Biclique Graphs and Biclique Matrices

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Abstract: A biclique of a graph G is a maximal induced complete bipartite subgraph of G. Given a graph G, the biclique matrix of G is a $\{0, 1, -1\}$ matrix having one row for each biclique and one column for each vertex of G, and such that a pair of 1, -1 entries in a same row corresponds exactly to adjacent vertices in the corresponding biclique. We describe a characterization of biclique matrices, in similar terms as those employed in Gilmore's characterization of clique matrices. On the other hand, the

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biclique graph of a graph is the intersection graph of the bicliques of G. Using the concept of biclique matrices, we describe a Krausz-type characterization of biclique graphs. Finally, we show that every induced P_3 of a biclique graph must be included in a diamond or in a 3-fan and we also characterize biclique graphs of bipartite graphs. J Graph Theory 63: 1–16, 2010

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1. INTRODUCTION

Bicliques of a graph have been considered in a variety of contexts. For example, in optimization problems as that of finding a maximum biclique of a graph, e.g. [21, 26, 27], in covering problems, e.g. [2, 25], among others. Also bicliques have been studied from the structural point of view, as in [13, 20, 22, 23]. In connection with the Helly Property, bicliques have been considered in [14, 15]. Algorithms for generating all the bicliques of a graph have been described in articles such as [1, 6].

On the other hand, intersection graphs of certain special subgraphs of a general graph have been studied extensively. Among others, we can mention the intersection graphs of the edges of a graph, intersection graphs of intervals of a line and of the maximal cliques of a graph, leading to the classes of line graphs, interval graphs and clique graphs, respectively. For example, see [4, 5, 7, 8, 10, 17, 18].

In this article, we study biclique graphs, the intersection graphs of the (maximal) bicliques of a graph. We describe a characterization for this class, using some similar ideas to those employed in the characterization of clique graphs, by Roberts and Spencer [24]. Afterwards, we show that in any biclique graph, every induced P_3 is contained in a diamond (K_4 minus an edge) or in a 3-fan (P_4 with an additional universal vertex). As a consequence, we describe a characterization for the graphs whose biclique graphs are diamond free. We also examine biclique graphs of bipartite graphs and characterize them.

The method of the proposed characterization of biclique graphs employs the concept of a biclique matrix of a graph. This has lead us to define biclique matrices and also characterize them. The latter is explicitly used in the study of biclique graphs.

We believe that the concept of biclique matrices might be of interest in general, independent of its use in biclique graphs. We describe a characterization for biclique matrices, in similar terms as those used for clique matrices. The latter have been characterized by Gilmore in 1960, and have been employed in several different contexts. For example, in the characterizations of interval graphs [12], Helly circular-arc graphs [9], self-clique graphs [3, 16], among others.

In Section 2, we present some definitions relevant to our purposes. The characterizations of biclique matrices and biclique graphs are described in Sections 3 and 4, respectively. In Section 5, we examine some special classes of biclique graphs, and prove a simple and useful necessary condition for a graph to be a biclique graph.

Section 6 considers biclique matrices of bipartite graphs and Section 7 describes biclique graphs of bipartite graphs. The last section contains some final remarks.

2. PRELIMINARIES

Denote by $\mathcal{H}$ a hypergraph, with vertex set $V(\mathcal{H})$ and hyperedge set $E(\mathcal{H})$. Write $V(\mathcal{H})=\{v_1,\ldots,v_n\}$ and $E(\mathcal{H})=\{E_1,\ldots,E_m\}$, $E_i \subseteq V(\mathcal{H})$. When $|E_i|=2$, for all $1\le i\le m$, then say that the hypergraph is a *graph* and the hyperedges are *edges*. Usually, we denote a graph by G. For a graph G, write $e_k=v_iv_j$, with the meaning of $E_k=\{v_i,v_j\}$ for some k, and say that vertices v_i, v_j are *adjacent*. The 2-*section* of a hypergraph $\mathcal{H}$ is a graph $G_2(\mathcal{H})$, where $V(G_2(\mathcal{H}))=V(\mathcal{H})$ and such that there is an edge $v_iv_j\in E(G_2(\mathcal{H}))$ precisely when there exists some hyperedge $E_k\supseteq\{v_i,v_j\}$, for all $1\le i\ne j\le n$. Say that $\mathcal{H}$ is *conformal* when each clique of $G_2(\mathcal{H})$ is contained in some hyperedge of $\mathcal{H}$. Say that $\mathcal{H}$ is *Helly* when every subfamily of intersecting hyperedges contains a common vertex. The neighborhood of a vertex v, $N(v)$, is the set of adjacent vertices to v. Given a graph G, when its family of neighborhoods is Helly, we say that G is *neighborhood-Helly*.

Finally, the *dual* of a hypergraph $\mathcal{H}$ is the hypergraph $\mathcal{H}^*$, where $V(\mathcal{H}^*)=E(\mathcal{H})$, $E(\mathcal{H}^*)=V(\mathcal{H})$, and such that for $v_i^*\in V(\mathcal{H}^*)$ and $E_j^*\in E(\mathcal{H}^*)$, $v_i^*\in E_j^*$ precisely when $v_j\in E_i\in E(\mathcal{H})$.

For a graph G, say that $V'\subseteq V(G)$ is a *complete set* when v_i, v_j are adjacent, for all $v_i,v_j\in V'$. A *complete bipartite set* is a subset $B\subseteq V(G)$, which admits a bipartition $V_1\cup V_2=B$, where $v_i,v_j\in B$ are adjacent exactly when v_i,v_j belong to distinct parts of the bipartition. We restrict to *proper* bipartitions, that is, $V_1,V_2\ne\emptyset$. A *clique* is a maximal complete set, while a *biclique* is a maximal complete bipartite set. Denote by $KB(G)$ the biclique graph of G, that is, the intersection graph of the bicliques of G. Figure 1 illustrates a graph G and its biclique graph $KB(G)$, while Figure 2 depicts some graphs which are not biclique graphs.

If G has c cliques $\{C_1,\ldots,C_c\}$ then the *clique matrix* of G is the $c\times n$ $\{0,1\}$ matrix A, defined as $a_{ki}=1$ if and only if $v_i\in C_k$. Finally, when G has d bicliques $B_1,\ldots,B_d\subseteq V(G)$, the *biclique matrix* of G is the $d\times n$ $\{0,1,-1\}$ matrix A, where $a_{ki}=-a_{kj}\ne 0$, precisely when $v_i,v_j\in B_k$ and v_i,v_j are adjacent, for all $1\le k\le d$ and $1\le i\ne j\le n$. The definition of biclique matrix suggests the concept of "row-similarity" between matrices, to be given later.

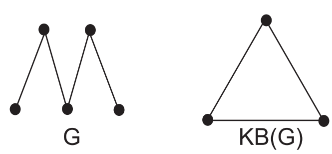

FIGURE 1. A graph and its biclique graph.

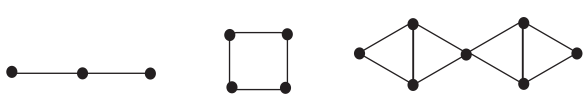

FIGURE 2. Graphs which are not biclique graphs.

A *coloring* c of a hypergraph $\mathcal{H}$ is an assignment of a color *white* or *black* to each of the occurrences of the vertices in the hyperedges of $\mathcal{H}$. That is, if vertex v belongs to hyperedges $E_1, \ldots, E_k$, then v has k independent colors *white* or *black*, one in each E_i. In this case, $\mathcal{H}$ is a *colored hypergraph*. Say that c is *compatible* when for any pair of vertices $v_i, v_j \in V(\mathcal{H})$ and any pair of hyperedges $E_k, E_l \supseteq \{v_i, v_j\}$, either v_i, v_j have the same color in E_k, E_l, or have different colors in E_k, E_l. If $\mathcal{H}$ is a hypergraph with a coloring c, then its dual hypergraph $\mathcal{H}^*$ has a coloring c^* defined as follows. Let $v_i \in V(\mathcal{H})$ and $E_j \in E(\mathcal{H})$, where $v_i \in E_j$. Denote by v_j^* and E_i^* the vertex and hyperedge of $\mathcal{H}^*$, corresponding to E_j and v_i, respectively. Then the color of v_j^* in E_i^* is precisely the same as the color of v_i in E_j.

Let $\mathcal{H}$ be a colored hypergraph and $E_i \in E(\mathcal{H})$. Denote by E_i^w the subset of E_i formed exactly by the white vertices of it. Similarly, define $E_i^b \subseteq E_i$, for the black vertices of E_i. Say that E_i is a *dominated* hyperedge when there exists a hyperedge $E_j \neq E_i$, such that $E_j^w \supseteq E_i^w$ and $E_j^b \supseteq E_i^b$, or $E_j^w \supseteq E_i^b$ and $E_j^b \supseteq E_i^w$. Finally, say that $E(\mathcal{H})$ *bicovers* $V(\mathcal{H})$ when for each vertex $v_i \in V(\mathcal{H})$ there are hyperedeges $E_j, E_k \supseteq \{v_i\}$, such that v_i is colored *white* in E_j and *black* in E_k.

Given a hypergraph $\mathcal{H}$ and a coloring c of it, a family of hyperedges $\mathcal{E} \subseteq E(\mathcal{H})$ is *monochromatically intersecting* if, for any two hyperedges $E_i, E_j \in \mathcal{E}$, either $E_i \cap E_j = \emptyset$ or each $v \in E_i \cap E_j$ has the same color in both E_i and E_j. Consider a bipartition $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ of $\mathcal{E}$. Say that $\mathcal{E}$ is *bipartite-intersecting* if $\mathcal{E}_1, \mathcal{E}_2$ are both monochromatically intersecting, and for every pair of hyperedges $E_1 \in \mathcal{E}_1$, $E_2 \in \mathcal{E}_2$, there exists a vertex $v \in E_1 \cap E_2$, such that v has different colors in E_1 and E_2. Finally, say that $\mathcal{H}$ is *bipartite Helly* if c is compatible and every bipartite-intersecting subfamily $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \subseteq E(\mathcal{H})$ contains a common vertex.

In Figure 3, there is an example of a colored hypergraph $\mathcal{H}$, using colors *white* and *black*, where v_w and v_b mean that vertex v is colored *white* and *black*, respectively. Observe that $E(\mathcal{H})$ bicovers $V(\mathcal{H})$. However, examining the coloring of the hyperedges E_1 and E_2, we conclude that the coloring is not compatible. On the other hand, the coloring restricted to the partial hypergraph formed by the hyperedges E_1 and E_3 is compatible. The subfamily $\{E_1, E_6\}$ is not monochromatically intersecting. On the other hand, $\{E_3\} \cup \{E_4, E_6\}$ and $\{E_5\} \cup \{E_1, E_4\}$ are examples of bipartite-intersecting subfamilies of $E(\mathcal{H})$. The latter contains a common element, while the former does not, meaning that $\mathcal{H}$ is not bipartite-Helly.

The coloring of a hypergraph also defines a coloring of the edges of the 2-section $G_2(\mathcal{H})$ of $\mathcal{H}$, using the colors *white* or *black*, as follows. Each $v_i v_j \in E(G_2(\mathcal{H}))$ is *black* when there exists some hyperedge $E_k \supseteq \{v_i, v_j\}$, where v_i and v_j have different colors in E_k; otherwise $v_i v_j$ is *white*. Define the *black section* of $\mathcal{H}$, as the subgraph $G_b(\mathcal{H})$ of $G_2(\mathcal{H})$, containing exactly the black edges of $G_2(\mathcal{H})$. Say that $\mathcal{H}$ is *bipartite-conformal*,

- V(H) = {v,t,s,r}, colors = {w,b}
- E(H) = $\{E_1, E_2, E_3, E_4, E_5, E_6\}$
- Hyperedges:$E_1 = \{v_w, t_b\}$, $E_2 = \{v_b, t_b\}$, $E_3 = \{s_b, t_b\}$, $E_4 = \{v_w, r_b, s_w\}$, $E_5 = \{v_b, r_w, t_w\}$, $E_6 = \{v_w, t_w\}$

FIGURE 3. Example of a colored hypergraph.

relative to c, when each biclique B of $G_b(\mathcal{H})$ is contained in some hyperedge of $\mathcal{H}$. That is, there is a hyperedge E_k such that v_iv_j is an edge of B precisely when v_i, v_j have different colors in E_k.

3. BICLIQUE MATRICES

In this section, we characterize biclique matrices of a graph.

Given a $\{0,1,-1\}$ matrix $A=(a_{ij})$, the *associated hypergraph* $\mathcal{H}_A$ of A is the hypergraph having one vertex v_i for each column i and one hyperedge E_k for each row k of A, such that $v_i \in E_k$ precisely when $a_{ki} \neq 0$. The *canonical coloring* of $\mathcal{H}_A$ is the coloring such that vertex $v_i \in V(\mathcal{H})$ is *white* in E_k when $a_{ki}=1$, while v_i is *black* in E_k when $a_{ki}=-1$.

Similarly, given a colored hypergraph $\mathcal{H}$, the *associated matrix* $A_{\mathcal{H}}=(a_{ij})$ of $\mathcal{H}$ is a $\{0,1,-1\}$ matrix defined as follows: $a_{ij}=0$, whenever $v_j \notin E_i$; otherwise, $a_{ij}=1$ when v_j is a *white* vertex of E_i, while $a_{ij}=-1$ when v_j is *black* in E_i.

For a family $\mathcal{C}$ of subsets C_i of some set, the associated hypergraph $\mathcal{H}_{\mathcal{C}}$ of $\mathcal{C}$ is defined as $V(\mathcal{H}_{\mathcal{C}})=\bigcup_{C_i\in\mathcal{C}} C_i$ and $E(\mathcal{H}_{\mathcal{C}})=\mathcal{C}$.

Let A be a $\{0,1,-1\}$ matrix. Denote by A_i the $\{0,1,-1\}$ vector consisting of row i of A. Call the vectors A_i and $-A_i$ *symmetric*. Say that row k is *dominated* by row l when, for all i, (i) $a_{ki}=1$ implies $a_{li}=1$ and $a_{ki}=-1$ implies $a_{li}=-1$ or (ii) $a_{ki}=1$ implies $a_{li}=-1$ and $a_{ki}=-1$ implies $a_{li}=1$. Let A, A' be two $\{0,1,-1\}$ matrices. Say that A is *row-similar* to A' when there is a bijection between the sets of rows of A and A', such that corresponding rows either coincide or are symmetric. Finally, say that a $\{0,1,-1\}$ matrix A is *compatible* when no pair of rows and no pair of columns, both not necessarily consecutive, form the matrix M_1 or any matrix which is row-similar to M_1. See Figure 4.

Figure 5 illustrates an example of a $\{0,1,-1\}$ matrix with dominated rows. The last row of A_1 is dominated by the first row. The hypergraphs $\mathcal{H}_1$, $\mathcal{H}_2$ associated to the matrices A_1 and A_2, respectively, have as vertex sets $V(\mathcal{H}_1)=V(\mathcal{H}_2)=\{v_1,v_2,w_1,w_2,w_3,w_4\}$, and hyperedges $\mathcal{H}_1=\{E_1,E_2,E_3\}$, $\mathcal{H}_2=\{E_1,E_2,E_3'\}$, where $E_1=\{v_1,w_2,w_3,w_4\}$, $E_2=\{v_2,w_1,w_2,w_3\}$, $E_3=\{v_1,w_2,w_4\}$ and $E_3'=\{v_1,v_2,w_2,w_3\}$. In Figure 6, we show the 2-section G_2 of $\mathcal{H}_2$ and the black section G_b of the hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$. Although A_1 and A_2 and their corresponding 2-sections are distinct, their black sections coincide. Observe that $\mathcal{H}_1$ is not bipartite-conformal, and that A_2 is a biclique matrix of G_b.

Notice that whenever A, A' are two row-similar matrices, then the 2-sections G_2, G_2' of their corresponding associated hypergraphs are isomorphic. Moreover, if $e\in E(G_2)$ and $e'\in E(G_2')$ are two corresponding edges in the isomorphism $G_2\cong G_2'$ then they have identical colors in the respective canonical colorings.

$$M_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

FIGURE 4. Forbidden submatrix.

$$A_1 = \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & w_1 & w_2 & w_3 & w_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 & -1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

$$A_2 = \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & w_1 & w_2 & w_3 & w_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 & -1 & 0 \\ 1 & 1 & 0 & -1 & -1 & 0 \end{pmatrix} \end{array}$$

FIGURE 5. $\{0, 1, -1\}$ matrices.

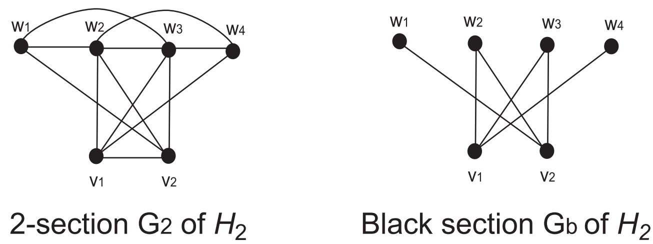

FIGURE 6. Graphs $G_2(\mathcal{H}_2)$ and $G_b(\mathcal{H}_2)$.

The characterization of clique matrices can be formulated in terms of the above concepts, applied to $\{0, 1\}$ matrices.

Theorem 3.1 (Gilmore [11])**.** *Let A be a $\{0, 1\}$ matrix and $\mathcal{H}$ its associated hypergraph. Then A is a clique matrix of some graph if and only if*

(i) *each row of A has at least one 1*,
(ii) *A has no dominated rows*, *and*
(iii) *$\mathcal{H}$ is conformal.*

The following theorem characterizes biclique matrices of graphs.

Theorem 3.2. *Let A be an $m \times n$ $\{0, 1, -1\}$ matrix*, *and $\mathcal{H}$ its associated hypergraph. Then A is a biclique matrix of some graph if and only if*

(i) *each row of A has at least one* 1 *and at least one* -1,
(ii) *A has no dominated rows*,
(iii) *A is compatible*, *and*
(iv) *$\mathcal{H}$ is bipartite-conformal*, *relative to its canonical coloring.*

Proof. Assume that A is a biclique matrix of some graph G. Let $V(G) = \{v_1, \ldots, v_n\}$, and denote its bicliques by $B_1, \ldots, B_m \subseteq V(G)$. We know that $a_{ki} = -a_{kj} \neq 0$, precisely when v_i, v_j are adjacent and belong to B_k. By definition, there is at least one edge $v_i v_j$ in biclique B_k. In this case, $a_{ki} = -a_{kj} \neq 0$, meaning that row k has at least one 1 and one -1. Then (i) holds. Next, observe that A is a biclique matrix of some graph if and only if any of the matrices row-similar to A is so. Consequently, row k cannot be dominated by any other row; otherwise, B_k would not be maximal. Hence (ii) holds.

For (iii), assume that A contains M_1 as a submatrix. Let k, l and i, j be the pairs of rows and columns of A, respectively, which contain M_1. Then row k implies that v_i, v_j are not adjacent, while row l implies that they are adjacent, which is a contradiction. The cases of matrices row-similar to M_1 are similar.

Next, examine (iv). Let B_k be a biclique of G, with bipartition $V_1 \cup V_2 = B_k$. Then row k of A has entries

$$a_{ki} = \begin{cases} 0 & \text{if } v_i \neq B_k \\ 1 & \text{if } v_i \in V_1 \\ -1 & \text{if } v_i \in V_2, \end{cases}$$

for all $1 \leq i \leq n$, where the choice of V_1, V_2 is arbitrary. By the construction of the associated hypergraph $\mathcal{H}$, the hyperedge $E_k \in E(\mathcal{H})$ contains all vertices v_i, such that $a_{ki} \neq 0$. Then $E_k \supseteq B_k$. Let G_2 be the 2-section of $\mathcal{H}$ and G_b its black section. We show that $G = G_b$. Clearly, $V(G_b) = V(G) = \{v_1, \ldots, v_n\}$. Let $v_i, v_j \in \{v_1, \ldots v_n\}$, $i \neq j$. First, suppose $v_i v_j \in E(G_b)$. Then $v_i v_j$ is a black edge of G_2, meaning that v_i, v_j are assigned different colors in some edge $E_l \in E(\mathcal{H})$. That is, $a_{li} = -a_{lj} \neq 0$. However, A is a biclique matrix of G. Then row l implies that v_i, v_j are adjacent also in G. Consequently, $E(G_b) \subseteq E(G)$. Finally, consider $v_i v_j \in E(G)$. Then $v_i v_j$ belong to some biclique B_r of G. That is, there is a row r of A, such that $a_{ri} = -a_{rj} \neq 0$. The latter implies that $v_i, v_j \in E_r \in E(\mathcal{H})$, meaning that $v_i v_j$ is a black edge of G_2, i.e. $v_i v_j \in E(G_b)$. Consequently $E(G) \subseteq E(G_b)$. That is, $G = G_b$. Then B_k is an arbitrary biclique of G_b. Since $E_k \supseteq B_k$, it follows that $\mathcal{H}$ is bipartite-conformal.

Conversely, assume that A satisfies (i)–(iv). We show that A is a biclique matrix. In fact, we show that A is a biclique matrix of the black section G_b of $\mathcal{H}$, relative to the canonical coloring.

To start, we show that every biclique B of G_b corresponds to a row of A. Let $V_1 \cup V_2 = B$ be the bipartition of B, $V_1, V_2 \neq \emptyset$. From (iv), we conclude that B is contained in some hyperedge E_k of $\mathcal{H}$. Let $v_i, v_j \in B$ and examine the possible alternatives. In the first alternative, suppose $v_i \in V_1$ and $v_j \in V_2$. Then $v_i v_j \in E(G_b)$. By definition, $v_i v_j$ is a black edge of G_2. Consequently, v_i, v_j have distinct colors in some hyperedge of $\mathcal{H}$. In addition, we know that v_i, v_j must have distinct colors in any hyperedge of $\mathcal{H}$ that contain both of these vertices. Otherwise A would contain as a submatrix, a matrix row-similar to M_1, contradicting (iii). Consequently, the row k of A, corresponding to E_k, is such that $a_{ki} = -a_{kj} \neq 0$. In the next alternative, let $v_i, v_j \in V_1$. Since $v_i, v_j \in E_k$, each of these vertices has a color in E_k. Because $v_i v_j$ is not a black edge of G_2, it follows that both vertices v_i, v_j must have identical colors in E_k. That is, $a_{ki} = a_{kj} \neq 0$. Finally, when $v_i \notin B_k$ it easily follows that $a_{ki} = 0$. The alternative $v_i, v_j \in V_2$ is similar. Consequently, B corresponds to E_k, hence to row k of A.

We now show that every row k of A corresponds to some biclique of G_b. Let $V_1 \subseteq V(\mathcal{H})$ be the set of vertices of $\mathcal{H}$ corresponding to the 1 entries of row k of A, and $V_2 \subset V(\mathcal{H})$, those corresponding to the -1 entries. From (i), it follows that $V_1, V_2 \neq \emptyset$. First, let $v_i \in V_1$ and $v_j \in V_2$. Then v_i, v_j are assigned distinct colors in the hyperedge $E_k \in E(\mathcal{H})$. Consequently, $v_i v_j$ is a black edge of G_2; hence, $v_i v_j \in E(G_b)$. Next, let

$v_i, v_j \in V_1$. Then v_i, v_j are both white in E_k. Again, we know that whenever v_i, v_j are both contained in some hyperedge $E_l \in E(\mathcal{H})$, then v_i, v_j have identical colors in E_l; otherwise, A would contain a forbidden submarix of (iii). Consequently, $v_i v_j$ is a white edge of G_2, meaning that $v_i v_j \notin E(G_b)$, the situation where $v_i, v_j \in V_2$ is similar. Consequently, $V_1 \cup V_2$ is a complete bipartite set of G_b included in a biclique B. Let l be the row corresponding to B. Because of (ii), row k is not dominated by any other row. Consequently, $l = k$ and $V_1 \cup V_2$ is indeed the biclique B of G_b, completing the proof. ■

The following property is a consequence of Theorem 3.2.

Corollary 3.1. *A matrix is a biclique matrix of some graph if and only if it is a biclique matrix of the black section of its associated hypergraph.*

4. CHARACTERIZATION OF BICLIQUE GRAPHS

In this section we give a characterization of biclique graphs.

The following observations are clear.

Observation 4.1. Let A be a $\{0, 1, -1\}$ matrix and $\mathcal{H}_A$ its associated hypergraph. Then row i is a dominated row of A if and only if $E_i \in E(\mathcal{H}_A)$ is a dominated hyperedege of $\mathcal{H}_A$.

Observation 4.2. Let A be a $\{0, 1, -1\}$ matrix and c the canonical coloring of its associated hypergraph $\mathcal{H}_A$. Then A is a compatible matrix if and only if c is a compatible coloring.

Observation 4.3. Let $\mathcal{H}$ be a hypergraph with a coloring c. Then c is compatible if and only if the coloring c^* of $\mathcal{H}^*$ is compatible.

Lemma 4.1. *Let $\mathcal{H}$ be a colored hypergraph relative to the coloring c and $\mathcal{H}^*$ its dual colored hypergraph. Then the coloring c of $\mathcal{H}$ is compatible and $\mathcal{H}$ is bipartite-conformal if and only if the family of hyperedges of $\mathcal{H}^*$ is bipartite-Helly.*

Proof. By observation 4.3, c is compatible if and only if c^* is also compatible. Then, it remains to prove that $\mathcal{H}$ is bipartite-conformal if and only every bipartite-intersecting subfamily of hyperedges of $\mathcal{H}^*$ has a common vertex.

Suppose $\mathcal{H}$ is bipartite-conformal and let $G_b(\mathcal{H})$ be its black section. Consider $\mathcal{E}_1 \cup \mathcal{E}_2$ a bipartite-intersecting family of hyperedges of $\mathcal{H}^*$, where $\mathcal{E}_1 = \{E^*_{i_1}, \ldots, E^*_{i_k}\}$ and $\mathcal{E}_2 = \{E^*_{i_{k+1}}, \ldots, E^*_{i_s}\}$.

Since $\mathcal{E}_1, \mathcal{E}_2$ are monochromatically intersecting families, both sets of vertices, $V_1 = \{v_{i_1}, \ldots, v_{i_k}\}$ and $V_2 = \{v_{i_{k+1}}, \ldots, v_{i_s}\}$, induce independent sets in $G_b(\mathcal{H})$. On the other hand, since for every i, j such that $E^*_i \in \mathcal{E}_1$ and $E^*_j \in \mathcal{E}_2$, E^*_i, E^*_j intersect in a vertex with different color, vertices $v_i \in V_1, v_j \in V_2$ are adjacent in G. It follows that V_1, V_2 induce a bipartite complete subgraph in G. Since $\mathcal{H}$ is bipartite-conformal, there is an hyperedge E_t which contains the vertices of $V_1 \cup V_2$. We conclude that E_t in $\mathcal{H}^*$ is a common element of the family $\mathcal{E}_1 \cup \mathcal{E}_2$.

Conversely, let B be a biclique of $G_b(\mathcal{H})$ and let $V_1 = \{v_{i_1}, \ldots, v_{i_s}\}$, $V_2 = \{v_{i_{s+1}}, \ldots, v_{i_t}\}$ be its bipartition. Consider $\mathcal{E}_1 = \{E^*_{i_1}, \ldots, E^*_{i_s}\}$, $\mathcal{E}_2 = \{E^*_{i_{s+1}}, \ldots, E^*_{i_t}\}$ hyperedges of $\mathcal{H}^*$. Since V_1, V_2 are independent sets, then $\mathcal{E}_1, \mathcal{E}_2$ are monochromatically intersecting families. Since every vertex of V_1 intersects every vertex of V_2, $\mathcal{E}_1 \cup \mathcal{E}_2$ is a bipartite-intersecting family in $\mathcal{H}^*$. Finally, as c^* is compatible and $\mathcal{H}^*$ is bipartite-Helly, by hypothesis there is a vertex E_t common to $\mathcal{E}_1, \mathcal{E}_2$. We conclude that the hyperedge E_t of $\mathcal{H}$ contains all the vertices of B and $\mathcal{H}$ is bipartite-conformal. ■

Observation 4.4. Let A be a $\{0, 1, -1\}$ compatible matrix. Let $\mathcal{H}$ be its associated colored hypergraph. Then $\mathcal{H}$ is bipartite-conformal if and only if the columns of A are bipartite-Helly.

The following is the main characterization.

Theorem 4.1. *Let G be a graph with no isolated vertices. Then G is a biclique graph if and only if G contains a family $\mathcal{C}$ of not necessarily distinct complete subgraphs covering the edges of G, whose associated hypergraph $\mathcal{H}_\mathcal{C}$ admits a coloring c satisfying*

(1) *$\mathcal{C}$ bicovers $V(G)$.*
(2) *$\mathcal{H}^*_\mathcal{C}$ has no dominated hyperedeges.*
(3) *c is a compatible coloring.*
(4) *$\mathcal{H}_\mathcal{C}$ is bipartite-Helly, relative to c.*

Proof. By hypothesis, $G = KB(H)$ for some graph H. Let A be a $\{0, 1, -1\}$ biclique matrix of H, $\mathcal{H}_A$ its associated hypergraph and c the canonical coloring of $\mathcal{H}_A$. Each biclique B_j of H corresponds to a hyperedge $E_j \in E(\mathcal{H}_A)$ and to a vertex $v_j \in V(G)$. Define a family $\mathcal{C}$ of subsets of $V(G)$ as follows. For each $w_i \in V(H)$, there is a subset $C_i \in \mathcal{C}$ satisfying $v_j \in C_i$ precisely when $w_i \in B_j$, for all $v_j \in V(G)$, that is, $\mathcal{C}$ is the family of columns of A. First, we show that each C_i is a complete subset. Let v_j, v_k be vertices of G, belonging to a common subset $C_i \in \mathcal{C}$. In this situation, $w_i \in B_j \cap B_k$, implying that the corresponding vertices v_j, v_k in G must be adjacent. Therefore $\mathcal{C}$ is a family of complete subsets.

Furthermore, because G is the biclique graph of H, each edge $v_i v_j \in E(G)$ corresponds to a pair of intersecting bicliques B_i, B_j of H. That is, some vertex $w_k \in V(H)$ belongs to both B_i, B_j. The latter implies that $v_i, v_j \in C_k$, meaning that $\mathcal{C}$ covers the edges of G.

Next, consider the associated hypergraph $\mathcal{H}_\mathcal{C}$ of $\mathcal{C}$ with a coloring c' defined as follows: the color of v_j in hyperedge C_i is the same as the color of vertex $w_i \in E_j$ in $\mathcal{H}_A$.

In other words, every vertex $v_i \in V(G)$ corresponds to a biclique B_i of H and every complete subset C_j of $\mathcal{C}$ corresponds to a vertex $w_j \in V(H)$, satisfying: $w_i \in B_j$ if and only if $v_j \in C_i$. Furthermore, w_i, w_j are adjacent in H precisely when $C_i \cap C_j$ contains at least one vertex having different color in these subsets.

Observe that $V(\mathcal{H}_A) \cong V(\mathcal{H}^*_\mathcal{C}) \cong V(H)$, while $E(\mathcal{H}_A) \cong E(\mathcal{H}^*_\mathcal{C}) \cong V(G)$. We conclude that $\mathcal{H}_A \cong \mathcal{H}^*_\mathcal{C}$ and $c'^* = c$.

Because A is a biclique matrix, it satisfies conditions (i)–(iv) of Theorem 3.2. Examine each of them.

From (i), for each row i of $A=(a_{ij})$, there are two columns j,k satisfying $a_{ij}=-a_{ik}$. This implies that $E_i \in E(\mathcal{H}_A)$ contains $w_j, w_k \in V(H)$ and the colors of w_j, w_k in E_i are different. Consequently, $v_i \in C_j \in \mathcal{C}$, $v_i \in C_k \in \mathcal{C}$ and v_i has distinct colors in C_j, C_k. Then $\mathcal{C}$ bicovers $V(G)$, and (1) holds.

From (ii), A has no dominated rows. By Observation 4.1, $\mathcal{H}_A$ has no dominated hyperedges, implying that $\mathcal{H}^*_{\mathcal{C}}$ also does not have dominated hyperedges. Hence (2) holds.

From (iii), A is a compatible matrix. By Observations 4.2 and 4.3, we conclude that c^* is also compatible, and (3) is valid.

Finally, from (iv), the isomorphism $\mathcal{H}_A \cong \mathcal{H}^*_{\mathcal{C}}$ and Lemma 4.1 imply (4), terminating the proof of necessity.

Conversely, by hypothesis, G contains a family $\mathcal{C}$ of complete subgraphs, which covers the edges of G and whose associated hypergraph $\mathcal{H}_{\mathcal{C}}$ admits a coloring c satisfying (1)–(4). First, consider $\mathcal{H}_{\mathcal{C}}$, its dual $\mathcal{H}^*_{\mathcal{C}}$ and the associated matrix $A_{\mathcal{H}^*_{\mathcal{C}}}$ of $\mathcal{H}^*_{\mathcal{C}}$. We prove that G is the biclique graph of $H=G_b(\mathcal{H}^*_{\mathcal{C}})$.

Examine matrix $A_{\mathcal{H}^*_{\mathcal{C}}}$. Each row i of it corresponds to a vertex $v_i \in V(G)$ and each column corresponds to a complete subgraph $C_j \in \mathcal{C}$. Because of condition (1), it follows that each row of $A_{\mathcal{H}^*_{\mathcal{C}}}$ has at least a 1 and a -1. Also, (2) implies that $A_{\mathcal{H}_{\mathcal{C}^*}}$ has no dominated rows. From (3) and Observation 4.2, we conclude that $A_{\mathcal{H}^*_{\mathcal{C}}}$ is compatible, and from (4) and Lemma 4.1, $\mathcal{H}^*_{\mathcal{C}}$ is bipartite-conformal, relative to c^*. Consequently, by Theorem 3.2, $A_{\mathcal{H}^*_{\mathcal{C}}}$ is the biclique matrix of the graph $H=G_b(\mathcal{H}^*_{\mathcal{C}})$. That is, every biclique B_i of H corresponds to a row v_i of A, and each complete subgraph C_j corresponds to a vertex w_i of H. We show below that $G=KB(H)$.

Let v_i, v_j be adjacent vertices of G. Because $\mathcal{C}$ covers the edges of G, there is a complete subgraph $C_k \in \mathcal{C}$ containing both v_i, v_j. Because $A_{\mathcal{H}^*_{\mathcal{C}}}$ is a biclique matrix of H, the bicliques B_i, B_j of H contain a common vertex $w_k \in V(H)$, i.e. $B_i \cap B_j \neq \emptyset$. Conversely, when B_i, B_j are intersecting bicliques of H, let $w_k \in B_i \cap B_j$. Then $C_k \supseteq \{v_i.v_j\}$. Since C_k is a complete subset, the latter implies $v_i v_j \in E(G)$. Consequently, $G=KB(H)$. The proof is complete. ■

The following observation is direct from the proof of Theorem 4.1.

Observation 4.5. Let G be a graph and $\mathcal{C}$ a family of complete subgraphs of it. Then $\mathcal{C}$ satisfies the conditions of Theorem 4.1 if and only if $A_{\mathcal{H}^*_{\mathcal{C}}}$ is the biclique matrix of some graph H. Furthermore, $G=KB(H)$ and $H \cong G_b(\mathcal{H}^*_{\mathcal{C}})$.

5. BICLIQUE GRAPHS AND DIAMONDS

In this section, we examine the question of finding classes of biclique graphs. We give a simple necessary condition for a graph to be a biclique graph, in terms of a 3-fan (Fig. 7) and a diamond (Fig. 8). As a consequence, we obtain a characterization for biclique graphs, restricted to the class of diamond-free graphs.

Start by examining complete graphs.

Lemma 5.1. *Complete graphs are biclique graphs of bipartite graphs.*

Proof. We will use an inductive argument. First, observe that $KB(P_4) = K_2$. Assume that $KB(G) = K_n$, G being a bipartite graph. We construct inductively a bipartite graph G' such that $K_B(G') = K_{n+1}$. Let V_1 and V_2 be the bipartition of G. Add to G vertices v_1' and v_2' and the set of edges $\{(v_1', w),\ w \in V_2\} \cup \{(v_1', v_2')\}$. The resulting bipartite graph G' verifies that $K(G') = K_{n+1}$. ■

Next, we give a necessary condition for biclique graphs.

Theorem 5.1. *Let G a biclique graph. Then, every induced P_3 of G is contained in a diamond or in a 3-fan.*

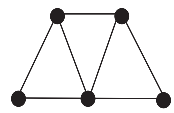

FIGURE 7. A 3-fan.

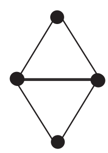

FIGURE 8. A diamond.

Proof. Suppose G contains an induced P_3 formed by vertices $v_1, v_2, v_3 \in V(G)$, where v_1, v_2 are not adjacent. Assume that G is a biclique graph of some graph H. Then G has a vertex v_i for each biclique B_i of H. By Theorem 4.1 and Observation 4.5, G contains a family $\mathcal{C}$ of complete subgraphs C_j satisfying (1)–(4), such that $G = KB(G_b(\mathcal{H}^*_{\mathcal{C}}))$. Let $G_b(\mathcal{H}^*_{\mathcal{C}}) = H$. Furthermore, we know that vertices $w_i, w_j \in V(H)$ are adjacent in H precisely when C_i, C_j both contain a vertex having different colors in these complete subgraphs. Moreover, $w_i \in B_j$ if and only if $v_j \in C_i$.

Because $\mathcal{C}$ covers the edges of G, there is a complete subset $C_1 \in \mathcal{C}$ satisfying $v_1, v_3 \in C_1$. From (2), there is a complete set $C_3 \in \mathcal{C}$ that contains v_1 but not v_3. Observe that $v_2 \notin C_3$.

Case 1. v_1 has different colors in C_1 and C_3, for some $C_3 \in \mathcal{C}$ that contains v_1 but not v_3.

Because $\mathcal{C}$ covers the edges of G, there exists a complete subgraph $C_2 \in \mathcal{C}$ that contains the edge v_3v_2. Examine the further alternatives.

Case 1.1. v_3 has different colors in C_1, C_2, for some $C_2 \in \mathcal{C}$ that contains edge v_3v_2. Also, there exists a complete subgraph $C_4 \in \mathcal{C}$, where $v_2 \in C_4$ has different colors in C_2 and C_4.

Then, the corresponding vertices w_3, w_1, w_2, w_4 form a path in H. Examine the possible additional adjacencies between these vertices of H. Observe that, if $v_3 \in C_4$, then the color of v_3 in C_4 is the same as in C_1, according to (3). Moreover, if w_1w_4

is edge then there exists a vertex $v \in C_1 \cap C_4$, $v \neq v_3$, such that v_1, v_2, v_3, v form a diamond. Analogously, if w_3w_2 is an edge, then the Theorem holds. If w_3w_4 is an edge of H, the complete bipartite set w_1, w_2, w_3, w_4 is included in some biclique B_4 and its corresponding vertex v_4 belongs to C_1, C_2, C_3, C_4. Then, a diamond containing P_3 is formed.

Next, consider the case w_3, w_1, w_2 form a P_3 in H, which is contained in a biclique B_5. Consider vertex v_5 of G corresponding to B_5.

It follows that $v_5 \in C_1, C_2, C_3$. Since $v_3 \notin C_3$, $v_3 \neq v_5$. Consequently, v_3, v_1, v_2, v_5 induce a diamond in G.

Case 1.2. v_3 has identical colors in C_1 and C_2, for all C_2.

Again, let C_4 be a complete subgraph, where $v_2 \in C_2 \cap C_4$ has different colors in C_2 and C_4.

We affirm that $v_3 \notin C_4$. Arguing by contradiction, suppose that $v_3 \in C_4$. By (3), v_3 has different colors in C_2, C_4. Then, C_4 is a complete subgraph that contains the edge v_3v_2, such that v_3 has different colors in C_1, C_4, a contradiction. Then, if $C_1 \cap C_4 \neq \emptyset$, then a diamond containing P_3 is formed.

Let C_5 be a complete subgraph that contains v_3 with a color different from the color that v_3 has in C_1.

Then, $v_2 \notin C_5$ (by the assumption of Case 1.2). Consider vertices w_3, w_1, w_5. Either they form an induced path included in a biclique or a triangle in H. In any of these cases, there exists a vertex $v_6 \in C_5$, $v_6 \neq v_3$, v_6 adjacent to v_3 and either adjacent or equal to v_1.

Similarly, considering the path w_4, w_2, w_5 we conclude that there is a vertex $v_7 \in C_5$ adjacent to v_3, v_2 and v_6. Finally, consider the following possibilities for v_6 and v_7:

- $v_6 = v_7$,
- $v_6 = v_1$,
- v_6 is adjacent to v_2,
- v_7 is adjacent to v_1.

If any of the above alternatives occur, then vertices v_1, v_3, v_2, v_7 or v_1, v_3, v_2, v_6 induce a diamond in G that contains the P_3 formed by v_1, v_2, v_3.

Otherwise, vertices v_1, v_6, v_7, v_2, v_3 induce a 3-fan in G that contains the P_3 formed by v_1, v_2, v_3.

Case 2. v_1 has identical colors in C_3 and C_1, for all $C_3 \in \mathcal{C}$ that contain v_1 but not v_3.

Let C_6 be a complete subgraph, $v_1 \in C_6$, such that v_1 has different colors in C_1, C_6. Then $v_3 \in C_6$ and, by (3), the color of v_3 in C_1 is different from the color of v_3 in C_6. If there exists a complete subgraph C_2, where $v_2, v_3 \in C_2$ such that the colors of v_3 in C_6 and C_2 are different, considering C_6 instead of C_1 in Case 1.1, we complete the proof. Otherwise, the color of v_3 in C_6 is identical to the color of v_3 in C_2, for every complete subgraph containing v_2, v_3. In this situation, following the proof of Case 1.2 we conclude that the P_3 formed by $v_1v_2v_3$ is contained in a diamond. ■

Observation 5.1 (Montero [19]). The converse of Theorem 5.1 is not true. The graph of Figure 9 is a counterexample.

It follows from the next corollary that the theorem is actually somewhat stronger.

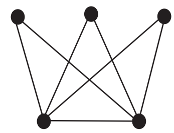

FIGURE 9. A Crown.

Corollary 5.1. *Let G be a connected diamond-free graph. Then G is a biclique graph if and only if G is a complete graph.*

Corollary 5.2. *The induced cycles $C_k, k \geq 4$, are not biclique graphs.*

Corollary 5.3. *Trees with more than* 2 *vertices are not biclique graphs.*

6. BICLIQUE MATRICES OF BIPARTITE GRAPHS

In this section we examine biclique matrices of bipartite graphs.

The following concept is useful. A $\{0, 1, -1\}$ matrix A is *bipartite* when it admits a row-similar matrix A', such that no column of A' has both entries 1 and -1. It is clear that a graph is bipartite if and only if its biclique matrix is bipartite. We observe that bipartite matrices can be recognized in polynomial-time.

As a direct corollary of Theorem 3.2, Corollary 6.1 is a characterization for biclique matrices of bipartite graphs.

Corollary 6.1. *Let A be a $m \times n$, $\{0, 1, -1\}$ matrix, and $\mathcal{H}$ its associated hypergraph. Then A is a biclique matrix of some bipartite graph if and only if*

(i) *each row of A has at least one* 1 *and at least one* -1,
(ii) *A has no dominated rows*,
(iii) *$\mathcal{H}$ is bipartite-conformal, relative to its canonical coloring*,
(iv) *A is bipartite.*

Next, we consider matrices to describe bicliques in the context of $\{0, 1\}$ matrices and not $\{0, 1, -1\}$ matrices. Given a graph G with d bicliques $B_1, \ldots, B_d \subseteq V(G)$, a *positive biclique* matrix A of G is a $d \times n$, $\{0, 1\}$ matrix such that $a_{ij} = 1$ if vertex v_j belongs to biclique B_i and $a_{ij} = 0$ otherwise. Clearly, a positive biclique matrix is the matrix obtained from a biclique matrix by replacing each -1 by 1. We need the following definitions. A *bicoloring* of G is a bipartition of the vertices of G into subsets V_1, V_2. A clique of G is bichromatic relative to a bicoloring V_1, V_2 if it contains at least a vertex of V_1 and a vertex of V_2. A *weak* 2*-coloring* of G is a bicoloring such that every clique of G is bichromatic, relative to V_1, V_2. When the considered matrix is a clique matrix, the following holds:

Theorem 6.1. *Let G be a graph, A be a clique matrix of G and $\mathcal{H}$ the associated hypergraph of A. Then, the following statements are equivalent*:

(1) *A is a positive biclique matrix of a bipartite graph H.*

(2) *A is a positive biclique matrix of a neighborhood-Helly bipartite graph H.*
(3) *G admits a weak 2-coloring $V_1, V_2 \subseteq V(G)$ and $\mathcal{H}$ is bipartite-conformal, relative to the bicoloring V_1, V_2.*

Proof. (1)$\Rightarrow$(2): Suppose A is a positive matrix of a bipartite graph H, with bipartition V_1, V_2. We prove that H is neighborhood-Helly. Let V' be a set of vertices of H whose neighborhoods pairwise intersect. Without loss of generality, $V' \subseteq V_1$. The columns of A corresponding to V' pairwise intersect, since any two vertices in V' belong to a common biclique. Then, columns of A corresponding to V pairwise intersect and V' is a complete subset of G, contained in some clique C. Finally, the columns of A corresponding to V' intersect at the row which corresponds to C in A.

(2)$\Rightarrow$(3): Let V_1, V_2 be the bipartition of H. Consider V_1, V_2 as a bicoloring of vertices of G. Since A is a positive biclique matrix of H, every clique of G is bichromatic, implying that V_1, V_2 is a weak 2-coloring of G. It is clear that $\mathcal{H}$ is bipartite conformal, by Corollary 6.1.

(3)$\Rightarrow$(1): Let V_1, V_2 be the bicoloring of G. Define the bipartite matrix B as follows: for every i, $b_{ij} = a_{ij}$ if $j \in V_1$ and $b_{ij} = -a_{ij}$ for $j \in V_2$. Since V_1, V_2 is a weak 2-coloring of G, every row of B has at least a 1 and a -1. Since A is a clique matrix, B has not dominated rows. Finally, by hypothesis, $\mathcal{H}$ is bipartite conformal. Corollary 6.1 says that B is a biclique matrix of a bipartite graph H, ie. A is a positive biclique matrix of H. ■

7. BICLIQUE GRAPHS OF BIPARTITE GRAPHS

In this section, we consider the question of characterizing biclique graphs of bipartite graphs. For formulating a full characterization of it, we need some additional notation.

A coloring of a hypergraph $\mathcal{H}$ is *stable* when each vertex of $\mathcal{H}$ receives the same color, in all the hyperedges containing it. Therefore, bipartite matrices are exactly those such that the coloring of the associated hypergraph is stable. The following lemma is clear.

Lemma 7.1. *Stable colorings are compatible.*

The following Theorem characterizes biclique graphs of bipartite graphs.

Theorem 7.1. *Let G be a graph with no isolated vertices. Then, G is a biclique graph of a bipartite graph if and only if G contains a family $\mathcal{C}$ of complete subgraphs covering the edges of G, and whose associated hypergraph $\mathcal{H}_{\mathcal{C}}$ admits a coloring c, satisfying*

(1) *$\mathcal{C}$ bicovers $V(G)$.*
(2) *$\mathcal{H}^*_{\mathcal{C}}$ has no dominated hyperedges.*
(3) *c^* is a stable coloring.*
(4) *$\mathcal{H}_{\mathcal{C}}$ is bipartite-Helly, relative to c.*

8. CONCLUSIONS

We have described a Krausz-type characterization of biclique graphs, that is, the intersection graphs of the bicliques of a graph. We have given some necessary conditions for a graph to be a biclique graph and we have presented some classes of graphs which are biclique graphs and some which are not.

Although the biclique graph characterization is not as short as the clique graph characterization by Roberts and Spencer, the study of iterated biclique graphs seems to be simpler than that of iterated clique graphs. In fact, unlike the latter, iterated biclique graphs can be characterized in a simple way [19].

In the characterization of biclique graphs, we have employed the concept of the biclique matrix of a graph. This has motivated us to study and characterize the biclique $\{0, 1, -1\}$ matrix of a graph, in similar terms as the well-known characterization of $\{0, 1\}$ clique matrices by Gilmore. However, we leave as an open question, to characterize positive $\{0, 1\}$ biclique matrices of a graph and determine the complexity of its recognition.

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