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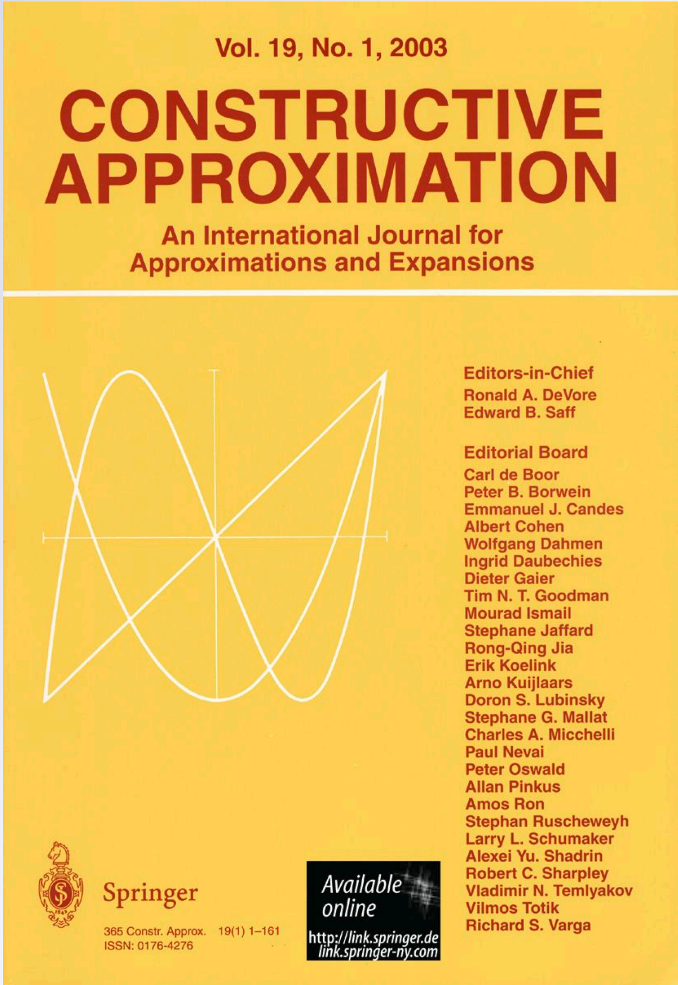

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Parabolic Besov Regularity for the Heat Equation

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Abstract We obtain parabolic Besov smoothness improvement for temperatures on cylindrical regions based on Lipschitz domains. The results extend those for harmonic functions obtained by S. Dahlke and R. DeVore using the wavelet description of Besov regularity.

Keywords Temperatures · Besov regularity · Wavelets · Lipschitz cylindrical regions · Nonlinear approximation

Mathematics Subject Classification Primary 35B65 · 35K05 · 46E35

1 Introduction

Dahlke and DeVore prove in [8] that a particular imbedding property in the scale of Besov spaces $B^{\alpha}_{p,q}(D)$ when $q = p$ ($B^{\alpha}_{p,p}(D) =: B^{\alpha}_{p}(D)$) holds when we restrict our attention to harmonic functions on a Lipschitz domain D of $\mathbb{R}^d$. Specifically, in Theorem 3.2 in [8], the authors prove that if D is a Lipschitz domain in $\mathbb{R}^d$, $1 < p < \infty$, $\lambda > 0$, $0 < \alpha < \frac{\lambda d}{d-1}$ and $\frac{1}{\tau} = \frac{1}{p} + \frac{\alpha}{d}$, then

$$\mathcal{H}(D) \cap B^{\lambda}_{p}(D) \subset B^{\alpha}_{\tau}(D), \tag{1.1}$$

where $\mathcal{H}(D)$ is the space of all harmonic functions defined on D.

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Since we can choose the parameter α to be larger than λ, the result is relevant in order to improve the rate of convergence of nonlinear methods of approximation. For details regarding this application of (1.1), we refer to [8].

In this paper, we aim to prove similar results for temperatures. We shall write Ω to denote the parabolic cylinder $D \times (0,T)$ with $T > 0$ and D a bounded Lipschitz domain in $\mathbb{R}^d$ ($d \geq 2$). Let $\Theta(\Omega)$ denote the space of all temperatures $u = u(x,t)$ in Ω. In other words, $\Theta(\Omega) := \{u : \frac{\partial u}{\partial t} = \Delta u \text{ in } \Omega\}$.

Let us briefly introduce the parabolic Besov scale $\mathbb{B}_r^\gamma(\Omega)$ for $0 < \gamma < 1$, $1 < r < \infty$. Among the several approaches to Besov spaces, we choose the interpolation approach. For $0 < \gamma < 1$, $1 < r, s < \infty$, define

$$B_{r,s}^{\gamma,\frac{\gamma}{2}}(\Omega) := \big(L_r(\Omega), W_r^{2,1}(\Omega)\big)_{\frac{\gamma}{2},s}$$

and $\mathbb{B}_r^\gamma(\Omega) = B_{r,r}^{\gamma,\frac{\gamma}{2}}(\Omega)$. Here, $W_r^{2,1}(\Omega)$ is the anisotropic Sobolev space defined by the norm

$$\|v\|_{W_r^{2,1}(\Omega)} := \|v\|_{L_r(\Omega)} + \sum_{i=1}^d \left\| \frac{\partial v}{\partial x_i} \right\|_{L_r(\Omega)} + \sum_{i=1}^d \sum_{j=1}^d \left\| \frac{\partial^2 v}{\partial x_i \partial x_j} \right\|_{L_r(\Omega)} + \left\| \frac{\partial v}{\partial t} \right\|_{L_r(\Omega)}.$$

For general $\gamma > 0$, we can define $\mathbb{B}_r^\gamma(\Omega)$ through the action of the derivatives $\frac{\partial}{\partial t}$ and $\frac{\partial^2}{\partial x_i \partial x_j}$.

We are now in a position to state our results, which give improvements of Besov regularity for temperatures in two different settings.

Theorem 1 *Let* $1 < p < \infty$, $\lambda > 0$, ℓ *the largest integer less than* $\lambda + d$, $0 < \alpha < \min\{\ell, \frac{\lambda d}{d-1}\}$, *and* $\frac{1}{\tau} = \frac{1}{p} + \frac{\alpha}{d}$. *Then*

$$\Theta(\Omega) \cap L_p\big((0,T); B_p^\lambda(D)\big) \subset L_\tau\big((0,T); B_\tau^\alpha(D)\big);$$

moreover, if u *is a temperature in* Ω, *we have that*

$$\|u\|_{L_\tau((0,T);B_\tau^\alpha(D))} \leq C \|u\|_{L_p((0,T);B_p^\lambda(D))}$$

for some constant C *depending on* Ω, d, p, *and* λ.

Theorem 2 *Let* $1 < p < \infty$, $\lambda > 0$, $0 < \alpha < \min\{d(1 - \frac{1}{p}), \frac{\lambda d}{d-1}\}$, *and* $\frac{1}{\tau} = \frac{1}{p} + \frac{\alpha}{d}$. *Then*

$$\Theta(\Omega) \cap \mathbb{B}_p^\lambda(\Omega) \subset \bigcap_{\alpha > \varepsilon > 0} \mathbb{B}_\tau^{\alpha - \varepsilon}(\Omega).$$

As in the elliptic case, a central tool for our results is the use of the characterization of the elliptic Besov spaces through Daubechies wavelet bases. For our first result, we shall use the parabolic analog of Theorem 3.1 in [8], essentially contained in Corollary 5.2 of [2]. Our second result is a corollary of the first one if we use Theorem 1.1 in [3].

In the case considered in [8], the improvement of Besov regularity for harmonic functions is applied, together with results in [12], to obtain Besov regularity for the Dirichlet and the Poisson problems in terms of Besov regularity of data. When the domain D is polyhedral, the results in [7] improve those in [8] for the Poisson problem. In [1], the authors, based in Theorem 1 and the results in [11], obtain parabolic Besov regularity of temperatures in terms of the Besov regularity of initial and boundary conditions. It could be expected that the techniques used here would be of application for temperatures when the basis D of the cylinder Ω is polyhedral.

In Sect. 2, we introduce the spaces involved in Theorems 1 and 2 and the wavelet characterization of the elliptic Besov spaces. Section 3 contains estimates for the space gradients of temperatures which are essentially proved in [2]. In Sect. 4, we introduce partitions of the wavelet index set at each scale and each time and we prove some central technical lemmas. Finally Sect. 5 is devoted to proving Theorems 1 and 2.

2 Besov Spaces

The literature on classical Besov spaces and their extensions to parabolic settings is abundant, see [5, 6, 15–17]. Let D be a bounded Lipschitz domain in $\mathbb{R}^d$. Under this regularity condition on ∂D, we have that the Besov space $B_r^\gamma(D)$ coincides with the space of the restrictions to D of $B_r^\gamma(\mathbb{R}^d)$. A basic tool in [8], which shall also be used here in the proof of Theorem 1, is the characterization of $B_r^\gamma(\mathbb{R}^d)$ in terms of Daubechies type wavelets. We shall closely follow, with minor changes, the notation in [8] in order to state the characterization results that we shall use subsequently. Let $\mathcal{D}_j$, $j \in \mathbb{Z}$, be the set of all dyadic cubes in $\mathbb{R}^d$ of measure 2^{-jd}. Set $\mathcal{D} = \bigcup_{j\in\mathbb{Z}} \mathcal{D}_j$ to denote the family of all dyadic cubes in $\mathbb{R}^d$ and $\mathcal{D}^+$ to denote those cubes in $\mathcal{D}$ with measure less than or equal to 1.

For any given positive integer n, there exists a set Ψ of $2^d - 1$ functions $\psi \in \mathcal{C}_0^n(\mathbb{R}^d)$, with n vanishing moments and $\phi \in \mathcal{C}_0^n(\mathbb{R}^d)$ all of them supported in a cube Q centered at the origin of $\mathbb{R}^d$, such that the system $\{\psi_I : \psi \in \Psi, I \in \mathcal{D}\}$ is an orthonormal basis for $L_2(\mathbb{R}^d)$ and $\{\psi_I : \psi \in \Psi, I \in \mathcal{D}^+\}$ is an orthonormal basis for the complement of S_0, the $L_2(\mathbb{R}^d)$ closure of the linear span of $\{\phi_I : I \in \mathcal{D}_0\}$. Here, as usual, ψ_I and ϕ_I denote the translation and scaling of ψ and ϕ defined by the dyadic cube $I \in \mathcal{D}$, $\psi_I(x) = 2^{\frac{jd}{2}}\psi(2^j x - k)$ with $I = \prod_{l=1}^d [\frac{k_l}{2^j}, \frac{k_l+1}{2^j}]$ and $k = (k_1, \dots, k_d)$. We point out here that the support of each ψ_I is contained in a cube $Q(I)$ containing I with $\frac{|Q(I)|}{|I|} = |Q|$.

Here and throughout this paper, we shall fix the restrictions of all the involved parameters. The space dimension d is larger than or equal to two. The starting integrability parameter p is finite and larger than one. The starting regularity parameter λ is positive. The improved regularity parameter α belongs to the open interval $(0, \min\{\ell, \frac{\lambda d}{d-1}\})$ with ℓ the largest integer less than $\lambda + d$. The final integrability, which is worse than the starting one, is given by the positive parameter τ defined by $\frac{1}{\tau} = \frac{1}{p} + \frac{\alpha}{d}$. Since we shall deal with these parameters when considering the elliptic Besov spaces, we will use their wavelet characterization for the integrability parameter less than one but only for positive regularity.

It will be useful to write out explicitly the characterizations of $B^{\lambda}_{p}(\mathbb{R}^d)$ and $B^{\alpha}_{\tau}(\mathbb{R}^d)$ with the above relationships among the parameters. We shall also adopt here the $L_p(\mathbb{R}^d)$ normalization of the wavelet basis introduced in [8]. For $\psi \in \Psi$, and $I \in \mathcal{D}$, with $\psi_{I,p}$, we define the function

$$\psi_{I,p} = |I|^{\frac{1}{2}-\frac{1}{p}}\psi_I.$$

Notice that $\psi_I = \psi_{I,2}$ and that $\|\psi_{I,p}\|_{L_p(\mathbb{R}^d)} = \|\psi\|_{L_p(\mathbb{R}^d)}$ for every $I \in \mathcal{D}$. Let us also observe that with this notation, the expansion of a function f in this wavelet basis is given by

$$f = \sum_{I\in\mathcal{D}}\sum_{\psi\in\Psi}\langle f,\psi_I\rangle\psi_I = \sum_{I\in\mathcal{D}}\sum_{\psi\in\Psi}\langle f,\psi_{I,p'}\rangle\psi_{I,p},$$

where p' is the Hölder conjugate of p. When the wavelet decomposition is only used at the scales defined by $|I| \le 1$, we only have the cubes in $\mathcal{D}^+$. And the representation of f becomes now

$$f = P_0 f + \sum_{I\in\mathcal{D}^+}\sum_{\psi\in\Psi}\langle f,\psi_{I,p'}\rangle\psi_{I,p}, \tag{2.1}$$

with P_0 the orthogonal projector on $S_0 = \overline{\operatorname{span}\{\phi_I : I \in \mathcal{D}_0\}}$, the closure in the L_2 sense.

Proposition 3 *Let d, p, λ, α, and τ be as before. Assume that $\Psi \subset \mathcal{C}^n(\mathbb{R}^d)$ with $n > \lambda + d$. Then*

(A) $f \in B^{\lambda}_{p}(\mathbb{R}^d)$ *if and only if*

$$\|P_0 f\|_{L_p(\mathbb{R}^d)} + \left(\sum_{I\in\mathcal{D}^+}\sum_{\psi\in\Psi}|I|^{-\frac{\lambda p}{d}}\left|\langle f,\psi_{I,p'}\rangle\right|^p\right)^{\frac{1}{p}} < \infty. \tag{2.2}$$

(B) $f \in B^{\alpha}_{\tau}(\mathbb{R}^d)$ *if and only if*

$$\|P_0 f\|_{L_\tau(\mathbb{R}^d)} + \left(\sum_{I\in\mathcal{D}^+}\sum_{\psi\in\Psi}\left|\langle f,\psi_{I,p'}\rangle\right|^\tau\right)^{\frac{1}{\tau}} < \infty. \tag{2.3}$$

For a proof, see [10, 14].

For characterizations of anisotropic Besov spaces, see [13] and [6].

3 Gradient Estimates

For a given bounded Lipschitz domain $D \subset \mathbb{R}^d$, $T > 0$, and $\Omega = D \times (0,T)$, define the parabolic distance of $(x,t) \in \Omega$ to the parabolic boundary of Ω, $\partial_{\text{par}}\Omega =$

$(D \times \{0\}) \cup (\partial D \times [0,T))$, by $\delta(x,t) = \inf\{\rho((x,t);(y,s)) : (y,s) \in \partial_{\mathrm{par}}\Omega\}$. Here ρ is the standard parabolic distance $\rho((x,t);(y,s)) = \max\{|x-y|, \sqrt{|t-s|}\}$ in $\mathbb{R}^{d+1}$.

Let us first introduce the two basic maximal functions involved in the pointwise estimate that can be proved following closely the lines of the proof of Corollary 5.2 in [2]. The one-sided Hardy-Littlewood maximal operator on $(0,T)$ is given by

$$M^- g(t) = \sup_{0<h<t} \frac{1}{h} \int_{t-h}^{t} \left|g(s)\right| ds$$

for $t \in (0,T)$, when g is a locally integrable function on $(0,T)$. On the other hand, for a given smooth function f on D, the local version of the Calderón maximal function of order λ is given at a point $x \in D$, by

$$M_D^{\#,\lambda} f(x) = \sup_{0<r<\tilde{\delta}(x)} \frac{1}{|B(x,r)|^{1+\frac{\lambda}{d}}} \int_{B(x,r)} \left|f(y) - P_x(y)\right| dy,$$

where $\tilde{\delta}(x) = \inf\{|x-y| : y \in \partial D\}$ and P_x is the Taylor polynomial of degree $\ell - 1$ for f at x, with ℓ the smallest integer larger than λ. With the above notation and the arguments in the proof of Corollary 5.2 in [2], we get the following result:

Theorem 4 *For $\lambda > 0$, there exists a constant C depending only on the dimension d and on λ such that for each $u \in \Theta(\Omega)$ the inequality*

$$\delta^{\ell-\lambda}\left|\nabla^\ell u\right| \le C M^-\left[M_D^{\#,\lambda} u\right]$$

holds in Ω, where $\nabla^\ell u$ denotes the vector of all the space derivatives of u of order ℓ.

From the above result, we shall obtain the L_p estimates which shall be used in the proof of Theorem 1. We denote by $L_p((0,T); B_p^\lambda(D))$ the space of all measurable functions $v(x,t)$ on Ω such that the norm

$$\|v\|_{L_p((0,T);B_p^\lambda(D))} := \left(\int_0^T \|v(\cdot,t)\|_{B_p^\lambda(D)}^p \, dt\right)^{\frac{1}{p}}$$

is finite.

Corollary 5 *For $\lambda > 0$ and $1 < p < \infty$, there exists a constant C depending on d, λ, p, and on the Lipschitz character of D such that the inequality*

$$\left\|\delta^{\ell-\lambda}\left|\nabla^\ell u\right|\right\|_{L_p(\Omega)} \le C\|u\|_{L_p((0,T);B_p^\lambda(D))}$$

holds for every $u \in \Theta(\Omega)$.

The proof of Corollary 5 follows from the $L_p(0,T)$ boundedness of M^- and the boundedness of $M_D^{\#,\lambda}$ as an operator from $B_p^\lambda(D)$ to $L_p(D)$ (see [9]).

For the proof of Theorem 2, we shall make use of the next result which is contained in Theorem 1.1 in [3] and remarks following it.

Theorem 6 *For* $\gamma > 0$, $1 < q < \infty$, *and* $0 < \varepsilon < \gamma$, *we have that*

$$\Theta(\Omega) \cap L_q\big((0,T); B_q^{\gamma}(D)\big) \subset \mathbb{B}_q^{\gamma-\varepsilon}(\Omega).$$

4 Partition of the wavelet index set at time *t*

As before, we consider the functions $\delta(x,t)$ defined in Ω and $\tilde{\delta}(x)$ defined in D, the parabolic and elliptic distance functions to the parabolic and elliptic boundaries of Ω and D respectively. It is not difficult to prove that $\delta(x,t) = \min\{\tilde{\delta}(x), \sqrt{t}\}$. This formula allows us to produce the following partition of Ω, $\Omega^1 = \{(x,t) \in \Omega : \delta(x,t) = \tilde{\delta}(x)\}$ and Ω^2 to be the interior of the set $\{(x,t) \in \Omega : \delta(x,t) = \sqrt{t}\}$.

The above partition of Ω induces a classification at each time level of the dyadic cubes in the family $\Gamma_j := \{I \in \mathcal{D}_j : Q(I) \cap \overline{D} \neq \varnothing\}$. Set $\Gamma_{j,0} := \{I \in \mathcal{D}_j : Q(I) \cap \partial D \neq \varnothing\}$ and $\overset{o}{\Gamma}_j := \Gamma_j \setminus \Gamma_{j,0}$. Now, for the inner cubes $I \in \overset{o}{\Gamma}_j$ and $0 < t < T$, two different positions $\overset{o}{\Gamma}{}_j^1(t)$ and $\overset{o}{\Gamma}{}_j^2(t)$ of the interval I raised to time t, $I(t) = I \times \{t\}$ are still possible according to the above decomposition of Ω. Specifically, define

$$\overset{o}{\Gamma}{}_j^1(t) := \big\{I \in \overset{o}{\Gamma}_j : I(t) \subset \Omega^1\big\} \quad \text{and} \quad \overset{o}{\Gamma}{}_j^2(t) := \overset{o}{\Gamma}_j \setminus \overset{o}{\Gamma}{}_j^1(t).$$

In Fig. 1, some typical possible situations for an interval $I \in \Gamma_j$ are sketched; $I_1 \in \overset{o}{\Gamma}{}_j^1(t_2) \cap \overset{o}{\Gamma}{}_j^2(t_1)$, $I_2 \in \overset{o}{\Gamma}{}_j^2(t_2)$ and $I_3 \in \overset{o}{\Gamma}{}_j^2(t_1)$.

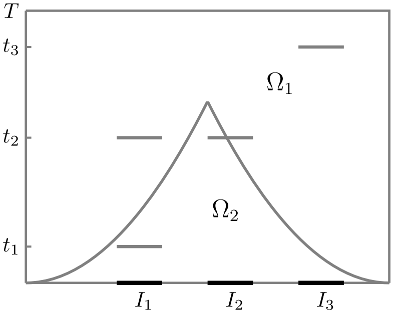
For each $k \in \mathbb{Z}^+$, set $\Gamma_{j,k} := \{I \in \Gamma_j : k2^{-j} \le \tilde{\delta}_{Q(I)} < (k+1)2^{-j}\}$, where $\tilde{\delta}_{Q(I)}$ denotes the elliptic Euclidean distance of $Q(I)$ to ∂D. By Γ, we shall denote the family of all cubes in any Γ_j, i.e., $\Gamma = \bigcup_{j \ge 0} \Gamma_j$. Notice also that $\Gamma_j = \bigcup_{k \ge 0} \Gamma_{j,k}$.

Let us state and prove some lemmas that we shall use in the next section.

Lemma 7 *There exist constants* c_1 *and* c_2 *such that with* $t > c_1 4^{-j}$ *and* $I \in \overset{o}{\Gamma}{}_j^2(t)$, *we have that* $\delta(x,t) \ge c_2\sqrt{t}$ *for every* $x \in Q(I)$.

Proof Since, once the regularity and support properties of the wavelets are fixed, the cube $Q(I)$ has measure comparable to that of I and $Q(I) \supset I$, there exists a constant ν such that $Q(I) \subset \nu I$, the dilation of I with respect to its center by a factor $\nu > 0$,

Fig. 1 Decomposition of Γ_j



for every $I \in \mathcal{D}$. On the other hand, since $I \in \overset{o}{\Gamma}{}^2_j(t)$, $I(t) \cap \Omega^2 \neq \varnothing$. So, there exists $z \in I$ such that $(z,t) \in \Omega^2$. In other words $\delta(z,t) = \sqrt{t}$. Pick $x \in Q(I)$; then

$$\begin{aligned}\sqrt{t} = \delta(z,t) &\le \rho\big((z,t);(x,t)\big) + \delta(x,t)\\ &= |x-z| + \delta(x,t)\\ &\le 2\nu 2^{-j} + \delta(x,t).\end{aligned}$$

Taking $c_1 = 16\nu^2$ and $c_2 = \frac{1}{2}$, we have the result, since

$$\sqrt{t} \le 2\nu 2^{-j} + \delta(x,t) < \frac{2\nu\sqrt{t}}{\sqrt{c_1}} + \delta(x,t) = \frac{\sqrt{t}}{2} + \delta(x,t).$$ □

For the intervals in $\overset{o}{\Gamma}{}^1_j(t)$ instead, we have the following result.

Lemma 8 *For every* $t > 0$, *every* $I \in \overset{o}{\Gamma}{}^1_j(t)$ *and every* $x \in Q(I)$, *we have that* $\delta(x,t) \ge \tilde{\delta}_{Q(I)}$.

Proof Since $I \in \overset{o}{\Gamma}{}^1_j(t)$, then for $x \in Q(I)$, we have that $\delta(x,t) = \tilde{\delta}(x) \ge \tilde{\delta}_{Q(I)}$. □

The next lemma contains properties of the families $\Gamma_{j,k}$ which will be important in the proof of our main result. The second one is a consequence of the regularity of the boundary of D, the first and the third of the boundedness of D. By $\#(E)$, we shall denote the number of elements of the set E.

Lemma 9 *There exist three constants* C_0, C_1, *and* C_2 *depending on* D *such that*

(9.1) $\#(\Gamma_j) \le C_0 2^{jd}$,
(9.2) $\#(\Gamma_{j,k}) \le C_1 2^{j(d-1)}$ *for every* $j, k \ge 0$,
(9.3) $\Gamma_{j,k} = \varnothing$ *for* $k > C_2 2^j$.

Proof The estimate (9.1) follows directly from the boundedness of D. Notice that (9.3) holds with $C_2 = \operatorname{diam} D$, since if $I \in \Gamma_{j,k}$, we necessarily have that

$$k 2^{-j} \le \tilde{\delta}_{Q(I)} \le \tilde{\delta}_I \le \operatorname{diam} D,$$

hence $k \le 2^j \operatorname{diam} D$.

Let us briefly sketch the proof of (9.2) for $k = 0$. Since D is a bounded Lipschitz domain in $\mathbb{R}^d$, we have that the $(d-1)$-Hausdorff measure $\sigma(\partial D)$ of the boundary of D is positive and finite. For $h > 0$, $0 < \alpha < \frac{\pi}{2}$, $|\nu| = 1$, $\nu \in \mathbb{R}^d$, and $x \in \mathbb{R}^d$, set $K(x,h,\alpha,\nu)$ to denote the cone $\{y : |(y-x)\cdot\nu| \ge |x-y|\cos\alpha \text{ and } |y-x| < h\}$. From the Lipschitz character of ∂D, there exist a positive number h and $\alpha \in (0, \frac{\pi}{2})$ such that for every $x \in \partial D$ there exists a unit vector $\nu(x)$, for which the cone $K(x) := K(x,h,\alpha,\nu(x))$ intersects ∂D only at x.

Since D is a bounded domain and the constant C_1 is expected to depend on D, we may assume that j is large enough in order to have $3|Q|^{\frac{1}{d}}2^{-j} < h$. For each $I \in \Gamma_{j,0}$, set $\tilde{Q}(I)$ to denote the cube concentric with $Q(I)$ with side length equal to three times the side length of $Q(I)$. Since $Q(I) \cap \partial D \neq \varnothing$ because $I \in \Gamma_{j,0}$, we can take a point $x \in Q(I) \cap \partial D$. Now since $(\partial D \cap \tilde{Q}(I)) \setminus \{x\} \subset (K(x))^c$, the complementary of $K(x)$, we have that $\sigma(\partial D \cap \tilde{Q}(I)) \geq \frac{1}{2}\sigma(\partial K(x) \cap \tilde{Q}(I)) \geq C_3 2^{-j(d-1)}$, where C_3 depends on α. Hence $\Gamma_{j,0} \subset \tilde{\Gamma}_{j,0} = \{I \in \Gamma_j : \sigma(\tilde{Q}(I) \cap \partial D) \geq C_3 2^{-j(d-1)}\}$, and it is enough to obtain an estimate for $\#(\tilde{\Gamma}_{j,0})$ of the form Lemma 9(9.2). Since the overlapping of the family $\{\tilde{Q}(I) : I \in \Gamma_j\}$ is bounded by a constant M which is independent of j, we have that

$$\begin{aligned}\sigma(\partial D) &= \int_{\mathbb{R}^d} \mathcal{X}_{\partial D}(y)\, d\sigma(y)\\ &\geq \frac{1}{M}\int_{\mathbb{R}^d}\Bigg(\sum_{I \in \tilde{\Gamma}_{j,0}} \mathcal{X}_{\partial D \cap \tilde{Q}(I)}(y)\Bigg)\, d\sigma(y)\\ &= \sum_{I \in \tilde{\Gamma}_{j,0}} \sigma\big(\partial D \cap \tilde{Q}(I)\big)\\ &\geq C_3 \#(\tilde{\Gamma}_{j,0}) 2^{-j(d-1)}. \qquad \square\end{aligned}$$

5 Proof of the Results

Proof of Theorem 2 Once we have Theorem 1, Theorem 2 follows readily from Theorem 6 in Sect. 3. In fact, since $\mathbb{B}^\lambda_p(\Omega) = (L_p(\Omega), W^{2,1}_p(\Omega))_{\frac{\lambda}{2},p}$, $L_p(\Omega) = L_p((0,T); L_p(D))$, and since $W^{2,1}_p(\Omega)$ is a Banach subspace of $L_p((0,T); W^2_p(D))$, with $W^2_p(D)$ the usual elliptic Sobolev space on D, a classical result by Lions and Peetre (see [4]) allows us to claim that $\mathbb{B}^\lambda_p(\Omega) \subseteq L_p((0,T); B^\lambda_p(D))$. Hence $\Theta(\Omega) \cap \mathbb{B}^\lambda_p(\Omega) \subset \Theta(\Omega) \cap L_p((0,T); B^\lambda_p(D))$.

From Theorem 1, we have that $\Theta(\Omega) \cap \mathbb{B}^\lambda_p(\Omega) \subseteq L_\tau((0,T); B^\alpha_\tau(D))$. Now, since $\alpha < d(1-\frac{1}{p})$, then $\tau > 1$, and Theorem 6 holds with $q = \tau$ and $\gamma = \alpha$. So $\Theta(\Omega) \cap \mathbb{B}^\lambda_p(\Omega) \subset \mathbb{B}^{\alpha-\varepsilon}_\tau(\Omega)$ for every $\varepsilon > 0$, which proves the result. $\square$

Proof of Theorem 1 Let us start by taking a temperature $u = u(x,t)$ in the space $L_p((0,T); B^\lambda_p(D))$ with $1 < p < \infty$ and $\lambda > 0$. We have to check that u belongs to $L_\tau((0,T); B^\alpha_\tau(D))$ with $\frac{1}{\tau} = \frac{1}{p} + \frac{\alpha}{d}$, $0 < \alpha < \min\{\ell, \frac{\lambda d}{d-1}\}$ and ℓ the largest integer less than $\lambda + d$.

For $0 < t < T$, set $U(t)$ to denote the function defined in D by $U(t)(x) = u(x;t)$. Since $u \in L_p((0,T); B^\lambda_p(D))$, then there exists a subset $Z \subset (0,T)$ of zero measure such that for every $t \notin Z$, $U(t) \in B^\lambda_p(D)$. For such a t, since D is a Lipschitz domain, $U(t)$ can be extended to the whole $\mathbb{R}^d$ as a $B^\lambda_p(\mathbb{R}^d)$ function $V(t)$ with $\|V(t)\|_{B^\lambda_p(\mathbb{R}^d)} \le C\|U(t)\|_{B^\lambda_p(D)}$.

Set $v(x,t) = V(t)(x)$ for $t \notin Z$ and $x \in \mathbb{R}^d$ and $v(x,t) = 0$ when $t \in Z$. Hence, $\int_0^T \|V(t)\|^p_{B^\lambda_p(\mathbb{R}^d)}\,dt \le C\int_0^T \|U(t)\|^p_{B^\lambda_p(D)}\,dt$ and $v \in L_p((0,T); B^\lambda_p(\mathbb{R}^d))$. Notice that if we apply at each $t \notin Z$ Stein's extension operator [18] (see also [12]) to $U(t)$ to obtain $V(t)$, we get a $v(x;t) = V(t)(x)$ which is measurable as a function of the variables $(x;t) \in \mathbb{R}^{d+1}$.

From (2.1) and for fixed $t \in (0,T)$, we have that

$$\begin{aligned} V(t) &= P_0 V(t) + \sum_{I\in\mathcal{D}^+}\sum_{\psi\in\Psi}\langle V(t), \psi_{I,p'}\rangle\psi_{I,p} \\ &= \sum_{I\in\mathcal{D}_0}\langle V(t), \varphi_I\rangle\varphi_I + \sum_{I\in\mathcal{D}^+}\sum_{\psi\in\Psi}\langle V(t), \psi_{I,p'}\rangle\psi_{I,p} \end{aligned}$$

in the $B^\lambda_p(\mathbb{R}^d)$ norm.

We can obtain another $B^\lambda_p(\mathbb{R}^d)$ function $W(t)$ by restriction of the above sums to those cubes I for which $Q(I)$ intersects D. Specifically, we define

$$W(t) = \sum_{I\in\Gamma_0}\langle V(t), \varphi_I\rangle\varphi_I + \sum_{I\in\Gamma}\sum_{\psi\in\Psi}\langle V(t), \psi_{I,p'}\rangle\psi_{I,p} =: W_0(t) + W_1(t),$$

which certainly belongs to $B^\lambda_p(\mathbb{R}^d)$ and $W(t) = V(t) = U(t)$ on D. To prove that $u \in L_\tau((0,T); B^\alpha_\tau(D))$, it is enough to show that $\int_0^T \|W(t)\|^\tau_{B^\alpha_\tau(\mathbb{R}^d)}$ is finite.

Let us start by estimating $\int_0^T \|W_0(t)\|^\tau_{B^\alpha_\tau(\mathbb{R}^d)}\,dt$. Notice that for each $t \in (0,T)$, $W_0(t)$ is a linear combination of a finite number of the φ_I's. From Hölder's inequality,

$$\begin{aligned} \|W_0(t)\|_{B^\alpha_\tau(\mathbb{R}^d)} &\le \sum_{I\in\Gamma_0}\left|\langle V(t), \varphi_I\rangle\right|\|\varphi_I\|_{B^\alpha_\tau(\mathbb{R}^d)} \\ &\le \|V(t)\|_{L_p(\mathbb{R}^d)}\sum_{I\in\Gamma_0}\|\varphi_I\|_{L_{p'}(\mathbb{R}^d)}\|\varphi_I\|_{B^\alpha_\tau(\mathbb{R}^d)} \\ &\le C\|V(t)\|_{B^\lambda_p(\mathbb{R}^d)} \\ &\le C\|U(t)\|_{B^\lambda_p(D)}. \end{aligned}$$

Hence

$$\int_0^T \|W_0(t)\|^\tau_{B^\alpha_\tau(\mathbb{R}^d)}\,dt \le C\int_0^T \|U(t)\|^\tau_{B^\lambda_p(D)}\,dt.$$

Since $\tau < p$, applying the Hölder inequality with $\frac{p}{\tau}$, we obtain that

$$\int_0^T \|W_0(t)\|^\tau_{B^\alpha_\tau(\mathbb{R}^d)}\,dt \le CT^{\frac{p-\tau}{p}}\left(\int_0^T \|U(t)\|^p_{B^\lambda_p(D)}\,dt\right)^{\frac{\tau}{p}},$$

which is finite since $u \in L_p((0,T); B^\lambda_p(D))$.

The difficult estimate, as in the elliptic case considered in [8], is actually the proof of the finiteness of $\int_0^T \|W_1(t)\|^\tau_{B^\alpha_\tau(\mathbb{R}^d)}\,dt$, where the characterization of $B^\alpha_\tau(\mathbb{R}^d)$ given by (2.3) in (B) of Proposition 3 becomes crucial. Applying (2.3) with $f = W_1$, we observe that it suffices to prove that

$$\int_0^T \sum_{I\in\Gamma}\sum_{\psi\in\Psi} \bigl|\bigl\langle V(t), \psi_{I,p'}\bigr\rangle\bigr|^\tau\,dt < \infty.$$

From now on we shall use the notation and the lemmas proved in Sect. 4. Hence, with c_1 the constant in Lemma 7, we can write the above integral as

$$\begin{aligned}
\int_0^T \sum_{I\in\Gamma}\sum_{\psi\in\Psi} \bigl|\bigl\langle V(t), \psi_{I,p'}\bigr\rangle\bigr|^\tau\,dt &= \sum_{j=0}^{\infty}\int_0^T \sum_{I\in\Gamma_j}\sum_{\psi\in\Psi} \bigl|\bigl\langle V(t), \psi_{I,p'}\bigr\rangle\bigr|^\tau\,dt \\
&= \sum_{j=0}^{\infty}\int_0^{c_1 4^{-j}} \sum_{I\in\Gamma_j}\sum_{\psi\in\Psi} \bigl|\bigl\langle V(t), \psi_{I,p'}\bigr\rangle\bigr|^\tau\,dt \\
&\quad + \sum_{j=0}^{\infty}\int_{c_1 4^{-j}}^T \sum_{I\in\Gamma_j}\sum_{\psi\in\Psi} \bigl|\bigl\langle V(t), \psi_{I,p'}\bigr\rangle\bigr|^\tau\,dt \\
&=: A + B.
\end{aligned}$$

Now, since for fixed $j \geq 0$ and $t \in (0, T)$, each Γ_j is the disjoint union of $\Gamma_{j,0}$, $\overset{o}{\Gamma}{}_j^1(t)$ and $\overset{o}{\Gamma}{}_j^2(t)$, we have that

$$\begin{aligned}
B &= \sum_{j=0}^{\infty}\int_{c_1 4^{-j}}^T \sum_{I\in\Gamma_{j,0}}\sum_{\psi\in\Psi} \bigl|\bigl\langle V(t), \psi_{I,p'}\bigr\rangle\bigr|^\tau\,dt \\
&\quad + \sum_{j=0}^{\infty}\int_{c_1 4^{-j}}^T \sum_{I\in\overset{o}{\Gamma}{}_j^1(t)}\sum_{\psi\in\Psi} \bigl|\bigl\langle V(t), \psi_{I,p'}\bigr\rangle\bigr|^\tau\,dt \\
&\quad + \sum_{j=0}^{\infty}\int_{c_1 4^{-j}}^T \sum_{I\in\overset{o}{\Gamma}{}_j^2(t)}\sum_{\psi\in\Psi} \bigl|\bigl\langle V(t), \psi_{I,p'}\bigr\rangle\bigr|^\tau\,dt \\
&=: B_0 + B_1 + B_2.
\end{aligned}$$

So we have to show that, under the assumption in the statement of Theorem 1 for d, λ, p, α, and τ, A, B_0, B_1 and B_2 are all finite. □

Estimate of A It follows from Lemma 9 and (2.2) in Propositon 3 after an adequate application of Hölder's inequality. In fact, since

$$A = \sum_{I\in\Gamma}\sum_{\psi\in\Psi}\int_0^T \mathcal{X}_{(0,c_1 4^{-j})}(t) 2^{-j\lambda\tau}\bigl(2^{j\lambda}\bigl|\bigl\langle V(t), \psi_{I,p'}\bigr\rangle\bigr|\bigr)^\tau\,dt,$$

we can apply Hölder's inequality with $\frac{p}{\tau}>1$, the measure on $\Gamma\times\Psi\times(0,T)$ obtained as product of the counting measures on Γ and Ψ, times the Lebesgue length on $(0,T)$ and the functions $f(I,\psi,t)=(2^{j(I)\lambda}|\langle V(t),\psi_{I,p'}\rangle|)^\tau$ and $g(I,\psi,t)=\mathcal{X}_{(0,c_1 4^{-j})}(t)2^{-j(I)\lambda\tau}$, where $j(I)$ is the resolution level of $I\in\Gamma$, in other words $I\in\Gamma_{j(I)}$. Hence A is bounded above by

$$\begin{aligned}
&\Bigl(\sum_{\Gamma}\sum_{\Psi}\int_0^T 2^{j(I)p\lambda}\bigl|\langle V(t),\psi_{I,p'}\rangle\bigr|^p\,dt\Bigr)^{\frac{\tau}{p}}\\
&\qquad\times\Bigl(\sum_j\sum_{\Gamma_j}\sum_{\Psi}\int_0^{c_1 4^{-j}} 2^{-j(I)\lambda\tau\frac{p}{p-\tau}}\,dt\Bigr)^{\frac{p-\tau}{p}}\\
&\quad\le\Bigl(\int_0^T\sum_{\mathcal{D}^+}\sum_{\Psi}2^{j(I)p\lambda}\bigl|\langle V(t),\psi_{I,p'}\rangle\bigr|^p\,dt\Bigr)^{\frac{\tau}{p}}\cdot C\Bigl(\sum_{j=0}^{\infty}2^{-j(\tau\lambda\frac{p}{p-\tau}+2-d)}\Bigr)^{\frac{p-\tau}{p}}\\
&\quad\le C\Bigl(\int_0^T\|V(t)\|^p_{B^\lambda_p(\mathbb{R}^d)}\,dt\Bigr)^{\frac{\tau}{p}}\\
&\quad\le C\Bigl(\int_0^T\|U(t)\|^p_{B^\lambda_p(D)}\,dt\Bigr)^{\frac{\tau}{p}},
\end{aligned}$$

since $\tau\lambda\frac{p}{p-\tau}+2-d=\frac{d\lambda}{\alpha}+2-d>0$ because $0<\alpha<\frac{\lambda d}{d-1}<\frac{\lambda d}{d-2}$.

Estimate of B_0 The series B_0 can be estimated like A by using Hölder's inequality. The basic difference is that, even when now the time integral is computed on a large interval, the families $\Gamma_{j,0}$ are smaller than the corresponding families Γ_j, as Lemma 9 shows. In fact, from Hölder's inequality and Lemma 9(9.2),

$$\begin{aligned}
B_0&=\sum_{j=0}^{\infty}\int_{c_1 4^{-j}}^T\Bigl(\sum_{I\in\Gamma_{j,0}}\sum_{\psi\in\Psi}\bigl|\langle V(t),\psi_{I,p'}\rangle\bigr|^\tau\Bigr)\,dt\\
&\le\sum_j\sum_{\Gamma_{j,0}}\sum_{\Psi}\int_0^T 2^{-j\lambda\tau}\bigl(2^{j\lambda}\bigl|\langle V(t),\psi_{I,p'}\rangle\bigr|\bigr)^\tau\,dt\\
&\le C\Bigl(\int_0^T\sum_j\sum_{\Gamma_{j,0}}\sum_{\Psi}2^{j\lambda p}\bigl|\langle V(t),\psi_{I,p'}\rangle\bigr|^p\,dt\Bigr)^{\frac{\tau}{p}}\cdot\Bigl(\sum_j 2^{-j\lambda\tau\frac{p}{p-\tau}}2^{j(d-1)}\Bigr)^{\frac{p-\tau}{p}}\\
&\le C\Bigl(\int_0^T\|U(t)\|^p_{B^\lambda_p(D)}\,dt\Bigr)^{\frac{\tau}{p}},
\end{aligned}$$

since, being $0<\alpha<\frac{d\lambda}{d-1}$, the exponent $\lambda\tau\frac{p}{p-\tau}-d+1$ of 2^{-j} in the last geometric series is positive.

Estimate of B_1 Now we have to estimate

$$B_1 = \sum_{j=0}^{\infty} \int_{c_1 4^{-j}}^{T} \sum_{I \in \overset{o}{\Gamma}{}_j^1(t)} \sum_{\psi \in \Psi} \left|\left\langle V(t), \psi_{I,p'} \right\rangle\right|^{\tau} dt,$$

where $\overset{o}{\Gamma}{}_j^1(t) = \{I \in \overset{o}{\Gamma}_j : I(t) \subset \Omega^1\}$ and $\Omega^1 = \{(x,t) \in \Omega : \delta(x,t) = \tilde{\delta}(x)\}$.

The next result contains an upper bound for $|\langle V(t), \psi_{I,p'}\rangle|$ which shall be used to estimate B_1.

Claim *There exists a constant C such that*

$$\left|\left\langle V(t), \psi_{I,p'} \right\rangle\right| \le C \left|Q(I)\right|^{\frac{\ell}{d}} \left(\int_{Q(I)} \left|\nabla^{\ell} V(t)(x)\right|^p dx \right)^{\frac{1}{p}}. \tag{5.1}$$

To prove it, we shall use the following Poincaré type inequality on each $Q(I)$ for $I \in \mathcal{D}^+$:

$$\left\| V(t) - P_{t,I} \right\|_{L_p(Q(I))} \le C \left|Q(I)\right|^{\frac{\ell}{d}} \left(\int_{Q(I)} \left|\nabla^{\ell} V(t)(x)\right|^p dx \right)^{\frac{1}{p}},$$

where, for each t and each $I \in \mathcal{D}^+$, $P_{t,I}(x)$ is a polynomial of degree less than ℓ in the variable x. Since each $\psi_{I,p'}$ has zero moments up to the order $n \ge \ell - 1$, and $L_{p'}$ norm independent of I, applying Hölder and Poincaré inequalities, we obtain the claim:

$$\begin{aligned} \left|\left\langle V(t), \psi_{I,p'} \right\rangle\right| &\le \left|\left\langle V(t) - P_{t,I}, \psi_{I,p'} \right\rangle\right| \\ &\le \left\| V(t) - P_{t,I} \right\|_{L_p(Q(I))} \|\psi_{I,p'}\|_{L_{p'}(\mathbb{R}^d)} \\ &\le C \left|Q(I)\right|^{\frac{\ell}{d}} \left(\int_{Q(I)} \left|\nabla^{\ell} V(t)(x)\right|^p dx \right)^{\frac{1}{p}}. \end{aligned}$$

From (5.1) when $I \in \overset{o}{\Gamma}{}_j^1(t)$, Lemma 8 gives

$$\left|\left\langle V(t), \psi_{I,p'} \right\rangle\right| \le C 2^{-j\ell} \, \tilde{\delta}_{Q(I)}^{(\lambda-\ell)} \left(\int_{Q(I)} \left|\delta^{\ell-\lambda}(x,t) \left|\nabla^{\ell} V(t)(x)\right|\right|^p dx \right)^{\frac{1}{p}}. \tag{5.2}$$

Hence

$$\begin{aligned} B_1 &\le C \sum_j \int_0^T \sum_{\overset{o}{\Gamma}_j} 2^{-j\ell\tau} \, \tilde{\delta}_{Q(I)}^{(\lambda-\ell)\tau} \left(\int_{Q(I)} \left|\delta^{\ell-\lambda}(x,t) \left|\nabla^{\ell} V(t)(x)\right|\right|^p dx \right)^{\frac{\tau}{p}} \\ &\le C \sum_j \left[\sum_{\overset{o}{\Gamma}_j} \left(2^{-j\ell\tau} \, \tilde{\delta}_{Q(I)}^{(\lambda-\ell)\tau}\right)^{\frac{p}{p-\tau}} \right]^{\frac{p-\tau}{p}} \end{aligned}$$

$$
\begin{aligned}
&\times\left[\int_0^T \sum_{\overset{o}{\Gamma}_j} \int_{Q(I)} \left|\delta^{\ell-\lambda}(x,t)\left|\nabla^{\ell} V(t)(x)\right|\right|^p dx\right]^{\frac{\tau}{p}} \\
&\le C\left(\sum_j\left[\sum_{\overset{o}{\Gamma}_j}\left(2^{-j\ell\tau}\,\tilde{\delta}_{Q(I)}^{(\lambda-\ell)\tau}\right)^{\frac{p}{p-\tau}}\right]^{\frac{p-\tau}{p}}\right) \\
&\times\left(\int_0^T\int_D \left|\delta^{\ell-\lambda}(x,t)\left|\nabla^{\ell} V(t)(x)\right|\right|^p dx\right)^{\frac{\tau}{p}},
\end{aligned} \tag{5.3}
$$

where in the first inequality we have used (5.2), in the second Hölder's inequality for the integral in t and the sum in I for j fixed with the exponents $\frac{p}{\tau}$ and $\frac{p}{p-\tau}$, and in the last one we have used that for fixed j the cubes $Q(I)\subset D$ and their overlapping is uniformly bounded.

Since on D we have that $V(t)=U(t)$, from Corollary 5 we see that the second term on the right-hand side of (5.3) is bounded above by a constant times $\|u\|^{\tau}_{L_p((0,T);B_p^{\lambda}(D))}$. Hence to finish the estimate of B_1, all we have to do is to prove that the series $\sum_j(\sum_{\overset{o}{\Gamma}_j}(2^{-j\ell\tau}\,\tilde{\delta}_{Q(I)}^{(\lambda-\ell)\tau})^{\frac{p}{p-\tau}})^{\frac{p-\tau}{p}}$ converges. Since $\overset{o}{\Gamma}_j=\bigcup_{k\ge 1}\Gamma_{j,k}$, from Lemma 9, we get

$$
\begin{aligned}
\left(\sum_{I\in\Gamma_j^o}\left(2^{-j\ell\tau}\,\tilde{\delta}_{Q(I)}^{(\lambda-\ell)\tau}\right)^{\frac{p}{p-\tau}}\right)^{\frac{p-\tau}{p}} &= \left(\sum_{k=1}^{C2^j}\sum_{I\in\Gamma_{j,k}}\left(2^{-j\ell\tau}\,\tilde{\delta}_{Q(I)}^{(\lambda-\ell)\tau}\right)^{\frac{p}{p-\tau}}\right)^{\frac{p-\tau}{p}} \\
&\le C\left(\sum_{k=1}^{C2^j}2^{j(d-1)}2^{-\frac{j\ell\tau p}{p-\tau}}\left(k2^{-j}\right)^{\frac{(\lambda-\ell)\tau p}{p-\tau}}\right)^{\frac{p-\tau}{p}} \\
&= C\left(2^{j(d-1-\frac{\lambda\tau p}{p-\tau})}\sum_{k=1}^{C2^j}k^{\frac{(\lambda-\ell)\tau p}{p-\tau}}\right)^{\frac{p-\tau}{p}} \\
&\le C\left(2^{j(d-1-\frac{\lambda\tau p}{p-\tau}+\frac{(\lambda-\ell)\tau p}{p-\tau}+1)}\right)^{\frac{p-\tau}{p}} \\
&= C2^{j(\frac{d(p-\tau)}{p}-\ell\tau)}.
\end{aligned}
$$

Now, since $\frac{d(p-\tau)}{p}-\ell\tau=\frac{dp}{\alpha p+d}(\alpha-\ell)$ is negative, the series converge.

Estimate of B_2 Even when the general scheme for the boundedness of B_2 looks similar to that for B_1, we would like to emphasize that they are quite different. This is reflected by the fact that for B_2, $t>c_1 4^{-j}$ is relevant to apply Lemma 7 to the intervals in $\overset{o}{\Gamma}{}_j^2(t)$. Since (5.1) holds for each cube I in $\mathcal{D}^+$, applying Lemma 7, we get

$$
\begin{aligned}
B_2 &\le \sum_{j=0}^{\infty}\int_{c_1 4^{-j}}^{T}\Bigg(\sum_{I\in\overset{o}{\Gamma}{}_j^2(t)}\sum_{\psi\in\Psi}\big|\langle V(t),\psi_{I,p'}\rangle\big|^{\tau}\Bigg)dt\\
&\le C\sum_{j=0}^{\infty}\int_{c_1 4^{-j}}^{T}\sum_{I\in\overset{o}{\Gamma}{}_j^2(t)}\big|Q(I)\big|^{\frac{\ell\tau}{d}}\bigg(\int_{Q(I)}\big|\nabla^{\ell}V(t)(x)\big|^{p}\,dx\bigg)^{\frac{\tau}{p}}dt\\
&\le C\sum_{j=0}^{\infty}\int_{c_1 4^{-j}}^{T}\sum_{I\in\overset{o}{\Gamma}{}_j^2(t)}2^{-j\ell\tau}t^{\frac{(\lambda-\ell)\tau}{2}}\bigg(\int_{Q(I)}\big(\delta^{\ell-\lambda}(x,t)\big|\nabla^{\ell}V(t)(x)\big|\big)^{p}\,dx\bigg)^{\frac{\tau}{p}}dt.
\end{aligned}
$$

Applying Hölder's inequality for the integral in t and the sum in I, for fixed j, with exponents $\frac{p}{\tau}$ and $\frac{p}{p-\tau}$, we obtain

$$
B_2 \le C\|u\|^{\tau}_{L_p((0,T);B_p^{\lambda}(D))}\sum_{j=0}^{\infty}\Bigg(\int_{c_1 4^{-j}}^{T}\sum_{I\in\overset{o}{\Gamma}{}_j^2(t)}\big(2^{-j\ell\tau}t^{\frac{(\lambda-\ell)\tau}{2}}\big)^{\frac{p}{p-\tau}}dt\Bigg)^{\frac{p-\tau}{p}}.
$$

Hence, to finish the proof of the theorem, all we have to do is show that the series

$$
\sum_{j=0}^{\infty}\Bigg(\int_{c_1 4^{-j}}^{T}\sum_{I\in\overset{o}{\Gamma}{}_j^2(t)}\big(2^{-j\ell\tau}t^{\frac{(\lambda-\ell)\tau}{2}}\big)^{\frac{p}{p-\tau}}dt\Bigg)^{\frac{p-\tau}{p}} =: S
$$

is finite,

$$
\begin{aligned}
S &= \sum_{j=0}^{\infty}2^{-j\ell\tau}\Bigg(\int_{c_1 4^{-j}}^{T}\sum_{I\in\overset{o}{\Gamma}{}_j}\mathcal{X}_{\overset{o}{\Gamma}{}_j^2(t)}(I)\,t^{\frac{(\lambda-\ell)\tau p}{2(p-\tau)}}\,dt\Bigg)^{\frac{p-\tau}{p}}\\
&\le \sum_{j=0}^{\infty}2^{-j\ell\tau}\Bigg(\sum_{k\ge 1}\int_{c_1 4^{-j}}^{T}\sum_{I\in\Gamma_{j,k}}\mathcal{X}_{\overset{o}{\Gamma}{}_j^2(t)}(I)\,t^{\frac{(\lambda-\ell)\tau p}{2(p-\tau)}}\,dt\Bigg)^{\frac{p-\tau}{p}}.
\end{aligned}
$$

Notice now that if $I\in\Gamma_{j,k}$ and $t>(k2^{-j})^2$, we have that $\sqrt{t}>\tilde{\delta}(Q(I))$, so that $I\in\overset{o}{\Gamma}{}_j^1(t)$. Hence in the last estimate obtained for S, the indicator function $\mathcal{X}_{\overset{o}{\Gamma}{}_j^2(t)}(I)$ is bounded above by $\mathcal{X}_{(0,(k2^{-j})^2)}(t)$. Then

$$
\begin{aligned}
S &\le \sum_{j=0}^{\infty}2^{-j\ell\tau}\Bigg(\sum_{k\ge\sqrt{c_1}}\sum_{I\in\Gamma_{j,k}}\int_{c_1 4^{-j}}^{(k2^{-j})^2}t^{\frac{(\lambda-\ell)\tau p}{2(p-\tau)}}\,dt\Bigg)^{\frac{p-\tau}{p}}\\
&\le C\sum_{j=0}^{\infty}2^{-j\ell\tau}\,2^{-2j[\frac{(\lambda-\ell)\tau}{2}+\frac{p-\tau}{p}]}\Bigg(\sum_{k\ge\sqrt{c_1}}\sum_{I\in\Gamma_{j,k}}\Big(k^{2[\frac{(\lambda-\ell)\tau p}{2(p-\tau)}+1]}-c_1^{[\frac{(\lambda-\ell)\tau p}{2(p-\tau)}+1]}\Big)\Bigg)^{\frac{p-\tau}{p}}
\end{aligned}
$$

$$\leq C\sum_{j=0}^{\infty}2^{-j\ell\tau}\,2^{-2j[\frac{(\lambda-\ell)\tau}{2}+\frac{p-\tau}{p}]}2^{j(d-1)\frac{p-\tau}{p}}\Bigg(\sum_{k\geq 1}k^{2[\frac{(\lambda-\ell)\tau p}{2(p-\tau)}+1]}\Bigg)^{\frac{p-\tau}{p}}$$

$$\leq C\sum_{j=0}^{\infty}2^{-j\ell\tau}\,2^{-2j[\frac{(\lambda-\ell)\tau}{2}+\frac{p-\tau}{p}]}\,2^{j(d-1)\frac{p-\tau}{p}}\,2^{j\frac{p-\tau}{p}}2^{2j[\frac{(\lambda-\ell)\tau}{2}+\frac{p-\tau}{p}]}$$

$$= C\sum_{j=0}^{\infty}2^{-j(\ell\tau-\frac{d(p-\tau)}{p})},$$

which is finite, since $\ell\tau-\frac{d(p-\tau)}{p}>0$. □

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