x\in\mathcal{X}\subseteq\mathbb{R}^n$ denotes the (constrained) state vector and $u\in\mathcal{U}\subseteq\mathbb{R}^m$ denotes the (constrained) impulsive control inputs. Both constraint sets, $\mathcal{X}$ and $\mathcal{U}$, are convex compact sets, containing the origin in their interior. Matrices $A_c\in\mathbb{R}^{n\times n}$ and $A_d\in\mathbb{R}^{n\times n}$ are the continuous and discrete transition matrices, and $B\in\mathbb{R}^{n\times m}$

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is the impulsive input matrix. Furthermore, as part of the system description, a target state set $\mathcal{X}^{\mathrm{Tar}} \subset \mathcal{X}$ is defined, which is the region where the system is desired to be driven to and kept. It is assumed that (A_c, A_d, B) is controllable ([22]).

Let $t_0 = 0$ be the initial time and $\mathcal{T} = \{0, \tau_1, \cdots, \tau_k, \cdots\}$ a set of time instants, with $T \triangleq \tau_{k+1} - \tau_k$, $k \in \mathbb{N}$, being the fixed time period. Let $\mathbf{u}$ be an input sequence of length N, $\mathbf{u} = \{u(\tau_1), u(\tau_2), \cdots, u(\tau_N)\}$, drawn from $\mathcal{U}$, and let τ_k be the largest impulse time such that $\tau_k \leq t$, with $k \leq N$, for some $t > 0$. Then, for $\tau_k < t \leq \tau_{k+1}$ the solution to (1) is $\varphi(t; x_0, \mathbf{u}) = e^{A_c(t-\tau_k)}\varphi(\tau_k^+; x_0, \mathbf{u})$, with $\varphi(0; x_0, \mathbf{u}) = x_0$, which can also be expressed as $x(t) = \varphi(t; x_0, \mathbf{u}) \triangleq e^{A_c(t-\tau_k)}\left(M^k x_0 + \sum_{j=1}^{k} M^{k-j} B u(\tau_j)\right)$, with $M = A_d e^{A_c T}$.

Let $\kappa(\cdot)$ be a control law, in such a way that $u(\tau_k) = \kappa(x(\tau_k))$, for $k \in \mathbb{N}$. Then, the closed-loop impulsive system is described by:

$$\dot{x}(t) = A_c x(t), \quad x(0) = x_0, \quad t \neq \tau_k, \tag{2a}$$
$$x(\tau_k^+) = A_d x(\tau_k) + B\kappa(x(\tau_k)), \qquad k \in \mathbb{N}. \tag{2b}$$

This way, the closed-loop trajectory is denoted by $x(t) = \phi_{cl}(t; x_0, \kappa(\cdot))$, for $t \geq 0$, with $\phi_{cl}(0; x_0, \kappa(\cdot)) = x_0$.

IV. Dynamic characterization of impulsive systems

It is known that advanced control strategies make an explicit use of the characterization of the system under control (equilibria, controllable sets, etc.). However, the dynamic characterization of non-zero equilibria has not received enough attention in the ICS literature, with the remarkable exception of [21]. In that work general invariant sets are defined, but they are in general difficult to compute (they are the fixed points of an iterative procedure). Here, a simpler form to obtain these kind of invariant sets is presented by defining two underlying discrete-time subsystems, which basically describe the ICS at times τ_k, and $t \to \tau_k^+$, for $k \in \mathbb{N}$.

A. Underlying discrete-time subsystems

The idea now is to define two underlying discrete-time subsystems (denoted as primary and secondary UDS, respectively) as follows:

$$x^\bullet(j+1) = A^\bullet x^\bullet(j) + B^\bullet u^\bullet(j), \; x^\bullet(0) = x(\tau_0), \tag{3a}$$
$$x^\circ(j+1) = A^\circ x^\circ(j) + B^\circ u^\circ(j), \; x^\circ(0) = x(\tau_0^+), \tag{3b}$$

where $A^\bullet \triangleq e^{A_c T} A_d$, $A^\circ \triangleq A_d e^{A_c T}$, $B^\bullet \triangleq e^{A_c T} B$ and $B^\circ \triangleq B$, and $u^\circ(j+1) = u^\bullet(j)$, for $j \in \mathbb{N}$.

The primary UDS (3a) is enough to characterize the ICS (1), at sampling times τ_k, because it takes into account both, the jumps and the free continuous-time evolution between the sampling times. In fact, the continuous-time response of the ICS (1), for a period $t \in (\tau_k, \; \tau_{k+1}]$, and a given state $x(\tau_k)$ and input $u(\tau_k)$, can be described by $x(t) = \phi(t; x(\tau_k), u(\tau_k)) = A^\bullet(t) x(\tau_k) + B^\bullet(t) u(\tau_k)$, for $t \in (0, T]$, where $A^\bullet(t) \triangleq e^{A_c t} A_d$ and $B^\bullet(t) \triangleq e^{A_c t} B$. The secondary UDS (3b) is useful to characterize the equilibrium region, as it can be seen in section IV-C.

B. Impulsive System Equilibrium Set Characterization

If matrices A_c and B are assumed to be full rank, the only formal equilibrium point of the ICS (1) is given by $(u_s, x_s) = (0, 0)$, which is the only pair verifying $\dot{x} = 0$ and $x(\tau_k^+) = x(\tau_k)$. However, by abstracting the general concept of equilibrium (invariance) and taking into account the ICS only at times τ_k, it is possible to find some generalization that accounts for equilibrium entities out of the origin [21].

Definition 1: (Generalized control equilibrium set of ICS) Consider a ICS (1), a period T and a non-empty convex set Ω. A set $\mathcal{X}_s \in \mathcal{X}$ is a generalized control equilibrium set with respect to Ω if for each $x_s \in \mathcal{X}_s$ there exists an input $u_s = u_s(x_s) \in \mathcal{U}$ such that

$$\phi(T; x_s, u_s(x_s)) \in \mathcal{X}_s, \tag{4}$$
$$o_s(x_s, u_s(x_s)) \subseteq \Omega, \tag{5}$$

where $o_s(x_s, u_s) \triangleq \{\phi(t; x_s, u_s), \;\; t \in (0, T]\}$. The set $\{u_s(x_s) \in \mathcal{U} : x_s \in \mathcal{X}_s\}$ is denoted as $\mathcal{U}_s(\mathcal{X}_s)$.

The state trajectory in a period T, $\phi(t; x_s, u_s)$, for $t \in (0, T]$, uniquely defines an orbit o_s. More precisely, the orbit is given by the free response after the jump, $\phi(t; x_s, u_s) = e^{A_c t}(A_d x_s + B u_s), t \in (0, T]$. The orbits $o_s(x_s, u_s)$, with $x_s \in \mathcal{X}_s$, are called equilibrium orbits. Fig. 1 shows a schematic plot of some equilibrium orbits.

Given that o_s is non-convex, a better characterization of the generalized equilibrium can be done by defining its convex hull.

Definition 2: (Equilibrium orbits set of ICS) Consider a ICS (1) and a generalized control equilibrium set $\mathcal{X}_s$, together with the equilibrium input set $\mathcal{U}_s(\mathcal{X}_s)$. Then, the equilibrium orbits set of ICS, $\mathcal{X}_{\mathcal{O}_s}(\mathcal{X}_s) = \mathcal{X}_{\mathcal{O}_s}(\mathcal{X}_s, \mathcal{U}_s(\mathcal{X}_s))$, is given by

$$\begin{aligned} \mathcal{X}_{\mathcal{O}_s}(\mathcal{X}_s) &\triangleq \mathrm{ch}\,\{\phi(t; x_s, u_s(x_s)), t \in [0, T], \forall x_s \in \mathcal{X}_s\} \\ &= \mathrm{ch}\,\{o_s(x_s, u_s(x_s)), \forall x_s \in \mathcal{X}_s\}\,. \end{aligned} \tag{6}$$

Now, according to Definitions 1 and 2, it is clear that each equilibrium set of the primary underlying subsystem (3a), $\mathcal{X}_s^\bullet$, is a (particular) generalized control equilibrium set.

Property 1: Let $\mathcal{X}_s^\bullet \subseteq \mathcal{X}$ be a set of states $x_s^\bullet \in \mathcal{X}$ for which there exists an input $u_s = u_s(x_s^\bullet) \in \mathcal{U}$ such that $x_s^\bullet = A^\bullet x_s^\bullet + B^\bullet u_s(x_s^\bullet)$. Then $\mathcal{X}_s^\bullet$ is a generalized control equilibrium set of ICS with respect to the associated equilibrium orbit set, $\mathcal{X}_{\mathcal{O}_s}(\mathcal{X}_s^\bullet) = \mathcal{X}_{\mathcal{O}_s}(\mathcal{X}_s^\bullet, \mathcal{U}_s(\mathcal{X}_s^\bullet))$, where $\mathcal{U}_s(\mathcal{X}_s^\bullet) = \{u_s(x_s^\bullet) \in \mathcal{U} : x_s^\bullet \in \mathcal{X}_s^\bullet\}$.

Proof: It is easy to see that for each $x_s^\bullet \in \mathcal{X}_s^\bullet$, $\phi(T; x_s^\bullet, u_s(x_s^\bullet)) = A^\bullet(T) x_s^\bullet + B^\bullet(T) u_s(x_s^\bullet) = x_s^\bullet \in \mathcal{X}_s^\bullet$ and $o_s(x_s^\bullet, u_s(x_s^\bullet)) \subseteq \mathcal{X}_{\mathcal{O}_s}(\{x_s^\bullet\}, \{u_s(x_s^\bullet)\}) \subseteq \mathcal{X}_{\mathcal{O}_s}(\mathcal{X}_s^\bullet)$. ■

Condition $x_s^\bullet = A^\bullet x_s^\bullet + B^\bullet u_s(x_s^\bullet)$ implies that there is a state $x_s^\circ = x_s^\circ(x_s^\bullet) \triangleq A_d x_s^\bullet + B u_s(x_s^\bullet)$, such that $x_s^\circ = A^\circ x_s^\circ + B^\circ u_s(x_s^\bullet)$ (i.e., x_s° is an equilibrium of (3b) associated to the same $u_s(x_s^\bullet)$).

Let us now introduce the counterpart of the generalized control equilibrium set, for the closed-loop ICS (2):

Definition 3: (Generalized equilibrium set of ICS) A set $\mathcal{X}_s$ is a generalized equilibrium set, with respect to $\bar{\Omega}$, for the closed-loop system (2), if $\mathcal{X}_s$ is a generalized control equilibrium set, with respect to $\bar{\Omega}$, for the open-loop system (1), with $u = \kappa(x)$.

C. Jump Set

The case $A_d = I_n$ in the ICS (1) is of particular interest in many practical applications as, for instance, drug administration problems. In this case, (1) can be viewed as a continuous-time system, $\dot{x}(t) = A_c x(t) + B u(t)$, controlled by impulsive inputs. Let now $\mathcal{X}_s^\bullet$ be the maximal equilibrium set of the UDS (3a) contained in $\mathcal{X}$. Given that system (1) is assumed to be controllable both UDS, (3a) and (3b) are also controllable. Then $\mathcal{X}_s^\bullet$ and $\mathcal{X}_s^\circ(\mathcal{X}_s^\bullet) \triangleq \{x_s^\circ \in \mathcal{X} : x_s^\circ = x_s^\bullet + B u_s(x_s^\bullet)\}$ (the equilibrium set of UDS (3b), corresponding to $\mathcal{X}_s^\bullet$) are compact sets contained in subspaces of dimension m of $\mathbb{R}^n$, and it is possible to define:

Definition 4: (Jump set of ICS) The jump set of ICS (1) is given by the convex hull:

$$\mathcal{X}_s(\mathcal{X}_s^\bullet) \triangleq \mathrm{ch}\,\{\mathcal{X}_s^\bullet, \mathcal{X}_s^\circ(\mathcal{X}_s^\bullet)\} \subseteq \mathbb{R}^n. \tag{7}$$

Next, the main property of $\mathcal{X}_s(\mathcal{X}_s^\bullet)$ is presented.

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Property 2: $\mathcal{X}_s(\mathcal{X}_s^\bullet)$ (and any subset of $\mathcal{X}_s(\mathcal{X}_s^\bullet)$ containing $\mathcal{X}_s^\bullet$) is a generalized control equilibrium set with respect to $\mathcal{X}_{\mathcal{O}_s}(\mathcal{X}_s(\mathcal{X}_s^\bullet))$, for system (1).

Proof: It will be shown that any state $x_s \in \mathcal{X}_s(\mathcal{X}_s^\bullet)$ can be feasibly steered to a state $x_s^\circ(x_s^\bullet) \in \mathcal{X}_s^\circ(\mathcal{X}_s^\bullet)$ in a jump, and then, after the free response, it reaches a state $x_s^\bullet \in \mathcal{X}_s^\bullet \subseteq \mathcal{X}_s(\mathcal{X}_s^\bullet)$. By Definition 4, every state $x_s \in \mathcal{X}_s(\mathcal{X}_s^\bullet)$ - not lying in $\mathcal{R}(B)$) - can be expressed as $x_s = \alpha x_s^\bullet + (1-\alpha)x_s^\circ(x_s^\bullet)$, for some $x_s^\bullet \in \mathcal{X}_s^\bullet$ and some $\alpha \in \mathbb{R}$ (not necessarily positive). Furthermore, since $\mathcal{X}_s(\mathcal{X}_s^\bullet)$ is bounded (given that $\mathcal{X}_s^\bullet$ and $\mathcal{X}_s^\circ$ are bounded), $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, for bounded minimal and maximal values, $\underline{\alpha}$ and $\overline{\alpha}$. So, if input $\hat{u} \triangleq \alpha u_s(x_s^\bullet)$ is injected to the system, then $\phi(T; x_s, \hat{u}) = e^{A_cT}(x_s + B\hat{u}) = e^{A_cT}(\alpha x_s^\bullet + (1-\alpha)x_s^\circ(x_s^\bullet) + \alpha B u_s(x_s^\bullet)) = e^{A_cT}((1-\alpha)x_s^\circ(x_s^\bullet) + \alpha x_s^\circ(x_s^\bullet)) = e^{A_cT}x_s^\circ(x_s^\bullet) = x_s^\bullet \in \mathcal{X}_s(\mathcal{X}_s^\bullet)$, where the equalities follow from the facts that $x_s^\circ(x_s^\bullet) = x_s^\bullet + Bu_s(x_s^\bullet)$ and $e^{A_cT}x_s^\circ(x_s^\bullet) = A^\bullet x_s^\circ(x_s^\bullet) = x_s^\bullet$. $\hat{u} = \alpha u_s(x_s^\bullet)$ is feasible, for $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, because of the convexity of $\mathcal{U}$ and $\mathcal{X}_s(\mathcal{X}_s^\bullet)$. Finally, it is trivial that the trajectories starting in $x_s \in \mathcal{X}_s(\mathcal{X}_s^\bullet)$ will remain in $\mathcal{X}_{\mathcal{O}_s}(\mathcal{X}_s(\mathcal{X}_s^\bullet))$. ■

Remark 1: (Computation of $\mathcal{X}_{\mathcal{O}_s}(\mathcal{X}_s(\mathcal{X}_s^\bullet))$) In practical scenarios set $\mathcal{X}_{\mathcal{O}_s}(\mathcal{X}_s(\mathcal{X}_s^\bullet))$ can be approximated by sampling in t the map $S(t) \triangleq e^{A_ct}\mathcal{X}_s^\circ(\mathcal{X}_s^\bullet)$. Also, it can be exactly over-approximated by a polytopic set, following methods as the one proposed in [7], which avoids the problem of selecting a sampling time.

D. Effect of the time period T

In order to better characterize the whole behavior of the equilibria of system (1), some properties that describe how the sets $\mathcal{X}_s^\bullet$, $\mathcal{X}_s^\circ(\mathcal{X}_s^\bullet)$, $\mathcal{X}_s(\mathcal{X}_s^\bullet)$ and $\mathcal{X}_{\mathcal{O}_s}(\mathcal{X}_s(\mathcal{X}_s^\bullet))$ behave as functions of T are presented.

Property 3: Let $\mathcal{E}_s^\bullet$ and $\mathcal{E}_s^\circ$ be the smallest affine sets containing $\mathcal{X}_s^\bullet$ and $\mathcal{X}_s^\circ(\mathcal{X}_s^\bullet)$, respectively (i.e., given that $\mathcal{X}_s^\bullet$ and $\mathcal{X}_s^\circ(\mathcal{X}_s^\bullet)$ contain the origin, the subspaces where these sets lie in). Let $\mathcal{X}_s^c$ be the maximum equilibrium set of the continuous-time system $\dot{x}(t) = A_cx(t) + Bu(t)$ (which does not depend on T). Then, (i) the angle between $\mathcal{E}_s^\bullet$ and $\mathcal{E}_s^\circ$ is an increasing function of T; i.e., $\pi/2 \geq \theta_{\mathcal{E}_s^\bullet,\mathcal{E}_s^\circ}(T_2) > \theta_{\mathcal{E}_s^\bullet,\mathcal{E}_s^\circ}(T_1) \geq 0$, for $T_2 > T_1$ and furthermore, $\lim_{T\to 0}\theta_{\mathcal{E}_s^\bullet,\mathcal{E}_s^\circ}(T) = 0$; (ii) $\lim_{T\to 0}\mathcal{X}_s^\bullet = \lim_{T\to 0}\mathcal{X}_s^\circ(\mathcal{X}_s^\bullet) = \mathcal{X}_s^c$, which means that $\mathcal{X}_s(\mathcal{X}_s^\bullet)$, also tends to $\mathcal{X}_s^c$ as $T \to 0$; (iii) $\lim_{T\to 0}\mathcal{X}_{\mathcal{O}_s}(\mathcal{X}_s(\mathcal{X}_s^\bullet)) = \mathcal{X}_s^c$.

Proof: (i) For every $x_s^\bullet \in \mathcal{E}_s^\bullet$, $x_s^\circ \in \mathcal{E}_s^\circ$, $x_s^\bullet = e^{A_cT}x_s^\circ(x_s^\bullet)$ (free response of the unconstrained system). Then, considering $m = 1$ (and denoting $x_s^\circ(x_s^\bullet) = x_s^\circ$) for simplicity, $\theta_{\mathcal{E}_s^\bullet,\mathcal{E}_s^\circ}(T_2) = \text{acos}\left(\left|x_s^{\bullet\prime}x_s^\circ\right| / \|x_s^\bullet\|\,\|x_s^\circ\|\right) = \text{acos}\left(\left|x_s^{\circ\prime}e^{(A_cT_2)'}x_s^\circ\right| / \|e^{A_cT_2}x_s^\circ\|\,\|x_s^\circ\|\right) < \text{acos}\left(\left|x_s^{\circ\prime}e^{(A_cT_1)'}x_s^\circ\right| / \|e^{A_cT_1}x_s^\circ\|\,\|x_s^\circ\|\right) = \theta_{\mathcal{E}_s^\bullet,\mathcal{E}_s^\circ}(T_1)$, for all $T_2 > T_1$. Furthermore, $\lim_{T\to 0}\left(\left|x_s^{\circ\prime}e^{(A_cT)'}x_s^\circ\right| / \|e^{A_cT}x_s^\circ\|\,\|x_s^\circ\|\right) = 1$, which implies that $\lim_{T\to 0}\theta_{\mathcal{E}_s^\bullet,\mathcal{E}_s^\circ}(T) = 0$.

(ii) The fact that $\lim_{T\to 0}\mathcal{X}_s^\bullet = \lim_{T\to 0}\mathcal{X}_s^\circ(\mathcal{X}_s^\bullet)$ follows from the fact that for every $x_s^\bullet \in \mathcal{X}_s^\bullet$, $x_s^\circ \in \mathcal{X}_s^\circ(\mathcal{X}_s^\bullet)$, $x_s^\bullet = e^{A_cT}x_s^\circ(x_s^\bullet)$. Now, assuming without loss of generality that A_c is invertible, the maximal equilibrium set $\mathcal{X}_s^\bullet$ is spanned by $G^\bullet \triangleq (I_n - A^\bullet)^{-1}B^\bullet = (I_n - e^{A_cT})^{-1}e^{A_cT}B$. Then, as $(I_n - e^{A_cT})^{-1}e^{A_cT}B$ approaches $(A_cT)^{-1}B$, as $T \to 0$, being $A_c^{-1}B \triangleq G$ the matrix spanning the continuous-time equilibrium set $\mathcal{X}_s^c$, and T a scalar, then $\lim_{T\to 0}\mathcal{X}_s^\bullet = \mathcal{X}_s^c$. The same happens with $\mathcal{X}_s^\circ(\mathcal{X}_s^\bullet)$, which implies that $\lim_{T\to 0}\mathcal{X}_s(\mathcal{X}_s^\bullet) = \mathcal{X}_s^c$.

(iii) $\lim_{T\to 0}\mathcal{X}_{\mathcal{O}_s}(\mathcal{X}_s(\mathcal{X}_s^\bullet)) = \mathcal{X}_s^c$ follows from the fact that, for $T \to 0$, the impulsive system (1) tends to the continuous-time system, $\dot{x}(t) = A_cx(t) + Bu(t)$ (from the sampling theory), and furthermore, by (ii), $\lim_{T\to 0}\mathcal{X}_s(\mathcal{X}_s^\bullet) = \mathcal{X}_s^c$. ■

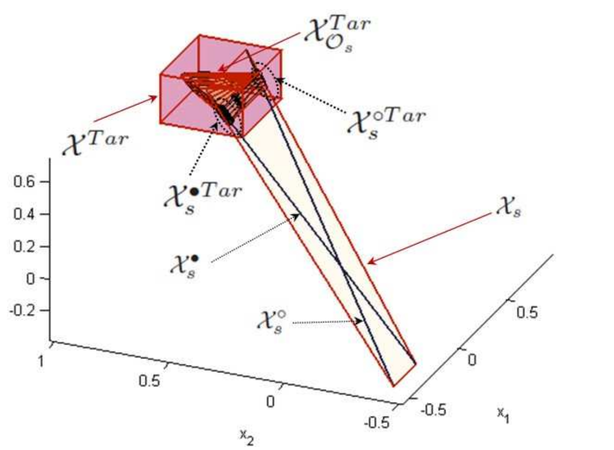

Fig. 1. State equilibrium orbit set, $\mathcal{O}_s^{\text{Tar}}$, state target set, $\mathcal{X}^{\text{Tar}}$, and state equilibrium sets for the UDS $\mathcal{X}_s^{\circ T}$ and $\mathcal{X}_s^{\bullet T}$. Each individual trajectory (in red) represents the equilibrium orbit o_s. Sets $\mathcal{X}_s$, $\mathcal{X}_s^\circ$ and $\mathcal{X}_s^\bullet$ are also shown.

E. Target equilibrium sets for control purposes

The control goal is to steer a LICS (1) to an arbitrary nonempty target set $\mathcal{X}^{\text{Tar}} \subseteq \mathcal{X}$, and once the system reaches this set, to keep it there indefinitely. To accomplish this, it is necessary to define the so-called target counterpart of the equilibrium sets of subsection IV-B. This way, $\mathcal{X}_s^{\bullet\text{Tar}}$ is defined as the maximal equilibrium set of UDS (3a) such that $\mathcal{X}_{\mathcal{O}_s}^{\text{Tar}}(\mathcal{X}_s^{\text{Tar}}(\mathcal{X}_s^{\bullet\text{Tar}})) \subseteq \mathcal{X}^{\text{Tar}}$, where $\mathcal{X}_s^{\text{Tar}}(\mathcal{X}_s^{\bullet\text{Tar}}) \triangleq \text{ch}\left\{\mathcal{X}_s^{\bullet\text{Tar}}, \mathcal{X}_s^{\circ\text{Tar}}(\mathcal{X}_s^{\bullet\text{Tar}})\right\}$, $\mathcal{X}_s^\circ(\mathcal{X}_s^{\bullet\text{Tar}}) = \{x_s^\circ(x_s^\bullet) \in \mathbb{R}^n : x_s^\bullet \in \mathcal{X}_s^{\bullet\text{Tar}}\}$ and $\mathcal{U}_s^{\text{Tar}} = \{u_s(x_s^\bullet) \in \mathbb{R}^m : x_s^\bullet \in \mathcal{X}_s^{\bullet\text{Tar}}\}$.

Remark 2: Notice that, according to Property 3, the parameter that decides if there exists a set $\mathcal{X}_{\mathcal{O}_s}^{\text{Tar}}(\mathcal{X}_s^{\text{Tar}}(\mathcal{X}_s^{\bullet\text{Tar}})) \subseteq \mathcal{X}^{\text{Tar}}$ is the time interval T. Given that the size of $\mathcal{X}_{\mathcal{O}_s}^{\text{Tar}}(\mathcal{X}_s^{\text{Tar}}(\mathcal{X}_s^{\bullet\text{Tar}}))$ increases when T increases, it could exist a maximal T, $T_{\max}$, for which the condition holds. Furthermore, $T_{\min}$ is usually given by practical restrictions (since maximal frequency of impulses is determined by the control problem itself), and so this minimum time interval should be checked to be smaller than $T_{\max}$; otherwise, the zone control problem would not be properly stated.

The procedure to compute $\mathcal{X}_s^{\bullet\text{Tar}}$ and to find $T_{\max}$ is as follows: (1) Compute the maximal set $\mathcal{X}_s^{\bullet\text{Tar}} \subseteq \mathcal{X}^{\text{Tar}}$, for $T = T_{\min}$ ($T_{\min}$ is assumed to be given), such that $\mathcal{X}_{\mathcal{O}_s}^{\text{Tar}}(\mathcal{X}_s^{\text{Tar}}(\mathcal{X}_s^{\bullet\text{Tar}})) \subseteq \mathcal{X}^{\text{Tar}}$. (2) If the set $\mathcal{X}_s^{\bullet\text{Tar}}$ is empty, then, the control problem is not properly formulated, and the target set $\mathcal{X}^{\text{Tar}}$ must be enlarged or $T_{\min}$ reduced. (3) If $\mathcal{X}_s^{\bullet\text{Tar}}$ is not empty, then, increase T up to a value such that the condition does not hold anymore. This value defines $T_{\max}$ for the given target $\mathcal{X}^{\text{Tar}}$. (4) The selected period T must be $T_{\min} < T < T_{\max}$. See Fig. 1 for a schematic plot of the orbit set, $\mathcal{X}_{\mathcal{O}_s}^{\text{Tar}}$, the target set, $\mathcal{X}^{\text{Tar}}$, and $\mathcal{X}_s^{\bullet\text{Tar}}$ in $\mathbb{R}^3$.

F. Attractivity of the equilibrium sets

Once the equilibrium sets are characterized, the next step is to establish some stability definitions. Based on [21], the attractivity of sets that not necessarily contain the origin will be discussed.

Definition 5: (Attractive sets) A nonempty, closed and convex set $\mathcal{X}_1 \subseteq \mathcal{X}$ is attractive for the closed-loop system (2), with respect to a (nonempty, closed and convex) set $\mathcal{X}_2 \supseteq \mathcal{X}_1$, with $\mathcal{X} \supseteq \mathcal{X}_2$, if there exists a vicinity of $\mathcal{X}_1$ such that $\lim_{k\to\infty}\text{dist}_{\mathcal{X}_1}(\phi(\tau_k; x_0, \kappa(\cdot))) =$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2017.2776598, IEEE Transactions on Automatic Control

0, and $\lim_{t\to\infty} \text{dist}_{\mathcal{X}_2}(\phi(t;x_0,\kappa(\cdot))) = 0$, for all x_0 in such vicinity. If $\mathcal{X}_2 \equiv \mathbb{R}^n$, then $\mathcal{X}_1$ is called weakly attractive.

Weak attractivity only accounts for the closed-loop system (2) at the impulsive times, while attractivity also specifies a second set where the continuous-time trajectories between jumps converge. This makes sense given that the trajectories between jumps are free responses (nothing can be done up to the next jump time) and they could escape. An asymptotic stability definition is also presented in [21], which basically requires the uniform boundedness of the solution trajectories. However, in this work only the attractivity of the control strategy will be considered. Next, some results regarding the attractivity of the generalized equilibrium sets are presented

Theorem 1: Let $\mathcal{X}_s^{\bullet\text{Tar}}$ be an attractive equilibrium set - in the usual sense of attractivity of discrete-time systems - for the closed-loop UDS (3a), $x^\bullet(k+1) = A^\bullet x^\bullet(k) + B^\bullet \kappa(x^\bullet(k))$, where $\kappa(x)$ is assumed to be continuous in x. Then, the generalized equilibrium set $\mathcal{X}_s^{\text{Tar}}(\mathcal{X}_s^{\bullet\text{Tar}})$ is attractive for the closed-loop system (2), controlled by $\kappa(\cdot)$, with respect to $\mathcal{X}_{\mathcal{O}_s}^{\text{Tar}}(\mathcal{X}_s^{\bullet\text{Tar}})$.

Proof: Given that $\mathcal{X}_s^{\bullet\text{Tar}}$ is an attractive equilibrium set for the UDS (3a), controlled by $\kappa(\cdot)$, then there exists a vicinity of this set such that $\lim_{k\to\infty} x^\bullet(k) = x_s^\bullet$, for some $x_s^\bullet \in \mathcal{X}_s^{\bullet\text{Tar}}$, and $\kappa(x_s^\bullet) = u_s(x_s^\bullet)$. Then, given that $x^\bullet(k) \triangleq \phi(\tau_k; x_0, \kappa(\cdot))$, with $x^\bullet(0) = \phi(0; x_0, \kappa(\cdot)) = x_0$, and $\mathcal{X}_s^{\bullet\text{Tar}} \subseteq \mathcal{X}_s^{\text{Tar}}(\mathcal{X}_s^{\bullet\text{Tar}})$, it follows that $\lim_{k\to\infty} \text{dist}_{\mathcal{X}_s^{\text{Tar}}(\mathcal{X}_s^{\bullet\text{Tar}})}(\phi(\tau_k; x_0, \kappa(\cdot))) = 0$. On the other hand, consider the sequence of trajectories $tr_k(t) \triangleq \phi(t; x^\bullet(k), \kappa(x^\bullet(k))) = e^{A_c t}(x^\bullet(k) + B\kappa(x^\bullet(k)))$, with $t \in (\tau_k, \tau_{k+1}]$, and $k \in \mathbb{N}$. These trajectories are continuous with respect to $x^\bullet(k)$, for every $k \in \mathbb{N}$ (note that $\kappa(\cdot)$ is assumed to be continuous). Then, as $\lim_{k\to\infty} x^\bullet(k) = x_s^\bullet$ for some $x_s^\bullet \in \mathcal{X}_s^{\bullet\text{Tar}}$, it follows that $o_s(t) \triangleq \phi(t; x_s^\bullet, \kappa(x_s^\bullet)) = e^{A_c t}(x_s^\bullet + Bu_s^\bullet)$, $t \in (\tau_k, \tau_{k+1}]$. Finally, by the definition of the equilibrium orbit set $\mathcal{X}_{\mathcal{O}_s}^{\text{Tar}}(\mathcal{X}_s^{\bullet\text{Tar}})$, $o_s(x_s^\bullet, u_s(x_s^\bullet)) \subseteq \mathcal{X}_{\mathcal{O}_s}^{\text{Tar}}(\mathcal{X}_s^{\bullet\text{Tar}})$ for every $x_s^\bullet \in \mathcal{X}_s^\bullet$, which means that $\lim_{t\to\infty} \text{dist}_{\mathcal{X}_{\mathcal{O}_s}^{\text{Tar}}(\mathcal{X}_s^{\bullet\text{Tar}})}(\phi(t; x_0, \kappa(\cdot))) = 0$, and so $\mathcal{X}_s^{\text{Tar}}(\mathcal{X}_s^{\bullet\text{Tar}})$ is attractive with respect to $\mathcal{X}_{\mathcal{O}_s}^{\text{Tar}}(\mathcal{X}_s^{\bullet\text{Tar}})$ for the closed-loop system (2). ■

This result permits a flexible design of the controllers, since steering the UDS (3a) to its corresponding equilibrium region, $\mathcal{X}^{\bullet\text{Tar}}$, implies to steer the ICS (1) to $\mathcal{X}_{\mathcal{O}_s}^{\text{Tar}}(\mathcal{X}_s^{\bullet\text{Tar}}) \subseteq \mathcal{X}^{\text{Tar}}$.

V. Control strategies

In this Section two control strategies are presented. The first one is a standard unconstrained feedback control adapted to the non-zero regulation problem, whose main advantage is its simplicity. The second one is an MPC, which is designed with an entire set as a target (zone MPC, [9]), and explicitly considers input and state constraints.

A. Affine feedback control for unconstrained systems

The control objective here is to steer the impulsive system (2) to a non-zero equilibrium inside the target set $\mathcal{X}^{\text{Tar}}$.

Theorem 2: Consider an equilibrium $x_s^\circ \in \mathcal{X}_s^{\circ\text{Tar}} \subseteq \mathcal{X}^{\text{Tar}}$, and assume that there exist parameters K and ζ such that $\rho(F) < 1$ and $B\zeta = (I-F)x_s^\circ$, with $F = (A_d - BK)e^{A_c T}$. Then, $x_s^\circ \in \mathcal{X}_s^{\text{Tar}}$ is attractive for system (2), with respect to $\mathcal{X}_{\mathcal{O}_s}^{\text{Tar}}$ and $\kappa(x) = -Kx + \zeta$.

Proof: Consider an equilibrium sate x_s° and the control law $\kappa(x) = -Kx+\zeta$, and define $F \triangleq (A_d - BK)e^{A_c T}$. Then, the evolution of system (1a) from an initial state x_0 can be written as $x(\tau_k^+) = F^k x_0 + \left(F^{k-1} + F^{k-2} + \cdots + F + I\right) B\zeta$. If K is chosen such that $\rho(F) < 1$, the geometric series $\left(F^{k-1} + F^{k-2} + \cdots + F + I\right)$ converges to $(I - F^k)(I-F)^{-1}$. Then, the system evolution becomes $x(\tau_k^+) = F^k x_0 + (I - F^k)(I-F)^{-1}B\zeta$. Then, $\lim_{k\to\infty} |x(\tau_k^+) - x_s^\circ| = |F^k x_0 + (I - F^k)(I-F)^{-1}B\zeta - x_s^\circ| = |(I-F)^{-1}B\zeta - x_s^\circ|$, where the last equality follows from the fact that $F^k \to 0$ as $k \to \infty$. If ζ is chosen to verify $B\zeta = (I-F)x_s^\circ$, then $\lim_{k\to\infty} |x(\tau_k^+) - x_s^\circ| = 0$, which implies that $x^\bullet(k) = e^{A_c T} x(\tau_k^+)$ converges to $e^{A_c T} x_s^\circ = x_s^\bullet \in \mathcal{X}_s^{\bullet\text{Tar}}$, or in other words, $\mathcal{X}_s^{\bullet\text{Tar}}$ is attractive for $x^\bullet(k+1) = A^\bullet x^\bullet(k) + B^\bullet \kappa(x^\bullet(k))$, with $\kappa(x) = -Kx+\zeta$. Now, since $\kappa(x)$ is continuous w.r.t. x, by Theorem 1, $\mathcal{X}_s^{\text{Tar}}$ is attractive for the closed-loop system with respect to $\mathcal{X}_{\mathcal{O}_s}^{\text{Tar}}$. ■

Notice that according to [22], the controllability of the pair (A_c, B) implies that there exists a feedback K such the closed-loop system eigenvalues can be placed in arbitrary locations. Besides, by the standard methods, if the underlying discrete system (1b) is considered, such an eigenvalue placement problem can be solved if the rank of $[\lambda I - A^\bullet, B^\bullet]$ is n, for all $\lambda \in \mathbb{C}$.

B. Zone model predictive control

In this subsection a zone MPC formulation which steers the impulsive system to an equilibrium target set defined by $\mathcal{X}_s^{\text{Tar}} \subset \mathcal{X}^{\text{Tar}}$ is described. The strategy is an extension of [9] to the impulsive case. In contrast to the feedback controller, this strategy takes the whole set $\mathcal{X}_s^{\text{Tar}}$ as a target and takes into account state and input constraints. The cost of the optimization problem that the MPC solves on-line is given by:

$$\begin{aligned} &V_N(x, \mathcal{X}_s^{\bullet\text{Tar}}, \mathcal{U}_s^{\text{Tar}}; \mathbf{u}, u_a, x_a) \\ &= V_{dyn}(x; \mathbf{u}, u_a, x_a) + V_f(\mathcal{X}_s^{\bullet\text{Tar}}, \mathcal{U}_s^{\text{Tar}}; u_a, x_a), \end{aligned} \tag{8}$$

where $V_{dyn}(x; \mathbf{u}, u_a, x_a) = \sum_{j=0}^{N-1} \|x(j) - x_a\|_Q^2 + \|u(j) - u_a\|_R^2$, with $Q > 0$ and $R > 0$, is a term devoted to steer the system to a certain artificial open-loop equilibrium given by $(u_a, x_a) \in \mathcal{U}_s \times \mathcal{X}_s^\bullet$; and $V_f(\mathcal{X}_s^{\bullet\text{Tar}}, \mathcal{U}_s^{\text{Tar}}; u_a, x_a) = p\left(\text{dist}_{\mathcal{X}_s^{\bullet\text{Tar}}}(x_a) + \text{dist}_{\mathcal{U}_s^{\text{Tar}}}(u_a)\right)$, with $p > 0$, is a terminal cost devoted to steer x_a to the whole sets $\mathcal{X}_s^{\bullet\text{Tar}}$ and u_a to $\mathcal{U}_s^{\text{Tar}}$, respectively. Notice that in the latter cost, the current state x and the sets $\mathcal{X}_s^{\bullet\text{Tar}}$ and $\mathcal{U}_s^{\text{Tar}}$ are parameters, while $\mathbf{u} = \{u(0), u(1), \cdots, u(N-1)\}$, u_a and x_a are the optimization variables (N being the control horizon).

The optimization problem to be solved at time k by the MPC is given by $P_{\text{MPC}}(x, \mathcal{X}_s^{\bullet\text{Tar}}, \mathcal{U}_s^{\text{Tar}})$:

$$\begin{aligned} \min_{\mathbf{u}, u_a, x_a} \quad & V_N(x, \mathcal{X}_s^{\bullet\text{Tar}}, \mathcal{U}_s^{\text{Tar}}; \mathbf{u}, u_a, x_a) \\ s.t. \quad & \\ & x(0) = x, \\ & x(j+1) = A^\bullet x(j) + B^\bullet u(j), \qquad j \in \mathbb{I}_{0:N-1} \\ & x(j) \in \mathcal{X},\ u(j) \in \mathcal{U}, \qquad j \in \mathbb{I}_{0:N-1} \\ & x(N) = x_a, \\ & A^\bullet x_a + B^\bullet u_a = x_a \qquad (x_a \in \mathcal{X}_s^\bullet,\ u_a \in \mathcal{U}_s). \end{aligned}$$

Constraint $x(N) = x_a$ is a terminal constraint that forces the state at the end of the control horizon N to reach the artificial equilibrium state x_a, while the last constraint forces the artificial variable to be an equilibrium pair in $\mathcal{X}_s^\bullet \times \mathcal{U}_s$. The control law, derived from the application of a receding horizon control policy (RHC), is given by $\kappa_{\text{MPC}}(x, \mathcal{X}_s^{\bullet\text{Tar}}, \mathcal{U}_s^{\text{Tar}}) = u^0(0; x)$, where $u^0(0; x)$ is the first element of the solution sequence $\mathbf{u}^0(x)$. The domain of attraction of the closed-loop is given by the controllable set in N steps to the maximum equilibrium set $\mathcal{X}_s^\bullet$, $\mathcal{X}_N(\mathcal{X}_s^\bullet)$ (because constraints $x(N) = x_a$ and $A^\bullet x_a + B^\bullet u_a = x_a$ force the current state x to reach any equilibrium in N time steps).

Remark 3: An approximate version of a continuous-time constraint of the form $A^\bullet(t)x(j) + B^\bullet(t)u(j) \in \mathcal{X}$, $t \in (0, T]$, $j \in \mathbb{I}_{0:N-1}$, can be included in Problem P_{MPC}, to ensure that the free response of system (1) remains in $\mathcal{X}$. This can be done by sampling in t the evolution $A^\bullet(t)x(j) + B^\bullet(t)u(j)$, with a sampling time small

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2017.2776598, IEEE Transactions on Automatic Control

enough to account for large escapes. Furthermore, in this case, an extra assumption is necessary to ensure that (at least) an input exists, for every $x \in \mathcal{X}_N^\bullet$, such that the condition is fulfilled.

Next, the main Theorem regarding the feasibility and attractivity of the proposed MPC is presented.

Theorem 3: Suppose that $\mathcal{X}_s^{\text{Tar}}(\mathcal{X}_s^{\bullet\text{Tar}})$ is a generalized equilibrium set with respect to $\mathcal{X}_{\mathcal{O}_s}^{\text{Tar}}(\mathcal{X}_s^{\bullet\text{Tar}})$, for the open loop system (1), as the one defined in Section IV-E. Then, (i) the MPC controller is recursively feasible, (ii) set $\mathcal{X}_s^{\text{Tar}}(\mathcal{X}_s^{\bullet\text{Tar}})$ is a generalized equilibrium set for system (1) controlled by the MPC, with respect to $\mathcal{X}_{\mathcal{O}_s}^{\text{Tar}}$, and (iii) set $\mathcal{X}_s^{\text{Tar}}(\mathcal{X}_s^{\bullet\text{Tar}})$ is attractive for system (1), with respect to $\mathcal{X}_{\mathcal{O}_s}^{\text{Tar}}(\mathcal{X}_s^{\bullet\text{Tar}})$.

Proof: (i) Given that any state $x \in \mathcal{X}_N^\bullet$ could be steered to $\mathcal{X}_s^\bullet$ fulfilling the constraint in the path, it is possible to follow the usual steps to show feasibility in MPC, and construct a feasible input sequence for the optimization problem at time step $k+1$, by shifting one time ahead the optimal solution to the same problem at time step k, and adding to the tail of the sequence the artificial input value ([9]).

(ii) Assume that the ICS is placed at some $x_s \in \mathcal{X}_s^{\text{Tar}}(\mathcal{X}_s^{\bullet\text{Tar}}) \subseteq \mathcal{X}_s$, at time $k = 0$. Given that $\mathcal{X}_s^{\text{Tar}}(\mathcal{X}_s^{\bullet\text{Tar}})$ is a generalized equilibrium set with respect to $\mathcal{X}_{\mathcal{O}_s}^{\text{Tar}}(\mathcal{X}_s^{\bullet\text{Tar}})$, for system (1), then, there exists a feasible input sequence $\hat{\mathbf{u}} = \{u(0), \ldots, u(N-1)\}$, with $u(0) = \alpha u_s(x_s^\bullet)$ and $u(j) = u_s(x_s^\bullet) \in \mathcal{U}_s^{\text{Tar}}$, for $j \in \mathbb{I}_{1:N-1}$, for a particular value of $\alpha \in [\underline{\alpha}, \overline{\alpha}]$ (see the Proof of Property 2), which produces a sequence of states $x(j) = x_s^\bullet$, for $j \in \mathbb{I}_{1:N}$, that belongs to $\mathcal{X}_s^{\bullet\text{Tar}} \subseteq \mathcal{X}_s^{\text{Tar}}$. Furthermore, by the MPC cost function definition (8), this input sequence, together with the artificial variables $u_a = u_s(x_s^\bullet)$ and $x_a = x_s^\bullet$, are the optimal solution to MPC optimization problem, since they produce a null dynamic and terminal cost (any input sequence different from the proposed one produces a positive cost). Then, given that $u(0) = \alpha u_s(x_s^\bullet)$ is injected to the ICS (1), $\phi(T; x_s^\bullet, u_s(x_s^\bullet)) = x_s^\bullet \in \mathcal{X}_s^{\text{Tar}}(\mathcal{X}_s^{\bullet\text{Tar}})$. In addition, the free trajectory corresponding to $x_s^\bullet$, $o_s(x_s^\bullet, u_s(x_s^\bullet)) = ch\{\phi(t; x_s^\bullet, u_s(x_s^\bullet)), t \in [0,T]\}$, is clearly in $\mathcal{X}_{\mathcal{O}_s}^{\text{Tar}}(\mathcal{X}_s^{\bullet\text{Tar}})$.

(iii) Consider a state $x \in \mathcal{X}_N \backslash \mathcal{X}_s^{\text{Tar}}$, at a given time k. Consider also the optimal solution for this state, $\mathbf{u}^0(x) = \{u^0(0;x), \cdots, u^0(N-1;x)\}$, $u_a^0(x)$ and $x_a^0(x)$, and the corresponding state sequence $\mathbf{x}^0(x) = \{x^0(0;x), \cdots, x^0(N;x)\}$, where $x^0(0;x) = x$ and $x^0(N;x) = x_a^0 \in \mathcal{X}_s^\bullet$. The cost function of Problem $P_{MPC}(x, \mathcal{X}_s^{\bullet\text{Tar}}, \mathcal{U}_s^{\text{Tar}})$, corresponding to $\mathbf{u}^0(x)$, $u_a^0(x)$ and $x_a^0(x)$ is given by $V_N^0(x) \triangleq \sum_{j=0}^{N-1} (\|x^0(j;x) - x_a^0(x)\|_Q^2 + \|u^0(j;x) - u_a^0(x)\|_R^2) + V_f(\mathcal{X}_s^{\bullet\text{Tar}}, \mathcal{U}_s^{\text{Tar}}; u_a^0(x), x_a^0(x))$. Now, consider the successor state $x^+ = A^\bullet x + B^\bullet u^0(0;x)$, at time $k+1$, which is obtained by applying the control law $\kappa_{\text{MPC}}(x) = u^0(0;x)$, and define the following input sequence and artificial variables candidates: $\hat{\mathbf{u}} = \{u^0(1;x), \cdots, u^0(N-1;x), u_a^0(x)\}$, $\hat{u}_a = u_a^0(x)$ and $\hat{x}_a = x_a^0(x)$, where $u_a^0(x)$ and $x_a^0(x)$ are the optimal artificial variables at time step k. Since no model mismatch is considered for predictions, the successor states x^+ is equal to $x^0(1;x)$. Solution $(\hat{\mathbf{u}}, \hat{u}_a, \hat{x}_a)$ has then an associated state sequence $\hat{\mathbf{x}} = \{x^0(1;x), \cdots, x^0(N;x), x_a^0(x)\}$. By the terminal constraint in Problem P_{MPC}, is $x^0(N;x) = x_a^0(x)$, which means that $x_a^0(x)$ fulfills $x_a^0(x) = A^\bullet x^0(N;x) + B^\bullet u_a^0(x)$. Furthermore, given that $\mathbf{u}^0(x)$ and $u_a^0(x)$ are part of a feasible solution to Problem P_{MPC} at time step k, then $\hat{\mathbf{u}}$ is feasible for Problem P_{MPC} at time step $k+1$. The cost function of Problem $P_{MPC}(x^+, \mathcal{X}_s^{\bullet\text{Tar}}, \mathcal{U}_s^{\text{Tar}})$, at $k+1$, for the solution $(\hat{\mathbf{u}}, \hat{u}_a, \hat{x}_a)$, is given by $\hat{V}_N(x^+) \triangleq V_N(x^+, \mathcal{X}_s^{\bullet\text{Tar}}, \mathcal{U}_s^{\text{Tar}}; \hat{\mathbf{u}}, \hat{u}_a, \hat{x}_a) = \sum_{j=0}^{N-1} (\|\hat{x}(j) - \hat{x}_a\|_Q^2 + \|u^0(j) - \hat{u}_a\|_R^2) + V_f(\mathcal{X}_s^{\bullet\text{Tar}}, \mathcal{U}_s^{\text{Tar}}; \hat{u}_a, \hat{x}_a)$, where $x^+ = x^0(1;x)$. So, this cost can be written as a function of x: $\hat{V}_N(x^+) = \sum_{j=1}^{N-1} (\|x^0(j;x) - x_a^0(x)\|_Q^2 + \|u^0(j;x) - u_a^0(x)\|_R^2) + \|x^0(N;x^+) - x_a^0(x)\|_Q^2 + \|u_a^0(x) - u_a^0(x)\|_R^2 + V_f(\mathcal{X}_s^{\bullet\text{Tar}}, \mathcal{U}_s^{\text{Tar}}; u_a^0(x), x_a^0(x))$ (where the second term of the right hand side is null). If the proposed feasible cost at time $k+1$ is compared with the optimal one at time k, it is obtained $\hat{V}_N(x^+) - V_N^0(x) = -\|x^0(0;x) - x_a^0(x)\|_Q^2 - \|u^0(0;x) - u_a^0(x)\|_R^2$. Now, by optimality of the solution to Problem $P_{MPC}(x^+, \mathcal{X}_s^{\bullet\text{Tar}}, \mathcal{U}_s^{\text{Tar}})$, at $k+1$, it follows that the optimal cost function $V_N^0(x^+)$ fulfils $V_N^0(x^+) \leq \hat{V}_N(x^+)$, and so $V_N^0(x^+) - V_N^0(x) \leq -\|x^0(0;x) - x_a^0(x)\|_Q^2 - \|u^0(0;x) - u_a^0(x)\|_R^2$. Since the stage cost is a positive definite function, by definition, this implies that $x^0(0;x)$ tends to $x_a^0(x)$ and $u^0(0;x)$ tends to $u_a^0(x)$ as $k \to \infty$. Now, by Lemma 1 in the Appendix, the fact that $(u(k), x(k)) \triangleq (u^0(0;x), x^0(0;x)) \to (u_a^0(x), x_a^0(x)) \triangleq (u_a(k), x_a(k))$, as $k \to \infty$ implies that $x^0(0;x)$ tends to $\mathcal{X}_s^{\bullet\text{Tar}}$ and $u^0(0;x)$ tends to $\mathcal{U}_s^{\text{Tar}}$ as $k \to \infty$, which means that $\mathcal{X}_s^{\bullet\text{Tar}}$ is an attractive set for the UDS (3a), controlled by the MPC. So, given that the control law $\kappa_{\text{MPC}}(x, \mathcal{X}_s^{\bullet\text{Tar}}, \mathcal{U}_s^{\text{Tar}})$ is continuous w.r.t. x, by Theorem 1, $\mathcal{X}_s^{\text{Tar}}$ is attractive, with respect to $\mathcal{X}_{\mathcal{O}_s}^{\text{Tar}}(\mathcal{X}_s^{\text{Tar}})$, for the ICS (1) controlled by the MPC. ■

VI. Numerical examples

A. Example 1: Lithium ions distribution in the human body

In [8] a physiological pharmacokinetic model based on experimental data, which describes the distribution of Lithium ions in the human body upon oral administration, is provided. The system state vector is given by $x(t) = [C_P(t)\ C_{RBC}(t)\ C_M(t)]^\top$, where $C_P(t)$ is the concentration of plasma (P), $C_{RBC}(t)$ is the concentration of the red blood cells (RBC), and $C_M(t)$ is the concentration of muscle cells (M). All these concentrations are given in nmol/L. The input u is given by the amount of the dose, in nmol. The administration period is initially fixed in $T = 3$ h. The dynamics of the drug distribution is described by an ICS as in (1), characterized by the matrices

$$A_c = \begin{pmatrix} -0.6137 & 0.1835 & 0.2406 \\ 1.2644 & -0.8 & 0 \\ 0.2054 & 0 & -0.19 \end{pmatrix}, B = \begin{pmatrix} 10.9 \\ 0 \\ 0 \end{pmatrix}, \quad (9)$$

and $A_d = I_{2x2}$. The state and input constraints are given by $\mathcal{X} = \{x : [0\ 0\ 0]^\top \leq x \leq [2\ 1.2\ 1.2]^\top\}$ and $\mathcal{U} = \{u : 0 \leq u \leq 5.95\}$, respectively. The state window target is defined by $\mathcal{X}^{\text{Tar}} = \{x : [0.4\ 0.6\ 0.5]^\top \leq x \leq [0.6\ 0.9\ 0.8]^\top\}$, as it is described in [8], [21]. The drug's concentration within the boundaries of $\mathcal{X}$ guarantees the effectiveness of the therapy.

According to the methodology proposed in Section IV-E, the maximal intake period T for the given therapeutic window $\mathcal{X}^{\text{Tar}}$, is given by $T_{\max} = 6$ h. In fact, for larger periods the set containing all the orbits starting at $\mathcal{X}_s^{\bullet\text{Tar}}$, $\mathcal{X}_{\mathcal{O}_s}^{\text{Tar}}(\mathcal{X}_s^{\bullet\text{Tar}})$, is not contained in $\mathcal{X}^{\text{Tar}}$. This analysis provides a practical way to find the maximal value of T, according to control system specifications. The intake period was then selected to be $T = 3$ h.

1) Affine Feedback Control: To implement the feedback control through Theorem 2, it is necessary to find K and ζ. K was chosen by a standard eigenvalue placement problem, and its value is $K = [0.05\quad 0.01\quad 0.02]$. The spectral radius was approximately 0.6807. The value $\zeta \approx 0.0513$ was computed from $B\zeta = (I - F)x_s^\circ$, with $x_s^\circ = [0.57\quad 0.78\quad 0.55]$. The control equilibrium is $u_s = 0.89$. The state and input time evolutions are shown in Fig. 2 (blue circles and dotted line). For this application, the performance of this strategy was good and the constraints in states and control were not violated. Its main advantage is, clearly, its simplicity. However, as it is shown in the second example,

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2017.2776598, IEEE Transactions on Automatic Control

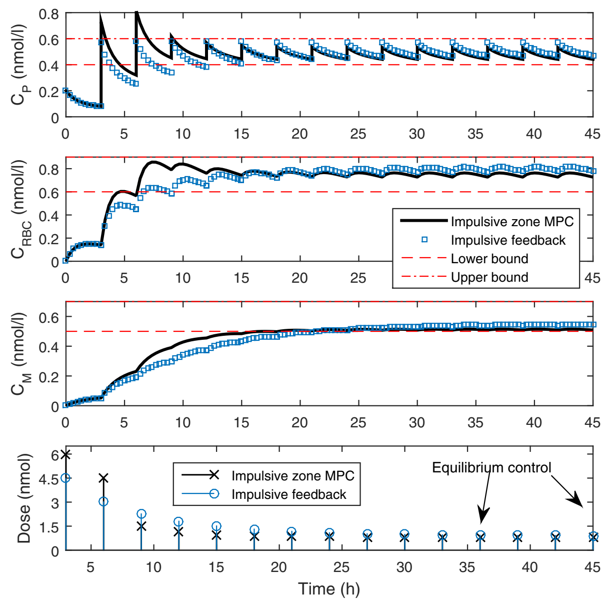

Fig. 2. State time evolution. Feedback control, dotted line; MPC, solid line

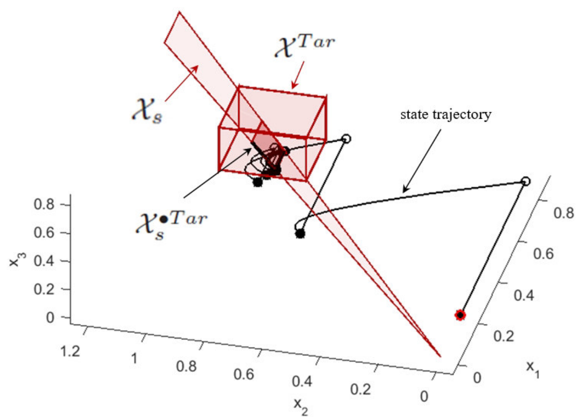

Fig. 3. State evolution in the state space, for the MPC controller.

constraints satisfaction cannot be ensured.

2) Zone MPC Control: The MPC controller is tuned as: $N = 5$, $Q = diag([1\ 1\ 1])$, $R = 2$ and $p = 100$. Notice that in contrast to the control horizon used in [21], which is $N = 15$, a reduced one is used here, because of the enlarged domain of attraction. Fig. 2 (black stars and solid line) shows the state and input time evolutions. As it is desired, each state is steered to its corresponding therapeutic window relatively fast. Besides, the input makes the main effort first and, after its settling time, it remains constant at the desired equilibrium value u_s. Notice that both, states and inputs, are feasible at any time. Fig. 3 shows the portrait phase in the state space for the evolution plotted in Fig. 2. As it can be seen, the state trajectory moves away from $\mathcal{X}_s(\mathcal{X}_s^{\bullet})$ and $\mathcal{X}_s^{\bullet\text{Tar}}$ first and then it converges to $\mathcal{X}_s^{\bullet\text{Tar}}$. Notice that the state trajectory enters $\mathcal{X}_s^{\text{Tar}}$ (and $\mathcal{X}^{\text{Tar}}$) from below, since the controller cost penalizes only the distance from the state trajectory to the entire set. In fact, no matter what state equilibrium $x_s^{\bullet} \in \mathcal{X}_s^{\bullet\text{Tar}}$ the system reaches, the controller objective will be null.

B. Example 2: HIV infection dynamics with treatment

This second example is '3D HIV model' (taken from [20]) which describes the virus infection dynamics and incorporates the interaction of the intake of drugs (w, u) ([13]). The complete impulsive model is given by:

$$\left\{\begin{array}{lcl} \dot{T}_c(t) & = & s - \delta T_c(t) - \beta T_c(t) z(t), \\ \dot{y}(t) & = & \beta T_c(t) z(t) - \mu y(t), \\ \dot{z}(t) & = & (1 - \frac{w(t)}{w(t)+w_{50}}) k y(t) - c z(t), \\ \dot{w}(t) & = & -K_w w(t), \\ w(\tau_k^+) & = & w(\tau_k) + u(\tau_k), \quad k \in \mathbb{N}, \end{array}\right. \tag{10}$$

where T_c represents the concentration of healthy CD4 cells ($^{\text{cell}}/_{\text{mm}^3}$) which are produced from the thymus at a constant rate s (cell mm^{-3} day^{-1}) and die with a half life time equal to $\frac{1}{\delta}$ (day). The healthy cells are infected by the virus at a rate proportional to the product of their population and the amount of free virus particles. Constant β (ml copies^{-1} day^{-1}) indicates the effectiveness of the infection process. The infected CD4+ cells (y) result from the infection of healthy CD4 cells and die at a rate μ (day^{-1}). Free virus particles (z) are produced from infected CD4 cells at a rate k (copies cells^{-1} mm^{-3} ml^{-1} day^{-1}) and die within a half life time equal to $\frac{1}{c}$ (day). The pharmacokinetics and pharmacodynamics phases of the drug administration are related to w (the amount of drug in the body at time t) and $\eta = \frac{w(t)}{w(t)+w_{50}}$ (the efficacy of an anti-HIV treatment, where w_{50} is the concentration of drug that lowers the viral load by 50%). Although a cocktail of drugs is generally used, only Zidovudine therapies will be considered. The parameters of the model are: $s = 9$, $\delta = 0.009$, $\beta = 4 \cdot 10^{-6}$, $\mu = 0.3$, $k = 80$, $c = 0.6$, $K_w = 8.4$ (day), $w_{50} = 89.6$ (mg). The *in vivo* parameters for Zidovudine given in [13] are used. For more details see [20].

This model has two equilibria, the first one (or the 'healthy' equilibrium) characterized by the absence of virus, *i.e.* $(T_{ch}, y_h, z_h, w_h) = \left(\frac{s}{\delta}, 0, 0, 0\right)$, and the second one (or the 'endemic' equilibrium) dominated by a virus concentration $(T_{ce}, y_e, z_e, w_e) = \left(\frac{(u_e + w_{50} K_w)\mu c}{\beta \kappa w_{50} K_w}, \frac{s - \delta T_{ce}}{\mu}, \frac{w_{50} K_w \kappa y_e}{c(u_e + w_{50} K_w)}, u_e/K_w\right)$. Notice that for $u_e = 0$ (equilibrium control), the maximum virus concentration is achieved. To design both impulsive control strategies, the system is linearized around the endemic equilibrium. The resulting matrices A_c and B are:

$$A_c = \begin{pmatrix} -\delta z_e & 0 & -\beta T_{ce} & 0 \\ \beta z_e & -\mu & \beta T_{ce} & 0 \\ 0 & \frac{k w_{50}}{w_e + w_{50}} & -c & \frac{-k w_{50} y_e}{(w_e + w_{50})^2} \\ 0 & 0 & 0 & -K_w \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The selected intake period is $T = 0.5$ day. The state and input constraints are imposed as $\mathcal{X} = \{x : [0\ 0\ 0\ 0]^\top \leq x \leq [1000\ 20\ 1500\ 100]^\top\}$ and $\mathcal{U} = \{u : 0 \leq u \leq 600\}$, respectively. The state window target is defined by $\mathcal{X}^{\text{Tar}} = \{x : [900\ 0\ 0\ 0]^\top \leq x \leq [1000\ 5\ 250\ 60]^\top\}$. As it is described in [20], the control goal is to steer the system from the endemic equilibrium to a healthy zone defined by $\mathcal{X}^{\text{Tar}}$. Besides, the anti-HIV treatment is considered successful if z is below the threshold of 50 $^{\text{copies}}/_{\text{ml}}$.

The initial state was selected to be $x_0 = [609.9\ 12\ 1508\ 6]^\top$, and to evaluate the performance of both strategies, at day 200 a disturbance was included (the drug dose is completely suspended for 36 days, which produces a rebound of the virus load).

1) Affine Feedback Control: According to Theorem 2, K and ζ can be computed as the impulsive system is controllable. K is selected by a standard eigenvalue placement problem, and its value is $K = \begin{bmatrix} 0.0349 & -33.8248 & -0.8414 & 0.9876 \end{bmatrix}$. The spectral radius was approximately 0.92, which ensures attractivity. ζ is computed from the condition $B\zeta = (I - F)x_s^{\circ}$, with $x_s^{\circ} = \begin{bmatrix} 968.3053 & 0.3636 & 29.5146 & 52.2726 \end{bmatrix}$ and its value is 600. The

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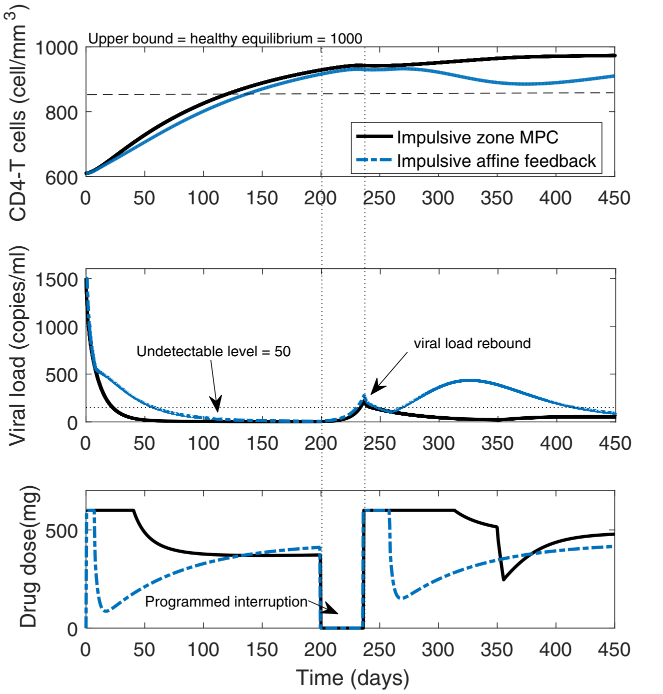

Fig. 4. State time evolution. Feedback control, dashed line; MPC, solid line

state and input time evolutions are shown in Fig. 4 (blue dashed line). The controller does not generate a feasible control at the beginning and after the disturbance; its value is greater than the upper limits of 600 mg of drug (it is around 650-1000 mg), which forces to manually saturate the dose. This is not a recommended practice, and generates an undesired viral load rebound.

2) Zone MPC Control: The MPC controller is tuned as: $N = 10$, $Q = diag([5\ 0\ 5\ 0])$, $R = 0.1$ and $p = 5 \times 10^5$. Fig. 4 (black solid line) shows the state and input time evolutions. In contrast to what happens with the affine feedback, the impulsive zone MPC ensures that both states and inputs are feasible at any time (no manual saturation is needed). This example clearly illustrates this well-known advantage of MPC over the standard methods.

VII. Conclusion

The problem of steering a linear impulsive system to a state window target that does not contain the origin has been tackled. To this aim, a new generalized equilibrium set is characterized, based on a discrete-time underlying subsystem. Two control strategies were proposed: (i) a simple-to-apply unconstrained affine feedback controller, that however needs additional conditions to fulfill the system constraints (mainly the positivity constraints), and (ii) a constrained zone MPC, that exploits the benefits of the use of artifical optimization variables. In contrast to other strategies ([21]), this latter MPC formulation only needs to compute a simple generalized equilibrium set as a target. This is an important benefit, given that complexity may prevent the use of such MPC strategies in some specific applications related to drug administration problems. Furthermore, the controller has an enlarged domain of attraction (for relatively small control horizons), because of the use of artificial intermediary variables ([9], [10]). Future works include a more detailed analysis of feasibility of the continuous-time closed-loop trajectories between jumps, and a complete stability proof (given that only attractivity was considered).

VIII. Appendix

Lemma 1 (Convergence to $\mathcal{X}_s^{\bullet \mathrm{Tar}}$): Consider the closed-loop system obtained by applying the MPC control law, κ_{MPC}, presented in Section V-B. Define the MPC closed loop current input and state as $z(k) = (u(k), x(k)) \triangleq (u^0(0; x(k)), x^0(0; x(k)))$, and the current artificial input and state variables as $z_a(k) = (u_a(k), x_a(k)) \triangleq (u_a^0(x(k)), x_a^0(x(k)))$. Then, $\lim_{k\to\infty} \|z(k) - z_a(k)\|_S^2 = 0$, with $S > 0$, implies that $\lim_{k\to\infty} \mathrm{dist}_{\mathcal{X}_s^{\bullet \mathrm{Tar}}}(x(k)) = 0$ and $\lim_{k\to\infty} \mathrm{dist}_{\mathcal{U}_s^{\mathrm{Tar}}}(u(k)) = 0$.

Proof: The elements $z(k) \triangleq (u(k), x(k))$, for $k \geq 0$, must fulfill $x(k+1) = A^{\bullet}x(k) + B^{\bullet}u(k)$, while $z_a(k) \triangleq (u_a(k), x_a(k))$ are forced to be in $\mathcal{Z}_s \triangleq \mathcal{U}_s \times \mathcal{X}_s^{\bullet}$, for $k \geq 0$ (last constraint in P_{MPC}). Then, once $z(k)$ reaches $z_a(k)$ for a large enough value of k both, $z(k)$ and $z(k+1)$ are in $\mathcal{Z}_s$ (equilibrium set). This means that $x(k) = A^{\bullet}x(k) + B^{\bullet}u(k)$ and $x(k+1) = A^{\bullet}x(k+1) + B^{\bullet}u(k+1)$. But, by the system evolution, it is $x(k+1) = A^{\bullet}x(k) + B^{\bullet}u(k)$, which means that $x(k+1){=}x(k)$. Replacing $x(k+1)$ by $x(k)$ in the second equilibrium equation, it follows that $x(k) = A^{\bullet}x(k) + B^{\bullet}u(k+1)$. Then subtracting both equation of $x(k)$, it is $B^{\bullet}(u(k) - u(k+1)) = 0$, and assuming that $rank(B^{\bullet}) = m$, this implies that $u(k+1){=}u(k)$, and so $z(k+1){=}z(k)$. Therefore $z(k)$ reaches $z_a(k)$ only at an equilibrium of the closed-loop system. Finally, according to Lemma 2, every closed-loop equilibrium pair of the MPC closed-loop is in $\mathcal{Z}_s^{\mathrm{Tar}} \triangleq \mathcal{U}_s^{\mathrm{Tar}} \times \mathcal{X}_s^{\bullet \mathrm{Tar}}$. This means that $\lim_{k\to\infty} \mathrm{dist}_{\mathcal{X}_s^{\bullet \mathrm{Tar}}}(x(k)) = 0$ and $\lim_{k\to\infty} \mathrm{dist}_{\mathcal{U}_s^{\mathrm{Tar}}}(u(k)) = 0$. ■

Lemma 2 (Uniqueness of $\mathcal{X}_s^{\bullet \mathrm{Tar}}$ as equilibrium set of the MPC closed-loop): Consider the closed-loop system obtained by applying the MPC control law, κ_{MPC}, presented in Section V-B. Then, every equilibrium pair $z_s = (u_s, x_s)$ is a minimizer of $V_f\left(\mathcal{X}_s^{\bullet \mathrm{Tar}}, \mathcal{U}_s^{\mathrm{Tar}}; u, x\right)$. Particularly, if $\mathcal{Z}_s^{\mathrm{Tar}} \subseteq \mathcal{Z}$, then every equilibrium pair $z_s = (u_s, x_s)$ of this closed-loop system is in $\mathcal{Z}_s^{\mathrm{Tar}}$.

Proof: Assume that $z_s = (u_s, x_s)$ is a minimizer of $V_f\left(\mathcal{X}_s^{\bullet \mathrm{Tar}}, \mathcal{U}_s^{\mathrm{Tar}}; u, x\right)$, and assume that the closed-loop system is placed at an equilibrium $z_i = (u_i, x_i) \neq (u_s, x_s) = z_s$. This means that $x_i = A^{\bullet}x_i + B^{\bullet}u_i$, $u_i = \kappa_{\mathrm{MPC}}(x_i)$, and so $u_i = \kappa_{\mathrm{MPC}}(A^{\bullet}x_i + B^{\bullet}u_i)$. Now, the solution of $P_{\mathrm{MPC}}(x_i, \mathcal{X}_s^{\bullet \mathrm{Tar}}, \mathcal{U}_s^{\mathrm{Tar}})$ is given by: $\mathbf{u}^0 = \{u^0(0), u^0(1), \cdots, u^0(N-1)\}$, u_a^0, x_a^0, with $u^0(0) = u_i = \kappa_{\mathrm{MPC}}(x_i)$, while the optimal state sequence is $\mathbf{x}^0 = \{x^0(0), x^0(1), \cdots, x^0(N-1), x^0(N)\}$, with $x^0(j) = x_i$, for $j \in \mathbb{I}_{0:1}$, and $x^0(N) = x_a^0$. The MPC optimal cost is then given by $V_N^0(x_i) = V_N(x_i, \mathcal{X}_s^{\bullet \mathrm{Tar}}, \mathcal{U}_s^{\mathrm{Tar}}; \mathbf{u}^0, u_a^0, x_a^0)$. As $(u_i, x_i) \neq (u_s, x_s)$, then, by Lemma 3, $(u_a^0, x_a^0) \neq (u_i, x_i)$. So, the optimal state sequence must go from x_i to x_a^0, in $N-1$ time steps.

A feasible input and artificial variable candidates to problem $P_{\mathrm{MPC}}(x_i, \mathcal{X}_s^{\bullet \mathrm{Tar}}, \mathcal{U}_s^{\mathrm{Tar}})$, which does not correspond to the closed-loop equilibrium condition of (u_i, x_i) is as follows: $\hat{\mathbf{u}} = \{u^0(1), u^0(2), \cdots, u^0(N-1), u_a^0\}$, u_a^0, x_a^0. This solution produces a state sequence given by $\hat{\mathbf{x}} = \{x^0(1), x^0(2), \cdots, x^0(N-1), x^0(N), x_a^0\}$, where $x^0(1) = x_i$, $x^0(N) = x_a^0$ and $x_a^0 = A^{\bullet}x_a^0 + B^{\bullet}u_a^0$ are the same artificial variables as before. Now, the cost corresponding to the solution $\mathbf{u}^0, u_a^0, x_a^0$ is compared with the one corresponding to $\hat{\mathbf{u}}, u_a^0, x_a^0$. The first one is given by $V_N^0(x_i) \triangleq V_N(x_i, \mathcal{X}_s^{\bullet \mathrm{Tar}}, \mathcal{U}_s^{\mathrm{Tar}}; \mathbf{u}^0, u_a^0, x_a^0) = \left\|x_i - x_a^0\right\|_Q^2 + \left\|u_i - u_a^0\right\|_R^2 + \left\|x^0(1) - x_a^0\right\|_Q^2 + \left\|u^0(1) - u_a^0\right\|_R^2 + \cdots + \left\|x^0(N-1) - x_a^0\right\|_Q^2 + \left\|u^0(N-1) - u_a^0\right\|_R^2 + V_f(\mathcal{X}_s^{\bullet \mathrm{Tar}}, \mathcal{U}_s^{\mathrm{Tar}}; u_a^0, x_a^0)$, while the second one is given by $\hat{V}_N(x_i) \triangleq V_N(x_i, \mathcal{X}_s^{\bullet \mathrm{Tar}}, \mathcal{U}_s^{\mathrm{Tar}}; \hat{\mathbf{u}}, u_a^0, x_a^0) = \left\|x^0(1) - x_a^0\right\|_Q^2 + \left\|u^0(1) - u_a^0\right\|_R^2 + \cdots + \left\|x^0(N-1) - x_a^0\right\|_Q^2 + \left\|u^0(N-1) - u_a^0\right\|_R^2 + \left\|x^0(N) - x_a^0\right\|_Q^2 + \left\|u_a^0 - u_a^0\right\|_R^2 + V_f(\mathcal{X}_s^{\bullet \mathrm{Tar}}, \mathcal{U}_s^{\mathrm{Tar}}; u_a^0, x_a^0)$. This way, given that $(u_i, x_i) \neq (u_a^0, x_a^0)$, it follows that $\hat{V}_N(x_i) - V_N^0(x_i) = -(\|x_i - x_a^0\|_Q^2 + \|u_i - u_a^0\|_R^2) < 0$. This means that the cost corresponding to the candidate solution is smaller than the optimal one, which is a contradiction, unless $(u_i, x_i) = (u_s, x_s)$, in which case every cost is null. So, every MPC closed-loop equilibrium pair minimize $V_f(\mathcal{X}_s^{\bullet \mathrm{Tar}}, \mathcal{U}_s^{\mathrm{Tar}}; u, x)$, and the unique equilibrium set is given by $\mathcal{X}_s^{\bullet \mathrm{Tar}}$. ∎

Lemma 3: Consider the MPC closed-loop system presented in Section V-B. Assume that it is placed at an equilibrium pair, $(u_i, x_i) \in \mathcal{U}_s \times \mathcal{X}_s^\bullet = \mathcal{Z}_s$, and $(u_i, x_i) \neq (u_s, x_s)$, being $(u_s, x_s) \in \mathcal{Z}_s$ any minimizer of $V_f(u, x)$. Then, $(u_a^0, x_a^0) \neq (u_i, x_i)$.

Proof: The proof proceeds by contradiction. Given that the system is placed at $(u_i, x_i) \in \mathcal{Z}_s$ the optimal solution, $(\mathbf{u}^0, u_a^0, x_a^0)$, steers the predicted trajectory from x_i to x_a^0 in N time steps. Suppose that $(u_a^0, x_a^0) = (u_i, x_i)$. Then, by convexity of V_{dyn}, $\mathbf{u}^0 = \mathbf{u}_i$, where $\mathbf{u}_i \triangleq \{u^0(0), \cdots, u^0(N-1)\}$, $u^0(j) = u_i$ for $j \in \mathbb{I}_{0:N-1}$. This input sequence produces a state sequence given by $\mathbf{x}_i = \{x^0(0), \cdots, x^0(N-1), x^0(N)\}$, with $x^0(j) = x_i$, for $j \in \mathbb{I}_{0:N}$. This way the optimal cost is given by $V_N^0 \triangleq V_N(x_i, \mathcal{X}_s^{\bullet \mathrm{Tar}}, \mathcal{U}_s^{\mathrm{Tar}}; \mathbf{u}_i, u_i, x_i) = V_{dyn}(x_i; \mathbf{u}_i, u_i, x_i) + V_f(\mathcal{X}_s^{\bullet \mathrm{Tar}}, \mathcal{U}_s^{\mathrm{Tar}}; u_i, x_i) = V_f(u_i, x_i)$. Consider now the feasible candidate, $(\mathbf{u}(\lambda), u_a(\lambda), x_a(\lambda))$, where $\mathbf{u}(\lambda) = CO_N^{\bullet\dagger}(x_s - x_i)\lambda + \mathbf{u}_i$, $u_a(\lambda) = (1-\lambda)u_i + \lambda u_s = u_i + \lambda(u_s - u_i)$, $x_a(\lambda) = (1-\lambda)x_i + \lambda x_s = x_i + \lambda(x_s - x_i)$, and $CO_j^\bullet = [A^{\bullet j-1}B^\bullet \;\; A^{\bullet j-2}B^\bullet \;\; \cdots \;\; A^\bullet B^\bullet \;\; B^\bullet]$ is the extended controllability matrix, † denotes the pseudo-inverse and $\lambda \in (0, 1]$. The feasible input and state sequences, which steer the initial state x_i to $x_a(\lambda)$, are given by $\mathbf{u}(\lambda) = CO_N^{\bullet\dagger}(x_s - x_i)\lambda + \mathbf{u}_i = \{u_\lambda(0), \cdots, u_\lambda(N-1)\}$, $\mathbf{x}(\lambda) = A_{aug}^{\bullet N}x_i + B_{aug}^{\bullet N}\mathbf{u}(\lambda) = A_{aug}^{\bullet N}x_i + B_{aug}^{\bullet N}[CO_N^{\bullet\dagger}(x_s - x_i)\lambda + \mathbf{u}_i] = A_{aug}^{\bullet N}x_i + B_{aug}^{\bullet N}CO_N^{\bullet\dagger}(x_s - x_i)\lambda + B_{aug}^{\bullet N}\mathbf{u}_i = B_{aug}^{\bullet N}CO_N^{\bullet\dagger}(x_s - x_i)\lambda + A_{aug}^{\bullet N}x_i + B_{aug}^{\bullet N}\mathbf{u}_i = B_{aug}^{\bullet N}CO_N^{\bullet\dagger}(x_s - x_i)\lambda + \mathbf{x}_i = \{x_\lambda(0), \cdots, x_\lambda(N-1), x_\lambda(N)\}$, where $x_\lambda(0) = x_a^0$, $x_\lambda(N) = x_a(\lambda)$ and $B_{aug}^{\bullet j} = \begin{bmatrix} 0 & \cdots & 0 \\ B^\bullet & \cdots & 0 \\ \vdots & \ddots & \vdots \\ A^{\bullet j-1}B^\bullet & \cdots & B^\bullet \end{bmatrix}$, $A_{aug}^{\bullet j} = \begin{bmatrix} I_n \\ A^\bullet \\ \vdots \\ A^{\bullet j} \end{bmatrix}$.

Given that $x_a(\lambda)$ is a convex combination of x_s and x_i, which are equilibria, and $\mathcal{X}_s^\bullet$ is convex, $x_a(\lambda) \in \mathcal{X}_s^\bullet$ is also an equilibrium. Now, consider the cost function of $P_{MPC}(x_i)$ corresponding to the feasible solution $\mathbf{u}(\lambda), u_a(\lambda), x_a(\lambda)$: $V_N(\lambda) \triangleq V_N(x_i, \mathcal{X}_s^{\bullet \mathrm{Tar}}, \mathcal{U}_s^{\mathrm{Tar}}; \mathbf{u}(\lambda), u_a(\lambda), x_a(\lambda)) = \sum_{j=0}^{N-1}\left(\|x_\lambda(j) - x_a(\lambda)\|_Q^2 + \|u_\lambda(j) - u_a(\lambda)\|_R^2\right) + V_f(u_a(\lambda), x_a(\lambda))$. Given the form of $\mathbf{u}(\lambda)$, $\mathbf{x}(\lambda)$, $u_a(\lambda)$ and $x_a(\lambda)$, the cost can be expressed as $V_N(\lambda) = \sum_{j=0}^{N-1}(\|(M_j^x + x_i - x_s)\lambda\|_Q^2 + \|(M_j^u + u_i - u_s)\lambda\|_R^2) + V_f(u_a(\lambda), x_a(\lambda)) = \sum_{j=0}^{N-1}\lambda^2(\|M_j^x + x_i - x_s\|_Q^2 + \|M_j^u + u_i - u_s\|_R^2) + V_f(u_a(\lambda), x_a(\lambda))$, where matrices M_j^x and M_j^u depend on the difference $(x_s - x_i) \neq 0$, and are given by the row blocks of $B_{aug}^{\bullet N}CO_N^{\bullet\dagger}(x_s - x_i)$ and $CO_N^{\bullet\dagger}(x_s - x_i)$, respectively. Now, it is shown that $V_N(\lambda)$ is smaller than V_N^0 for some small value of λ. Lets take the derivative of the former w.r.t. λ: $\frac{\partial V_N(\lambda)}{\partial \lambda} = \sum_{j=0}^{N-1}2\lambda\left(\|M_j^x + x_i - x_s\|_Q^2 + \|M_j^u + u_i - u_s\|_R^2\right) + \frac{\partial V_f(u_a(\lambda), x_a(\lambda))}{\partial \lambda}$; and evaluate it for $\lambda = 0$: $\left.\frac{\partial V_N(\lambda)}{\partial \lambda}\right|_{\lambda=0} = \left.\frac{\partial V_f(u_a(\lambda), x_a(\lambda))}{\partial \lambda}\right|_{\lambda=0}$. By convexity, it is $V_f(u_a(\lambda), x_a(\lambda)) \leq (1-\lambda)V_f(u_i, x_i) + \lambda V_f(u_s, x_s)$, and by optimality of $(u_s, x_s) \neq (u_i, x_i)$, it is $V_f(u_i, x_i) > V_f(u_s, x_s)$. Then, $V_f(u_a(\lambda), x_a(\lambda)) \leq (1-\lambda)V_f(u_i, x_i) + \lambda V_f(u_i, x_i) = V_f(u_i, x_i)$, for any $\lambda \in [0, 1]$. This means that $\frac{\partial V_N(\lambda)}{\partial \lambda} = \frac{\partial V_f(u_a(\lambda), x_a(\lambda))}{\partial \lambda} < 0$, for $\lambda = 0$, and so $\bar{\lambda} \in (0, 1]$ exists such that $V_N(\bar{\lambda}) < V_N^0$ or, $V_N(x_i, \mathcal{X}_s^{\bullet \mathrm{Tar}}, \mathcal{U}_s^{\mathrm{Tar}}; \mathbf{u}(\bar{\lambda}), u_a(\bar{\lambda}), x_a(\bar{\lambda})) < V_N(x_i, \mathcal{X}_s^{\bullet \mathrm{Tar}}, \mathcal{U}_s^{\mathrm{Tar}}; \mathbf{u}_i, u_i, x_i)$. Therefore, $(\mathbf{u}^0, u_a^0, x_a^0) = (\mathbf{u}_i, u_i, x_i)$, which is the only solution for which $(u_a^0, x_a^0) = (u_i, x_i)$ is not the optimal one, contradicting the initial assumption. Therefore, $(u_a^0, x_a^0) \neq (u_i, x_i)$. ∎

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