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Supersymmetric soliton solution in a dimensionally reduced Schrödinger-Chern-Simons model

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We obtain, by dimensional reduction, a (1 + 1) supersymmetric system introduced in the description of ultracold quantum gases. The correct supercharges are identified and their algebra is constructed. Finally, novel solitonic equations emerge and their solution is constructed for the bosonic case.

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I. INTRODUCTION

Since its birth in the early 1970's in the context of high energy physics and mathematical physics, supersymmetry has found a growing number of applications strongly influencing many areas both in experimental and theoretical physics.

Originally proposed as a graded extension of the Poincaré algebra [1], it was soon recognized that it can also be considered in the systems that exhibit Galilean invariance, and this leads to the construction of a graded super-Galilean algebra [2] in $d = 3 + 1$ spacetime dimensions. Constructing the supersymmetric extension of the Galilean invariant $d = 2 + 1$ Jackiw-Pi model [3], Leblanc *et al.* [4] discovered the existence of 2 graded superalgebras and related this to the possibility of finding BPS equations in the bosonic sector. After that, there were several developments related to Galilean supersymmetry in diverse contexts [5].

In a (1 + 1) dimension, Galilean supersymmetry was considered to study ultra cold quantum gases [5–7]. Ultracold quantum gases not only are interesting by the physics that they describe, but also are a useful tool in the modeling of others branches in physics [8]. One interesting example of this modeling was considered in Ref. [5,6]. There, the authors propose to combine a vortex line in a one-dimensional optical lattice with a fermionic gas bound to the vortex core, and it is possible to tune the laser parameters such that a nonrelativistic supersymmetric string is created. This could allow to test experimentally several aspects of the superstring and supersymmetry theory.

From the theoretical point of view, the model that describe that proposal, presents a supersymmetric structure. Despite that the theory has interactions, the authors only found the generators of the super-Galilei algebra for the free theory. This fact contrasts with a basic feature of supersymmetry theory, where the full Hamiltonian is generated by the supercharge algebra. Later, Ref. [7] shows the existence of a supersymmetry charge whose algebra generates the Hamiltonian with quartic interactions. Nevertheless, a supersymmetry transformation associated to a charge to be able to generate the full Hamiltonian of the theory was not found. In this paper we will show the existence of a supersymmetry related to a charge generating the full Hamiltonian of the theory presented in Ref. [5,6]. Also, we will construct the complete supersymmetry algebra. In order to discus these aspects we will present the (1 + 1) model study found in Ref. [6] as the dimensional reduction of a Maxwell-Chern-Simons model proposed by Manton [9]. This has no influence in the derivation of the correct supersymmetry generators but leads us to the presences of interesting soliton solutions in the system.

II. THE MODEL

Let us start by considering some features of the model proposed by Manton. This model is governed by a (2 + 1)-dimensional action consisting of a mixture from the standard Landau-Ginzburg and the Chern-Simons model, where the matter is represented by a complex scalar field $\phi(x)$:

$$S_{(2+1)} = \int d^3x\Big(-\frac{1}{2}B^2 + i\gamma(\phi^\dagger\partial_t\phi + iA_0|\phi|^2) - \frac{1}{2m}(D_i\phi)^\dagger D_i\phi + \kappa(A_0B + A_2\partial_0A_1) + \gamma A_0 + \lambda(|\phi|^2 - 1)^2 - A_iJ_i^T\Big). \tag{1}$$

Here γ, κ and λ are real constants, $D_\mu = \partial_\mu + iA_\mu$ ($\mu = 0, 1, 2$) is the covariant derivative and $B = \partial_1A_2 - \partial_1A_2$ the magnetic field. The term γA_0 is related to the possibility of a condensate in the ground state [10] and J_i^T is a constant transport current. It was shown by Manton in Ref. [9] that this theory presents Galilean invariance with the requirement that the transport current transforms as $J_i^T \to J_i^T + \gamma v_i$ under a boost. With this consideration, we can choose a frame where $J_i^T = 0$. The field equations in this frame takes the form

$$i\gamma D_0\phi = -\frac{1}{2}D_iD_i\phi - 2\lambda(|\phi|^2 - 1)\phi$$
$$\epsilon_{ij}\partial_jB = J_i + \kappa\epsilon_{ij}E_j \qquad \kappa B = \gamma(|\phi|^2 - 1), \tag{2}$$

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where $E_i = \partial_i A_0 - \partial_0 A_i$ is the electric field and J_i is the supercurrent defined by

$$J_i = -\frac{i}{2}(\phi^\dagger D_i \phi - \phi(D_i\phi)^\dagger). \tag{3}$$

The first equation of this system is the nonlinear Schrödinger equation. The second is the Ampère's law in two dimensions. The last equation is the Chern-Simons version of Gauss's law, which takes a different form here from the one presented in the Jackiw-Pi model [3]. The energy of the system for static field configurations reads as

$$E = \int d^3x\left(\frac{1}{2}B^2 + \frac{1}{2m}(D_i\phi)^\dagger D_i\phi - \lambda(|\phi|^2 - 1)^2\right). \tag{4}$$

For finiteness we require that the energy vanishes asymptotically. This fixes the asymptotic behavior of the fields

$$\lim_{r\to\infty}\phi(x) = e^{i\alpha(\phi)}, \qquad \lim_{r\to\infty}A_i = \partial_i\alpha \tag{5}$$

where α is the common phase angle. With these conditions the magnetic flux reads

$$\Phi = \int d^2xB = \oint_{|x|=\infty} A_i dx^i = 2\pi N \tag{6}$$

where N is a topological invariant which takes only integer values. Following Hassaïne *et al.* [11], we can rewrite the expression (4) as

$$E = \int d^3x\left(\frac{1}{2m}|(D_1 \pm iD_2)\phi|^2 + \left(\mp\frac{\gamma}{2\kappa m} + \frac{\gamma^2}{2\kappa^2} - \lambda\right)(|\phi|^2 - 1)^2 \mp \frac{1}{2m}B\right) \tag{7}$$

where we have used Gauss's law shown in Eq. (2) and the identity $|D_i\phi|^2 = |(D_1 \pm iD_2)\phi|^2 \mp B|\phi|^2 \pm \epsilon^{ij}\partial_i J_j$. For $\lambda = \mp\frac{\gamma}{2\kappa m} + \frac{\gamma^2}{2\kappa^2}$ the potential terms cancel, and we see that the energy is bounded below by a multiple of the magnitude of the magnetic flux (for positive flux we choose the lower signs, and for negative flux we choose the upper signs):

$$E \geq \frac{1}{2m}|\Phi|. \tag{8}$$

So this bound is saturated by fields obeying the self-duality equations

$$(D_1 \pm iD_2)\phi = 0 \qquad \kappa B = \gamma(|\phi|^2 - 1). \tag{9}$$

Motivated by these results and the previous works on the (1 + 1)-dimensional supersymmetry in ultracold quantum gases [5–7] we are interested here in the dimensional reduction of the supersymmetric extension of the model (1). Such extension can be carried out by considering the inclusion of nonrelativistic (down-spinor) fermion ψ [4]:

$$\begin{aligned} S_{(2+1)} = \int d^3x\Big(&-\frac{1}{2}B^2 + i\gamma(\phi^\dagger\partial_t\phi + \psi^\dagger\partial_t\psi \\ &+ iA_0[|\phi|^2 + |\psi|^2]\Big) - \frac{1}{2m}(D_i\phi)^\dagger D_i\phi \\ &- \frac{1}{2m}(D_i\psi)^\dagger D_i\psi + \kappa(A_0B + A_2\partial_0A_1) + \gamma A_0 \\ &- \frac{1}{2m}\psi^\dagger B\psi + \lambda_1(|\phi|^2 - 1)^2 + \lambda_2(|\phi|^2 - 1)|\psi|^2\Big), \end{aligned} \tag{10}$$

where the coupling constants are given by

$$\lambda_1 = \frac{\gamma}{2m\kappa} + \frac{\gamma^2}{2\kappa^2}, \qquad \lambda_2 = \frac{3\gamma}{2m\kappa} + \frac{\gamma^2}{\kappa^2} \tag{11}$$

and we have included a Pauli term for the fermion corresponding to a down spinor. This action is invariant under the following supersymmetry transformation:

$$\begin{aligned} &\delta_1\phi = \sqrt{2m}\eta_1^\dagger\psi, \qquad \delta_1\psi = -\sqrt{2m}\eta_1\phi, \\ &\delta_1\mathbf{A} = 0, \qquad \delta_1 A^0 = \frac{1}{\sqrt{2m}c\kappa}(\eta_1\phi\psi^\dagger - \eta_1^\dagger\psi\phi^\dagger) \end{aligned} \tag{12}$$

if the coupling constants satisfy

$$\frac{\gamma}{2m\kappa} + 2\lambda_1 - \lambda_2 = 0, \tag{13}$$

where η_1, appearing in (12), is a complex Grassmann variable. In order to analyze the lineal problem [7,12], it is natural to consider a dimensional reduction of the action (10) by suppressing dependence on the second spacial coordinate, renaming A_y as B. Then, the action (10) becomes

$$\begin{aligned} S_{(1+1)} = \int d^2x\Big(&-\frac{1}{2}(\partial_xB)^2 + i\gamma(\phi^\dagger\partial_t\phi + \psi^\dagger\partial_t\psi + iA_0\rho) \\ &- \frac{1}{2m}(D_x\phi)^\dagger D_x\phi - \frac{1}{2m}B^2\rho - \frac{1}{2m}(D_x\psi)^\dagger D_x\psi \\ &+ \kappa(A_0\partial_xB + B\partial_0A_1) + \gamma A_0 \\ &+ \lambda_1(\rho_b - 1)^2 + \lambda_2(\rho_b - 1)\rho_f\Big). \end{aligned} \tag{14}$$

Where we have introduced the matter densities

$$\rho_b = |\phi|^2, \qquad \rho_f = |\psi|^2, \qquad \rho = \rho_b + \rho_f, \tag{15}$$

Gauss's law constraint for this action is

$$\partial_xB = \frac{\gamma}{\kappa}(\rho - 1). \tag{16}$$

Note that this constraint has an additional constant term from those appearing in Ref. [7,12]. The equation can be solved as

$$B(x) = \frac{\gamma}{2\kappa}\int dz\epsilon(x - z)(\rho(z) - 1). \tag{17}$$

Using these expressions for the magnetic field and its derivative, the action (14) takes the form

$$S_{(1+1)} = \int d^2x\Big(i\gamma(\phi^\dagger\partial_t\phi + \psi^\dagger\partial_t\psi) - \frac{1}{2m}(D_x\phi)^\dagger D_x\phi - \frac{1}{2m}(D_x\psi)^\dagger D_x\psi - \frac{1}{2m}B^2\rho + \kappa\Big(\frac{\gamma}{2\kappa}\int dz\epsilon(x-z)(\rho(z)-1)\Big)\partial_0 A_1 + \lambda_1'(\rho_b-1)^2 + \lambda_2'(\rho_b-1)\rho_f\Big). \tag{18}$$

Here the action contains new coupling constants defined as

$$\lambda_1' = \lambda_1 - \frac{\gamma^2}{2\kappa^2}, \qquad \lambda_2' = \lambda_2 - \frac{\gamma^2}{\kappa^2} - \frac{\gamma}{2m\kappa}. \tag{19}$$

Following the method proposed in Ref. [12], the gauge field A_x may be eliminated from the action (18) via a gauge transformation. Indeed, after transforming the matter fields as

$$\phi(x) \to e^{-i\alpha(x)}\phi(x), \qquad \psi(x) \to e^{-i\alpha(x)}\psi(x) \tag{20}$$

with

$$\alpha(x) = \frac{1}{2}\int dz\epsilon(x-z)A_x(z) \tag{21}$$

the action can be written simply as

$$S_{(1+1)} = \int d^2x\Big(i\gamma(\phi^\dagger\partial_t\phi + \psi^\dagger\partial_t\psi) - \frac{1}{2m}(\partial_x\phi)^\dagger\partial_x\phi - \frac{1}{2m}(\partial_x\psi)^\dagger\partial_x\psi + \lambda_1'(\rho_b-1)^2 + \lambda_2'(\rho_b-1)\rho_f - \frac{1}{2m}B^2\rho\Big). \tag{22}$$

The last term appearing in this action is a constant of motion:

$$\begin{aligned}\frac{1}{2m}\int dx_1 B^2(x_1)\rho(x_1) &= \frac{\gamma^2}{8m\kappa^2}\int dx_1dx_2dx_3\epsilon(x_1-x_2)\epsilon(x_1-x_3)(\rho(x_1)-1)\\ &\quad\times(\rho(x_2)-1)\rho(x_3)\\ &= \frac{\gamma^2}{24m\kappa^2}\int dx_1dx_2dx_3(\rho(x_1)-1)(\rho(x_2)-1)\rho(x_3)\\ &= \frac{(N^3-2N^2+N)\gamma^2}{24m\kappa^2},\end{aligned} \tag{23}$$

where $N = \int dx\rho(x)$, and use has been made of the identity

$$\epsilon(x_1-x_2)\epsilon(x_1-x_3) + \epsilon(x_2-x_3)\epsilon(x_2-x_1) + \epsilon(x_3-x_1)\epsilon(x_3-x_2) = 1. \tag{24}$$

Thus, dropping this term, the action can be written as

$$S_{(1+1)} = \int d^2x\Big(i\gamma(\phi^\dagger\partial_t\phi + \psi^\dagger\partial_t\psi) - \frac{1}{2m}(\partial_x\phi)^\dagger\partial_x\phi - \frac{1}{2m}(\partial_x\psi)^\dagger\partial_x\psi + \lambda_1'(\rho_b-1)^2 + \lambda_2'(\rho_b-1)\rho_f\Big). \tag{25}$$

This is the model studied in Ref. [5,6], in the context of the description of ultracold gases dynamics. The action (25) is similar to the action derived by the dimensional reduction of the Jackiw-Pi model [7]. The difference is that our model also includes the chemical potential terms.

Let us now consider the possible supersymmetry transformations that leave unchanged the action (25). One obvious supersymmetry of the system (25) takes place when bosons and fermions are interchanged according to

$$\delta_1\phi = \sqrt{2m}\eta_1^\dagger\psi, \qquad \delta_1\psi = -\sqrt{2m}\eta_1\phi, \tag{26}$$

and the coupling constants are related by

$$\lambda_2' = 2\lambda_1'. \tag{27}$$

It is interesting to note that the transformation (26) is also a supersymmetry of the model studied in Ref. [7].

The fact that is less evident is the existence of a second supersymmetry. In discussing this supersymmetry, note that the action (25) is invariant under the following transformation:

$$\begin{aligned}\delta_2\phi &= \frac{i}{\sqrt{2m}}\eta_2^\dagger(\partial_x\psi - B_1\psi) + i\delta_2\alpha\phi,\\ \delta_2\psi &= -\frac{i}{\sqrt{2m}}\eta_2(\partial_x\phi + B_1\phi) + i\delta_2\alpha\psi,\end{aligned} \tag{28}$$

provided that

$$\lambda_1' = \frac{\gamma}{2m\kappa}, \qquad \lambda_2' = 2\lambda_1' \tag{29}$$

and

$$\begin{aligned}B_1(x) &= \frac{\gamma}{2\kappa}\int dz\epsilon(x-z)\rho(z)\\ \delta_2\alpha &= \frac{-\gamma}{2\kappa\sqrt{2m}}\int dz\epsilon(x-z)(\eta_2\phi\psi^\dagger - \eta_2^\dagger\psi\phi^\dagger).\end{aligned} \tag{30}$$

Notice that combining Eqs. (19) and (29) we obtain Eq. (11). As the transformation (26), the expression (28) is also a supersymmetry of the model explored in Ref. [7]. Another interesting fact is that the condition for the coupling constants imposed in (29) is a particular case of Eq. (27). This implies that the following combination of the precedent two supersymmetries:

$$\begin{aligned}\delta_2\phi &= \frac{i}{\sqrt{2m}}\eta_2^\dagger(\partial_x\psi - B_1\psi) + i\delta_2\alpha\phi - \frac{i\eta_2^\dagger\sqrt{\gamma}}{\sqrt{\kappa m}}\psi,\\ \delta_2\psi &= -\frac{i}{\sqrt{2m}}\eta_2(\partial_x\phi + B_1\phi) + i\delta_2\alpha\psi + \frac{i\eta_2\sqrt{\gamma}}{\sqrt{\kappa m}}\phi\end{aligned} \tag{31}$$

is also a supersymmetry of the action in(25) if the condition (29) is kept.

III. THE SUPERSYMMETRY ALGEBRA

In this section we shall study the algebra of the generators associated to the transformations (26) and (31).

In the representation of the supersymmetry algebra the generator associated to the supersymmetry (26) can easily be fond to be

$$Q_1 = -i\sqrt{2m}\int dx\psi^\dagger\phi. \tag{32}$$

In order to write the supersymmetry algebra we define the Poisson brackets for the functions of the matter field as

$$\{F, G\}_{PB} = i\int dr\bigg(\frac{\delta F}{\delta\phi^\dagger(r)}\frac{\delta G}{\delta\phi(r)} - \frac{\delta F}{\delta\phi(r)}\frac{\delta G}{\delta\phi^\dagger(r)} - \frac{\delta^r F}{\delta\psi^\dagger(r)}\frac{\delta^l G}{\delta\psi(r)} - \frac{\delta^r F}{\delta\psi(r)}\frac{\delta^l G}{\delta\psi^\dagger(r)}\bigg) \tag{33}$$

where the subscripts r and l refer to right and left derivatives and, in particular, we have

$$\begin{aligned}\{\phi(x_1,t), \phi^*(x_2,t)\} &= -i\delta(x_1 - x_2)\\ \{\psi(x_1,t), \psi^*(x_2,t)\} &= -i\delta(x_1 - x_2).\end{aligned} \tag{34}$$

Using the definition of the Poisson bracket it is easy to get

$$\{Q_1, Q_1^\dagger\} = -2im\int dx\rho \equiv -2iM. \tag{35}$$

Following Ref. [6] we can define a second generator:

$$R = -\frac{1}{\sqrt{2m}}\int dx\psi^\dagger\partial_x\phi, \tag{36}$$

which generates the free part of the Hamiltonian,

$$\{R, R^\dagger\} = \frac{i}{2m}\int dx(\phi^\dagger\partial_x^2\phi + \psi^\dagger\partial_x^2\psi) = -iH_{\text{free}}. \tag{37}$$

Nevertheless, as we mentioned in the introduction, the previous works do not propose a supercharge be able to generate the full Hamiltonian of the model explored in Ref. [6], that is, the Hamiltonian derived from the action (25)

$$H = \int d^2x\bigg(\frac{1}{2m}(\partial_x\phi)^\dagger\partial_x\phi + \frac{1}{2m}(\partial_x\psi)^\dagger\partial_x\psi - \frac{\gamma}{2m\kappa}(\rho_b - 1)^2 - \frac{\gamma}{m\kappa}(\rho_b - 1)\rho_f\bigg). \tag{38}$$

In addition, we know from Ref. [7] that the supersymmetry (28) is generated by the supercharge

$$\begin{aligned}Q_2^{(1)} &= -\frac{1}{\sqrt{2m}}\int dx\psi^\dagger(\partial_x + B_1)\phi\\ &= -\frac{1}{\sqrt{2m}}\int dx\psi^\dagger(x)\bigg(\partial_x + \frac{\gamma}{2\kappa}\int dz\epsilon(x-z)\rho(z)\bigg)\phi(x),\end{aligned} \tag{39}$$

which generates a Hamiltonian only with potential terms quartic in the fields. Based on this idea and with the fact that the transformation (31) is a combination of the supersymmetry (26) and (28), it seems natural to define a supercharge Q_2 so that it is a linear combination of Q_1 and $Q_2^{(1)}$:

$$Q_2 = -\frac{1}{\sqrt{2m}}\int dx\psi^\dagger(\partial_x + B_1)\phi + \frac{\sqrt{\gamma}}{\sqrt{\kappa m}}\int dx\psi^\dagger\phi. \tag{40}$$

We will show that this charge generates the Hamiltonian (38). Using this charge we can calculate

$$\begin{aligned}\Big\{Q_2, Q_2^\dagger\Big\} &= \Big\{(K_1 + K_2), (K_1^\dagger + K_2^\dagger)\Big\}\\ &= \Big\{K_1, K_1^\dagger\Big\} + \Big\{K_1, K_2^\dagger\Big\} + \Big\{K_2, K_1^\dagger\Big\} + \{K_2, K_2^\dagger\}\end{aligned} \tag{41}$$

where

$$K_1 = Q_2^{(1)} \qquad K_2 = \frac{i\sqrt{\gamma}}{\sqrt{2\kappa m}}Q_1. \tag{42}$$

The first of the brackets were calculated in Ref. [7] and the result are

$$\begin{aligned}\{K_1, K_1^\dagger\} = -\frac{i}{2m}\int d^2x\bigg(&\frac{1}{2m}(\partial_x\phi)^\dagger\partial_x\phi + \frac{1}{2m}(\partial_x\psi)^\dagger\partial_x\psi + B_1^2\rho - \frac{2}{\kappa}\rho_f\rho_b\\ &- \partial_x B_1\rho_b + \partial_x B_1\rho_f\bigg),\end{aligned} \tag{43}$$

which can be reduced, after using the definition of the B_1 and eliminating the constant term $\int d^2xB_1^2\rho$, to

$$\{K_1, K_1^\dagger\} = -\frac{i}{2m}\int d^2x\bigg(\frac{1}{2m}(\partial_x\phi)^\dagger\partial_x\phi + \frac{1}{2m}\times(\partial_x\psi)^\dagger\partial_x\psi - \frac{2\gamma}{\kappa}\rho_f\rho_b - \frac{\gamma}{\kappa}\rho_b^2\bigg). \tag{44}$$

The second bracket can be developed as follows:

$$\begin{aligned}\Big\{K_1, K_2^\dagger\Big\} = -\frac{1}{m\sqrt{2\kappa}}\int dx_1dx_2\bigg(&\Big\{\psi^\dagger(x_1), \psi(x_2)\Big\}\phi^\dagger(x_2)\\ &\times(\partial_x + B_1(x_1))\phi(x_1) + \psi^\dagger(x_1)\\ &\times\Big\{\partial_x\phi(x_1), \phi^\dagger(x_2)\Big\}\psi(x_2) + \psi^\dagger(x_1)\\ &\times\Big\{B_1(x_1), \phi^\dagger(x_2)\psi(x_2)\Big\}\phi(x_1) + \psi^\dagger(x_1)B_1(x_1)\\ &\times\Big\{\phi(x_1), \phi^\dagger(x_2)\Big\}\psi(x_2)\bigg).\end{aligned} \tag{45}$$

It can be easily checked that

$$\left\{B_1(x_1), \phi^\dagger(x_2)\psi(x_2)\right\} = 0. \tag{46}$$

Thus,

$$\left\{K_1, K_2^\dagger\right\} = \frac{i}{m\sqrt{2\kappa}} \int dx\Big(\phi^\dagger(x)(\partial_x + B_1(x))\phi(x) + \psi^\dagger(x)(\partial_x + B_1(x))\psi(x)\Big). \tag{47}$$

In similar form we have for the third bracket

$$\left\{K_2, K_1^\dagger\right\} = -\frac{i}{m\sqrt{2\kappa}} \int dx\Big(\phi^\dagger(x)(\partial_x - B_1(x))\phi(x) + \psi^\dagger(x)(\partial_x - B_1(x))\psi(x)\Big). \tag{48}$$

From (47) and (48) we get

$$\left\{K_1, K_2^\dagger\right\} + \left\{K_2, K_1^\dagger\right\} = \frac{i}{m\sqrt{2\kappa}} \int dx B_1(x)\rho = \frac{i}{2m\kappa}\int dx dz \epsilon(x-z)\rho(x)\rho(z) = 0. \tag{49}$$

The last bracket gives

$$\left\{K_2, K_2^\dagger\right\} = -\frac{i\gamma}{\kappa m}\int \rho dx. \tag{50}$$

Then, the full bracket takes the form

$$\left\{Q_2, Q_2^\dagger\right\} = -\frac{i}{2m}\int dx\Big((\partial_x\phi)^\dagger\partial_x\phi + (\partial_x\psi)^\dagger\partial_x\psi - \frac{2\gamma}{\kappa}\rho_f\rho_b - \frac{\gamma}{\kappa}\rho_b^2 + \frac{2\gamma}{\kappa}\rho\Big) = -iH. \tag{51}$$

The algebra is completed by the following bracket:

$$\left\{Q_1, Q_2^\dagger\right\} = \left\{Q_1, K_1^\dagger\right\} + \left\{Q_1, K_2^\dagger\right\}. \tag{52}$$

The first of these brackets can be calculated to give

$$\left\{Q_1, K_1^\dagger\right\} = -\frac{1}{2}\int d^2x\Big(\phi^\dagger\partial_x\phi - \partial_x\phi^\dagger\phi + \psi^\dagger\partial_x\psi - \partial_x\psi^\dagger\psi\Big), \tag{53}$$

where we have used

$$\int dx B_1(x)\rho = 0. \tag{54}$$

The second bracket is

$$\left\{Q_1, K_2^\dagger\right\} = -\sqrt{\frac{2\gamma}{\kappa}}\int dx\rho. \tag{55}$$

Therefore

$$\left\{Q_1, Q_2^\dagger\right\} = -\frac{1}{2}\int d^2x\Big(\phi^\dagger\partial_x\phi - \partial_x\phi^\dagger\phi + \psi^\dagger\partial_x\psi - \partial_x\psi^\dagger\psi\Big) - \sqrt{\frac{2\gamma}{\kappa}}\int dx\rho. \tag{56}$$

In order to show that this expression may be identified with the linear momentum of the system we use Noether's theorem. Let $\{\theta_c\} = \{\phi, \phi^\dagger, \psi, \psi^\dagger\}$ be the set of fields of our system, where c runs from 1 to 4. The theorem establishes that if under a variation of the fields $\delta\theta_c$, the variation of the Lagrangian density is a surface term, $\delta\mathcal{L} = \partial_\mu X^\mu$, and then there exists a conserved current associated with such variation of the fields. The Noether current, assuming the summation convention over the index c, is

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\theta_c)}\delta\theta_c - X^\mu. \tag{57}$$

We are interested in the zero component of this current associated to the transformations

$$\{\delta\theta_c\} = \{\partial_x\phi, \partial_x\phi^\dagger, \partial_x\psi, \partial_x\psi^\dagger\} \tag{58}$$

where

$$X^0 = -i\gamma\sqrt{\frac{2\gamma}{\kappa}}\rho, \qquad X^1 = 0. \tag{59}$$

Note that, from the Noether theorem, the only restriction for X^μ is that $\partial_\mu X^\mu$ be a surface term. Using (57) the linear momentum is

$$P = \int j^0 dx = -i\gamma\{Q_1, Q_2^\dagger\}. \tag{60}$$

The energy-momentum tensor may then be defined as

$$T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\theta_c)}\partial_\nu\theta_c - \mathcal{L}\delta^\mu_\nu + i\gamma\sqrt{\frac{2\gamma}{\kappa}}\rho\tilde{\delta}^\mu_\nu \tag{61}$$

where $\tilde{\delta}^\mu_\nu = 1$ if $\mu \neq \nu$ and $\tilde{\delta}^\mu_\nu = 0$ if $\mu = \nu$.

Finally, it is easy to check that the remaining brackets are zero:

$$\{Q_\alpha, Q_\beta\} = \left\{Q_\alpha^\dagger, Q_\beta^\dagger\right\} = 0. \tag{62}$$

IV. THE SOLITON SOLUTION

Consider now the derivation of the self-dual equations. As discussed in Ref. [13] the field B plays an important role in the derivation of self-dual equations. Indeed, the expression (17) of B involves the existence of a novel soliton. The action (22) may be easily re-expressed as

$$S = \int d^2x \Big(i\gamma\{\phi^\dagger \partial_t \phi + \psi^\dagger \partial_t \psi\} - \frac{1}{2m}|(\partial_x + \zeta B)\phi|^2 - \frac{1}{2m}|(\partial_x + \zeta B)\psi|^2 - \frac{\zeta}{2m}\partial_x B\rho + \lambda_1'(\rho_b - 1)^2 + \lambda_2'(\rho_b - 1)\rho_f \Big), \tag{63}$$

where $\zeta = \pm 1$. Using Gauss's law, (16) and after a bit of algebra, we have

$$S = \int d^2x \Big(i\gamma\{\phi^\dagger \partial_t \phi + \psi^\dagger \partial_t \psi\} - \frac{1}{2m}|(\partial_x + \zeta B)\phi|^2 - \frac{1}{2m}|(\partial_x + \zeta B)\psi|^2 + (\lambda_1' - \frac{\zeta\gamma}{2m\kappa})(\rho_b - 1)^2 + (\lambda_2' - \frac{\zeta\gamma}{m\kappa})(\rho_b - 1)\rho_f - \frac{\zeta}{2m}\partial_x B \Big), \tag{64}$$

which leads, in the static field configuration, to the Hamiltonian of the form,

$$H = \int d^2x \Big(\frac{1}{2m}|(\partial_x + \zeta B)\phi|^2 + \frac{1}{2m}|(\partial_x + \zeta B)\psi|^2 - (\lambda_1' - \frac{\zeta\gamma}{2m\kappa})(\rho_b - 1)^2 - (\lambda_2' - \frac{\zeta\gamma}{m\kappa})(\rho_b - 1)\rho_f + \frac{\zeta}{2m}\partial_x B \Big). \tag{65}$$

We can choose $\lambda_1' = \frac{\zeta\gamma}{2m\kappa}$ and $\lambda_2' = \frac{\zeta\gamma}{m\kappa}$ so that our Hamiltonian becomes

$$H = \int d^2x \Big(\frac{1}{2m}|(\partial_x + \zeta B)\phi|^2 + \frac{1}{2m}|(\partial_x + \zeta B)\psi|^2 + \frac{\zeta}{2m}\partial_x B \Big). \tag{66}$$

The last integral vanishes since B must be zero in the boundary. Then, at the minimum of the energy configurations, the self-dual equations are satisfied:

$$(\partial_x + \zeta B)\phi = 0 \tag{67}$$

$$(\partial_x + \zeta B)\psi = 0. \tag{68}$$

Notice that for the particular choice $\zeta = 1$, we recover the supersymmetric case. This equations can be explicitly written by using the Eq. (17):

$$\partial_x \phi(x) + \frac{\zeta\gamma}{2\kappa}\Big(\int dz \epsilon(x - z)\rho(z)\phi(x) - \int dz \epsilon(x - z)\phi(x) \Big) = 0 \tag{69}$$

$$\partial_x \psi + \frac{\zeta\gamma}{2\kappa}\Big(\int dz \epsilon(x - z)\rho(z)\psi(x) - \int dz \epsilon(x - z)\psi(x) \Big) = 0, \tag{70}$$

which present an additional linear term from those found in Ref. [7]. When ψ is set to zero, the above set of equations reduces to

$$\partial_x \phi(x) + \frac{\zeta\gamma}{2\kappa}\int dz \epsilon(x - z)(\rho_b(z) - 1)\phi(x) = 0. \tag{71}$$

Assuming a solution of the form $\phi = \sqrt{\rho_b}$, we arrive at

$$\frac{1}{2}\partial_x(\log\rho_b(x)) + \frac{\zeta\gamma}{2\kappa}\int dz \epsilon(x - z)(\rho_b(z) - 1) = 0. \tag{72}$$

Differentiating with respect to x, we get the following one-dimensional Liouville type equation:

$$\frac{1}{2}\partial_x^2(\log\rho_b(x)) + \frac{\zeta\gamma}{2\kappa}(\rho_b(x) - 1) = 0. \tag{73}$$

We propose as the solution to this equation the following series:

$$\rho = 1 + \sum_{n=1}^{\infty} a_n \,\mathrm{sech}^n(bx). \tag{74}$$

Here a_n are the real coefficients of series, b is a real constant and we have renamed ρ as $\rho_b(x)$. In order to check that this is really a solution we rewrite Eq. (73) as

$$-(\partial_x\rho)^2 + (\partial_x^2\rho)\rho + \upsilon\rho^2(\rho - 1) = 0 \tag{75}$$

where $\upsilon = \frac{\zeta\gamma}{\kappa}$. When the series (74) is introduced into Eq. (75) we obtain

$$\sum_{n,m=1}^{\infty}\Big[-a_n a_m mnb^2 + n^2 a_n a_m b^2 + 2\upsilon a_n a_m\Big]\mathrm{sech}^{n+m}(bx) + \sum_{n,m=1}^{\infty}\Big[a_n a_m nmb^2 - n^2 a_n a_m b^2 - na_n a_m b^2\Big]\mathrm{sech}^{n+m+2}(bx) - \sum_{n=1}^{\infty}(n^2 + n)a_n b^2 \mathrm{sech}^{2+n}(bx) + \sum_{n=1}^{\infty}\Big[n^2 a_n b^2 + \upsilon a_n\Big]\mathrm{sech}^n(bx) + \upsilon\sum_{n,i,m=1}^{\infty} a_n a_m a_i \mathrm{sech}^{n+m+i}(bx) = 0, \tag{76}$$

where, for arriving to this expression, we have used the relation

$$\tanh^2(bx) = 1 - \mathrm{sech}^2(bx). \tag{77}$$

So we have an expansion of powers of sech(bx) which must be equal to zero. This implies that the coefficient of each power must vanish separately. From the coefficient of sech(bx) we obtain

$$b^2 = -\upsilon, \tag{78}$$

whereas from the coefficients of $\mathrm{sech}^2(bx)$ and $\mathrm{sech}^3(bx)$ we have

$$a_2 = \frac{2}{3} a_1^2 \tag{79}$$

and

$$a_3 = \frac{3a_1^3 + 2a_1}{8}. \tag{80}$$

This method can be continued in order to determine the rest of the coefficients.

V. CONCLUSION

In this article we have studied a $(1 + 1)$-dimensional model introduced in the description of the supersymmetric-ultracold gases. This model and its supersymmetries were previously studied in Ref. [5–7]. However, the problem of finding the supersymmetry algebra that generates the full theory was unresolved. In this paper we started by extending supersymmetrically a model proposed by Manton [9] and related it to the theory of ultracold gases. Then, the correct supercharges that generate the full theory were identified and their algebra was constructed. In addition, the solitonic structure was analyzed and novel solitons were found.

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