Periodic motions in forced problems of Kepler type

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Abstract

A Newtonian equation in the plane is considered. There is a central force (attractive or repulsive) and an external force $\lambda h(t)$, periodic in time. The periodic second primitive of $h(t)$ defines a planar curve and the number of periodic solutions of the differential equation is linked to the number of loops of this curve, at least when the parameter λ is large.
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1 Introduction and main results

Consider the second order equation in the plane

$$\ddot{z} \pm \frac{z}{|z|^{q+1}} = \lambda h(t), \quad z \in \mathbb{C} \setminus \{0\} \tag{1}$$

where $q \geq 2$, $\lambda \geq 0$ is a parameter and $h : \mathbb{R} \to \mathbb{C}$ is a continuous and 2π-periodic function satisfying

$$\int_0^{2\pi} h(t)dt = 0.$$

This equation models the motion of a particle under the action of a central force $F(z) = \mp\frac{z}{|z|^{q+1}}$ and an external force $\lambda h(t)$. The force F can be attractive or repulsive depending on the sign $+$ or $-$ in the equation (1). For $q = 2$ the vector field F becomes the classical gravitational or Coulomb force. For general information on this type of problems we refer to [1].

For the repulsive case it is known that (1) has no 2π-periodic solutions when λ is small enough (see [8] and [2]). In this paper we will discuss the existence of 2π-periodic solutions when λ is large. Before stating the main result we recall the notion of index as it is usually employed in Complex Analysis (see [5]). Given a continuous and 2π-periodic function $\gamma : \mathbb{R} \to \mathbb{C}$ and a point z lying in $\mathbb{C} \setminus \gamma(\mathbb{R})$, the index of z with respect to the circuit γ is an integer denoted by $j(z, \gamma)$. When γ is smooth, say C^1, this index can be expressed as an integral,

$$j(z,\gamma) = \frac{1}{2\pi\imath}\int_\gamma \frac{d\xi}{z-\xi} = \frac{1}{2\pi\imath}\int_0^{2\pi} \frac{\dot\gamma(t)}{z-\gamma(t)}dt.$$

It is well known that $z \mapsto j(z,\gamma)$ is constant on each connected component Ω of $\mathbb{C}\setminus\gamma(\mathbb{R})$. From now we write $j(\Omega,\gamma)$ for this index. Let $\phi(t)$ be a 2π-periodic solution of (1), the index $j(0,\phi)$ is well defined and can be interpreted as the winding number of the solution ϕ around the singularity $z=0$.

Theorem 1.1. *Let $H(t)$ be a 2π-periodic solution of*

$$\ddot H(t) = -h(t)$$

and let $\Omega_1,\ldots,\Omega_r$ be bounded components of $\mathbb{C}\setminus H(\mathbb{R})$. Then there exists $\lambda_* > 0$ such that the equation (1) has at least r different solutions $\phi_1(t),\ldots,\phi_r(t)$ of period 2π if $\lambda \geq \lambda_*$. Moreover,

$$j(0,\phi_k) = j(\Omega_k, H), \quad k = 1,\ldots,r.$$

Next we discuss the applicability of the theorem in three simple cases.

Example 1.2. $h(t) \equiv 0$.

We also have $H(t)\equiv 0$ and so $\mathbb{C}\setminus H(\mathbb{R}) = \mathbb{C}\setminus\{0\}$. This set has no bounded components and so the theorem is not applicable. This is reasonable since the equation $\ddot z - \frac{z}{|z|^{q+1}} = 0$ has no periodic or even bounded solutions. This is easily checked since all solutions satisfy

$$\frac{1}{2}\frac{d^2}{dt^2}(|z|^2) = |\dot z|^2 + \frac{1}{|z|^{q-1}} > 0\ .$$

On the contrary, in the attractive case the equation (1) has many periodic solutions for $h\equiv 0$. Notice that $\phi(t) = e^{\imath(t+c)}$ is a 2π periodic solution for any $c\in\mathbb{R}$.

Example 1.3. $h(t) = e^{\imath t}$.

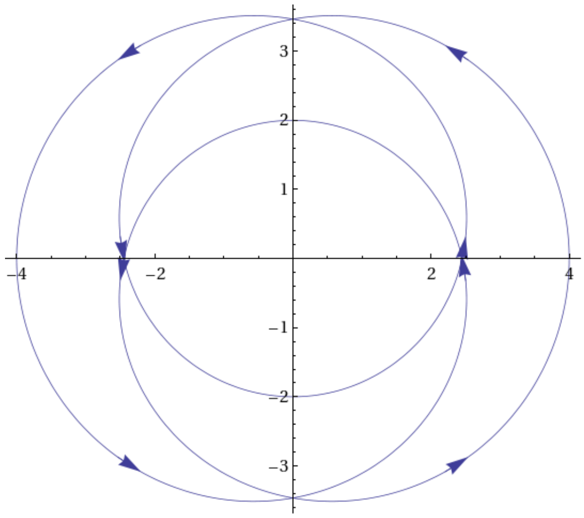
The second primitive of $-h$ is $H(t) = e^{\imath t}$ and $\mathbb{C}\setminus H(\mathbb{R})$ has one bounded component, the open disk $\{|z|<1\}$. The theorem asserts the existence of a 2π-periodic solution $\phi_1(t)$ with $j(0,\phi_1) = 1$ for λ large enough. Indeed this result can be obtained using very elementary techniques. The change of variables $z = e^{\imath t}w$ transforms (1) into

$$\ddot w + 2\imath\dot w - w \pm \frac{w}{|w|^{q+1}} = \lambda\ .$$

This equation has, for large λ, two equilibria. These equilibria become 2π-periodic solutions with index one in the z-plane. After lengthy computations it is possible to find the spectrum of the linearization of the w equation around the equilibria. This allows to apply Lyapunov center theorem in some cases to deduce the existence of sub-harmonic and quasi-periodic solutions in the z-plane (see [7] for more details on this technique).

Example 1.4. $h(t) = e^{\imath t} + 27e^{3\imath t}$.

The function $H(t) = e^{\imath t} + 3e^{3\imath t}$ is a parametrization of the epicycloid.



We observe that $\mathbb{C} \setminus H(\mathbb{R})$ has five bounded connected components with corresponding indices $3, 2, 2, 1, 1$. Hence we obtain five 2π-periodic solutions.

For some forcings $h(t)$ the set $\mathbb{C} \setminus H(\mathbb{R})$ has infinitely many bounded components. In such a case the previous result implies that the number of 2π-periodic solutions grows arbitrarily as $\lambda \to \infty$.

2 Brouwer degree and weakly nonlinear systems

This section is devoted to describe a well known result on the existence of periodic solutions of the system

$$\dot{x} = \varepsilon g(t, x; \varepsilon), \quad x \in U \subseteq \mathbb{R}^d \tag{2}$$

where U is an open and connected subset of $\mathbb{R}^d$, $\varepsilon \in [0, \varepsilon_*]$ is a small parameter and $g : \mathbb{R} \times U \times [0, \varepsilon_*] \to \mathbb{R}^d$ is continuous and 2π-periodic with respect to t. Later it will be shown that our original system (1) can be transformed into a system of the type (2). Following the ideas of the averaging method, we define the function

$$G(c) = \frac{1}{2\pi} \int_0^{2\pi} g(t, c; 0) dt, \quad c \in U \ .$$

Next we assume that G does not vanish on the boundary of a certain open set W, whose closure $\overline{W}$ is compact and contained in U. In such a case the degree of G on W is well defined.

Proposition 2.1. *In the above conditions assume that*

$$deg(G, W, 0) \neq 0.$$

Then the system (2) has at least one 2π-periodic solution $x_\varepsilon(t)$ lying in W for $\varepsilon > 0$ sufficiently small.

This result is essentially contained in Cronin's book [4]. We also refer to the more recent paper by Mawhin [6] containing more general results and some history.

Before applying this Proposition to (1) it will be convenient to have some information on the behaviour of $x_\varepsilon(t)$ as $\varepsilon \searrow 0$. The function g is bounded on the compact set $[0, 2\pi] \times \overline{W} \times [0, \varepsilon_*]$ and so

$$\|\dot{x_\varepsilon}\|_\infty = O(\varepsilon) \text{ as } \varepsilon \searrow 0.$$

Let $\varepsilon_n \searrow 0$ be a sequence such that $x_{\varepsilon_n}(0)$ converges to some point c in $\overline{W}$. Then $x_{\varepsilon_n}(t)$ converges uniformly to the constant c in $[0, 2\pi]$. Integrating the equation (2) over a period we obtain

$$\int_0^{2\pi} g(t, x_{\varepsilon_n}(t); \varepsilon_n)dt = 0$$

and letting $n \to \infty$ we deduce that $G(c) = 0$. In other words, as $\varepsilon \searrow 0$ the solutions $x_\varepsilon(t)$ given by the previous Proposition must accumulate on $G^{-1}(0)$, the set of zeros of G.

3 Reduction to a problem with small parameters

Let us start with the original equation (1) and consider the change of variables

$$z = \lambda(w - H(t))$$

where $w = w(t)$ is the new unknown. Then (1) is transformed into

$$\ddot{w} = \mp\varepsilon^2 \frac{w - H(t)}{|w - H(t)|^{q+1}} \tag{3}$$

with $\varepsilon^2 = \frac{1}{\lambda^{q+1}}$.

In principle this equation can have solutions passing through $H(\mathbb{R})$ but we will look for solutions lying in one of the components Ω_k of $\mathbb{C} \setminus H(\mathbb{R})$. On this domain the equation (3) is equivalent to a first order system of the type (2) with $x = (w, \xi) \in \mathbb{C}^2$, $U = \Omega_k \times \mathbb{C}$ and

$$\dot{w} = \varepsilon\xi, \quad \dot{\xi} = \mp\varepsilon \frac{w - H(t)}{|w - H(t)|^{q+1}}.$$

The averaging function is

$$G(c_1, c_2) = (c_2, \Phi(c_1)), \;\; c_1 \in \Omega_k, \; c_2 \in \mathbb{C}$$

and

$$\Phi(c_1) = \mp\frac{1}{2\pi}\int_0^{2\pi} \frac{c_1 - H(t)}{|c_1 - H(t)|^{q+1}} dt \; .$$

In the next section we will prove the following

Claim 3.1. *For each $k = 1, \dots, r$ there exists an open and bounded set Ω_k^*, whose closure is contained in Ω_k, and such that*

$$\Phi(c_1) \neq 0 \; \textit{if} \; c_1 \in \partial\Omega_k^*, \quad deg(\Phi, \Omega_k^*, 0) = 1.$$

Assuming for the moment that this claim holds, we notice that G does not vanish on the boundary of $W = \Omega_k^* \times B$ where B is the unit disk $|c_2| < 1$. Moreover G can be expressed as

$$G = L \circ (\Phi \times id)$$

where $L : \mathbb{C}^2 \to \mathbb{C}^2$ is the linear map $(c_1, c_2) \mapsto (c_2, c_1)$ and id is the identity in $\mathbb{C}$. The general properties of degree imply that

$$\begin{aligned} deg(G, W, (0,0)) &= sign(det L) \cdot deg(\Phi \times id, \Omega_k^* \times B, (0,0)) \\ &= deg(\Phi, \Omega_k^*, 0) = 1. \end{aligned}$$

In consequence Proposition 2.1 is applicable and we have proved the first part of the theorem 1.1. Namely, the existence of 2π-periodic solutions $\phi_1(t), \dots, \phi_r(t)$ for large λ (or small ε).

Notice that $\phi_k(t) = \lambda(\psi_k(t) - H(t))$, where ψ_k is a 2π-periodic solution of (3) lying in Ω_k^*. For convenience we make explicit the dependence of ϕ_k with respect to ε and write $\phi_k(t) = \phi_k(t, \varepsilon)$.

To prove the identity

$$j(0, \phi_k(., \varepsilon)) = j(\Omega_k, H)$$

when ε is small enough, we proceed by contradiction. Let us assume that for some sequence $\varepsilon_n \searrow 0$, $j(0, \phi_k(., \varepsilon_n)) \neq j(\Omega_k, H)$. After extracting a subsequence of ε_n we can assume that $\psi_k(t, \varepsilon_n) \to z$, $\dot{\psi}_k(t, \varepsilon_n) \to 0$, uniformly in t, where z is some point in $\Omega_k^* \subset \Omega_k$ with $\Phi(z) = 0$. This is a consequence of the discussion after Proposition 2.1. Computing indexes via integrals and passing to the limit

$$\begin{aligned} j(0, \phi_k(\cdot, \varepsilon_n)) = -\frac{1}{2\pi\imath}\int_0^{2\pi} \frac{\dot{\psi}(t, \varepsilon_n) - \dot{H}(t)}{\psi(t, \varepsilon_n) - H(t)} dt \to \\ \to \frac{1}{2\pi\imath}\int_0^{2\pi} \frac{\dot{H}(t)}{z - H(t)} dt = j(z, H) = j(\Omega_k, H). \end{aligned}$$

Since we are dealing with integer numbers, $j(0, \phi_k(., \varepsilon_n))$ and $j(\Omega_k, H)$ must coincide for large n. This is a contradiction with the definition of ε_n. By now the proof of the main theorem is complete excepting for the above claim.

4 Degree of gradient vector fields

The purpose of this section is to prove the claim concerning the function Φ. To do this we first prove a result valid for general gradient maps in the plane.

Proposition 4.1. *Let Ω be a bounded, open and simply connected subset of $\mathbb{C}$ and let $V : \Omega \to \mathbb{R}$ be a C^1 function (in the real sense). In addition assume that*

$$V(z) \to +\infty \text{ as } z \to \partial\Omega \ . \tag{4}$$

Then there exists an open set Ω^*, whose closure is contained in Ω, such that

1. $\nabla V(z) \neq 0$ *for each* $z \in \partial\Omega^*$
2. $deg(\nabla V, \Omega^*, 0) = 1$.

Remark. The condition (4) says that V blows up in the boundary of Ω. More precisely, given $r > 0$ there exist $\delta > 0$ such that if $z \in \Omega$ with $dist(x, \partial\Omega) < \delta$ then $V(z) > r$.

Notice also that, by the properties of degree in two dimensions,

$$deg(\nabla V, \Omega^*, 0) = deg(-\nabla V, \Omega^*, 0) \ .$$

Proof. By Sard lemma we know that V has many regular values in the interval $]\min_\Omega V, +\infty[$. Let us pick one of these values, say α. Then the set $M = V^{-1}(\alpha)$ is a one-dimensional manifold of class C^1. Since V blows up at the boundary, M is compact and so it has to be composed by a finite number of disjoint Jordan curves. Let γ be one of these Jordan curves and let us define Ω^* as the bounded component of $\mathbb{C} \setminus \gamma$. Notice that the closure of Ω^* is contained in Ω because Ω is simply connected.

We know that

$$V(z) = \alpha \text{ and } \nabla V(z) \neq 0 \text{ if } z \in \gamma$$

and so $\nabla V(z)$ must be colinear to $n(z)$, the outward unitary normal vector to the curve γ. This implies that $\langle \nabla V(z), n(z) \rangle$ does not vanish on the curve γ. Assume for instance that

$$\langle \nabla V(z), n(z) \rangle > 0 \text{ if } z \in \gamma,$$

the other case being similar. Then it is easy to prove that $\nabla V(z)$ is linearly homotopic to any continuous vector field which is tangent to γ on every point of this curve. The proof is complete because it is well known that these tangent vector fields have degree one. See for instance Th. 4.3 (Ch. 15) of [3]. $\square$

We are ready to prove the claim concerning the function

$$\Phi : \Omega_k \to \mathbb{C},\ \Phi(z) = \pm \frac{1}{2\pi} \int_0^{2\pi} \frac{z - H(t)}{|z - H(t)|^{q+1}} dt$$

where Ω_k is a bounded component of $\mathbb{C} \setminus H(\mathbb{R})$.

To do this we will apply Proposition 4.1 and the crucial observation is that Φ is a gradient vector field. Namely

$$\Phi = \mp \nabla V \text{ on } \Omega_k$$

where V is the real analytic function on Ω_k,

$$V(z) = \frac{1}{2\pi(q-1)} \int_0^{2\pi} \frac{dt}{|z - H(t)|^{q-1}}.$$

Using very standard arguments of planar topology one can prove that Ω_k is simply connected and so we only have to check that (4) holds. We finish this paper with a proof of this fact.

Lemma 4.1. *In the above setting,*

$$V(z) \to +\infty \textit{ as } z \to \partial\Omega_k.$$

Proof. By a contradiction argument assume the existence of a sequence $\{z_n\}$ in Ω_k with $dist(z_n, \partial\Omega_k) \to 0$ and such that $V(z_n)$ remains bounded. Since Ω_k is bounded it is possible to extract a subsequence (again z_n) converging to some point $p \in \partial\Omega_k$. Let us define the set $A = \{t \in [0, 2\pi] : H(t) = p\}$ and the function

$$\mu(t) = \begin{cases} \frac{1}{|H(t)-p|^{q-1}}, & t \in [0, 2\pi] \setminus A \\ +\infty, & t \in A. \end{cases} \tag{5}$$

Then the sequence of functions $\frac{1}{|H(t)-z_n|^{q-1}}$ converges to μ pointwise. By Fatou's Lemma

$$\int_0^{2\pi} \mu(t)dt \le \liminf_{n\to\infty} \int_0^{2\pi} \frac{dt}{|H(t) - z_n|^{q-1}} = 2\pi(q-1) \liminf_{n\to\infty} V(z_n) < \infty.$$

Hence $\mu(t)$ is integrable in the Lebesgue sense. In particular the set A has measure zero. Since the boundary of Ω_k is contained in $H(\mathbb{R})$, the set A is non-empty and we can fix $\tau \in [0, 2\pi]$ with $H(\tau) = p$. The previous discussion shows that

$$\mu(t) = \frac{1}{|H(t) - p|^{q-1}}, \text{ a.e. } t \in [0, 2\pi].$$

Let $L > 0$ be a Lipschitz constant for H, then

$$\mu(t) \ge \frac{1}{L^{q-1}|t - \tau|^{q-1}} \text{ a.e. } t \in [0, 2\pi].$$

At this point the condition $q \geq 2$ plays a role,

$$\int_0^{2\pi} \mu(t)dt \geq \frac{1}{L^{q-1}} \int_0^{2\pi} \frac{dt}{|t-\tau|^{q-1}} = +\infty$$

and this is a contradiction with the integrability of μ. $\square$

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