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THE ONE-SIDED A_p CONDITIONS AND LOCAL MAXIMAL OPERATOR

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Abstract We introduce the one-sided local maximal operator and study its connection to the one-sided A_p conditions. We get a new characterization of the boundedness of the one-sided maximal operator on a quasi-Banach function space. We obtain applications to weighted Lebesgue spaces and variable-exponent Lebesgue spaces.

Keywords: one-sided A_p conditions; one-sided local maximal operator; quasi-Banach function spaces; variable-exponent Lebesgue spaces

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1. Introduction

For a function f on the real line $\mathbb{R}$, the maximal function Mf at x is defined by

$$Mf(x) := \sup_{a<x<b} \frac{1}{b-a} \int_a^b |f(t)| \, \mathrm{d}t.$$

In [**8**] Muckenhoupt characterized, for $1 < p < \infty$, the weights w on $\mathbb{R}$ satisfying the weighted norm inequality

$$\int_{-\infty}^{\infty} Mf(x)^p w(x) \, \mathrm{d}x \leqslant C \int_{-\infty}^{\infty} |f(x)|^p w(x) \, \mathrm{d}x$$

with a positive constant C independent of f. This result left open the characterization of the corresponding weighted norm inequalities for the original maximal functions of Hardy and Littlewood, namely

$$M^{-}f(x) := \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f(t)| \,\mathrm{d}t$$

and its counterpart

$$M^{+}f(x) := \sup_{h>0} \frac{1}{h} \int_{x}^{x+h} |f(t)| \,\mathrm{d}t$$

that were later called the *one-sided maximal operators*. This problem was solved later, in [**13**], by Sawyer, who also pointed out that such results are, for example, indispensable in applications to estimates of the ergodic maximal function, while, on the other hand, they are quite deep and often require the introduction of new techniques that are not analogous to their two-sided counterparts. An elementary proof of his main theorem was later given in [**7**].

In [**6**], Lerner and Pérez characterized boundedness of M on a general quasi-Banach function space (not necessarily rearrangement invariant) by several criteria. The necessary and sufficient conditions they established were expressed in terms of the norm of the so-called local maximal operator, and also in terms of a generalized upper Boyd index that they introduced for this purpose. Among the applications of this result they established a new characterization of the A_p class of weight functions, the new feature being a certain bump function inserted to the estimate that defines the condition $A_{p,1}$. They presented applications to weighted Lebesgue and Lorentz spaces and to variable-exponent spaces. In particular, for the latter application they developed a new approach to the theorem of Nekvinda [**10**] on the boundedness of M on variable-exponent Lebesgue spaces.

Our main aim in this paper is to study one-sided versions of the problems considered in [**6**]. Analogously to [**6**, Theorem 1.2], we prove a new characterization of the boundedness of the one-sided maximal operator on a quasi-Banach function space by three other equivalent statements (Theorem 3.5). To this end, we first build up an appropriate theory including the introduction of the *one-sided local maximal operator*. We study this operator in detail and, interestingly, we show some of its key properties that do not have double-sided analogues. Next, we introduce a one-sided upper Boyd index. We then present applications to weighted Lebesgue spaces and to variable-exponent spaces. The latter application is of particular interest since it requires a very special sufficient condition for the boundedness of the one-sided maximal operator on variable-exponent spaces, which had not been known for a long time and which was only very recently obtained by Nekvinda [**11**]. Our techniques are based on a key one-sided lemma that relies on a certain local version of a reverse weak-type inequality for the one-sided maximal operator, which, again, reveals features that do not exist in the two-sided world. In our proofs we mostly use techniques and results that have been obtained only very recently, although their two-sided versions have been known for several decades.

2. The one-sided local maximal operator

The key role in our investigation is played by the one-sided local maximal operator. In this section we introduce this operator and study its basic properties. First, we need to recall the notion of the non-increasing rearrangement of a function.

For a measurable function f on $\mathbb{R}$, we define its *non-increasing rearrangement*, $f^*\colon [0,\infty) \to [0,\infty)$, by

$$f^*(t) = \sup\{s \geqslant 0\colon |\{x \in \mathbb{R}\colon |f(x)| > s\}| > t\}, \quad t \in [0,\infty),$$

where, as usual, $|E|$ denotes the Lebesgue measure of E.

In what follows we denote, as usual, by χ_E the characteristic function of a measurable set $E \subset \mathbb{R}$.

Definition 2.1. Given $\lambda \in (0,1)$, $h > 0$, a measurable function f on $\mathbb{R}$ and $x \in \mathbb{R}$, we define the *one-sided local maximal operator* m_λ^+ by

$$m_\lambda^+ f(x) = \sup_{h>0} (f\chi_{(x,x+h)})^*(\lambda h).$$

Remark 2.2. Let $\lambda \in (0,1)$, $\alpha, \beta > 0$ and let $E \subset \mathbb{R}$ be measurable. Then, the following facts follow immediately from the definitions:

$$m_\lambda^+ f(x) > \alpha \quad \Longleftrightarrow \quad M^+\chi_{\{|f|>\alpha\}}(x) > \lambda; \tag{2.1}$$

$$m_\lambda^+(\chi_E)(x) = \chi_{\{M^+(\chi_E)>\lambda\}}(x); \tag{2.2}$$

$$(m_\lambda^+ f(x))^\beta = m_\lambda^+(|f|^\beta)(x); \tag{2.3}$$

$$m_\lambda^+(f+g)(x) \leqslant m_{\lambda/2}^+ f(x) + m_{\lambda/2}^+ g(x); \tag{2.4}$$

$$f(x) \leqslant m_\lambda^+ f(x) \text{ almost everywhere (a.e.);} \tag{2.5}$$

$$\text{if } f \text{ is non-increasing, then } f = m_\lambda^+ f \text{ for every } \lambda > 0. \tag{2.6}$$

We shall now point out a reverse weak-type inequality for the one-sided maximal operator. This result is essentially contained, though not stated explicitly, in [**13**]. We present its simple proof for the sake of completeness. Notice that in the two-sided case the reverse weak-type inequality is stated with $1/2\lambda$ [**14**] instead of the best factor $1/\lambda$, which appears in the next result for the one-sided case.

Lemma 2.3. *Let $I = (a,b) \subset \mathbb{R}$ be a fixed interval. Then*

$$|\{x \in I\colon M^+(f\chi_I)(x) > \lambda\}| \geqslant \frac{1}{\lambda} \int_{\{x\in I\colon f(x)>\lambda\}} f(x)\,\mathrm{d}x$$

for every $\lambda \geqslant M^+(f\chi_I)(a)$ and every non-negative function $f \in L^1_{\mathrm{loc}}(\mathbb{R})$.

Proof. It is well known [**4**, Lemma 21.75, p. 423] that

$$|\{x \in \mathbb{R}\colon M^+ f(x) > \lambda\}| = \frac{1}{\lambda} \int_{\{x\in\mathbb{R}\colon M^+ f(x)>\lambda\}} f(x)\,\mathrm{d}x. \tag{2.7}$$

Since $f(x) \leqslant M^+f(x)$ a.e. we obtain

$$|\{x \in \mathbb{R}\colon M^+f(x) > \lambda\}| = \frac{1}{\lambda}\int_{\{x\in\mathbb{R}\colon M^+f(x)>\lambda\}} f(x)\,\mathrm{d}x \geqslant \frac{1}{\lambda}\int_{\{x\in\mathbb{R}\colon f(x)>\lambda\}} f(x)\,\mathrm{d}x.$$

Applying this result to $f\chi_I$, we get

$$|\{x \in \mathbb{R}\colon M^+(f\chi_I)(x) > \lambda\}| \geqslant \frac{1}{\lambda}\int_{\{x\in I\colon f(x)>\lambda\}} f(x)\,\mathrm{d}x.$$

Since $\lambda > M^+(f\chi_I)(a)$, one has $\{x \in \mathbb{R}\colon M^+(f\chi_I)(x) > \lambda\} \subset I$, and the assertion follows. □

We shall now show an important pointwise lower-type estimate for the rearrangement of the local maximal operator.

Lemma 2.4. *Let $x \in \mathbb{R}$, $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, $h > 0$, $I = (x, x + h)$ and $\lambda \in (0, 1)$. If $t \in (0, h)$ and*

$$(\chi_I f)^*(\lambda t) > m^+_\lambda f(x), \tag{2.8}$$

then

$$(\chi_I f)^*(\lambda t) \leqslant [\chi_I(m^+_\lambda f)]^*(t). \tag{2.9}$$

Remark 2.5. We point out an interesting significant difference between (2.9) and the corresponding two-sided inequality: it is shown in [**6**, (3.7)] that

$$(\chi_I f)^*(2\lambda t) \leqslant [\chi_I(m_\lambda f)]^*(t),$$

where m_λ is the two-sided local maximal operator defined in [**6**] by

$$m_\lambda f(x) = \sup_{h,k>0} (f\chi_{(x-k,x+h)})^*(\lambda(h + k)).$$

The factor 2 does not appear in (2.9) due to its absence from the inequality asserted in Lemma 2.3.

Proof of Lemma 2.4. By (2.1) we know that, for every $\alpha > 0$,

$$\{y \in I\colon m^+_\lambda f(y) > \alpha\} = \{y \in I\colon M^+(\chi_{\{|f|>\alpha\}})(y) > \lambda\}. \tag{2.10}$$

Take $\varepsilon > 0$ such that $(\chi_I f)^*(\lambda t) - \varepsilon > m^+_\lambda f(x)$. Setting $\alpha = (\chi_I f)^*(\lambda t) - \frac{1}{2}\varepsilon$ in (2.10) and defining

$$E := \{y \in I\colon |f(y)| > (\chi_I f)^*(\lambda t) - \tfrac{1}{2}\varepsilon\},$$

we get

$$|\{y \in I\colon m^+_\lambda f(y) > (\chi_I f)^*(\lambda t) - \tfrac{1}{2}\varepsilon\}| = |\{y \in I\colon M^+(\chi_E)(y) > \lambda\}|.$$

Observe that, for $s < h$,

$$\begin{aligned}\frac{1}{s}\int_x^{x+s}\chi_E(y)\,\mathrm{d}y &= \frac{1}{s}|\{y\in(x,x+s)\colon |f(y)|>(\chi_I f)^*(\lambda t)-\tfrac{1}{2}\varepsilon\}|\\ &\leqslant \frac{1}{s}|\{y\in(x,x+s)\colon |f(y)|>m_\lambda^+ f(x)+\tfrac{1}{2}\varepsilon\}|\\ &\leqslant \frac{1}{s}|\{y\in(x,x+s)\colon |f(y)|>(f\chi_{(x,x+s)})^*(\lambda s)+\tfrac{1}{2}\varepsilon\}|\\ &\leqslant \lambda.\end{aligned}$$

Then $M^+(\chi_E)(x) = M^+(\chi_E\chi_I)(x) \leqslant \lambda$ and we can apply the reverse inequality to $f = \chi_E$ and λ. Therefore,

$$|\{y\in I\colon m_\lambda^+ f(y)>(\chi_I f)^*(\lambda t)-\tfrac{1}{2}\varepsilon\}| \geqslant \frac{1}{\lambda}|\{y\in I\colon |f(y)|>(\chi_I f)^*(\lambda t)-\tfrac{1}{2}\varepsilon\}| > \frac{1}{\lambda}\lambda t = t,$$

and, by the definition of the non-increasing rearrangement,

$$[\chi_I(m_\lambda^+ f)]^*(t) \geqslant (\chi_I f)^*(\lambda t)-\tfrac{1}{2}\varepsilon.$$

On letting $\varepsilon\to 0$, we obtain the claim. □

Now we are in a position to formulate a proposition that constitutes a key step in our analysis.

Proposition 2.6. *Let $\lambda\in(0,1)$ and let f be a measurable function. Then*

(i) *for every $t>0$, we have*

$$f^*(\lambda t) = (m_\lambda^+ f)^*(t); \tag{2.11}$$

(ii) *for every $\xi\in(0,1)$, we have*

$$m_{\lambda\xi}^+ f(x) \leqslant m_\xi^+(m_\lambda^+ f)(x) \quad \text{for almost every } x\in\mathbb{R}. \tag{2.12}$$

Remark 2.7. It is worth noticing that, while in the one-sided case one has an equality in (2.11), the corresponding two-sided statement [**6**, Lemma 3.1] reads as follows:

$$f^*(2\lambda t) \leqslant (m_\lambda f)^*(t) \leqslant f^*(\tfrac{1}{3}\lambda t).$$

Again, this phenomenon is caused by the absence of multiplicative factors 2 and 3 in Lemma 2.3 and (2.7), respectively.

Another, even more dramatic, difference between the one-sided and two-sided environments is illustrated by Proposition 2.6 (ii), which provides us with an important estimate for a composition of two local maximal operators with possibly different parameters. It shows, in fact, that the (quasi-)norm of m_λ^+, $\|m_\lambda^+\|_X$, in an arbitrary quasi-Banach function space X (see Definition 3.2, below), is a submultiplicative function of λ, in contrast to the two-sided case, where the corresponding function does not necessarily have this property (although it is comparable to a submultiplicative one). This is caused, once again, by the fact that the one-sided world allows a sharper reverse weak-type inequality (Lemma 2.3).

Proof of Proposition 2.6. We may assume that $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. From (2.1) and the weak-type $(1,1)$ inequality for M^+ we get

$$\begin{aligned}|\{x\colon m^+_\lambda f(x) > \alpha\}| &= |\{x\colon M^+(\chi_{\{|f|>\alpha\}})(x) > \lambda\}| \\ &= \frac{1}{\lambda}|\{x\colon M^+(\chi_{\{|f|>\alpha\}})(x) > \lambda\} \cap \{x\colon |f(x)| > \alpha\}| \\ &= \frac{1}{\lambda}|\{x\colon |f(x)| > \alpha\}|.\end{aligned}$$

Then (i) follows from the above inequalities and the definition of the non-increasing rearrangement.

As for (ii), let $h > 0$ and $t = \xi h$. Assume that (2.8) holds for t and $I = (x, x+h)$. Then, by Lemma 2.4, we get

$$(f\chi_{(x,x+h)})^*(\xi\lambda h) \leqslant [\chi_{(x,x+h)}(m^+_\lambda f)]^*(\xi h) \leqslant m^+_\xi(m^+_\lambda f)(x) \quad \text{for almost every } x.$$

On the other hand, if (2.8) does not hold for $t = \xi h$, then, using (2.5), we get

$$(f\chi_{(x,x+h)})^*(\xi\lambda h) \leqslant m^+_\lambda f(x) \leqslant m^+_\xi(m^+_\lambda f)(x) \quad \text{for almost every } x.$$

Taking the supremum over all $h > 0$, we obtain (2.12). □

We finish this section by establishing a pointwise inequality between the local and the ordinary one-sided maximal operators.

Proposition 2.8. *Let $\lambda \in (0,1)$ and let f be a measurable function. Then*

$$m^+_\lambda f(x) \leqslant \frac{4}{\lambda \log(1/\lambda)} M^+(M^+ f)(x), \quad x \in \mathbb{R}. \tag{2.13}$$

Proof. Let $h > 0$ and $I = (x, x+h)$. As in the two-sided case, we have the inequality

$$f^{**}(t) \leqslant 2(M^+ f)^*(t), \quad t \in (0,\infty),$$

where

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s$$

(see [**1**, p. 122] for the two-sided case). Then

$$\begin{aligned}(f\chi_I)^*(\lambda|I|) &\leqslant \frac{1}{\lambda|I|}\int_0^{\lambda|I|} (f\chi_I)^*(\tau)\,\mathrm{d}\tau \\ &\leqslant \frac{1}{\lambda\log(1/\lambda)|I|}\int_0^{\lambda|I|} (f\chi_I)^*(\tau)\log\left(\frac{|I|}{\tau}\right)\mathrm{d}\tau \\ &\leqslant \frac{1}{\lambda\log(1/\lambda)|I|}\int_0^{|I|} (f\chi_I)^{**}(\tau)\,\mathrm{d}\tau \\ &\leqslant \frac{2}{\lambda\log(1/\lambda)|I|}\int_0^{|I|} (M^+(f\chi_I))^*(\tau)\,\mathrm{d}\tau.\end{aligned}$$

Next,

$$\begin{aligned}\int_0^{|I|} (M^+(f\chi_I))^*(\tau)\,\mathrm{d}\tau &= \sup_{|E|=|I|} \int_E M^+(f\chi_I)(y)\,\mathrm{d}y \\ &\leqslant \sup_{|E|=h} \int_{E\cap(-\infty,x)} M^+(f\chi_{(x,x+h)})(y)\,\mathrm{d}y \\ &\quad + \sup_{|E|=h} \int_{E\cap(x,x+h)} M^+(f\chi_{(x,x+h)})(y)\,\mathrm{d}y \\ &\leqslant hM^+f(x) + \int_x^{x+h} M^+f(y)\,\mathrm{d}y \\ &\leqslant hM^+f(x) + hM^+(M^+f)(x) \\ &\leqslant 2hM^+(M^+f)(x).\end{aligned}$$

Therefore, we have

$$(f\chi_I)^*(\lambda|I|) \leqslant \frac{4}{\lambda\log(1/\lambda)} M^+(M^+f)(x), \quad x \in \mathbb{R},$$

and the desired inequality (2.13) follows by taking the supremum over all such intervals on the left. □

3. The main results

We shall work in the environment of the so-called quasi-Banach function spaces on $\mathbb{R}$.

Definition 3.1. We say that a linear space X of measurable functions on $\mathbb{R}$, equipped with a complete quasi-norm $\|\cdot\|_X$, is a *quasi-Banach function space* if the following three conditions are satisfied:

- if $|f| \leqslant |g|$ almost everywhere in $\mathbb{R}$, then $\|f\|_X \leqslant c\|g\|_X$ for some absolute $c > 0$;
- if $0 \leqslant f_n \nearrow f$ almost everywhere in $\mathbb{R}$, then $\|f_n\|_X \nearrow \|f\|_X$;
- $\chi_E \in X$ for every measurable E such that $|E| < \infty$.

Definition 3.2. Let X be a quasi-Banach space of functions on $\mathbb{R}$. We define

$$\Phi_X^+(\lambda) := \|m_\lambda^+\|_X = \sup_{\|f\|_X \leqslant 1} \|m_\lambda^+ f\|_X, \quad \lambda \in (0,1).$$

Observe that Φ_X^+ is non-increasing on $(0,1)$ and $\Phi_X^+(\lambda) \geqslant 1$. Moreover, by (2.12), Φ_X^+ is a submultiplicative function of λ. Consequently, if $\Phi_X^+(\lambda) < +\infty$ for some $\lambda \in (0,1)$ then it is so for all such λ.

Definition 3.3. Let X be a quasi-Banach space of functions on $\mathbb{R}$. Then the upper one-sided Boyd index of X is defined as

$$\alpha_X^+ := \begin{cases} \displaystyle\lim_{\lambda\to 0} \frac{\log \Phi_X^+(\lambda)}{\log(1/\lambda)} = \inf_{0<\lambda<1} \frac{\log \Phi_X^+(\lambda)}{\log(1/\lambda)} & \text{if } \Phi_X^+(\lambda) < +\infty \text{ for all } \lambda \in (0,1), \\ +\infty & \text{if } \Phi_X^+(\lambda) = +\infty \text{ for all } \lambda \in (0,1). \end{cases}$$

In order to see that the one-sided Boyd index is well defined we proceed as in [**1**, p. 147] and [**6**]. We sketch the argument. First, we need the following lemma.

Lemma 3.4. *Let ω be a real-valued, non-decreasing, non-negative and subadditive function on $(0, +\infty)$. Then $\omega(s)/s$ tends to a finite limit α as $s \to +\infty$ and*

$$\alpha = \lim_{s\to+\infty} \frac{\omega(s)}{s} = \inf_{s>0} \frac{\omega(s)}{s}.$$

We omit the proof since it is completely analogous to that of [**1**, Chapter 3, Lemma 5.8].

Now we can justify that α_X^+ is well defined. Assume that $\Phi_X^+(\lambda) < +\infty$ for all $\lambda \in (0,1)$. Then Lemma 3.4 can be applied to the function $\omega(s) = \log \Phi_X^+(\exp(-s))$, $s > 0$. Therefore, the finite limit

$$\alpha = \lim_{s\to+\infty} \frac{\log \Phi_X^+(\exp(-s))}{s} = \inf_{s>0} \frac{\log \Phi_X^+(\exp(-s))}{s}$$

exists. Finally, it is obvious that the change of variable $\lambda = \exp(-s)$ gives

$$\alpha = \lim_{\lambda\to 0} \frac{\log \Phi_X^+(\lambda)}{\log(1/\lambda)} = \inf_{0<\lambda<1} \frac{\log \Phi_X^+(\lambda)}{\log(1/\lambda)}.$$

In conclusion, α_X^+ is well defined.

We shall now characterize the action of the one-sided maximal operator on quasi-Banach function spaces.

Theorem 3.5. *Let X be a quasi-Banach function space on $\mathbb{R}$. Then the following statements are equivalent:*

(i) *M^+ is bounded on X;*

(ii) *$\alpha_X^+ < 1$;*

(iii) *$\Phi_X^+ \in L^1(0,1)$;*

(iv) *$\lim_{\lambda\to 0} \lambda \Phi_X^+(\lambda) = 0$.*

Proof. The proof follows the pattern of the proof of [**6**, Theorem 1.2] and uses ideas from [**1**]. We shall show the following implications: (iii) $\Rightarrow$ (iv) $\Rightarrow$ (ii) $\Rightarrow$ (iii), (i) $\Rightarrow$ (iv) and (ii) $\Rightarrow$ (i).

(iii) $\Rightarrow$ (iv) Since Φ_X^+ is non-increasing, we have

$$\frac{\lambda}{2} \Phi_X^+(\lambda) \leqslant \int_{\lambda/2}^{\lambda} \Phi_X^+(t)\,\mathrm{d}t \leqslant \int_0^{\lambda} \Phi_X^+(t)\,\mathrm{d}t.$$

It is clear that

$$\lim_{\lambda\to 0} \int_0^{\lambda} \Phi_X^+(t)\,\mathrm{d}t = 0$$

because Φ_X^+ is integrable. This implies (iv).

(iv) ⇒ (ii) It follows from (iv) that there exists $\delta \in (0,1)$ such that $\lambda \Phi_X^+(\lambda) < \frac{1}{2}$ for all $\lambda \in (0,\delta)$. Then

$$\frac{\log \Phi_X^+(\lambda)}{\log(1/\lambda)} < 1 + \frac{\log 2}{\log \lambda} < 1$$

for all $\lambda \in (0,\delta)$. Consequently, $\Phi_X^+(\lambda) < +\infty$ for all $\lambda \in (0,\delta)$ and

$$\alpha_X^+ = \inf_{0<\lambda<1} \frac{\log \Phi_X^+(\lambda)}{\log(1/\lambda)} < 1.$$

(ii) ⇒ (iii) Since $\alpha_X^+ < 1$ we have that $\Phi_X^+(\lambda) < +\infty$ for all λ and

$$\lim_{\lambda \to 0} \frac{\log \Phi_X^+(\lambda)}{\log(1/\lambda)} < 1.$$

Then there exist $\varepsilon, \delta \in (0,1)$ such that

$$\frac{\log \Phi_X^+(\lambda)}{\log(1/\lambda)} < 1 - \varepsilon$$

or, equivalently,

$$\Phi_X^+(\lambda) < \frac{1}{\lambda^{1-\varepsilon}}$$

for all $\lambda \in (0,\delta)$. Thus,

$$\int_0^1 \Phi_X^+(\lambda)\,\mathrm{d}\lambda \leqslant \int_0^\delta \frac{1}{\lambda^{1-\varepsilon}}\,\mathrm{d}\lambda + \Phi_X^+(\delta)(1-\delta) < +\infty,$$

which yields (iii).

(i) ⇒ (iv) By Proposition 2.8, we have

$$\|m_\lambda^+ f\|_X \leqslant \frac{4C}{\lambda \log(1/\lambda)} \|f\|_X$$

and

$$\Phi_X^+(\lambda) \leqslant \frac{4C}{\lambda \log(1/\lambda)},$$

and (iv) follows.

(ii) ⇒ (i) We know that there exist $C > 0$ and $\delta \in (0,1)$ such that

$$\|m_\lambda^+ f\|_X \leqslant C\lambda^{-\delta} \|f\|_X.$$

Now, for every interval $I \subset \mathbb{R}$,

$$\frac{1}{|I|} \int_I f(y)\,\mathrm{d}y \leqslant \int_0^1 (f\chi_I)^*(\lambda|I|)\,\mathrm{d}\lambda.$$

Thus,

$$M^+f(x) \leqslant \int_0^1 m_\lambda^+ f(x)\,\mathrm{d}\lambda \leqslant \sum_{i=1}^\infty 2^{-i} m_{2^{-i}}^+ f(x).$$

Hence, using a version of the Aoki–Rolewicz Theorem (see, for example, [**5**, p. 3]), for some $p \leqslant 1$,

$$\begin{aligned}\|M^+f\|_X &\leqslant \Big\|\sum_{i=1}^\infty 2^{-i} m_{2^{-i}}^+ f\Big\|_X \leqslant 4^{1/p}\biggl(\sum_{i=1}^\infty \|2^{-i} m_{2^{-i}}^+ f\|_X^p\biggr)^{1/p} \\ &\leqslant C\biggl(\sum_{i=1}^\infty 2^{-(1-\delta)pi}\biggr)^{1/p}\|f\|_X \leqslant C\|f\|_X.\end{aligned}$$

□

Corollary 3.6. *Let X be a quasi-Banach function space on $\mathbb{R}$. Then M^+ is bounded on X if and only if the operator M_r^+, defined by*

$$M_r^+ f := (M^+|f|^r)^{1/r},$$

is bounded on X for some $r > 1$.

Proof. We first define the space X_r by

$$\|f\|_{X_r} := \||f|^{1/r}\|_X^r.$$

Note that the boundedness of M_r^+ on X is equivalent to that of M^+ on X_r. Therefore, by Theorem 3.5, it suffices to establish that $\alpha_X^+ < 1$ if and only if there is an $r > 1$ such that $\alpha_{X_r}^+ < 1$. It follows from (2.3) that

$$\varPhi_{X_r}^+(\lambda) = \varPhi_X^+(\lambda)^r, \quad \lambda \in (0,1).$$

Then $\alpha_{X_r}^+ = r\alpha_X^+$, and the assertion follows. □

Remark 3.7. If a quasi-Banach function space X on $\mathbb{R}$ is rearrangement invariant, then α_X^+ coincides with the upper-Boyd index of X denoted by $\bar{\alpha}_X$. An analogous result for the two-sided index can be found in [**6**, Theorem 1.2], where $\bar{\alpha}_X$ is defined as

$$\bar{\alpha}_X = \inf_{1<t<\infty} \frac{\log h_X(t)}{\log t} = \lim_{t\to\infty} \frac{\log h_X(t)}{\log t}, \tag{3.1}$$

with

$$h_X(t) = \sup_{\|f\|_X \leqslant 1} \|D_{(1/t)} f\|_X \quad \text{and} \quad D_{(1/t)}f(x) = f\Big(\frac{x}{t}\Big).$$

Let us show that $\alpha_X^+ = \bar{\alpha}_X$. Let $f^\star$ be the symmetric rearrangement of a measurable function f, that is,

$$f^\star(x) = f^*(2|x|).$$

The functions f and $f^\star$ are equimeasurable. It follows from $(D_\lambda f)^*(t) = f^*(\lambda t)$ and (2.11) that

$$(D_\lambda f)^\star(x) = (D_\lambda f)^*(2|x|) = f^*(2\lambda|x|) = (m_\lambda^+ f)^*(2|x|) = (m_\lambda^+ f)^\star(x).$$

Therefore,

$$\|D_\lambda f\|_X = \|m_\lambda^+ f\|_X.$$

Thus, $h_X(t) = \Phi_X^+(1/t)$, and the desired identity $\alpha_X^+ = \bar{\alpha}_X$ follows from (3.1) and Definition 3.3.

Remark 3.8. One can define, with obvious modifications only, the functions m_λ^-, Φ_X^- and the index α_X^- associated to the left one-sided Hardy–Littlewood maximal operator M^-. It is not difficult to prove then that the following relations hold:

$$\max(m_\lambda^-, m_\lambda^+) \leqslant m_\lambda \leqslant m_\lambda^- + m_\lambda.$$

It follows that

$$\max(\Phi_X^-(\lambda), \Phi_X^+(\lambda)) \leqslant \Phi_X(\lambda) \leqslant 2\max(\Phi_X^-(\lambda), \Phi_X^+(\lambda)),$$

where $\Phi_X(\lambda) = \|m_\lambda\|_X$. Therefore,

$$\frac{\max\{\log \Phi_X^-(\lambda), \log \Phi_X^+(\lambda)\}}{\log(1/\lambda)} \leqslant \frac{\log \Phi_X(\lambda)}{\log(1/\lambda)} \leqslant \frac{\log 2}{\log(1/\lambda)} + \max\left\{\frac{\log \Phi_X^-(\lambda)}{\log(1/\lambda)}, \frac{\log \Phi_X^+(\lambda)}{\log(1/\lambda)}\right\}.$$

On letting $\lambda \to 0$, we have

$$\max\{\alpha_X^-, \alpha_X^+\} = \alpha_X,$$

where

$$\alpha_X = \lim_{\lambda \to 0} \frac{\log \Phi_X(\lambda)}{\log(1/\lambda)}$$

is the two-sided index introduced in [**6**]. Then, for instance, Theorem 1.2 from [**6**] in dimension 1 is a consequence of Theorem 3.5 and the corresponding one for the left one-sided case.

4. An application to weighted Lebesgue spaces

Let u be a weight, that is, a non-negative measurable function on $\mathbb{R}$. We shall give a characterization of the boundedness of M^+ on the weighted Lebesgue space L_u^p by using the main result of the preceding section (Theorem 3.5).

One of the principal results of this section is Theorem 4.6, in which we compute the one-sided Boyd index and the function Φ^+ for a weighted Lebesgue space. Once equipped with this result, combining it with Theorem 3.5, we get, as an application, a new characterization of the boundedness of the one-sided maximal operator on a weighted Lebesgue space. Moreover, as a corollary, we get a new proof of the earlier, celebrated Sawyer characterization [**13**].

In this connection it is of interest to recall the corresponding restricted weak-type inequality, which had been characterized in [**12**] by the $A^+_{p,1}$ condition. Again, we shall obtain a new proof of this result as a corollary of our main results. We note that our characterization of it is similar to $A^+_{p,1}$, but it contains a bump-function factor.

We start by computing $\Phi^+_{L^p_u}$ and establishing some results that will be needed in the proofs of the main theorems of this section.

Lemma 4.1. *Let $p > 0$ and define*

$$\Psi^+_p(\lambda) := \sup_E \left(\frac{u(\{x \colon M^+\chi_E(x) > \lambda\})}{u(E)} \right)^{1/p},$$

where the supremum is taken over all measurable sets such that $u(E) > 0$. Then, we have

$$\Phi^+_{L^p_u}(\lambda) = \Psi^+_p(\lambda), \quad \lambda \in (0,1). \tag{4.1}$$

Proof. By (2.2),

$$\Phi^+_{L^p_u}(\lambda) \geqslant \frac{\|m^+_\lambda(\chi_E)\|_{L^p_u}}{\|\chi_E\|_{L^p_u}} = \left(\frac{u(\{x \colon M^+\chi_E(x) > \lambda\})}{u(E)} \right)^{1/p}.$$

Taking the supremum over all E, we obtain

$$\Psi^+_p(\lambda) \leqslant \Phi^+_{L^p_u}(\lambda), \quad \lambda \in (0,1).$$

As for the converse inequality, we use (2.1) in order to get

$$\begin{aligned} u(\{x \colon m^+_\lambda f(x) > \alpha\}) &= u(\{x \colon M^+\chi_{\{|f|>\alpha\}}(x) > \lambda\}) \\ &\leqslant [\Psi^+_p(\lambda)]^p u(\{x \colon |f(x)| > \alpha\}), \end{aligned}$$

Thus,

$$\|m^+_\lambda f\|_{L^p_u} \leqslant \Psi^+_p(\lambda)\|f\|_{L^p_u}.$$

Therefore,

$$\Phi^+_{L^p_u}(\lambda) \leqslant \Psi^+_p(\lambda), \quad \lambda \in (0,1),$$

finishing the proof. □

In what follows, we will use the following notation: if $I = (a,c)$ and $a < b < c$, then $I^- = (a,b)$ and $I^+ = (b,c)$. By saying that a certain statement holds 'for all intervals I, I^- and I^+' we mean that it holds for all intervals I and all possible partitions of I into intervals I^- and I^+.

Proposition 4.2. *Let φ be a non-increasing function on $\mathbb{R}$ and let ψ be a non-decreasing function on $\mathbb{R}$.*

(i) *If*

$$u(\{x\colon M^+f(x)>\lambda\})\leqslant C\varphi(\lambda)\int_{\mathbb{R}}\psi(|f(x)|)u(x)\,\mathrm{d}x$$

holds with some $C>0$ independent of f and λ, then

$$u(I^-)\biggl(\varphi\biggl(\frac{1}{|I|}\int_{I^+}|f(y)|\,\mathrm{d}y\biggr)\biggr)^{-1}\leqslant C\int_{I^+}\psi(|f(y)|)u(y)\,\mathrm{d}y$$

holds with some $C>0$ for all functions f on $\mathbb{R}$, all intervals I, I^- and I^+.

(ii) *If there exists a $C>0$ such that*

$$u(I^-)\biggl(\varphi\biggl(\frac{4}{|I|}\int_{I^+}|f(y)|\,\mathrm{d}y\biggr)\biggr)^{-1}\leqslant C\int_{I^+}\psi(|f(y)|)u(y)\,\mathrm{d}y$$

for all functions f on $\mathbb{R}$, all intervals I, I^- and I^+, then

$$u(\{x\colon M^+f(x)>\lambda\})\leqslant C\varphi(\lambda)\int_{\mathbb{R}}\psi(|f(x)|)u(x)\,\mathrm{d}x$$

holds for all f and λ.

Proof. Assertion (i) follows by testing the inequality on the functions $f=\chi_{I^+}$. For the proof of (ii), as usual, we write

$$\{x\in\mathbb{R}\colon M^+f(x)>\lambda\}=\bigcup_{i=1}^{\infty}(a_i,b_i),$$

where

$$\lambda=\frac{1}{b_i-a_i}\int_{a_i}^{b_i}|f(x)|\,\mathrm{d}x$$

and

$$\lambda<\frac{1}{b_i-x}\int_{x}^{b_i}|f(y)|\,\mathrm{d}y,\quad x\in(a_i,b_i).$$

For a fixed $i\in\mathbb{N}$, we define the sequence $\{x_{i,n}\}$ by

$$x_{i,0}=a_i;$$

and when $x_{i,n-1}$ is established, we define $x_{i,n}\in(x_{i,n-1},b_i)$ so that

$$\int_{x_{i,n-1}}^{x_{i,n}}|f|=\int_{x_{i,n}}^{b_i}|f|.$$

We also define $I_{i,n}:=(x_{i,n},x_{i,n+1})$. Then, of course,

$$\lambda<\frac{1}{b_i-x_{i,n}}\int_{x_{i,n}}^{b_i}|f(y)|\,\mathrm{d}y=\frac{4}{b_i-x_{i,n}}\int_{x_{i,n+1}}^{x_{i,n+2}}|f(y)|\,\mathrm{d}y.$$

Consequently,

$$\begin{aligned} u(\{x \in \mathbb{R} : M^+ f(x) > \lambda\}) &= \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} u(I_{i,n}) \\ &\leqslant \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} u(I_{i,n})\varphi(\lambda)\biggl(\varphi\biggl(\frac{4}{b_i - x_{i,n}}\int_{x_{i,n+1}}^{x_{i,n+2}} |f(y)|\,\mathrm{d}y\biggr)\biggr)^{-1} \\ &\leqslant C\varphi(\lambda)\sum_{i=1}^{\infty}\sum_{n=1}^{\infty} u(I_{i,n})\biggl(\varphi\biggl(\frac{4}{b_i - x_{i,n}}\int_{x_{i,n+1}}^{x_{i,n+2}} |f(y)|\,\mathrm{d}y\biggr)\biggr)^{-1} \\ &\leqslant C\varphi(\lambda)\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\int_{I_{i,n+1}} \psi(|f(x)|)u(x)\,\mathrm{d}x \\ &\leqslant C\varphi(\lambda)\int_{\mathbb{R}} \psi(|f(x)|)u(x)\,\mathrm{d}x. \end{aligned}$$

□

The next result can be obtained in the same way as Proposition 4.2.

Proposition 4.3. *Let φ be a non-increasing function on $\mathbb{R}$ and let ψ be a non-decreasing function on $\mathbb{R}$.*

(i) *If*

$$u(\{x\colon M^+\chi_E(x) > \lambda\}) \leqslant C\varphi(\lambda)u(E)$$

holds with some $C > 0$ independent of the set $E \subset I^+$ and $\lambda > 0$, then

$$\frac{u(I^-)}{\varphi(|E|/|I|)} \leqslant Cu(E)$$

holds with some $C > 0$ for all intervals I, I^-, I^+ and all $E \subset I^+$.

(ii) *If there exists a $C > 0$ such that*

$$\frac{u(I^-)}{\varphi(4|E|/|I|)} \leqslant Cu(E)$$

holds with some $C > 0$ for all intervals I, I^-, I^+ and all $E \subset I^+$, then

$$u(\{x\colon M^+\chi_E(x) > \lambda\}) \leqslant C\varphi(\lambda)u(E)$$

holds with some $C > 0$ independent of the set $E \subset \mathbb{R}$ and $\lambda > 0$.

It is worth noticing that Proposition 4.3 immediately yields the characterization of the restricted weak-type (p, p) of M^+ by the $A^+_{p,1}$ condition, proved earlier in [**12**]. For a weight u on $\mathbb{R}$ and a measurable set $E \subset \mathbb{R}$, we write $u(E)$ for $\int_E u(y)\,\mathrm{d}y$.

Corollary 4.4. *Let u be a weight on $\mathbb{R}$ and let $1 < p < \infty$. Then the following statements are equivalent.*

(i) *There exists a $C > 0$ such that*

$$u(\{x\colon M^+\chi_E(x) > \lambda\}) \leqslant \frac{C}{\lambda^p} u(E)$$

for all sets set $E \subset \mathbb{R}$ and all $\lambda > 0$.

(ii) *There exists a $C > 0$ such that*

$$\frac{|E|}{|I|} \leqslant C\left(\frac{u(E)}{u(I^-)}\right)^{1/p}$$

for all intervals I, I^-, I^+ and all $E \subset I^+$.

Definition 4.5. Let u be a weight on $\mathbb{R}$. We define

$$\nu_u^+(\lambda) := \inf_{I, I^-, I^+} \inf_{|E| = \lambda|I|, E \subset I^+} \frac{u(E)}{u(I^-)}, \quad 0 < \lambda < 1,$$

where the infima are taken over all intervals I, I^- and I^+ and all subsets $E \subset I^+$ with $|E| = \lambda|I|$.

The principal result in this section is the following theorem.

Theorem 4.6. *For any $p > 0$, we have*

$$\frac{1}{(\nu_u^+)^{1/p}(\lambda)} \leqslant \Phi^+_{L^p_u}(\lambda) \leqslant \frac{C_2}{(\nu_u^+)^{1/p}(\frac{1}{4}\lambda)} \tag{4.2}$$

and

$$\alpha^+_{L^p_u} = \frac{1}{p} \lim_{\lambda \to 0} \frac{\log(1/\nu_u^+(\lambda))}{\log 1/\lambda}. \tag{4.3}$$

Proof. By Lemma 4.1, we have

$$\left(\Phi^+_{L^p_u}\left(\frac{|E|}{|I|}\right)\right)^{-1} \leqslant \left(\frac{u(E)}{u(I^-)}\right)^{1/p}$$

whenever $E \subset I^+$ and $|E| = \lambda|I|$. Therefore,

$$\frac{1}{(\nu_u^+)^{1/p}(\lambda)} \leqslant \Phi^+_{L^p_u}(\lambda).$$

By the definition of ν_u^+, we know that

$$\nu_u^+\left(\frac{|E|}{|I|}\right) \leqslant \frac{u(E)}{u(I^-)}$$

whenever $E \subset I^+$. Clearly,

$$\varphi(\lambda) := \frac{1}{\nu_u^+(\frac{1}{4}\lambda)}$$

is non-increasing. Hence, by Proposition 4.3,

$$u(\{x\colon M^+(\chi_E)(x) > \lambda\}) \leqslant C\varphi(\lambda)u(E).$$

From this and from Lemma 4.1, we obtain

$$\Phi^+_{L^p_u}(\lambda) \leqslant C\frac{1}{(\nu_u^+)^{1/p}(\frac{1}{4}\lambda)}, \quad 0 < \lambda < 1.$$

This proves (4.2). Finally, (4.3) follows on taking the appropriate limits in (4.2). □

Our next aim is to apply Theorem 4.6 to get a new description of boundedness of M^+ on weighted Lebesgue spaces. To this end, we need to introduce the notion of a bump function.

Definition 4.7. We say that a function ψ on $[1,\infty)$ is a *bump function* and write $\psi \in \mathcal{A}$ if ψ is non-decreasing, positive, $\lim_{t\to\infty}\psi(t) = \infty$ and $\psi(t) = O(t^\varepsilon)$ for every $\varepsilon > 0$.

Now we can state and prove the main application theorem.

Theorem 4.8. *Let $1 < p < \infty$. Given a weight u on $\mathbb{R}$, then the following statements are equivalent:*

(i) *M^+ is bounded on L^p_u;*

(ii) $\displaystyle\lim_{\lambda\to 0+}\frac{\nu_u^+(\lambda)}{\lambda^p} = \infty$;

(iii) $\displaystyle\lim_{\lambda\to 0+}\frac{1}{\log 1/\lambda}\log\left(\frac{1}{\nu_u^+(\lambda)}\right) < p$;

(iv) *if $\psi \in \mathcal{A}$, then there exists a positive constant C such that, for all intervals I, I^-, I^+ and every $E \subset I^+$,*

$$\frac{|E|}{|I|}\psi\left(\frac{|I|}{|E|}\right) \leqslant C\left(\frac{u(E)}{u(I^-)}\right)^{1/p}; \tag{4.4}$$

(v) *if $\psi \in \mathcal{A}$, then there exists a positive constant C such that, for all intervals I, I^-, I^+ such that $|I^-| = |I^+|$ and every $E \subset I^+$,*

$$\frac{|E|}{|I|}\psi\left(\frac{|I|}{|E|}\right) \leqslant C\left(\frac{u(E)}{u(I^-)}\right)^{1/p}.$$

Proof. The equivalences (i) ⇔ (ii) ⇔ (iii) follow from Theorems 3.5 and 4.6. Since (iv) ⇒ (v) is obvious, it will suffice to show (iv) ⇒ (ii), (iii) ⇒ (iv) and (v) ⇒ (iv).

We have from (4.4) that

$$\psi\left(\frac{1}{\lambda}\right)^p \leqslant C\frac{\nu_u^+(\lambda)}{\lambda^p}.$$

Since $\lim_{t\to\infty}\psi(t) = \infty$, the implication (iv)⇒(ii) follows.

Let us assume that (iii) holds. Then there exists a $\delta \in (0,1)$ such that

$$\lambda^\delta \leqslant C\nu_u^+(\lambda)^{1/p}.$$

Therefore,

$$\left(\frac{|E|}{|I|}\right)^\delta \leqslant C\left(\frac{u(E)}{u(I^-)}\right)^{1/p},$$

that is,

$$\frac{|E|}{|I|}\left(\frac{|I|}{|E|}\right)^{1-\delta} \leqslant C\left(\frac{u(E)}{u(I^-)}\right)^{1/p}.$$

Since $\psi(t) = O(t^\varepsilon)$ for every $\varepsilon > 0$, we have

$$\frac{|E|}{|I|}\psi\left(\frac{|I|}{|E|}\right) \leqslant C\left(\frac{u(E)}{u(I^-)}\right)^{1/p},$$

as desired.

Finally, we shall prove (v) $\Rightarrow$ (iv). Assume that (v) holds and let $I = (a,c)$, $I^- = (a,b)$, $I^+ = (b,c)$ and $E \subset I^+$. If $|I^-| \leqslant |I^+|$, we can choose $\bar{a} \leqslant a$ such that if $J = (\bar{a}, c)$, $J^- = (\bar{a}, b)$ and $J^+ = (b,c)$. Then we have $|J^-| = |J^+|$. Applying (v), we get

$$\frac{|E|}{|J|}\psi\left(\frac{|J|}{|E|}\right) \leqslant C\left(\frac{u(E)}{u(J^-)}\right)^{1/p}.$$

Since $|I| \leqslant |J| \leqslant 2|I|$, $I^- \subset J^-$ and ψ is non-decreasing,

$$\frac{|E|}{|I|}\psi\left(\frac{|I|}{|E|}\right) \leqslant 2C\left(\frac{u(E)}{u(I^-)}\right)^{1/p}.$$

If $|I^+| \leqslant |I^-|$, we proceed in a similar way choosing $\bar{c} \geqslant c$, $J = (a, \bar{c})$, $J^- = (a,b)$ and $J^+ = (b, \bar{c})$. □

The following corollary is immediate from Theorem 4.8 (iii).

Corollary 4.9. *Let $1 < p < \infty$. Given a weight u, if M^+ is bounded on L_u^p, then M^+ is bounded on L_u^q for some $q < p$.*

We shall finish this section by pointing out that Theorem 4.8 leads to a new proof of the equivalence of the condition A_p^+ to the boundedness of M^+ on L_u^p proved first by Sawyer in [**13**] (more precisely, its sufficiency part).

We start with proving an auxiliary technical assertion.

Lemma 4.10. *If $u \in A_p^+$, then there exists $C > 0$ such that for any function f and any interval $I = (a,d)$,*

$$\frac{1}{|I|}\int_I |f(x)|\,\mathrm{d}x \leqslant C[M_u^+(|f|^p\chi_I)(a)]^{1/p}. \tag{4.5}$$

Proof. Let $\{x_i\}$ be a sequence such that $x_0 = d$ and $u(x_{i+1}, x_i) = u(a, x_{i+1})$. Since $u \in A_p^+$,

$$\begin{aligned}\int_{x_{i+1}}^{x_i} |f(y)|\,\mathrm{d}y &\leqslant \left(\int_{x_{i+1}}^{x_i} |f(y)|^p u(y)\,\mathrm{d}y\right)^{1/p}\left(\int_{x_{i+1}}^{x_i} u(y)^{-1/(p-1)}\,\mathrm{d}y\right)^{(p-1)/p}\\ &\leqslant C\left(\int_{x_{i+1}}^{x_i} |f(y)|^p u(y)\,\mathrm{d}y\right)^{1/p}\left(\int_{x_{i+2}}^{x_{i+1}} u(y)\,\mathrm{d}y\right)^{-1/p}(x_i - x_{i+2})\\ &\leqslant C\left[\frac{1}{u(a,x_i)}\int_a^{x_i} |f(y)|^p u(y)\,\mathrm{d}y\right]^{1/p}(x_i - x_{i+2})\\ &\leqslant C[M_u^+(|f|^p\chi_I)(a)]^{1/p}(x_i - x_{i+2}).\end{aligned}$$

Summing over i, we obtain the lemma. □

The following proposition is the key to our application goal.

Proposition 4.11. *Let $1 < p < \infty$. If $u \in A_p^+$, then there exists $C > 0$ such that, for every interval $I = (a, c)$ and $E \subset I^+ = (b, c)$,*

$$\frac{|E|}{|I|}\log^{1-1/p}\left(\mathrm{e} + \frac{|I|}{|E|}\right) \leqslant C\left(\frac{u(E)}{u(I^-)}\right)^{1/p}.$$

In other words, the A_p^+ condition implies (4.4) with $\psi(t) := \log^{1-1/p}(\mathrm{e} + t)$.

Proof. By the equivalence (iv) ⇔ (v) in Theorem 4.8, we may assume that $|I^-| = |I^+|$. The proof follows the lines of [**6**]. However, we have to overcome several technical obstacles caused by the nature of the one-sided setting.

Claim 4.12. *If $u \in A_p^+$, then there exists $C > 0$ such that for every interval $I = (a, c)$ and all $E \subset I^+$*

$$\frac{1}{|I|}\int_I M^+(\chi_E)(x)\,\mathrm{d}x \leqslant C[M_u^+(M_u^+(\chi_E))(a)]^{1/p}, \tag{4.6}$$

where

$$M_u^+ f(x) := \sup_{h>0}\frac{1}{u(x, x+h)}\int_x^{x+h} |f(y)|u(y)\,\mathrm{d}y,$$

where the quotient is understood as zero when $u(x, x+h) = 0$.

Proof of Claim 4.12. It follows from the lemma that

$$M^+(f\chi_I)(x) \leqslant C[M_u^+(|f|^p\chi_I)(x)]^{1/p} \tag{4.7}$$

for all $x \in I$. Now, by (4.5), with f replaced by $M^+(f\chi_I)$ and applying (4.7), we get

$$\frac{1}{|I|}\int_I M^+(f\chi_I)(x)\,\mathrm{d}x \leqslant C[M_u^+(M_u^+(|f|^p))(a)]^{1/p}. \tag{4.8}$$

Taking $f = \chi_E$, with $E \subset I^+$ in (4.8), we get the claim. □

Claim 4.13. *There exists a $C>0$ such that for every interval $I=(a,c)$ and all $E\subset I^+$*

$$\frac{1}{|I|}\int_I M^+(\chi_E)(x)\,\mathrm{d}x \geqslant C\frac{|E|}{|I|}\log\left(\mathrm{e}+\frac{|I|}{|E|}\right).$$

Proof of Claim 4.13. We have

$$\begin{aligned}\int_I M^+(\chi_E)(x)\,\mathrm{d}x &= \int_0^\infty |\{x\in I\colon M^+(\chi_E)(x)>\lambda\}|\,\mathrm{d}\lambda\\ &= \int_0^{|E|/|I^+|} |\{x\in I\colon M^+(\chi_E)(x)>\lambda\}|\,\mathrm{d}\lambda\\ &\quad + \int_{|E|/|I^+|}^\infty |\{x\in I\colon M^+(\chi_E)(x)>\lambda\}|\,\mathrm{d}\lambda\\ &=: \mathrm{I}+\mathrm{II}.\end{aligned}$$

Notice that if $x\in I^-$ and $\lambda<|E|/2|I^+|$, then $M^+(\chi_E)(x)>\lambda$, so that

$$\mathrm{I}\geqslant \int_0^{|E|/2|I^+|} |\{x\in I^-\colon M^+(\chi_E)(x)>\lambda\}|\,\mathrm{d}\lambda = \tfrac{1}{2}|E|.$$

In order to estimate II, we use a version of the reverse inequality for the one-sided maximal operator from [**2**, Lemma 3.3]. We get

$$\begin{aligned}\mathrm{II} &\geqslant C\int_{|E|/|I^+|}^\infty \frac{1}{\lambda}\int_{\{x\in I^+\colon \chi_E(x)>\lambda\}} \chi_E(x)\,\mathrm{d}x\,\mathrm{d}\lambda\\ &= C\int_E\int_{|E|/|I^+|}^1 \frac{1}{\lambda}\,\mathrm{d}\lambda\,\mathrm{d}x\\ &= C|E|\log\left(\frac{|I^+|}{|E|}\right);\end{aligned}$$

hence

$$\frac{1}{|I|}\int_I M^+(\chi_E)(x)\,\mathrm{d}x \geqslant \frac{|E|}{2|I|}+C\frac{|E|}{|I|}\log\left(\frac{|I|}{2|E|}\right)\geqslant C\frac{|E|}{|I|}\log\left(\mathrm{e}+\frac{|I|}{|E|}\right).$$

This proves Claim 4.13. □

Claim 4.14. *There exists a $C>0$ such that*

$$M_u^+(M_u^+(f))(a)\leqslant CM^+_{u,L\log L}(f)(a),$$

where

$$M^+_{u,L\log L}(f)(a):=\sup_{b>a}\|f\|_{u,L\log L,(a,b)}$$

and

$$\|f\|_{u,L\log L,(a,b)}:=\inf\left\{\lambda>0\colon \frac{1}{u(a,b)}\int_a^b\frac{|f(x)|}{\lambda}\log\left(\mathrm{e}+\frac{|f(x)|}{\lambda}\right)u(x)\,\mathrm{d}x\leqslant 1\right\}.$$

Proof of Claim 4.14. First, we shall prove that

$$\frac{1}{u(J)}\int_J (M_u^+ f)(x)u(x)\,\mathrm{d}x \leqslant C\|f\|_{u,L\log L,J} \tag{4.9}$$

for all f such that $\operatorname{supp}(f) \subset J$, where J is an interval. In fact, by a homogeneity argument we may assume that $\|f\|_{u,L\log L,J} = 1$ and, thus,

$$\frac{1}{u(J)}\int_J |f(x)|\log(\mathrm{e}+|f(x)|)u(x)\,\mathrm{d}x \leqslant 1.$$

Now, using the weak-type $(1,1)$ inequality for M_u^+ with respect to the measure $u(x)\,\mathrm{d}x$, we get

$$\begin{aligned}\int_J (M_u^+ f)(x)u(x)\,\mathrm{d}x &= \int_0^\infty u(\{x\in J\colon M_u^+ f(x) > \lambda\})\,\mathrm{d}\lambda \\ &\leqslant u(J) + C\int_1^\infty \frac{1}{\lambda}\int_{\{x\in J\colon |f(x)|>1/2\}} |f(x)|u(x)\,\mathrm{d}x\,\mathrm{d}\lambda \\ &= u(J) + C\int_{\{x\in J\colon |f(x)|>1/2\}} |f(x)|u(x)\int_1^{2|f(x)|}\frac{1}{\lambda}\,\mathrm{d}\lambda\,\mathrm{d}x \\ &= u(J) + C\int_{\{x\in J\colon |f(x)|>1/2\}} |f(x)|u(x)\log(2|f(x)|)\,\mathrm{d}x \\ &\leqslant u(J) + C\int_J |f(x)|\log(\mathrm{e}+|f(x)|)u(x)\,\mathrm{d}x. \\ &\leqslant Cu(J).\end{aligned}$$

Hence, (4.9) follows for f such that $\operatorname{supp} f \subset J$.

Let $d > a$ and let $\tilde{d} > d$ be such that $u(a,d) = u(d,\tilde{d})$. (The existence of such $\tilde{d}$ is guaranteed by the condition A_p^+.) We write $f = f_1 + f_2$ with $f_1 = f\chi_J$, where $J = (a,\tilde{d})$. Then

$$\begin{aligned}&\frac{1}{u(a,d)}\int_a^d (M_u^+ f)(x)u(x)\,\mathrm{d}x \\ &\qquad \leqslant \frac{1}{u(a,d)}\int_a^d (M_u^+ f_1)(x)u(x)\,\mathrm{d}x + \frac{1}{u(a,d)}\int_a^d (M_u^+ f_2)(x)u(x)\,\mathrm{d}x \\ &\qquad =: \mathrm{I} + \mathrm{II}.\end{aligned}$$

By (4.9),

$$\mathrm{I} \leqslant \frac{2}{u(J)}\int_J (M_u^+ f_1)(x)u(x)\,\mathrm{d}x \leqslant C\|f_1\|_{u,L\log L,J} \leqslant CM_{u,L\log L}^+(f)(a).$$

On the other hand, since $(M_u^+ f_2)(x) \leqslant 2(M_u^+ f_2)(a)$ for every $x \in (a,d)$, we get

$$\mathrm{II} \leqslant 2(M_u^+ f_2)(a) \leqslant 2M_{u,L\log L}^+(f)(a).$$

The claim now follows on taking the supremum over $d > a$. □

Claim 4.15. *There exists a $C > 0$ such that, for every interval $I = (a, c)$ and all $E \subset I^+$,*

$$M_u^+(M_u^+(\chi_E))(a) \leqslant C \frac{u(E)}{u(I^-)} \log\left(\mathrm{e} + \frac{u(I^-)}{u(E)}\right).$$

Proof of Claim 4.15. Let $h > 0$ such that $\lambda_h := \|\chi_E\|_{u, L\log L, (a,a+h)} > 0$. Then

$$\frac{1}{\lambda_h} \log\left(\mathrm{e} + \frac{1}{\lambda_h}\right) = \frac{u(a, a+h)}{u(E \cap (a, a+h))} \geqslant \frac{u(I^-)}{u(E)}. \tag{4.10}$$

Define $\phi(t) := t \log(\mathrm{e} + t)$. Then

$$\phi\left(\frac{1}{\lambda_h}\right) \geqslant \frac{u(I^-)}{u(E)};$$

hence

$$\frac{1}{\lambda_h} \geqslant \phi^{-1}\left(\frac{u(I^-)}{u(E)}\right).$$

Since $\phi^{-1}(t) \approx t/\log(\mathrm{e} + t)$ for large t, we have

$$\lambda_h \leqslant C \frac{u(E)}{u(I^-)} \log\left(\mathrm{e} + \frac{u(I^-)}{u(E)}\right)$$

and taking supremum in $h > 0$ we get the claim by applying Claim 4.14 to $f = \chi_E$. This shows Claim 4.15. □

Now, if $B(t) := t \log(\mathrm{e} + 1/t)$, then we obtain from Claims 4.12, 4.13 and 4.15 that

$$B\left(\frac{|E|}{|I|}\right) \leqslant C\left[B\left(\frac{u(E)}{u(I^-)}\right)\right]^{1/p}. \tag{4.11}$$

The rest of the proof is exactly the same as that in [**6**]. In fact, since

$$B^{-1}(B(t)^p) \sim t^p \log^{p-1}\left(\mathrm{e} + \frac{1}{t}\right), \quad 0 < t < 1,$$

from (4.11)

$$\left(\frac{|E|}{|I|}\right)^p \log^{p-1}\left(\mathrm{e} + \frac{|I|}{|E|}\right) \leqslant C \frac{u(E)}{u(I^-)},$$

and we obtain the assertion of the proposition. □

Finally, combining Theorem 4.8 and Proposition 4.11, we get the (sufficiency part of the) following result of Sawyer.

Corollary 4.16. *Let u be a weight on $\mathbb{R}$ and let $1 < p < \infty$. Then M^+ is bounded on L_u^p if and only if there exists $C > 0$ such that*

$$\int_{I^-} u \left(\int_{I^+} u^{-1/(p-1)}\right)^{p-1} \leqslant C|I|^p$$

for all intervals I, I^- and I^+.

5. An application to variable-exponent Lebesgue spaces

In this final section, we shall present applications of Theorem 3.5 to variable-exponent Lebesgue spaces.

Let $p\colon \mathbb{R} \to [1,\infty)$ be a measurable function. We denote by $L^{p(\cdot)}(\mathbb{R})$ the space of all measurable functions f on $\mathbb{R}$ such that, for some $\lambda > 0$,

$$\int_{\mathbb{R}} \left|\frac{f(x)}{\lambda}\right|^{p(x)} \mathrm{d}x < \infty,$$

endowed with the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R})} = \inf\left\{\lambda > 0\colon \int_{\mathbb{R}} \left|\frac{f(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \leqslant 1\right\}.$$

The space $L^{p(\cdot)}(\mathbb{R})$ is a particular instance of the so-called Musiełak–Orlicz space [**9**].

We denote by $\mathcal{P}^+(\mathbb{R})$ the class of all measurable functions $p\colon \mathbb{R} \to [1,\infty)$, for which M^+ is bounded on $L^{p(\cdot)}(\mathbb{R})$. We further denote by $\mathcal{B}$ the set of all measurable functions $p\colon \mathbb{R} \to [1,\infty)$ such that

$$1 < p_- := \operatorname{ess\,inf}\{p(x);\ x \in \mathbb{R}\} \leqslant \operatorname{ess\,sup}\{p(x);\ x \in \mathbb{R}\} =: p_+ < \infty.$$

It has been proved in [**3**] that if p satisfies the following uniform continuity condition

$$p(x) - p(y) \leqslant \frac{K}{-\log(|x-y|)} \tag{5.1}$$

for $x, y \in \mathbb{R}$, $0 < y - x \leqslant \frac{1}{2}$ and if p is constant outside some large interval, then $p \in \mathcal{P}^+(\mathbb{R})$. Their condition thus contains two separate requirements (a control over the local behaviour and constancy near infinity). Recently, Nekvinda [**11**, Theorem 1] improved this result by finding certain finer conditions at ∞, still sufficient for the boundedness of the one-sided maximal operator. Precisely, he proved that if p satisfies (5.1) and there exists a non-increasing function $q \in \mathcal{B}(\mathbb{R})$ and a constant $c > 0$ such that

$$\int_{\{x\colon |p(x)-q(x)|\neq 0\}} c^{1/(|p(x)-q(x)|)}\, \mathrm{d}x < \infty, \tag{5.2}$$

then, again, M^+ is bounded on $L^{p(\cdot)}(\mathbb{R})$.

We will give an alternative proof of the Nekvinda's Theorem, based on Theorem 3.5. This step is done in the spirit of [**6**].

Theorem 5.1. *Let p be a bounded positive function with $p_- > 0$, satisfying (5.1) and (5.2). Then*

$$\|m_\lambda^+ f\|_{L^{p(\cdot)}(\mathbb{R})} \leqslant \frac{C}{\lambda^{1/p_-}} \|f\|_{L^{p(\cdot)}(\mathbb{R})}, \quad 0 < \lambda < 1,$$

where C depends on p.

In the case when $p_- > 1$ the conditions of Theorem 5.1 coincide with those of [**11**, Theorem 1]. In this case, Theorem 5.1 clearly yields

$$\alpha^+_{L^{p(\cdot)}} \leqslant \frac{1}{p_-} < 1$$

and thus, by Theorem 3.5, the boundedness of M^+ on $L^{p(\cdot)}(\mathbb{R})$.

Proof of Theorem 5.1. The statement of the theorem is equivalent to saying that there exists a constant $C > 0$, independent of f and λ, such that

$$\int_{\mathbb{R}} (\lambda^{1/p_-} m^+_\lambda f(x))^{p(x)} \, \mathrm{d}x \leqslant C \tag{5.3}$$

whenever

$$\int_{\mathbb{R}} |f(x)|^{p(x)} \, \mathrm{d}x \leqslant 1.$$

We fix a function f and set

$$f_1 = f\chi_{\{|f| \geqslant 1\}} \quad \text{and} \quad f_2 = f - f_1.$$

Let us show that, for any x,

$$(\lambda^{1/p_-} m^+_{\lambda/2} f_1(x))^{p(x)} \leqslant C\lambda m^+_{\lambda/2}(f_1^{p(\cdot)})(x), \tag{5.4}$$

$$(\lambda^{1/p_-} m^+_{\lambda/2} f_2(x))^{p(x)} \leqslant C(\psi(x) + \lambda m^+_{\lambda/4}(f_2^{p(\cdot)})(x)), \tag{5.5}$$

with some $\psi \in L^1$, where C and $\|\psi\|_{L^1}$ depend on p.

Assume, for the time being, that (5.4) and (5.5) are satisfied. We note that these estimates easily imply (5.3). Indeed, since (2.11) implies

$$\|m^+_\lambda f\|_{L^1} = \frac{1}{\lambda}\|f\|_{L^1}, \tag{5.6}$$

and, next,

$$\int_{\mathbb{R}} |f(x)|^{p(x)} \, \mathrm{d}x \leqslant 1,$$

we conclude that (5.6) shows that the L^1-norms of the right-hand sides of (5.4) and (5.5) are bounded by constants depending only on p. Observing also that, by (2.4),

$$(\lambda^{1/p_-} m^+_\lambda f(x))^{p(x)} \leqslant 2^{p_+ - 1}((\lambda^{1/p_-} m^+_{\lambda/2} f_1(x))^{p(x)} + (\lambda^{1/p_-} m^+_{\lambda/2} f_2(x))^{p(x)}),$$

we obtain that (5.4) and (5.5) imply (5.3). To prove (5.4), fix arbitrary x and $h > 0$. We claim that

$$F(x, h) := [\lambda^{1/p_-}(f_1\chi_{(x,x+h)})^*(\tfrac{1}{2}\lambda h)]^{p(x) - p_-(x,x+h)} \leqslant C, \tag{5.7}$$

where we define $p_-(x, x + h) := \inf_{y \in (x,x+h)} p(y)$.

For $h > \frac{1}{2}$, we get, using the Chebyshev inequality,

$$\begin{aligned}\lambda^{1/p_-}(f_1\chi_{(x,x+h)})^*\left(\frac{\lambda h}{2}\right) &\leqslant \left(\frac{2}{h}\right)^{1/p_-}\|f_1\|_{p_-}\\ &\leqslant \left(\frac{2}{h}\right)^{1/p_-}\left(\int_{\mathbb{R}}|f_1(x)|^{p(x)}\,\mathrm{d}x\right)^{1/p_-}\\ &\leqslant \left(\frac{2}{h}\right)^{1/p_-},\end{aligned}$$

whence $F(x,h) \leqslant 4^{(p_+-p_-)/p_-}$, while, if $h \leqslant \frac{1}{2}$, by (5.1) we get

$$\begin{aligned}F(x,h) &\leqslant \left(\frac{2}{h}\right)^{(p(x)-p_-(x,x+h))/p_-}\\ &\leqslant \left(\frac{2}{h}\right)^{K/(-p_-\log h)}\\ &= 2^{K/(p_-\log 1/h)}\left(\frac{1}{h}\right)^{K/(p_-\log 1/h)}\\ &\leqslant 2^{K/(p_-\log 2)}\left(\frac{1}{h}\right)^{K/(p_-\log 1/h)}\\ &= 2^{\log_2\exp(K/p_-)}\left(\frac{1}{h}\right)^{\log_{1/h}\exp(K/p_-)}\\ &= \exp\left(\frac{2K}{p_-}\right),\end{aligned}$$

which shows (5.7). Now, (5.7) combined with (2.3) yields

$$\begin{aligned}\left(\lambda^{1/p_-}(f_1\chi_{[x,x+h)})^*\left(\frac{\lambda h}{2}\right)\right)^{p(x)} &\leqslant C\left(\lambda^{1/p_-}(f_1\chi_{[x,x+h)})^*\left(\frac{\lambda h}{2}\right)\right)^{p_-(x,x+h)}\\ &= C\lambda^{p_-(x,x+h)/p_-}(f_1^{p_-(x,x+h)}\chi_{[x,x+h)})^*\left(\frac{\lambda h}{2}\right)\\ &\leqslant C\lambda m^+_{\lambda/2}(f_1^{p(\cdot)})(x),\end{aligned}$$

and (5.4) follows.

To prove (5.5), we apply [**6**, Lemma 5.5] together with (2.3) and (2.4), which yields

$$\begin{aligned}&(\lambda^{1/p_-}m^+_{\lambda/2}f_2(x))^{p(x)}\\ &\quad\leqslant \alpha^{1/(|p(x)-q(x)|)} + \left(\left(\frac{1}{\alpha}\right)^{1/p_-}+1\right)(\lambda^{1/p_-}m^+_{\lambda/2}f_2(x))^{q(x)}\\ &\quad\leqslant \alpha^{1/(|p(x)-q(x)|)} + \left(\left(\frac{1}{\alpha}\right)^{1/p_-}+1\right)(\lambda^{q(x)/p_-}m^+_{\lambda/2}(f_2^{q(\cdot)})(x))\end{aligned}$$

$$\leqslant \alpha^{1/(|p(x)-q(x)|)} + \left(\left(\frac{1}{\alpha}\right)^{1/p_-} + 1\right)(\lambda^{q(x)/p_-} m^+_{\lambda/4}(\alpha^{1/(|p(x)-q(x)|)})(x)) + \lambda^{q(x)/p_-}\left(\left(\frac{1}{\alpha}\right)^{1/p_-} + 1\right) m^+_{\lambda/4}(|f_2|^{p(\cdot)})(x).$$

By (5.2) and the monotonicity of q, we get $q(x) > p_-$. Thus,

$$(\lambda^{1/p_-} m^+_{\lambda/2} f_2(x))^{p(x)} \leqslant \alpha^{1/(|p(x)-q(x)|)} + \left(\left(\frac{1}{\alpha}\right)^{1/p_-} + 1\right)\lambda m^+_{\lambda/4}(\alpha^{1/(|p(x)-q(x)|)})(x) + \lambda\left(\left(\frac{1}{\alpha}\right)^{1/p_-} + 1\right) m^+_{\lambda/4}(|f_2|^{p(\cdot)})(x),$$

and (5.5) follows with

$$\psi(x) = \alpha^{1/(|p(x)-q(x)|)} + \lambda\left(\left(\frac{1}{\alpha}\right)^{1/p_-} + 1\right) m^+_{\lambda/4}(\alpha^{1/(|p(x)-q(x)|)})(x),$$

by (5.6) and (5.2). It is clear that $\|\psi\|_{L^1}$ depends only on p.

The proof of the theorem is complete. □

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