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Confluence and combinatorics in finitely generated unital lattice-ordered abelian groups

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Abstract. A unital ℓ-group (G, u) is an abelian group G equipped with a translation-invariant lattice-order and a distinguished element u, called order-unit, whose positive integer multiples eventually dominate each element of G. It is shown that, for direct systems $\mathcal{S}$ and $\mathcal{T}$ of finitely presented unital ℓ-groups, confluence is a necessary condition for $\lim \mathcal{S} \cong \lim \mathcal{T}$. (Sufficiency is an easy byproduct of a general result). When (G, u) is finitely generated we equip it with a sequence $\mathcal{W}_{(G,u)} = (W_0, W_1, \ldots)$ of weighted abstract simplicial complexes, where W_{t+1} is obtained from W_t either by the classical Alexander binary stellar operation, or by deleting a maximal simplex of W_t. We show that the map $(G, u) \mapsto \mathcal{W}_{(G,u)}$ has an inverse. A confluence criterion is given to recognize when two sequences arise from isomorphic unital ℓ-groups.

Keywords. Lattice-ordered abelian group, rational polyhedron, order-unit, confluence, direct system, confluent direct system, simplicial complex, abstract simplicial complex, weighted abstract simplicial complex, stellar subdivision, Alexander starring, regular fan, De Concini–Procesi theorem, piecewise linear function, Elliott classification, AF C^*-algebra.

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1 Introduction

This paper deals with an abelian group G equipped with a translation-invariant lattice-order and a distinguished *order-unit* u, i.e., an element whose positive integer multiples eventually dominate each element of G. For brevity, (G, u) will be called a *unital ℓ-group*. We refer to [5, 15] for background. Morphisms in the category of unital ℓ-groups are known as *unital ℓ-homomorphisms*: they preserve the lattice, the group structure and map order-units into order-units. Whenever the context is clear, we will write for short "isomorphism" instead of "unital ℓ-isomorphism".

Since a categorical equivalence Γ exists between unital ℓ-groups and the equational class of MV-algebras (see [20]) one can naturally define free unital ℓ-groups

(Theorem 2.1), as well as finitely presented unital ℓ-groups. The latter are defined as usual as the quotients of free finitely generated unital ℓ-groups modulo a finitely generated congruence. Then every finitely generated unital ℓ-group is the direct limit (= filtered colimit, in categorical language) of a countable direct system of finitely presented unital ℓ-groups with surjective connecting unital ℓ-homomorphisms. And conversely, the direct limit of any such sequence is a finitely generated unital ℓ-group.

Two sequences of unital ℓ-groups

$$(G_0, u_0) \twoheadrightarrow (G_1, u_1) \twoheadrightarrow \cdots \quad \text{and} \quad (H_0, v_0) \twoheadrightarrow (H_1, v_1) \twoheadrightarrow \cdots \tag{1.1}$$

are said to be *confluent* if there are indices $i(1) < j(1) < i(2) < j(2) < \cdots$ and surjective unital ℓ-homomorphisms

$$f_{i(k)}\colon (G_{i(k)}, u_{i(k)}) \twoheadrightarrow (H_{j(k)}, v_{j(k)}),$$
$$g_{j(k)}\colon (H_{j(k)}, v_{j(k)}) \twoheadrightarrow (G_{i(k+1)}, u_{i(k+1)})$$

such that the composite map $g_{j(k)} \circ f_{i(k)}$ coincides with the map $(G_{i(k)}, u_{i(k)}) \twoheadrightarrow (G_{i(k+1)}, u_{i(k+1)})$ and conversely, $f_{i(k+1)} \circ g_{j(k)}$ coincides with $(H_{j(k)}, v_{j(k)}) \twoheadrightarrow (H_{j(k+1)}, v_{j(k+1)})$ in (1.1). Then by a standard argument [12, 2, VIII, 4.13–4.15], the confluence of the two sequences above is *sufficient* for their direct limits to be isomorphic. While in general categories confluence is not a necessary condition for direct limits to be isomorphic, in Theorems 3.1 and 3.3 it is proved that direct systems of unital ℓ-groups and unital ℓ-homomorphisms with isomorphic limits are necessarily confluent.

We next deal with finitely generated unital ℓ-groups. In Section 4 we recall the definition of a weighted abstract simplicial complex, i.e., an (always finite) abstract simplicial complex K enriched with a weight function from the set of vertices of K into $\{1, 2, 3, \ldots\}$. Using Alexander stellar operations we introduce suitable sequences of weighted abstract simplicial complexes, called *stellar sequences*. In Section 5 we construct a map assigning to each stellar sequence a unital ℓ-group, and in Theorem 5.1 we prove that, up to isomorphism, every finitely generated unital ℓ-group arises from some stellar sequence. In Corollary 5.4 a necessary and sufficient condition is given to recognize when two stellar sequences yield isomorphic unital ℓ-groups.

2 Unital ℓ-groups, polyhedra and regular complexes

Lattice-ordered abelian groups with order-unit

A *lattice-ordered abelian group* (*ℓ-group*) is a structure $(G, +, -, 0, \vee, \wedge)$ such that $(G, +, -, 0)$ is an abelian group, $(G, \vee, \wedge)$ is a lattice, and $x + (y \vee z) =$

$(x+y) \vee (x+z)$ for all $x, y, z \in G$. An *order-unit* in G ("unité forte" in [5]) is an element $u \in G$ having the property that for every $g \in G$ there is $0 \leq n \in \mathbb{Z}$ such that $g \leq nu$. A *unital* ℓ-group (G, u) is an ℓ-group G with a distinguished order-unit u.

By an *ℓ-ideal* I of (G, u) we mean the kernel of a unital ℓ-homomorphism. Any such I determines the (quotient) unital ℓ-homomorphism $(G, u) \to (G, u)/I$ in the usual way [5, 15].

We let $\mathcal{M}([0,1]^n)$ denote the unital ℓ-group of piecewise linear continuous functions $f\colon [0,1]^n \to \mathbb{R}$ such that each piece of f has integer coefficients, with the constant function 1 as a distinguished order-unit. The number of pieces is always finite; "linear" is to be understood in the affine sense.

More generally, for any nonempty subset $X \subseteq [0,1]^n$ we denote by $\mathcal{M}(X)$ the unital ℓ-group of restrictions to X of the functions in $\mathcal{M}([0,1]^n)$, with the constant function 1 as the order-unit. For every $f \in \mathcal{M}(X)$ we let $\mathcal{Z}(f) = f^{-1}(0)$. For every ℓ-ideal I of $\mathcal{M}(X)$ we let $\mathcal{Z}(I) = \{Y \subseteq X \mid \exists g \in I \text{ with } Y = \mathcal{Z}(g)\}$.

The coordinate functions $\pi_i\colon [0,1]^n \to \mathbb{R}$ $(i = 1, \ldots, n)$, together with the unit 1, generate the unital ℓ-group $\mathcal{M}([0,1]^n)$. They are said to be a *free generating set* of $\mathcal{M}([0,1]^n)$ because they have the following universal property:

Theorem 2.1 ([20, 4.16]). *Let $\{g_1, \ldots, g_n\} \subseteq [0, u]$ be a set of generators of a unital ℓ-group (G, u). Then the* map $\pi_i \mapsto g_i$ *can be uniquely extended to a unital ℓ-homomorphism of* $\mathcal{M}([0,1]^n)$ *onto* (G, u).

Corollary 2.2. *Up to isomorphism, every finitely generated unital ℓ-group has the form* $\mathcal{M}([0,1]^n)\,/I$ *for some $n = 1, 2, \ldots$ and ℓ-ideal I of* $\mathcal{M}([0,1]^n)$.

Rational polyhedra, complexes and regularity

Following [25, p. 4], by a *polyhedron* $P \subseteq \mathbb{R}^n$ we mean a finite union of convex hulls of finite sets of points in $\mathbb{R}^n$. A *rational polyhedron* is a finite union of convex hulls of finite sets of rational points in $\mathbb{R}^n$, $n = 1, 2, \ldots$. An example of a rational polyhedron $P \subseteq [0,1]^n$ is given by the zeroset $\mathcal{Z}(f)$ of any $f \in \mathcal{M}([0,1]^n)$. In Propositions 2.4 and 2.6 below we will see that this is the most general possible example.

As an immediate consequence of the definitions we have

Lemma 2.3. *If $\mathcal{P} = P_1 \supseteq P_2 \supseteq P_3 \supseteq \cdots$ is a sequence of nonempty rational polyhedra in the n-cube, then the set*

$$\langle \mathcal{P} \rangle = \{f \in \mathcal{M}([0,1]^n) \mid \mathcal{Z}(f) \supseteq P_i \textit{ for some } i = 1, 2, \ldots\}$$

is an ℓ-ideal of $\mathcal{M}([0,1]^n)$.

For any rational point $y \in \mathbb{R}^n$ we denote by $\mathrm{den}(y)$ the least common denominator of the coordinates of y. The integer vector $\tilde{y} = \mathrm{den}(y)(y, 1) \in \mathbb{Z}^{n+1}$ is called the *homogeneous correspondent* of y. For every rational m-simplex $T = \mathrm{conv}(v_0, \ldots, v_m) \subseteq \mathbb{R}^n$, we will use the notation

$$T^{\uparrow} = \mathbb{R}_{\geq 0}\, \tilde{v}_0 + \cdots + \mathbb{R}_{\geq 0}\, \tilde{v}_m$$

for the positive span of $\tilde{v}_0, \ldots, \tilde{v}_m$ in $\mathbb{R}^{n+1}$.

We refer to [14] for background on simplicial complexes. Unless otherwise specified, every complex $\mathcal{K}$ in this paper will be simplicial, and the adjective "simplicial" will be omitted. For every complex $\mathcal{K}$, its *support* $|\mathcal{K}|$ is the pointset union of all simplexes of $\mathcal{K}$. We say that the complex $\mathcal{K}$ is *rational* if all simplexes of $\mathcal{K}$ are rational: in this case, the set

$$\mathcal{K}^{\uparrow} = \{T^{\uparrow} \mid T \in \mathcal{K}\}$$

is known as a simplicial fan [14].

A rational m-simplex $T = \mathrm{conv}(v_0, \ldots, v_m) \subseteq \mathbb{R}^n$ is *regular* if $\{\tilde{v}_0, \ldots, \tilde{v}_m\}$ is part of a basis in the free abelian group $\mathbb{Z}^{n+1}$. A rational complex Δ is said to be *regular* if every simplex $T \in \Delta$ is regular. In other words, the fan $\Delta^{\uparrow}$ is regular [14, V, §4].

For later use we recall here some results about regular complexes and rational polyhedra. For the proofs we refer to [18] and [22], where regular complexes are called "unimodular triangulations".

Proposition 2.4 ([18, 5.1]). *A set $X \subseteq [0, 1]^n$ coincides with the support of some regular complex Δ iff $X = \mathcal{Z}(f)$ for some $f \in \mathcal{M}([0, 1]^n)$.*

Proposition 2.5 ([18, 5.2]). *A unital ℓ-group (G, u) is finitely presented iff there exists a rational polyhedron $P \in [0, 1]^n$ such that $(G, u) \cong \mathcal{M}(P)$ for some $n \in \{1, 2, \ldots\}$.*

Proposition 2.6 ([22, p. 539]). *Any rational polyhedron $P \subseteq [0, 1]^n$ is the support of some regular complex Δ.*

Subdivision, blow-up, Farey mediant

Given complexes $\mathcal{K}$ and $\mathcal{H}$ with $|\mathcal{K}| = |\mathcal{H}|$ we say that $\mathcal{H}$ is a *subdivision* of $\mathcal{K}$ if every simplex of $\mathcal{H}$ is contained in a simplex of $\mathcal{K}$. For any point $p \in |\mathcal{K}| \subseteq \mathbb{R}^n$, the *blow-up $\mathcal{K}_{(p)}$ of $\mathcal{K}$ at p* is the subdivision of $\mathcal{K}$ given by replacing every simplex $T \in \mathcal{K}$ that contains p by the set of all simplexes of the form $\mathrm{conv}(F \cup \{p\})$, where F is any face of T that does not contain p (see [14, III, 2.1] or [26, p. 376]).

For any regular 1-simplex $E = \mathrm{conv}(v_0, v_1) \subseteq \mathbb{R}^n$, the *Farey mediant* of E is the rational point v of E whose homogeneous correspondent $\tilde{v}$ coincides with $\tilde{v}_0 + \tilde{v}_1$. If E belongs to a regular complex Δ and v is the Farey mediant of E, then the blow-up $\Delta_{(v)}$, called *binary Farey blow-up*, is a regular complex.

Proposition 2.7. *Suppose we are given rational polyhedra $Q \subseteq P \subseteq [0,1]^n$ and a regular complex Δ with support P. Then there is a subdivision $\Delta^\natural$ of Δ obtained by binary Farey blow-ups such that $Q = \bigcup\{T \in \Delta^\natural \mid T \subseteq Q\}$.*

Proof. We closely follow the argument of the proof in [22, p. 539]. Let us write $Q = T_1 \cup \cdots \cup T_t$ for suitable rational simplexes. Let $\mathcal{H} = \{H_1, \ldots, H_h\}$ be a set of rational half-spaces in $\mathbb{R}^n$ such that every T_j is the intersection of half-spaces of $\mathcal{H}$. Using the De Concini–Procesi theorem [14, p. 252], we obtain a sequence of regular complexes $\Delta = \Delta_0, \Delta_1, \ldots, \Delta_r$ where each Δ_{k+1} is obtained by blowing-up Δ_k at the Farey mediant of some 1-simplex of Δ_k, and for each $i = 1, \ldots, h$, the convex polyhedron $H_i \cap [0,1]^n$ is a union of simplexes of Δ_r. It follows that each simplex $T_1, \ldots, T_t$ is a union of simplexes of Δ_r. Now $\Delta^\natural = \Delta_r$ yields the desired subdivision of Δ. □

The following proposition states that every ℓ-ideal I of $\mathcal{M}(P)$ is uniquely determined by the zerosets of all functions in I:

Proposition 2.8. *Suppose that $P \subseteq [0,1]^n$ is a rational polyhedron and I is an ℓ-ideal of $\mathcal{M}(P)$. Then for every $f \in \mathcal{M}(P)$ we have $f \in I$ iff $\mathcal{Z}(f) \supseteq \mathcal{Z}(g)$ for some $g \in I$.*

Proof. For the nontrivial direction, suppose $\mathcal{Z}(f) \supseteq \mathcal{Z}(g)$ and without loss of generality, $g \geq 0$, and $f \geq 0$. We must find $0 \leq m \in \mathbb{Z}$ such that $mg \geq f$. By Proposition 2.6, P is the support of some regular complex Λ. By suitably subdividing Λ, we obtain a rational (simplicial but not necessarily regular) complex Δ with $|\Delta| = P$ such that over every $T \in \Delta$ both f and g are linear. Let $\{v_1, \ldots, v_s\}$ be the vertices of Δ. For each $i = 1, \ldots, s$, since $f(v_i) \neq 0$ implies $g(v_i) \neq 0$, there exists an integer $m_i > 0$ such that $m_i g(v_i) \geq f(v_i)$. Letting $m = \max(m_1, \ldots, m_s)$, the desired result follows from the linearity of f and g over each simplex of Δ. □

Proposition 2.9. *Let I be an ℓ-ideal of $\mathcal{M}([0,1]^m)$ and $P \in \mathcal{Z}(I)$. Let further*

(i) $I \upharpoonright P = \{f \upharpoonright P \mid f \in I\}$,

(ii) $\mathcal{Z}(I)_{\cap P} = \{X \cap P \mid X \in \mathcal{Z}(I)\}$,

(iii) $\mathcal{Z}(I)_{\subseteq P} = \{X \in \mathcal{Z}(I) \mid X \subseteq P\}$.

Then $\mathcal{Z}(I \upharpoonright P) = \mathcal{Z}(I)_{\cap P} = \mathcal{Z}(I)_{\subseteq P}$.

Proof. $[\mathcal{Z}(I)_{\cap P} \subseteq \mathcal{Z}(I)_{\subseteq P}]$: Let $X \in \mathcal{Z}(I)_{\cap P}$. By definition of $\mathcal{Z}(I)_{\cap P}$, there exists $f \in I$ such that $X = \mathcal{Z}(f) \cap P$. Combining Propositions 2.4 and 2.6 there exists $g \in \mathcal{M}([0,1]^m)$ such that $P = \mathcal{Z}(g)$. Since $P \in \mathcal{Z}(I)$, $g \in I$ by Proposition 2.8. Therefore, $|f| + |g| \in I$ and $X = \mathcal{Z}(f) \cap P = \mathcal{Z}(|f|) \cap \mathcal{Z}(|g|) = \mathcal{Z}(|f| + |g|) \in \mathcal{Z}(I)_{\subseteq P}$.

The inclusions $[\mathcal{Z}(I)_{\subseteq P} \subseteq \mathcal{Z}(I)_{\cap P}]$, $[\mathcal{Z}(I \upharpoonright P) \subseteq \mathcal{Z}(I)_{\cap P}]$, and $[\mathcal{Z}(I)_{\cap P} \subseteq \mathcal{Z}(I \upharpoonright P)]$ immediately follow by definition. □

3 $\mathbb{Z}$-homeomorphism of rational polyhedra

Given rational polyhedra $P \subseteq \mathbb{R}^m$ and $Q \subseteq \mathbb{R}^n$, a piecewise linear homeomorphism η of P onto Q is said to be a $\mathbb{Z}$-*homeomorphism*, in symbols, $\eta\colon P \cong_{\mathbb{Z}} Q$, if all linear pieces of η and η^{-1} have integer coefficients.

The following first main result of this paper highlights the mutual relations between $\mathbb{Z}$-homeomorphisms of polyhedra and isomorphisms of finitely generated unital ℓ-groups, as represented by Corollary 2.2:

Theorem 3.1. *For any ℓ-ideals I of $\mathcal{M}([0,1]^m)$ and J of $\mathcal{M}([0,1]^n)$ the following conditions are equivalent:*

(i) $\mathcal{M}([0,1]^m)/I \cong \mathcal{M}([0,1]^n)/J$.

(ii) *For some $P \in \mathcal{Z}(I)$, $Q \in \mathcal{Z}(J)$ and $\mathbb{Z}$-homeomorphism η of P onto Q, the map $X \mapsto \eta(X)$ sends $\mathcal{Z}(I)_{\cap P}$ one-one onto $\mathcal{Z}(J)_{\cap Q}$.*

Proof. (i) $\Rightarrow$ (ii) Let $\iota\colon \mathcal{M}([0,1]^m)/I \cong \mathcal{M}([0,1]^n)/J$, and $\epsilon = \iota^{-1}$. Let id_m denote the identity $(\pi_1, \ldots, \pi_m)$ over the m-cube, and id_n the identity over the n-cube. Each element $\pi_i/I \in \mathcal{M}([0,1]^m)/I$ is sent by ι to some element a_i/J of $\mathcal{M}([0,1]^n)/J$. Writing $[0,1] \ni ((a_i/J) \vee 0) \wedge 1 = ((a_i \vee 0) \wedge 1)/J$, and replacing, if necessary, a_i by $(a_i \vee 0) \wedge 1$, it is no loss of generality to assume that a_i belongs to the unit interval of $\mathcal{M}([0,1]^n)$, i.e., the range of a_i is contained in the unit interval $[0,1]$. Thus for a suitable m-tuple $a = (a_1, \ldots, a_m)$ of functions $a_i \in \mathcal{M}([0,1]^n)$ we have $a\colon [0,1]^n \to [0,1]^m$. Symmetrically, for some $b = (b_1, \ldots, b_n)\colon [0,1]^m \to [0,1]^n$, $b_j \in \mathcal{M}([0,1]^m)$, we can write

$$\iota\colon \mathrm{id}_m/I \mapsto a/J \quad \text{and} \quad \epsilon\colon \mathrm{id}_n/J \mapsto b/I. \tag{3.1}$$

For any $f \in \mathcal{M}([0,1]^m)$ and $g \in \mathcal{M}([0,1]^n)$, arguing by induction on the number of operations in f and g in the light of Theorem 2.1, we get the following generalization of (3.1):

$$\iota\colon f/I \mapsto (f \circ a)/J \quad \text{and} \quad \epsilon\colon g/J \mapsto (g \circ b)/I. \tag{3.2}$$

It follows that

$$\frac{f}{I} = (\epsilon \circ \iota)\frac{f}{I} = \epsilon\left(\iota\left(\frac{f}{I}\right)\right) = \epsilon\left(\frac{f \circ a}{J}\right) = \frac{f \circ a \circ b}{I}.$$

By definition of the congruence induced by I, for each $i = 1, \ldots, m$ the function $|\pi_i - a_i \circ b| = |\pi_i - \pi_i \circ a \circ b|$ belongs to I. Here, as usual, $|\cdot|$ denotes absolute value. It follows that the function $e = \sum_{i=1}^{m} |\pi_i - a_i \circ b|$ belongs to I, and its zeroset $\mathcal{Z}(e)$ belongs to $\mathcal{Z}(I)$. The set $P = \mathcal{Z}(e)$ satisfies the identity $P = \{x \in [0,1]^m \mid (a \circ b)(x) = x\}$. One similarly notes that the set $Q = \{y \in [0,1]^n \mid (b \circ a)(y) = y\}$ belongs to $\mathcal{Z}(J)$. By construction, the restriction of b to P provides a $\mathbb{Z}$-homeomorphism η of P onto Q, whose inverse θ is the restriction of a to Q. In symbols,

$$b \restriction P = \eta\colon P \cong_{\mathbb{Z}} Q, \quad a \restriction Q = \theta\colon Q \cong_{\mathbb{Z}} P. \tag{3.3}$$

Suppose $X \in \mathcal{Z}(I)_{\cap P}$, with the intent of proving $\eta(X) \in \mathcal{Z}(J)_{\cap Q}$. By Proposition 2.9, we can write $X = \mathcal{Z}(k \restriction P)$ for some $k \in I$. By (3.2), the composite function $k \circ a$ belongs to J. Thus $\eta(X) = \eta(\mathcal{Z}(k \restriction P)) = \mathcal{Z}((k \restriction P) \circ \theta) = \mathcal{Z}(k \circ a \restriction Q) = Q \cap \mathcal{Z}(k \circ a) \in \mathcal{Z}(J)_{\cap Q}$. Reversing the roles of η and θ we have the required one-one correspondence $X \mapsto \eta(X)$ between $\mathcal{Z}(I)_{\cap P}$ and $\mathcal{Z}(J)_{\cap Q}$.

(ii) $\Rightarrow$ (i) Let I_P (resp., let J_Q) be the ℓ-ideal of $\mathcal{M}([0,1]^m)$ (resp., of $\mathcal{M}([0,1]^n)$) given by all functions identically vanishing over P (resp., over Q). By [18, 5.2], we have isomorphisms

$$\alpha\colon \mathcal{M}(P) \cong \mathcal{M}([0,1]^m)\,/I_P \quad \text{with} \quad \alpha(I \restriction P) = I/I_P \tag{3.4}$$

and

$$\beta\colon \mathcal{M}(Q) \cong \mathcal{M}([0,1]^n)\,/J_Q \quad \text{with} \quad \beta(J \restriction Q) = J/J_Q. \tag{3.5}$$

As a particular case of a general algebraic result (sometimes called "the second isomorphism theorem"), the map $\frac{f/I_P}{I/I_P} \mapsto \frac{f}{I}$ is an isomorphism of $\frac{\mathcal{M}([0,1]^m)\,/I_P}{I/I_P}$ onto $\frac{\mathcal{M}([0,1]^m)}{I}$. From (3.4)–(3.5) we have isomorphisms

$$\frac{\mathcal{M}([0,1]^m)}{I} \cong \frac{\mathcal{M}([0,1]^m)\,/I_P}{I/I_P} \cong \frac{\mathcal{M}(P)}{I \restriction P} \tag{3.6}$$

and

$$\frac{\mathcal{M}([0,1]^n)}{J} \cong \frac{\mathcal{M}([0,1]^n)\,/J_Q}{J/J_Q} \cong \frac{\mathcal{M}(Q)}{J \restriction Q}. \tag{3.7}$$

Letting $\theta = \eta^{-1}$, we have $\theta\colon Q \cong_{\mathbb{Z}} P$ and the map $\lambda\colon k \mapsto k \circ \theta$ is an isomorphism of $\mathcal{M}(P)$ onto $\mathcal{M}(Q)$. Further, the map $Y \mapsto \theta(Y)$ sends $\mathcal{Z}(J)_{\cap Q} = \mathcal{Z}(J \restriction Q)$ one-one onto $\mathcal{Z}(I)_{\cap P} = \mathcal{Z}(I \restriction P)$.

Claim. *The restriction of λ to the ℓ-ideal $I \upharpoonright P$ of $\mathcal{M}(P)$ maps $I \upharpoonright P$ one-one onto $J \upharpoonright Q$. Thus the map*

$$\frac{k}{I \upharpoonright P} \mapsto \frac{\lambda(k)}{\lambda(I \upharpoonright P)}$$

defines an isomorphism of $\mathcal{M}(P)/(I \upharpoonright P)$ onto $\mathcal{M}(Q)/(J \upharpoonright Q)$.

By Proposition 2.9, for each $l \in \mathcal{M}(P)$ if $l \in I \upharpoonright P$, then $\mathcal{Z}(l) \in \mathcal{Z}(I \upharpoonright P) = \mathcal{Z}(I)_{\cap P}$. Thus by definition of λ, $\mathcal{Z}(\lambda(l)) = \mathcal{Z}(l \circ \theta) = \eta(\mathcal{Z}(l)) \in \mathcal{Z}(J)_{\cap Q}$. By Proposition 2.8, $\lambda(l) \in J \upharpoonright Q$. Reversing the roles of λ and λ^{-1}, our claim is settled.

Combining (3.6)–(3.7) and our claim above we have isomorphisms

$$\frac{\mathcal{M}([0,1]^m)}{I} \cong \frac{\mathcal{M}(P)}{I \upharpoonright P} \cong \frac{\mathcal{M}(Q)}{J \upharpoonright Q} \cong \frac{\mathcal{M}([0,1]^n)}{J},$$

as required to conclude the proof. □

Using Theorem 3.1, in Theorem 3.3 below we will show that confluence is a necessary condition for two direct systems of finitely presented unital ℓ-groups to have isomorphic direct limits. For the proof we prepare

Corollary 3.2. *Let $P \subseteq [0,1]^m$ and $Q \subseteq [0,1]^n$ be rational polyhedra.*

(i) *$\mathcal{M}(P) \cong \mathcal{M}(Q)$ if and only if $P \cong_{\mathbb{Z}} Q$.*

(ii) *If η is a $\mathbb{Z}$-homeomorphism of Q onto some rational polyhedron $R \subseteq P$, the map $f \mapsto f \circ \eta$ is a unital ℓ-homomorphism of $\mathcal{M}(P)$ onto $\mathcal{M}(Q)$.*

(iii) *For every unital ℓ-homomorphism h of $\mathcal{M}(P)$ onto $\mathcal{M}(Q)$ there exists a unique $\mathbb{Z}$-homeomorphism θ of Q onto some rational polyhedron $R \subseteq P$ such that $h(f) = f \circ \theta$ for each $f \in \mathcal{M}(P)$.*

Proof. (i) Let $I_P = \{f \in \mathcal{M}([0,1]^m) \mid \mathcal{Z}(f) \supseteq P\}$ and $J_Q = \{g \in \mathcal{M}([0,1]^n) \mid \mathcal{Z}(g) \supseteq Q\}$. By [18, 5.2], the maps $\alpha\colon f \upharpoonright P \mapsto f/I_P$ and $\beta\colon g \upharpoonright Q \mapsto g/J_Q$ are isomorphisms of $\mathcal{M}(P)$ onto $\mathcal{M}([0,1]^m)\,/I_P$ and of $\mathcal{M}(Q)$ onto $\mathcal{M}([0,1]^n)\,/J_Q$, respectively. An application of Theorem 3.1 now settles (i).

(ii) By (i), $\mathcal{M}(R) \cong \mathcal{M}(Q)$. Let us define now the map $\iota\colon \mathcal{M}(R) \to \mathcal{M}(Q)$ by

$$\iota\colon f \mapsto f \circ \eta.$$

Then the proof of Theorem 3.1 shows that ι is an isomorphism of $\mathcal{M}(R)$ onto $\mathcal{M}(Q)$. The map $\lambda\colon g \mapsto g \upharpoonright R$ is an ℓ-homomorphism of $\mathcal{M}(P)$ onto $\mathcal{M}(R)$. Thus

$$\iota \circ \lambda(f) = (f \upharpoonright R) \circ \eta = f \circ \eta$$

for each $f \in \mathcal{M}(P)$, and the map $f \mapsto f \circ \eta$ is a unital ℓ-homomorphism of $\mathcal{M}(P)$ onto $\mathcal{M}(Q)$.

(iii) With reference to (i), let the unital ℓ-homomorphism h' of $\mathcal{M}([0,1]^m)$ onto $\mathcal{M}([0,1]^n)\,/\,J_Q$ be defined by $h'(f) = \beta(h(f \upharpoonright P))$. Letting I denote the kernel of h', it follows that $I_P \subseteq I$ and the map $\iota\colon f/I \mapsto h'(f)$ is an isomorphism of $\mathcal{M}([0,1]^m)\,/\,I$ onto $\mathcal{M}([0,1]^n)\,/\,J_Q$. By Theorem 3.1, there exist $S \in \mathcal{Z}(I)$, $T \in \mathcal{Z}(J_Q)$ and a $\mathbb{Z}$-homeomorphism η of S onto T such that the map $X \mapsto \eta(X)$ sends $\mathcal{Z}(I)_{\cap S}$ one-one onto $\mathcal{Z}(J_Q)_{\cap T}$. By definition of J_Q and Proposition 2.6, Q is the smallest element of $\mathcal{Z}(J_Q)$, whence $R = \eta^{-1}(Q)$ is the smallest element of $\mathcal{Z}(I)_{\cap S}$. In the proof of Theorem 3.1, a map $a\colon [0,1]^n \to [0,1]^m$ is introduced having the property that $\iota(f/I) = (f \circ a)/J_Q$ and $\eta^{-1} = a \upharpoonright T$ for each $f \in \mathcal{M}([0,1]^m)$. Since $Q \subseteq T$, for each $f \in \mathcal{M}([0,1]^m)$ we can write

$$\begin{aligned} h(f \upharpoonright P) &= \beta^{-1}(h'(f)) = \beta^{-1}(\iota(f/I)) = \beta^{-1}((f \circ a)/J_Q) = (f \circ a) \upharpoonright Q \\ &= f \circ (\eta^{-1} \upharpoonright Q). \end{aligned}$$

Let us define $\theta = \eta^{-1} \upharpoonright Q$. Then $\theta\colon Q \cong_{\mathbb{Z}} R$, $R \subseteq P \cap S \subseteq P$ and $h(f \upharpoonright P) = f \circ \theta$. Finally, the uniqueness of θ follows from the separation property [20, 4.17], stating that for any two distinct points $x, y \in P$ there is $f \in \mathcal{M}(P)$ with $f(x) = 0$ and $f(y) > 0$. □

Theorem 3.3. *Given direct systems $\mathcal{S}$ and $\mathcal{T}$ of finitely presented unital ℓ-groups with surjective connecting unital ℓ-homomorphisms*

$$\mathcal{S} = (G_0, u_0) \overset{f_1}{\twoheadrightarrow} (G_1, u_1) \overset{f_2}{\twoheadrightarrow} (G_2, u_2) \cdots,$$
$$\mathcal{T} = (H_0, v_0) \overset{g_1}{\twoheadrightarrow} (H_1, v_1) \overset{g_2}{\twoheadrightarrow} (H_2, v_2) \cdots,$$

let (G, u) and (H, v) denote their respective direct limits. Then the following conditions are equivalent:

(i) $(G, u) \cong (H, v)$.

(ii) *$\mathcal{S}$ and $\mathcal{T}$ are confluent.*

Proof. (ii) $\Rightarrow$ (i) was dealt with in the Introduction. For the converse implication, Proposition 2.5 yields rational polyhedra $P_0, P_1, \ldots$ such that $\mathcal{M}(P_i) \cong (G_i, u_i)$ for each $i = 0, 1, 2, \ldots$. Let $\theta_i\colon P_i \cong_{\mathbb{Z}} \theta_i(P_i) \subseteq P_{i-1}$ be the $\mathbb{Z}$-homeomorphism associated to each f_i, as given by Corollary 3.2. Let the sequence $\mathcal{P}$ be defined by

$$\mathcal{P} = P'_0 \supseteq P'_1 \supseteq P'_2 \supseteq \cdots,$$

where $P'_0 = P_0 \subseteq [0,1]^m$ and $P'_i = \theta_1 \circ \cdots \circ \theta_i(P_i)$ for each $i = 1, 2, \ldots$. Once more from Corollary 3.2 we get

$$(G_i, u_i) \cong \mathcal{M}(P_i) \cong \mathcal{M}(P'_i). \tag{3.8}$$

It follows that $(G, u) \cong \mathcal{M}([0,1]^m) / \langle \mathcal{P} \rangle$. Applying the same construction to $\mathcal{T}$ we obtain a sequence

$$[0,1]^n \supseteq Q_0 \overset{\eta_1}{\leftarrow} Q_1 \overset{\eta_2}{\leftarrow} Q_2 \overset{\eta_3}{\leftarrow} \cdots,$$

where for each i, $(H_i, v_i) \cong \mathcal{M}(Q_i)$ and η_i is a $\mathbb{Z}$-homeomorphism of Q_i onto $\eta_i(Q_i) \subseteq Q_{i-1}$. Let $\mathcal{Q} = Q'_0 \supseteq Q'_1 \supseteq Q'_2 \supseteq \cdots$, where $Q'_0 = Q_0$ and $Q'_i = \eta_1 \circ \cdots \circ \eta_i(Q_i)$ for each $i = 1, 2, \ldots$. It follows that

$$(H_i, v_i) \cong \mathcal{M}(Q_i) \cong \mathcal{M}(Q'_i) \quad \text{and} \quad (H, v) \cong \mathcal{M}([0,1]^n) / \langle \mathcal{Q} \rangle. \tag{3.9}$$

By hypothesis, $\mathcal{M}([0,1]^m) / \langle \mathcal{P} \rangle \cong (G, u) \cong (H, v) \cong \mathcal{M}([0,1]^n) / \langle \mathcal{Q} \rangle$. By Theorem 3.1, there exist $P \in \langle \mathcal{P} \rangle$, $Q \in \langle \mathcal{Q} \rangle$ and a $\mathbb{Z}$-homeomorphism $\phi: P \cong_{\mathbb{Z}} Q$ sending $\mathcal{Z}(\langle \mathcal{P} \rangle)_{\cap P}$ one-one onto $\mathcal{Z}(\langle \mathcal{Q} \rangle)_{\cap Q}$. By definition of $\langle \mathcal{P} \rangle$ and $\langle \mathcal{Q} \rangle$, there exist P'_k and Q'_l such that $P'_k \subseteq P$ and $Q'_l \subseteq Q$. Thus, for each $i \geq k$ there exists i' such that $\phi^{-1}(Q'_{i'}) \subseteq P'_i$. Reversing the roles of ϕ and ϕ^{-1} it follows that for each $j \geq l$ there exists j' such that $\phi(P'_{j'}) \subseteq Q'_j$. Summing up, there are indices $i(1) < j(1) < i(2) < j(2) < \cdots$ such that $\phi(P'_{i(k)}) \subseteq Q'_{j(k)}$ and $\phi^{-1}(Q'_{j(k)}) \subseteq P'_{i(k+1)}$ for each $k = 1, 2, \ldots$. The desired result now follows from (3.8) and (3.9), in view of Corollary 3.2. □

4 Weighted abstract simplicial complexes

Let us recall that a *(finite) abstract simplicial complex* is a pair $H = (\mathcal{V}, \Sigma)$ where $\mathcal{V}$ is a finite nonempty set, whose elements are called the *vertices* of H, and Σ is a collection of subsets of $\mathcal{V}$ whose union is $\mathcal{V}$, and with the property that every subset of an element of Σ is again an element of Σ. Following Alexander [2, p. 298], given a two-element set $\{v, w\} \in \Sigma$ and $a \notin \mathcal{V}$ we define the *binary subdivision* $(\{v, w\}, a)$ of H as the abstract simplicial complex $(\{v, w\}, a)H$ obtained by adding a to the vertex set, and replacing every set $\{v, w, u_1, \ldots, u_t\} \in \Sigma$ by the two sets $\{v, a, u_1, \ldots, u_t\}$ and $\{a, w, u_1, \ldots, u_t\}$ and their subsets. A *weighted abstract simplicial complex* is a triple $W = (\mathcal{V}, \Sigma, \omega)$ where $(\mathcal{V}, \Sigma)$ is an abstract simplicial complex and ω is a map of $\mathcal{V}$ into the set $\{1, 2, 3, \ldots\}$. For $\{v, w\} \in \Sigma$ and $a \notin \mathcal{V}$, the *binary subdivision* $(\{v, w\}, a)W$ is the abstract simplicial complex $(\{v, w\}, a)(\mathcal{V}, \Sigma)$ equipped with the weight function $\tilde{\omega}: \mathcal{V} \cup \{a\} \to \{1, 2, 3, \ldots\}$ given by $\tilde{\omega}(a) = \omega(v) + \omega(w)$ and $\tilde{\omega}(u) = \omega(u)$ for all $u \in \mathcal{V}$.

For every regular complex Λ, the *skeleton* of Λ is the weighted abstract simplicial complex $W_\Lambda = (\mathcal{V}, \Sigma, \omega)$ given by the following stipulations:

(i) $\mathcal{V} =$ vertices of Λ.

(ii) For every vertex v of Λ, $\omega(v) = \mathrm{den}(v)$.

(iii) For every subset $W = \{w_1, \ldots, w_k\}$ of $\mathcal{V}$, $W \in \Sigma$ iff $\mathrm{conv}(w_1, \ldots, w_k) \in \Lambda$.

Given two weighted abstract simplicial complexes $W = (\mathcal{V}, \Sigma, \omega)$ and $W' = (\mathcal{V}', \Sigma', \omega')$ we write

$$\gamma\colon W \cong W',$$

and we say that γ is a *combinatorial isomorphism* between W and W', if γ is a one-one map from $\mathcal{V}$ onto $\mathcal{V}'$ such that $\omega'(\gamma(v)) = \omega(v)$ for all $v \in \mathcal{V}$, and $\{w_1, \ldots, w_k\} \in \Sigma$ iff $\{\gamma(w_1), \ldots, \gamma(w_k)\} \in \Sigma'$ for each subset $\{w_1, \ldots, w_k\}$ of $\mathcal{V}$.

Definition 4.1. Let W be a weighted abstract simplicial complex and ∇ a regular complex. Then a ∇-*realization* of W is a combinatorial isomorphism ι between W and the skeleton W_∇ of ∇. We write $\iota\colon W \to \nabla$ to mean that ι is a ∇-realization of W.

For any regular complex Λ, the identity function over the set of vertices of Λ is a Λ-realization of W_Λ, called the *trivial realization of the skeleton* W_Λ.

Symmetrically, let $W = (\mathcal{V}, \Sigma, \omega)$ be a weighted abstract simplicial complex with vertex set $\mathcal{V} = \{v_1, \ldots, v_n\}$. For $e_1, \ldots, e_n$ the standard basis vectors of $\mathbb{R}^n$, let Δ_W be the complex whose vertices are

$$v_1' = e_1/\omega(v_1), \ldots, v_n' = e_n/\omega(v_n),$$

and whose k-simplexes ($k = 0, \ldots, n$) are given by

$$\mathrm{conv}(v'_{i(0)}, \ldots, v'_{i(k)}) \in \Delta_W \quad \text{iff} \quad \{v_{i(0)}, \ldots, v_{i(k)}\} \in \Sigma.$$

Note that Δ_W is a regular complex and $|\Delta_W| \subseteq [0,1]^n$. The function

$$\tilde{\iota}\colon v_i \in \mathcal{V} \mapsto v_i' \in [0,1]^n \tag{4.1}$$

is a Δ_W-realization of W, called the *canonical realization* of W. The dependence on the order in which the elements $\{v_1, \ldots, v_n\}$ are listed, is tacitly understood.

For later purposes, we record here the following trivial property of *linear* $\mathbb{Z}$-homeomorphisms.

Lemma 4.2. *Let $T = \mathrm{conv}(v_0, \ldots, v_k) \subseteq \mathbb{R}^m$ and $U = \mathrm{conv}(w_0, \ldots, w_k) \subseteq \mathbb{R}^n$ be regular k-simplexes. If $\mathrm{den}(v_i) = \mathrm{den}(w_i)$ for all $i = 0, \ldots, k$, then there is precisely one linear $\mathbb{Z}$-homeomorphism η_T of T onto U such that $\eta_T(v_i) = w_i$ for all i.*

Lemma 4.3. *Let Λ and ∇ be regular complexes, with $|\Lambda| \subseteq \mathbb{R}^m$ and $|\nabla| \subseteq \mathbb{R}^n$. We then have:*

(i) *If $\theta\colon W_\Lambda \cong W_\nabla$ is a combinatorial isomorphism between the skeletons of Λ and ∇, then there is a $\mathbb{Z}$-homeomorphism η_θ of $|\Lambda|$ onto $|\nabla|$ such that $\eta_\theta(v) = \theta(v)$ for each vertex v of Λ, and η_θ is linear over each simplex of Λ.*

(ii) *Letting $\nabla = \Delta_{W_\Lambda}$, it follows that the combinatorial isomorphism $\tilde{\iota}$ of* (4.1) *between W_Λ and W_∇ uniquely extends to a $\mathbb{Z}$-homeomorphism $\eta_{\tilde{\iota}}$ of $|\Lambda|$ onto $|\nabla|$ such that $\eta_{\tilde{\iota}}$ is linear over each simplex of Λ.*

Stellar transformations

Let $W = (\mathcal{V}, \Sigma, \omega)$ and W' be two weighted abstract simplicial complexes. A map $\flat : W \to W'$ is called a *stellar transformation* if $\flat$ is either a deletion of a maximal set of Σ, or a binary subdivision, or else $\flat$ is the identity map.

A sequence $\mathcal{W} = (W_0, W_1, \ldots)$ of weighted abstract simplicial complexes is *stellar* if W_{j+1} is obtained from W_j by a stellar transformation.

Recalling Definition 4.1 we have

Lemma 4.4. *Let $W = (\mathcal{V}, \Sigma, \omega)$ and $W' = (\mathcal{V}', \Sigma', \omega')$ be two weighted abstract simplicial complexes, Δ a regular complex, and ι a Δ-realization of W, $\iota\colon W \to \Delta$. Suppose that $\flat\colon W \to W'$ is a stellar transformation.*

(i) *In case $\flat$ deletes a maximal set $M \in \Sigma$, let $\flat(\iota)\colon \Delta \to \Delta'$ delete from Δ the corresponding maximal simplex* $\mathrm{conv}(\iota(M))$. *Then the map $\iota' = \iota \upharpoonright \mathcal{V}'$ is a Δ'-realization of W'.*

(ii) *In case $\flat$ is the binary subdivision $W' = (\{a, b\}c)W$ at some two-element set $E = \{a, b\} \in \Sigma$, and $c \notin \mathcal{V}$, let e be the Farey mediant of the* 1*-simplex* $\mathrm{conv}(\iota(E))$. *Let $\flat(\iota)$ be the Farey blow-up $\Delta' = \Delta_{(e)}$ of Δ at e. Then the map $\iota' = \iota \cup \{(c, e)\}$ is a Δ'-realization of W'.*

Further, we have a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\flat} & W' \\ \downarrow \iota & & \downarrow \iota' \\ \Delta & \xrightarrow{\flat(\iota)} & \Delta'. \end{array}$$

We say that $\flat(\iota)$ is the Δ-transformation of $\flat$. *(It is tacitly understood that if $\flat$ is the identity map, then $\flat(\iota) : \Delta \to \Delta'$ is the identity function.)*

5 Construction of the map $\mathcal{W} \mapsto \mathcal{G}(\mathcal{W})$

In this section we will construct a map $\mathcal{W} \mapsto \mathcal{G}(\mathcal{W})$, from stellar sequences to unital ℓ-groups and prove that the map is onto all finitely generated unital ℓ-groups.

Main construction

Let $\mathcal{W} = W_0, W_1, \ldots$ be a stellar sequence. For each $j = 0, 1, \ldots$ let $\flat_j$ be the corresponding stellar transformation sending W_j to W_{j+1}. For some $n \geq 1$ and regular complex Δ_0 in the n-cube let ι_0 be a Δ_0-realization of W_0. Then Lemma 4.4 yields a commutative diagram

$$\begin{array}{ccccccc} W_0 & \xrightarrow{\flat_0} & W_1 & \xrightarrow{\flat_1} & W_2 & \ldots \\ \downarrow{\scriptstyle \iota_0} & & \downarrow{\scriptstyle \iota_1} & & \downarrow{\scriptstyle \iota_2} & \\ \Delta_0 & \xrightarrow{\flat_0(\iota_0)} & \Delta_1 & \xrightarrow{\flat_1(\iota_1)} & \Delta_2 & \ldots . \end{array} \tag{5.1}$$

The sequence of supports $|\Delta_0| \supseteq |\Delta_1| \supseteq \cdots$ is called the Δ_0-*orbit of* $\mathcal{W}$ and is denoted $\mathcal{O}(\mathcal{W}, \Delta_0)$ (the role of ι_0 being tacitly understood). As in Lemma 2.3, the filtering set $\mathcal{O}(\mathcal{W}, \Delta_0)$ determines the ℓ-ideal $\mathcal{J}(\mathcal{W}, \Delta_0) = \langle \mathcal{O}(\mathcal{W}, \Delta_0) \rangle$ of $\mathcal{M}([0,1]^n)$, as well as the unital ℓ-group $\mathcal{G}(\mathcal{W}, \Delta_0) = \mathcal{M}([0,1]^n)/\mathcal{J}(\mathcal{W}, \Delta_0)$. In the particular case when ι_0 is the canonical realization of W_0 we write $\mathcal{O}(\mathcal{W})$, $\mathcal{J}(\mathcal{W})$, $\mathcal{G}(\mathcal{W})$ instead of $\mathcal{O}(\mathcal{W}, \Delta_{W_0})$, $\mathcal{J}(\mathcal{W}, \Delta_{W_0})$, $\mathcal{G}(\mathcal{W}, \Delta_{W_0})$.

Theorem 5.1. *For every finitely generated unital ℓ-group (G, u) there is a stellar sequence $\mathcal{W}$ such that $\mathcal{G}(\mathcal{W}) \cong (G, u)$.*

As a preliminary step for the proof we need the following immediate consequence of the definitions:

Lemma 5.2. *For any weighted abstract simplicial complex W and regular complexes ∇ and Δ, let ι be a ∇-realization of W, and ϵ a Δ-realization of W. Let $\eta_\gamma : |\nabla| \to |\Delta|$ be the $\mathbb{Z}$-homeomorphism of Lemma 4.3 corresponding to the combinatorial isomorphism $\gamma = \epsilon \circ \iota^{-1}$. Suppose the stellar transformation $\flat$ transforms W into W'. Let the commutative diagram*

$$\begin{array}{ccc} \Delta & \xrightarrow{\flat(\epsilon)} & \Delta' \\ \uparrow{\scriptstyle \epsilon} & & \uparrow{\scriptstyle \epsilon'} \\ W & \xrightarrow{\flat} & W' \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota'} \\ \nabla & \xrightarrow{\flat(\iota)} & \nabla' \end{array} \tag{5.2}$$

be as in Lemma 4.4. Let further $\gamma' = \epsilon' \circ \iota'^{-1}$, and $\eta_{\gamma'}$ be the $\mathbb{Z}$-homeomorphism of $|\nabla'|$ onto $|\Delta'|$ given by Lemma 4.3. Then $\eta_\gamma \upharpoonright |\nabla'| = \eta_{\gamma'}$, whence in particular $\eta_{\gamma'}$ is linear over each simplex of $|\nabla'|$.

We next prove

Lemma 5.3. *Let $\mathcal{W} = W_0, W_1, \ldots$ be a stellar sequence. Let ϵ_0 be a Δ_0-realization of W_0 and ι_0 be a ∇_0-realization of W_0. Then $\mathcal{G}(\mathcal{W}, \Delta_0) \cong \mathcal{G}(\mathcal{W}, \nabla_0)$.*

Proof. Let us write for short $I = \mathcal{I}(\mathcal{W}, \Delta_0)$, $J = \mathcal{I}(\mathcal{W}, \nabla_0)$. By definition of realization, there is a combinatorial isomorphism ξ of W_{Δ_0} onto W_{∇_0}. By Lemma 4.3 (i), ξ can be extended to a $\mathbb{Z}$-homeomorphism η of $|\Delta_0|$ onto $|\nabla_0|$, which is linear over each simplex of Δ_0. Lemma 5.2 now yields $\mathbb{Z}$-homeomorphisms

$$\eta_i = \eta \upharpoonright |\Delta_i| : |\Delta_i| \cong_{\mathbb{Z}} |\nabla_i|, \quad i = 0, 1, 2, \ldots,$$

with $\eta \upharpoonright |\Delta_i|$ linear on every simplex of Δ_i. In other words, we have a commutative diagram

$$\begin{array}{ccccccc}
|\Delta_0| & \overset{i_1}{\hookleftarrow} & |\Delta_1| & \overset{i_2}{\hookleftarrow} & |\Delta_2| & \ldots \\
\eta_0 \big\downarrow \big\uparrow \eta_0^{-1} & & \eta_1 \big\downarrow \big\uparrow \eta_1^{-1} & & \eta_2 \big\downarrow \big\uparrow \eta_2^{-1} & \\
|\nabla_0| & \overset{j_1}{\hookleftarrow} & |\nabla_1| & \overset{j_2}{\hookleftarrow} & |\nabla_2| & \ldots
\end{array}$$

where, for each $k = 1, 2, \ldots$, $i_k : |\Delta_k| \hookrightarrow |\Delta_{k-1}|$ and $j_k : |\nabla_k| \hookrightarrow |\nabla_{k-1}|$ are the inclusion maps. Corollary 3.2 ensures that the following diagram is commutative:

$$\begin{array}{ccccccc}
\mathcal{M}(|\Delta_0|) & \overset{g_1}{\twoheadrightarrow} & \mathcal{M}(|\Delta_1|) & \overset{g_2}{\twoheadrightarrow} & \mathcal{M}(|\Delta_2|) & \ldots \\
\alpha_0^{-1} \big\downarrow \big\uparrow \alpha_0 & & \alpha_1^{-1} \big\downarrow \big\uparrow \alpha_1 & & \alpha_2^{-1} \big\downarrow \big\uparrow \alpha_2 & \\
\mathcal{M}(|\nabla_0|) & \overset{h_1}{\twoheadrightarrow} & \mathcal{M}(|\nabla_1|) & \overset{h_2}{\twoheadrightarrow} & \mathcal{M}(|\nabla_2|) & \ldots .
\end{array} \tag{5.3}$$

Here $g_k : \mathcal{M}(|\Delta_{k-1}|) \twoheadrightarrow \mathcal{M}(|\Delta_k|)$ (resp., $h_k : \mathcal{M}(|\nabla_{k-1}|) \twoheadrightarrow \mathcal{M}(|\nabla_k|)$) are defined by $g_k(f) = f \upharpoonright |\Delta_k|$ (resp., $h_k(f) = f \upharpoonright |\nabla_k|$), and $\alpha_k : \mathcal{M}(|\Delta_k|) \cong \mathcal{M}(|\nabla_k|)$ are the isomorphisms defined by $\alpha_k(f) = f \circ \eta_k = f \circ \eta \upharpoonright |\Delta_k|$.

To conclude the proof we observe that $\mathcal{G}(\mathcal{W}, |\Delta_0|)$ and $\mathcal{G}(\mathcal{W}, |\nabla_0|)$ respectively are the direct limits of the direct systems

$$\mathcal{M}(|\Delta_0|) \overset{g_1}{\twoheadrightarrow} \mathcal{M}(|\Delta_1|) \overset{g_2}{\twoheadrightarrow} \mathcal{M}(|\Delta_2|) \ldots$$

and

$$\mathcal{M}(|\nabla_0|) \overset{h_1}{\twoheadrightarrow} \mathcal{M}(|\nabla_1|) \overset{h_2}{\twoheadrightarrow} \mathcal{M}(|\nabla_2|) \ldots .$$

From (5.3) it follows that $\mathcal{G}(\mathcal{W}, |\Delta_0|) \cong \mathcal{G}(\mathcal{W}, |\nabla_0|)$, and the proof is complete. □

Proof of Theorem 5.1. By Corollary 2.2, there exists an integer $n > 0$ such that (G, u) is isomorphic to $\mathcal{M}([0,1]^n)/I$ for some ℓ-ideal I of $\mathcal{M}([0,1]^n)$. We list the elements of I in a sequence $f_0, f_1, \ldots$. Let $P_i = \bigcap_{j=0}^{i} \mathcal{Z}(f_i)$, for each $i = 0, 1, 2, \ldots$.

Since $\mathcal{Z}(f_i) \in \mathcal{Z}(I)$ and $\mathcal{Z}(I)$ is closed under finite intersections, P_i belongs to $\mathcal{Z}(I)$. Moreover, for each $f \in I$ there is $j = 0, 1, 2, \ldots$ such that $P_j \subseteq \mathcal{Z}(f)$. Thus,

$$\langle\{P_0, P_1, \ldots\}\rangle = I. \tag{5.4}$$

By Proposition 2.6, P_0 is the support of a regular complex Δ_0. Proposition 2.7 yields a finite sequence of regular complexes $\Delta_{0,0}, \Delta_{0,1}, \ldots, \Delta_{0,k_0}$ having the following properties:

(i) $\Delta_{0,0} = \Delta_0$;

(ii) for each $t = 1, 2, \ldots$, $\Delta_{0,t}$ is obtained by blowing-up $\Delta_{0,t-1}$ at the Farey mediant of some 1-simplex $E \in \Delta_{0,t-1}$;

(iii) P_1 is a union of simplexes of Δ_{0,k_0}.

Let the sequence of regular complexes $\Delta_{0,k_0}, \Delta_{0,k_0+1}, \ldots, \Delta_{0,r_0}$ be obtained by the following procedure: for each $i > 0$, delete in Δ_{0,k_0+i-1} a maximal simplex T which is not contained in P_1; denote by Δ_{0,k_0+i} the resulting complex; stop when no such T exists. Then the sequence of skeletons $W_{\Delta_{0,0}}, \ldots, W_{\Delta_{0,k_0}}, \ldots, W_{\Delta_{0,r_0}}$ is a finite initial segment of a stellar sequence and $|\Delta_{0,r_0}| = P_1$. Let us write $\Delta_{1,0}$ instead of Δ_{0,r_0}. Proceeding inductively, we obtain a sequence $\mathcal{S}$ of regular complexes

$$\mathcal{S} = \Delta_{0,0}, \ldots, \Delta_{1,0}, \ldots, \Delta_{2,0}, \ldots, \Delta_{j,0}, \ldots$$

such that $P_j = |\Delta_{j,0}|$ for each $j = 0, 1, 2, \ldots$.

To conclude the proof, let $\mathcal{W}$ be the stellar sequence given by the skeletons of the regular complexes in $\mathcal{S}$. Let ρ be the trivial Δ_0-realization of the skeleton W_{Δ_0} of Δ_0. Recalling (5.4) we get

$$\begin{aligned}\mathcal{I}(\mathcal{W}, \Delta_0) &= \langle \mathcal{O}(\mathcal{W}, \Delta_{0,0})\rangle \\ &= \langle\{|\Delta_{0,0}|, \ldots, |\Delta_{1,0}|, \ldots,\}\rangle = \langle\{P_0, P_1, \ldots\}\rangle = I.\end{aligned}$$

An application of Lemma 5.3 yields

$$\mathcal{G}(\mathcal{W}) \cong \mathcal{G}(\mathcal{W}, \Delta_0) = \mathcal{M}([0,1]^n)/\mathcal{I}(\mathcal{W}, \Delta_0) = \mathcal{M}([0,1]^n)/I \cong (G, u),$$

which concludes the proof of Theorem 5.1. □

The following is an immediate consequence of Theorem 3.3:

Corollary 5.4. *For any two stellar sequences $\mathcal{W}$ and $\bar{\mathcal{W}}$ let us write $\mathcal{O}(\mathcal{W}) = |\Delta_0| \supseteq |\Delta_1| \supseteq \cdots$, and $\mathcal{O}(\bar{\mathcal{W}}) = |\bar{\Delta_0}| \supseteq |\bar{\Delta_1}| \supseteq \ldots$. Then the following conditions are equivalent:*

(i) $\mathcal{G}(\mathcal{W}) \cong \mathcal{G}(\bar{\mathcal{W}})$.

(ii) *For some integer $i \geq 0$ there is a $\mathbb{Z}$-homeomorphism η of $|\Delta_i|$ such that* $\langle\{\eta(|\Delta_i|), \eta(|\Delta_{i+1}|), \ldots\}\rangle = \langle\mathcal{O}(\bar{\mathcal{W}})\rangle$.

6 Concluding remarks

6.1 Relations with Beynon's work

In his Ph.D. thesis, [4, Lemma 1, pp. 173–174], Beynon proves that confluence is a necessary condition for the isomorphism of the direct limit of two sequences of finitely presented ℓ-groups. From the 20 lines of his self-contained proof we have been unable to extract any simplifying argument for our Theorems 3.1–3.3. This should come as no surprise: the proofs of several results in the theory of finitely presented ℓ-groups need not have an analog for finitely presented unital ℓ-groups—and vice-versa. Here are some typical examples:

• By Baker–Beynon duality theory, finitely generated projective ℓ-groups are the same as finitely presented ℓ-groups. As shown in [8], finitely generated projective unital ℓ-groups are a tiny fragment of finitely presented ones.

• Baker–Beynon duality also yields a correspondence between abstract simplicial complexes A and finitely presented ℓ-groups G, such that G is isomorphic to G' iff A and A' are connected by a path of Alexander stellar moves. This follows from the main result of Alexander's classical paper [2]. Stellar moves are a generalization of the binary subdivisions considered in this paper, and their inverses. By contrast, the results of this paper yield, as a particular case, a correspondence between finitely presented unital ℓ-groups (G, u) and weighted abstract simplicial complexes W, in such a way that (G, u) is isomorphic to (G', u') iff the regular fans corresponding to W and W' are connected by a path of regular blow-ups and blow-downs. This follows from the proof of the weak Oda conjecture by Włodarczyk–Morelli, [26, 19].

• As proved in [22], every finitely presented unital ℓ-group has a faithful invariant positive unital homomorphism into $\mathbb{R}$, but no finitely presented ℓ-group G has a faithful invariant positive homomorphism into $\mathbb{R}$, unless G is a finite product of integers with the product ordering.

• The isomorphism problem of finitely presented ℓ-groups is undecidable. The (un)decidability of the isomorphism problem for finitely presented unital ℓ-groups is open. As shown in [1] for finitely presented unital ℓ-groups with one-dimensional maximal spectral space, weighted abstract simplicial complexes and their connectability may be a key tool to settle this problem (also see [23]).

6.2 Relations with Elliott classification

Up to isomorphism, every stellar sequence $\mathcal{W}$ determines a unique AF C*-algebra $A = A_{\mathcal{W}}$ via the map

$$\mathcal{W} \mapsto \mathcal{G}(\mathcal{W}) \mapsto K_0^{-1}(\mathcal{G}(\mathcal{W})),$$

where $K_0(A)$ is the unital dimension group of A, [16]. Combining Elliott classification [13, 16] with Theorem 5.1, it follows that the range of the map $\mathcal{W} \mapsto A_{\mathcal{W}}$ coincides (up to isomorphism) with the class of unital AF C*-algebras A whose dimension group $K_0(A)$ is lattice-ordered and finitely generated. Various important AF C*-algebras existing in the literature belong to this class, including the Behnke–Leptin algebra with a two-point dual [3], the Effros–Shen algebras [11], and various algebras considered in [9] and [24], the universal AF C*algebra $\mathfrak{M}_1$ of [21] (= the algebra $\mathfrak{A}$ of [6], see [23]). Corollary 5.4 provides a simple criterion to recognize when two stellar sequences $\mathcal{W}$ and $\mathcal{W}'$ determine isomorphic AF C*-algebras $A_{\mathcal{W}}$ and $A_{\mathcal{W}'}$. This criterion is a simplification of the equivalence criterion for Bratteli diagrams, [7, 2.7]. The proof of Theorem 5.1 crucially uses Proposition 2.7, which is an affine variant of the De Concini–Procesi theorem on the elimination of points of indeterminacy in toric varieties.

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