A NOTE ON THE CONVERGENCE TO INITIAL DATA OF HEAT AND POISSON EQUATIONS

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Abstract. We characterize the weighted Lebesgue spaces, $L^p(\mathbb{R}^n, v(x)dx)$, for which the solutions of the Heat and Poisson problems have limit a.a. when the time t tends to zero.

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1. Introduction

Consider the following classical problems in the upper half plane,

$$(A)\begin{cases} \frac{\partial u}{\partial t}(x,t) = -\Delta_x u(x,t) \\ \\ u(x,0) = f(x) \end{cases} \qquad (B)\begin{cases} \frac{\partial^2 w}{\partial t^2}(x,t) = -\Delta_x w(x,t) \\ \\ w(x,0) = g(x), \end{cases}$$

$x \in \mathbb{R}^n$, $t > 0$.

It is well known that under mild size conditions of the initial data f and g, for example $f, g \in L^p(\mathbb{R}^n, dx)$, $1 \le p < \infty$, the following limits hold

$$\lim_{t\to 0} u(x,t) = f(x) \qquad \lim_{t\to 0} w(x,t) = g(x) \qquad \text{for almost all } x. \tag{1.1}$$

The aim of this paper is to obtain optimal weighted Lebesgue spaces $L^p(\mathbb{R}^n, v(x)dx)$, $1 < p < \infty$, for which the limits in (1.1) still hold.

We find two classes D_p^W and D_p^P (see Definition 2.2) of weights v (strictly positive and finite functions for almost all x) such that

$$\lim_{t\to 0} u(x,t) = f(x) \ a.a.x \text{ for all } f \in L^p(\mathbb{R}^n, v(x)dx) \ \text{ if and only if } \ v \in D_p^W, \tag{1.2}$$

and

$$\lim_{t\to 0} w(x,t) = g(x) \ a.a.x \ \text{ for all } g \in L^p(\mathbb{R}^n, v(x)dx) \text{ if and only if } v \in D_p^P. \tag{1.3}$$

These two statements are included in Theorem 2.3, which states the existence of optimal spaces $L^p(\mathbb{R}^n, v(x)dx)$ adapted, either to statements (1.2) or (1.3).

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Along this note the wording **"weighted inequality"** for an operator T means to find conditions in a given weight v in order to assure the existence of a weight u for which T maps $L^p(\mathbb{R}^n, v(x)dx)$ into $L^p(\mathbb{R}^n, u(x)dx)$.

Theorem 2.3 involves some weighted inequalities for local maximal operators associated to Problems (A) and (B), namely, $\sup_{t<R}|u(x,t)|$ and $\sup_{t<R}|w(x,t)|$, respectively. Even more, the finitude almost everywhere of each of these maximal operators is equivalent to the almost everywhere convergence stated either in (1.2) or (1.3).

These weighted inequalities are proved in this work by using a non constructive method due to J.L. Rubio de Francia, see [RdF]. In proving them we shall need some weighted inequalities for the local Hardy-Littlewood maximal operator that we believe are of independent interest (see Lemma 3.4). For the (global) Hardy-Littlewood maximal function, some classes of weights for the weighted inequalities were obtained by L. Carleson and P. Jones, [C-J], and Rubio de Francia, [RdF] indepedently. These results are shown in Theorem 3.2.

Finally in Theorem 2.5 we compare all the classes of weights that appear along this note.

It is worth mentioning that the characterization of the weights v such that the Hardy-Littlewood maximal function maps $L^p(\mathbb{R}^n, v(x)dx)$, $1 < p < \infty$, into itself was done by B. Muckenhoupt in the celebrated paper [M]. A dual weighted inequality for the same maximal operator was also proved by C. Gutiérrez and E. Gatto in [G-G], while the problem of characterization of the pairs (u, v) for which the Hardy-Littlewood function maps $L^p(\mathbb{R}^n, v(x)dx)$ into $L^p(\mathbb{R}^n, u(x)dx)$ was solved by E. Sawyer in [S].

2. Preliminaries and main results

The solutions to problems (A) and (B) can be described via the Heat and Poisson semigroups. In fact, if the functions f and g belong to the Lebesgue space $L^p(\mathbb{R}^n, dx)$ it is well known that the solutions of those problemas are

$$u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y)dy = W_t * f(x), \qquad t > 0, \tag{2.4}$$

and

$$w(x,t) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} \frac{t}{(t^2 + |x-y|^2)^{\frac{n+1}{2}}} g(y)dy = P_t * f(x), \qquad t > 0, \tag{2.5}$$

where $W(x) = (4\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4}}$, $W_t(x) = t^{-\frac{n}{2}} W(t^{-\frac{1}{2}}x)$, $P(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}(1+|x|^2)^{-\frac{n+1}{2}}$ and $P_t(x) = t^{-n}P(t^{-1}x)$.

Moreover, the maximal operators

$$f \to \sup_{t>0} u(\cdot, t) \qquad \text{and} \qquad g \to \sup_{t>0} w(\cdot, t)$$

are bounded on $L^p(\mathbb{R}^n, dx)$.

A first reflection shows that any weight v for which the maximal operators $\sup_{t>0}|u(x,t)|$ or $\sup_{t>0}|w(x,t)|$ have good boundedness properties would be a good weight for our problem. A further analysis reveals that in order to have the limits in (1.1) it is not necessary to consider the global maximal operators $\sup_{t>0}|u(x,t)|$ or $\sup_{t>0}|w(x,t)|$ but only local versions of them. Namely

$$W_R^*f(x) := \sup_{t<R}|u(x,t)| = \sup_{t<R}|W_t * f(x)|$$

and

$$P_R^*g(x) := \sup_{t<R}|w(x,t)| = \sup_{t<R}|P_t * g(x)|,$$

for some $R > 0$.

The first arising question is about boundedness properties of the operators $W_t * f(x)$ and $P_t * f(x)$. The following Proposition gives the answer

PROPOSITION **2.1.** *Let v be a weight in $\mathbb{R}^n$, $1 < p < \infty$ and let $\{\phi_t\}_t$ be either the Heat, $\{W_t\}_t$, or the Poisson, $\{P_t\}_t$, semigroup (see (2.4) and (2.5)).*

The following statements are equivalent:

(a) *There exists $t_0 > 0$ and a weight u such that the operator $f \to \phi_{t_0} * f$ maps $L^p(\mathbb{R}^n, v(x)dx)$ into $L^p(\mathbb{R}^n, u(x)dx)$.*

(b) *There exists $t_0 > 0$ and a weight u such that the operator $f \to \phi_{t_0} * f$ maps $L^p(\mathbb{R}^n, v(x)dx)$ into weak-$L^p(\mathbb{R}^n, u(x)dx)$.*

(c) *There exists $t_0 > 0$ such that $\phi_{t_0} * f(x) < \infty$ a.a. x for all $f \in L^p(\mathbb{R}^n, v(x)dx)$.*

(d) *There exists $t_0 > 0$ such that*

$$0 < \int_{\mathbb{R}^n} \phi_{t_0}^{p'}(y) v^{-\frac{p'}{p}}(y)dy < \infty.$$

Motivated by the above Proposition we give the following definition

DEFINITION **2.2.** *Let $1 < p < \infty$ and $\{\phi_t\}_{t>0}$ be the Heat, $\{W_t\}_{t>0}$, (respectively Poisson, $\{P_t\}_{t>0}$) semigroup.*
We say that the weight v belongs to the class D_p^W (respectively D_p^P) if there exists $t_0 > 0$ such that

$$\int_{\mathbb{R}^n} \phi_{t_0}^{p'}(y) v^{-\frac{p'}{p}}(y)dy < \infty.$$

The result dealing directly with the aim of this note is the following.

THEOREM **2.3.** *Let v be a weight in $\mathbb{R}^n$, $1 < p < \infty$, and $\{\phi_t\}_t$ be either the Heat, $\{W_t\}_t$, or the Poisson, $\{P_t\}_t$, semigroup. Let denote*

$$\Phi_R^*f(x) = \sup_{t<R}|\phi_t * f(x)|,$$

for some R, $0 < R < \infty$.
The following statements are equivalent:

(1) *There exist $0 < R < \infty$ and a weight u such that the operator*
$$f \to \Phi_R^* f$$
maps $L^p(\mathbb{R}^n, v(x)dx)$ into $L^p(\mathbb{R}^n, u(x)dx)$.

(2) *There exist $0 < R < \infty$ and a weight u such that the operator*
$$f \to \Phi_R^* f$$
maps $L^p(\mathbb{R}^n, v(x)dx)$ into weak-$L^p(\mathbb{R}^n, u(x)dx)$.

(3) *The limit*
$$\lim_{t \to 0} \phi_t * f(x)$$
exists a.a. x for all $f \in L^p(\mathbb{R}^n, v(x)dx)$.

(4) *There exist $0 < R < \infty$ such that*
$$\Phi_R^* f(x) < \infty,$$
a.e. x, for all $f \in L^p(\mathbb{R}^n, v(x)dx)$.

(5) *The weight*
$$v \in D_p^\phi$$
(see Definition 2.2).

Along this paper more classes of weights will appear motivating the following definitions.

Definition **2.4.** *Let $1 < p < \infty$. We say that the weight*

- *v belongs to the class D_p^* if v satisfies* (II) *in Theorem 3.2.*
- *v belongs to the class D_p^{loc} if $v^{-\frac{p'}{p}} \in L^1_{loc}(\mathbb{R}^n, dx)$.*

The relationship among the classes of weights in Definitions 2.2 and 2.4 is given by the next Theorem.

Theorem **2.5.** *The chain of inclusions*
$$D_p^* \subsetneq D_p^P \subsetneq D_p^W \subsetneq D_p^{loc},$$
holds for $1 < p < \infty$.

3. Proofs

We need the following Lemma to prove Proposition 2.1.

Lemma **3.1.** *Let v be a weight in $\mathbb{R}^n$, $1 < p < \infty$, and $\{\phi_t\}_t$ be either the Heat or the Poisson semigroup.*
The following statements are equivalent:

(i) *The weight*
$$v \in D_p^\phi$$

(ii) There exists $t_1 > 0$ such that

$$0 < \int_{\mathbb{R}^n} \phi_{t_1}^{p'}(x-y)v^{-\frac{p'}{p}}(y)dy < \infty$$

for all $x \in \mathbb{R}^n$.

Proof of Lemma 3.1

Assume that $\{\phi_t\}$ is the Heat semigroup and $v \in D_p^{\phi}$. Hence there exist $t_0 > 0$ and a positive constant C_0 such that $C_0 = \displaystyle\int_{\mathbb{R}^n} \phi_{t_0}^{p'}(y)v^{-\frac{p'}{p}}(y)dy < \infty$ and $v^{-\frac{p'}{p}} \in L^1_{loc}$.
Given x we consider the ball $B_x = \{y : |x-y| < |x|\}$, hence for $t > 0$ we have

$$\int_{\mathbb{R}^n} \phi_t^{p'}(x-y)v^{-\frac{p'}{p}}(y)dy \leq \int_{B_x} \phi_t^{p'}(x-y)v^{-\frac{p'}{p}}(y)dy + \int_{\{y:|x-y|>|x|} \phi_t^{p'}(x-y)v^{-\frac{p'}{p}}(y)dy.$$

If $|x-y| < |x|$ then $|y| < |x-y| + |x| \leq 2|x|$, hence

$$e^{-\frac{1}{4t}|x-y|^2} \leq 1 \leq e^{\frac{1}{4t}|x|^2}e^{-\frac{1}{4t}|x|^2} \leq e^{\frac{1}{4t}|x|^2}e^{-\frac{1}{4t}(\frac{|y|}{2})^2}.$$

If, on the other hand, $|x-y| > |x|$ then $|y| < |x-y| + |x| \leq 2|x-y|$, thus

$$e^{-\frac{1}{4t}|x-y|^2} \leq e^{-\frac{1}{4t}(\frac{|y|}{2})^2}.$$

Choosing $t = t_0/4$, we get the result.
The proof in the case of the Poisson semigroup is analogous.□

Proof of Proposition 2.1 Clearly (a) implies (b) and this implies (c).

Let assume now that (c) holds. Hence, given a positive function $f \in L^p(\mathbb{R}^n, v(x)dx)$, there exists a set N (which depends on f) of measure zero such that $\phi_{t_0} * f(x) < \infty$ for $x \in \mathbb{R}^n \setminus N$.

Let $x_0 = (x_0^1, \ldots, x_0^n) \in N$ such that $\phi_{t_0} * f(x_0) = \infty$. Without lost of generality we can assume that the support of f is included in the set

$$A_1 = \{y = (y_1, \ldots, y_n) : y_1 - x_0^1 \geq 0, \ldots, y_n - x_0^n \geq 0.$$

Also consider the set $Q_0 = \{x : |x - x_0| < t_0\}$. If $x \in Q_0 \cap A_1$ then for all $y \in A_1$ is

$$|x-y|^2 = |x-x_0|^2 + |x_0-y|^2 + 2(x-x_0)(x_0-y) \leq |x-x_0|^2 + |x_0-y|^2 \leq t_0^2 + |x_0-y|^2.$$

Thus, $\phi_{t_0}(x-y) \geq C_0\, \phi_{t_0}(x_0-y)$ for some positive constant C_0 and all $y \in A_1$, therefore,

$$\phi_{t_0} * f(x) \geq C_0\, \phi_{t_0} * f(x_0) = \infty, \quad \text{for all} \quad x \in Q_0 \cap A_1,$$

contradicting the fact that N is of null measure. Hence the statement in (c) is valid for every point $x \in \mathbb{R}^n$ and the functional

$$f \to \int_{\mathbb{R}^n} \phi_{t_0}(x-y)f(y)dy$$

is well defined for all $f \in L^p(\mathbb{R}^n, v(x)dx)$ and for every $x \in \mathbb{R}^n$.
By Landau's Principle of Resonance the mapping

$$y \to \phi_{t_0}(x-y)$$

belongs to $L^{p'}(\mathbb{R}^n, v^{-\frac{p'}{p}}(x)dx)$ for every $x \in \mathbb{R}^n$ thus obtaining (d).

Finally if (d) holds then by Hölder's inequality we get that

$$\int |\phi_t * f(x)|^p u(x)dx \leq \int |f(y)|^p v(y)\, dy \int \Big(\int \phi_t{}^{p'}(|x-y|) v^{-\frac{p'}{p}}(y)dy \Big)^{\frac{p}{p'}} u(x)dx.$$

Applying Lemma 3.1 there exists $t > 0$ such that

$$\psi(x) = \Big(\int \phi_{t_0}{}^{p'}(|x-y|) v^{-\frac{p'}{p}}(y)dy \Big)^{\frac{p}{p'}}$$

is finite for all x, then it is enough to choose $u \in L^1_{loc}$ such that $\psi\, u \in L^1$ to obtain (a). This ends the proof of Proposition 2.1.$\square$

Since ϕ is a radial and integrable function, the maximal operator $\Phi^* f(x) = \sup_t \phi_t * f(x)$ is bounded by a constant times the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{r>0} \frac{1}{r^n} \int_{B(x,r)} |f(y)|dy.$$

Since W and P are radial and integrable functions, any good weight for the operator M would be good for our purposes.

Seeking good weights for the operator M we recall some results going back to the 80's, due independently to J.L. Rubio de Francia [RdF] and to L. Carleson and P. Jones [C-J].

THEOREM **3.2.** *Let v be a weight in $\mathbb{R}^n$ and $1 < p < \infty$.*
The following statements are equivalent:

(I) There exists a weight u such that the Hardy-Littlewood maximal operator M is bounded from $L^p(\mathbb{R}^n, v(x)dx)$ to $L^p(\mathbb{R}^n, u(x)dx)$.

(II) There exists a constant C such that

$$\sup_{R>1} \frac{1}{R^{n\,p'}} \int_{B(0,R)} v^{-\frac{p'}{p}}(y)dy \leq C,$$

i.e. $v \in D_p^*$, (see Definition 2.4).

REMARK **3.3.** *Statement* (II) *in Theorem 3.2 can be replaced by*

(II') *For any $a > 0$, there exists a constant C_a such that*

$$\sup_{R>a} \frac{1}{R^{n\,p'}} \int_{B(0,R)} v^{-\frac{p'}{p}}(y)dy \leq C_a.$$

To see this claim, we observe that if $a < 1$ and $a < S < 1$ then

$$\begin{aligned} \frac{1}{S^{n\,p'}} \int_{B(0,S)} v^{-\frac{p'}{p}}(y)dy &\leq \frac{1}{a^{n\,p'}} \int_{B(0,1)} v^{-\frac{p'}{p}}(y)dy \\ &\leq \frac{1}{a^{n\,p'}} \sup_{R>1} \frac{1}{R^{n\,p'}} \int_{B(0,R)} v^{-\frac{p'}{p}}(y)dy. \end{aligned}$$

Even more, statement (II) can be replaced by

(II'') *For any $x \in \mathbb{R}^n$, and any $a > 0$, there exists a constant $C_{a,x}$ such that*

$$\sup_{R>a} \frac{1}{R^{n\,p'}} \int_{B(x,R)} v^{-\frac{p'}{p}}(y)dy \leq C_{x,a}.$$

In order to prove this claim, we observe that

$$\begin{aligned} \frac{1}{R^{n\,p'}} \int_{B(x,R)} v^{-\frac{p'}{p}}(y)dy &\leq \frac{1}{R^{n\,p'}} \int_{B(0,|x|+R)} v^{-\frac{p'}{p}}(y)dy \\ &\leq \frac{(|x|+R)^{n\,p'}}{R^{n\,p'}} (|x|+R)^{n\,p'-1} \int_{B(0,|x|+R)} v^{-\frac{p'}{p}}(y)dy, \end{aligned}$$

then, we use (II'). $\square$

PROOF OF THEOREM 2.5

Given $t > 0$,

$$\begin{aligned} P_t(x-y) &= C_n \frac{1}{t^n\Big(1+\frac{|x-y|^2}{t^2}\Big)^{\frac{n+1}{2}}} \\ &\leq C_n\Big(\frac{1}{t^n}\chi_{\{|x-y|<t\}}(y) + \sum_{j=0}^{\infty} \frac{1}{t^n(2^j)^{n+1}}\chi_{\{2^j\,t<|x-y|<2^{j+1}\,t\}}(y)\Big) \\ &\leq C_n\Big(\frac{1}{t^n}\chi_{\{|x-y|<t\}}(y) + \sum_{j=0}^{\infty} 2^{-j}\frac{1}{(2^{j+1}\,t)^n}\chi_{\{|x-y|<2^{j+1}\,t\}}(y)\Big). \end{aligned}$$

Thus,

$$\int P_t(x-y)^{p'} v^{-\frac{p'}{p}}(y)dy \leq C \sup_{R\geq t} \frac{1}{R^{n\,p'}} \int_{B(x,R)} v^{p'/p}(y)dy.$$

From Remark 3.3 it follows that $D_p^* \subset D_p^P$.

Since $W_{t^2}(x) \leq C P_t(x)$ then $D_p^P \subset D_p^W$.
The following chain of inequalities proves $D_p^W \subset D_p^{loc}$:

$$\begin{aligned} \int_{\{|x-y|<R^{1/2}\}} v^{-\frac{p'}{p}}(y)\,dy &\leq e^{p'} \int_{\{|x-y|<R^{1/2}\}} e^{-\frac{|x-y|^2}{R}p'} v^{-\frac{p'}{p}}(y)dy \\ &\leq e^{p'} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{R}p'} v^{-\frac{p'}{p}}(y)dy. \end{aligned}$$

To finish the proof of Theorem 2.5 it remains to show that each class is strictly included in the bigger class. We leave to the reader to check the following assertions:

(a) The weight $v_1(y) = e^{-|x|^3 p}$ belongs to D_p^{loc} but $v_1 \notin D_p^W$.
(b) The weight $v_2(y) = |x|^{-(n+1)\,p}$ belongs to D_p^W but $v_2 \notin D_p^P$.
(c) The weight $v_3(y) = |x|^{-np-\varepsilon p}$ with $\frac{n}{p'}+1 > 1-\varepsilon > \frac{n}{p'}$, belongs to D_p^P but $v_3 \notin D_p^*$.

This ends the proof of Theorem 2.5.□

In order to prove Theorem 2.3 we need a technical result about the local Hardy-Littlewood maximal function. Given $R > 0$ the local Hardy-Littlewood maximal function $\mathcal{M}_R f$ is defined by

$$\mathcal{M}_R f(x) = \sup_{0<s\leq R} \mathcal{A}_s f(x),$$

where $\mathcal{A}_s f(x) = \frac{1}{s^n} \int_{|x-y|<s} f(y)dy$.

Lemma **3.4.** *Let v be a weight in $\mathbb{R}^n$. Let $1 < p < \infty$ and $R > 0$ fix..*
The following statements are equivalent:

(i) There exists a weight u such that $\mathcal{M}_R$ is bounded from $L^p(\mathbb{R}^n, v(x)dx)$ to $L^p(\mathbb{R}^n, u(x)dx)$.
(ii) There exists a weight u such that $\mathcal{M}_R$ is bounded from $L^p(\mathbb{R}^n, v(x)dx)$ to weak-$L^p(\mathbb{R}^n, u(x)dx)$.
(iii) There exists a weight u such that $\mathcal{A}_R$ is bounded from $L^p(\mathbb{R}^n, v(x)dx)$ to weak-$L^p(\mathbb{R}^n, u(x)dx)$.
(iv) The weight

$$v^{-\frac{p'}{p}} \in L^1_{loc},$$

i.e. $v \in D_p^{loc}$. (see Definition 2.4).

To prove the above Lemma we need the following technical Lemma due to J.L. Rubio de Francia in [RdF]. It can be found in the form we need in [F-T].

Lemma **3.5.** *Let (X, μ) a measurable space, $\mathcal{B}$ a Banach space and T a sublinear operator from $T : \mathcal{B} \to L^s(X)$, for some $s < p$, satisfying*

$$\|(\sum_{j\in\mathbb{Z}} |Tf_j|^p)^{1/p}\|_{L^s(X)} \leq C_{p,s}(\sum_{j\in\mathbb{Z}} \|f_j\|^p_{\mathcal{B}})^{1/p},$$

where $C_{p,s}$ is a constant depending on p and s.
Then there exists a function u such that $u^{-1} \in L^{s/p}(X)$, $\|u^{-1}\|_{s/p} \leq 1$ and

$$\int_{\mathbb{R}^n} |Tf(x)|^p u(x) d\mu(x) \leq C\|f\|_{\mathcal{B}}$$

for some constant C.

Proof of Lemma 3.4

We shall prove (iii) ⇒ (iv) ⇒ (i), the rest of the implications are obvious. Assume that (iii) holds.
Let $x_0 \in \mathbb{R}^n$ and $R > 0$ fix. We consider the sets $S_j = \{y : v(y) > j^{-1}\}$.
Since $B(x_0, R/2) \subset B(x, R)$ for $x \in B(x_0, R/2)$ then, for any nonnegative f, we have

$$\mathcal{A}_R f(x) = \frac{1}{R^n} \int_{B(x,R)} f(y)dy \geq \frac{1}{R^n} \int_{B(x_0,R/2)} f(y)dy$$

Therefore

$$\begin{aligned} u(B(x_0,R/2)) &\le \int_{\{x:\mathcal{A}_R f(x)>\frac{1}{R^n}\int_{B(x_0,R/2)} f(y)dy\}} u(x)dx \\ &\le \frac{R^{np}}{\left(\int_{B(x_0,R/2)} f(y)dy\right)^p} \int f^p(y)v(y)dy. \end{aligned}$$

In the last inequality we choose $f(x) = v^{-\frac{p'}{p}}(x)\chi_{B(x_0,R/2)}(x)\chi_{S_j}(x)$ and conclude that

$$\int_{S_j\cap B(x_0,R/2)} v^{-\frac{p'}{p}}(y)dy \le \frac{C_R}{u(B(x_0,R/2))^{\frac{p'}{p}}} \le C_{R,x_0},$$

with the constant C_{R,x_0} independent of j. Hence

$$\int_{B(x_0,R/2)} v^{-\frac{p'}{p}}(y)dy \le C_{R,x_0}.$$

Finally since $B(x_0,R)$ can be covered by a finite number of balls of radius $R/2$ we get (iv).

Let now assume that (iv) holds. To prove (i) we define the sets $E_0 = B(0,R), E_k = \{x : 2^{k-1}R \le |x| < 2^k R\}$.
For each k fixed we split $f = f' + f''$, where $f'(x) = f(x)\chi_{B(0,R2^{k+1})}(x)$.
Using Kolmogorov's inequality and the continuity of the maximal operator in the vector valued setting, see [RdFRT], given $0 < s < 1 < p$ for each k is

$$\begin{aligned} \Big\|\Big(\sum_j |\mathcal{M}_R f'_j|^p\Big)^{1/p}\Big\|_{L^s(E_k)} &\le |E_k|^{1/s-1}\Big\|\Big(\sum_j |\mathcal{M}_R f'_j|^p\Big)^{1/p}\Big\|_{L^1_*(E_k)} \\ &\le C|E_k|^{1/s-1}\Big\|\Big(\sum_j |f'_j|^p\Big)^{1/p}\Big\|_{L^1} \\ &\le C|E_k|^{1/s-1}\int_{B(0,R2^{k+1})}\Big(\sum_j |f_j(x)|^p\Big)^{1/p}dx \\ &\le C|E_k|^{1/s-1}\Big(\int_{B(0,R2^{k+1})}\sum_j |f_j(x)|^p v(x)\,dx\Big)^{1/p}\Big(\int_{B(0,R2^{k+1})} v^{-\frac{p'}{p}}(x)dx\Big)^{1/p'} \\ &\le C_{k,v}|E_k|^{1/s-1}\Big(\int\sum_j |f_j(x)|^p v(x)dx\Big)^{1/p}. \end{aligned} \tag{3.6}$$

On the other hand, if $x \in E_k$ and $y \notin B(0,R\,2^{k+1})$ then,

$$R2^{k+1} < |y| \le |y-x| + |x| \le |y-x| + R2^k$$

and, thus, $|y-x| > R2^k$. Hence

$$\mathcal{M}_R f''_j(x) = 0, \text{ for all } j \in \mathbb{N}, x \in E_k. \tag{3.7}$$

Pasting together (3.6) and (3.7), we see that the operator satisfies Lemma 3.5 in each set E_k. Hence a family of weights U_k, each one with support in E_k, can be found satisfying the statements in that Lemma.

The weight $u(x) = \sum_k \frac{1}{2^k} U_k(x)\chi_{E_k}(x)$ satisfies (i). We ended the proof of Lemma 3.4.$\square$

Proof of Theorem 2.3

The density of continuous functions with compact support on $L^p(\mathbb{R}^n, v(x)dx)$ gives (2) $\Rightarrow$ (3). On the other hand Proposition 2.1 together with the arguments in its proof give (4) $\Rightarrow$ (5). Hence the implication to be proved is (5)$\Rightarrow$ (1). We shall give the proof in the case $\phi = W$, see 2.4. Given $R > 0$ and $0 < t < R$ we split

$$W_t = W_t^1 + W_t^2$$

where $W_t^1 = W_t \chi_{\{|x| \le (2nR)^{1/2}\}}$.
If $j_0 \in \mathbb{Z}$ is such that $2^{j_0} t < R < 2^{j_0+1} t$ then

$$\begin{aligned} W_t^1(x) &\le W_t(x)\Big(\chi_{\{|x|\le(2nt)^{1/2}\}}(x) + \sum_{j=0}^{j_0} \chi_{\{(2n2^jt)^{1/2}\le|x|\le(2n2^{j+1}t)^{1/2}\}}(x)\Big) \\ &\le C\Big(\frac{1}{t^{\frac{n}{2}}}\chi_{\{|x|\le(2nt)^{1/2}\}}(x) + \sum_{j=0}^{j_0}(2n2^j)^{\frac{n}{2}} e^{-\frac{n}{2}2^j}\frac{1}{(2n2^jt)^{\frac{n}{2}}}\chi_{\{|x|\le(2n2^{j+1}t)^{1/2}\}}(x)\Big). \end{aligned}$$

Thus for $f \ge 0$

$$\sup_{t<R} W_t^1 * f(x) \le C_n \mathcal{M}_{(2nR)^{1/2}} f(x) \tag{3.8}$$

with $C_n = C\Big((2n)^{\frac{n}{2}} + \sum_{j=0}^{\infty}(2n2^j)^{\frac{n}{2}} e^{-\frac{n}{2}2^j}\Big) < \infty$.
On the other hand, since $W_t^2(x)$ is increasing in the time interval $(0, R)$ we also have

$$\sup_{0<t<R} W_t^2 * f(x) = W_R^2 * f(x) \le W_R * f(x) \tag{3.9}$$

Thus, by (3.8) and (3.9)

$$W_R^* f(x) \le C\left(\mathcal{M}_{(2nR)^{1/2}} f(x) + W_R * f(x)\right) \tag{3.10}$$

then the result follows by using Proposition 2.1, Theorem 2.5 and Lemma 3.4.

The proof in the case $\phi = P$ follows choosing $P_t^1 = P_t \chi_{\{|x|\le(n)^{1/2}R\}}$ and repeating the above argument.

References

[C-J] L. Carleson and P. Jones, Weighted norm inequalities and a Theorem of Koosis. Mittag-Leffler Institut, Report n. 2, 1981.

[F-T] L. M. Fernández-Cabrera and J.L. Torrea, Vector-alued inequalities with weights, Pub. Mat. 37, 1993, 177-208.

[G-G] A.E. Gatto and C. Gutiérrez, On weighted norm inequalities for the maximal function, Studia Math.,76, N.1, 1983, 59-62.

[M] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function. Trans. Amer. Math. Soc. 165, 1972, 207-226.

[RdF] J. L. Rubio de Francia, Weighted norm inequalities and vector valued inequalities. Harmonic analysis (Minneapolis, Minn., 1981), Lecture Notes in Math., 908, Springer, Berlin-New York, 1982, 86–101.

[RdFRT] J. L. Rubio de Francia, F. J. Ruiz and J.L. Torrea, Calderón-Zygmund Theory for operators-valued kernels. Adv. in Math., 62, 1986, 7-48.

[S] E. Sawyer, Two weight norm inequalities for certain maximal and integral operators, Harmonic Analysis, Lecture Notes in Mathematics, volume 908, 1982, 102-127.

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