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Measuring the level sets of anisotropic homogeneous functions

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Abstract In this note we investigate some basic properties of the level sets of functions which are homogeneous with respect to nonisotropic dilations. In particular we obtain a formula for the volume of the level sets in terms of the area on the level surfaces. We relate the results to some well known mean value formulas for solutions of PDE's.

Keywords Homogeneous functions · Volume of level sets · Mean values formula · Euler formula

Mathematics Subject Classification (2000) 31C99 · 26B15

1 Introduction and statement of the results

For harmonic functions on $\mathbb{R}^n$, the integrals involved in the well known mean value formulas

$$u(x) = \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} u(y)dy = \frac{1}{\sigma(\partial B(x,r))} \int\limits_{\partial B(x,r)} u(y)d\sigma(y)$$

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can be regarded as volume and surface integrals on the level sets

$$B(x,r) = \left\{ y : \Gamma_e(x-y) > r^{-n+2} \right\} \quad \text{and} \quad \partial B(x,r) = \left\{ y : \Gamma_e(x-y) = r^{-n+2} \right\}$$

of the fundamental solution $\Gamma_e(x) = |x|^{-n+2}$ for the Laplacian ($n \geq 3$).

A little less standard, but equally well known is the case of the mean values for the solutions of the heat equation. Now, the fundamental solution is given by

$$\Gamma_p(x,t) = \begin{cases} (\sqrt{4\pi t})^{-d} e^{-\frac{|x|^2}{4t}} & t > 0 \\ 0 & t \leq 0 \end{cases} \tag{1.1}$$

in $\mathbb{R}^{d+1} = \{(x,t) : x \in \mathbb{R}^d; t \in \mathbb{R}\}$. Following [2] (see also [3] and [7]) let us define the heat balls $E((x,t);r)$ as the set of all the points $(y,s) \in \mathbb{R}^{d+1}$ for which $\Gamma_p(x-y, t-s) > r^{-d}$. The corresponding mean value for a temperature $u(x,t)$ takes now the following form

$$u(x,t) = \frac{1}{4r^d} \iint\limits_{E((x,t);r)} u(y,s) \frac{|x-y|^2}{|t-s|^2} dyds. \tag{1.2}$$

For the parabolic case a mean value on the boundaries of the heat balls is also available (see [3] and [7]). We would like to mention also the results in [8] and [6] where mean values for more general linear hypoelliptic PDE's are considered.

The elliptic and parabolic situations briefly described above share a common pattern. In fact both, $\Gamma_e(x)$ and $\Gamma_p(x,t)$ are homogeneous functions. Of course not with respect to the same dilations. While $\Gamma_e(x)$ is homogeneous of degree $-n+2$ with respect to the usual dilations in $\mathbb{R}^n$: $\Gamma_e(\lambda x) = \lambda^{-n+2}\Gamma_e(x)$ for every $x \in \mathbb{R}^n - \{0\}$ and $\lambda > 0$, the function $\Gamma_p(x,t)$ is parabolically homogeneous of degree $-d$. Precisely $\Gamma_p(\lambda x, \lambda^2 t) = \lambda^{-d}\Gamma_p(x,t)$ for every $x \in \mathbb{R}^d$, $t \in \mathbb{R}$ and $\lambda > 0$.

Notice that both Γ_e and Γ_p are $\mathscr{C}^\infty$ away from the origin. In an equivalent way, we have that both Γ_e and Γ_p are smooth when restricted to the corresponding unit sphere, S^{n-1} for Γ_e and $S^{(d+1)-1}$ for Γ_p. Aside from the above mentioned difference of types of homogeneities in Γ_e and Γ_p, they have another essential difference. In fact, while Γ_e never vanishes on S^{n-1}, $\Gamma_p \equiv 0$ on $S_-^{(d+1)-1} = \{(x,t) \in \mathbb{R}^{d+1} : |x|^2 + t^2 = 1 \text{ and } t \leq 0\}$. This fact has relevant geometric consequences. In particular, all the level surfaces of Γ_p have the origin as a limit point, while the distance of any level surface of Γ_e to the origin is positive.

In the applications to some problems in PDE (see [4]), it is sometimes important to have as smooth as possible versions of the mean value formulas. Smoothness in the elliptic case means that the indicator functions of the Euclidean ball can be substituted by compactly supported $\mathscr{C}^\infty$ functions whose level surfaces are Euclidean spheres. In the elliptic case this is easy to accomplish from the mean value formula on spherical shells. In the parabolic situation instead, the construction is impossible since all the level surfaces of Γ_p collapse at the origin. Nevertheless in the parabolic case (see [1])

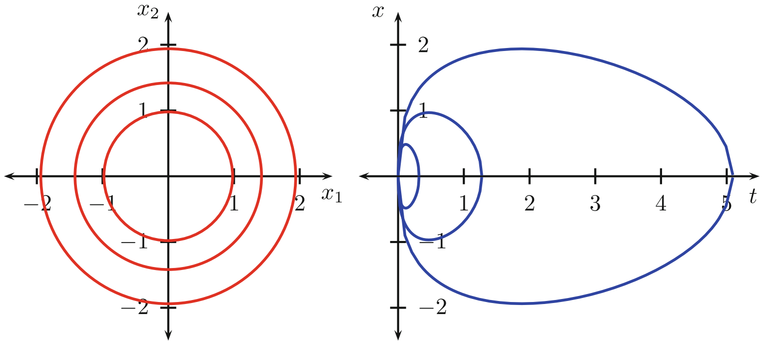

Fig. 1 Level sets of Γ_e and Γ_p for 2 dimension

the indicator function of $E((x,t);r)$ can be substituted by a function which is smooth of the space variables for fixed time.

In both cases, elliptic and parabolic, the mean value formulas on the solid balls are equivalent to the corresponding mean value formulas on the spherical shells. In this note we show that this behavior is a general fact.

Precisely we aim to compute the volume of level sets of generalized homogeneous functions in terms of the radial integral of some measures supported on the corresponding level surfaces. Let us point out here that for the isotropic case, i.e. when we only deal with the usual dilations of the space, closely related results are contained in [5].

We shall consider generalized nonisotropic dilations induced by an $n \times n$ diagonal matrix A with eigenvalues $1 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ of the type $T_\lambda = e^{A \log \lambda}$ for $\lambda > 0$. So that for a given point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and a given positive λ, $T_\lambda x$ denotes the point $(\lambda^{a_1} x_1, \ldots, \lambda^{a_n} x_n)$. For a fixed $x \neq 0$ in $\mathbb{R}^n$ the curve $\{T_\lambda x : \lambda > 0\}$ is the natural generalization of the ray through x starting at the origin when A is the identity matrix.

We say that a function $\Gamma : \mathbb{R}^n - \{0\} \to \mathbb{R}^+ \cup \{0\}$ is A-homogeneous of degree m if the identity $\Gamma(T_\lambda x) = \lambda^m \Gamma(x)$ holds for each $x \in \mathbb{R}^n - \{0\}$ and each $\lambda > 0$. When $T_\lambda = \lambda I$ we recover the classical isotropic homogeneity. Generically, as in the isotropic case, we can only expect smooth level surfaces for nonvanishing Γ when $m \neq 0$.

It is easy to see that $\Gamma > 0$ is A-homogeneous of degree m if and only if $\frac{1}{\Gamma}$ is A-homogeneous of degree $-m$. Notice also that $\Gamma > 0$ is A-homogeneous of degree $-m$ $(m > 0)$ if and only if $\Gamma^{1/m}$ is A-homogeneous of degree -1. Following the usual pattern described above for the elliptic and parabolic cases, $B(x,r) = \left\{y : \Gamma_e^{1/(n-2)}(x-y) > \frac{1}{r}\right\}$ and $E((x,t);r) = \left\{(y,s) : \Gamma_p^{1/d}(x-y, t-s) > \frac{1}{r}\right\}$, we shall keep assuming that Γ is a nonnegative A-homogeneous function of degree -1.

Let Γ be a continuously differentiable, nonnegative and A-homogeneous function of degree -1 defined on $\mathbb{R}^n - \{0\}$. For $\alpha > 0$ let us considerer the three level sets of Γ given by $E_\alpha = \{x \in \mathbb{R}^n - \{0\} : \Gamma(x) > \alpha^{-1}\}$, $\Sigma_\alpha = \left\{x \in \mathbb{R}^n - \{0\} : \Gamma(x) = \alpha^{-1}\right\}$ and $G_\alpha = \{x \in \mathbb{R}^n - \{0\} : \Gamma(x) < \alpha^{-1}\}$.

The following lemma, which shall be proved in Sect. 2 contains some basic properties of these level sets of Γ.

Lemma 1.1 *Let $\Gamma \in \mathscr{C}^1(\mathbb{R}^n - \{0\})$ be a nontrivial and nonnegative A-homogeneous function of degree -1. Then*

(1.1.1) *The sets E_α, Σ_α and G_α produce a partition of $\mathbb{R}^n - \{0\}$.*
(1.1.2) *$T_\alpha(\Sigma_1) = \Sigma_\alpha$, $T_\alpha(E_1) = E_\alpha$ and $T_\alpha(G_1) = G_\alpha$ for every $\alpha > 0$.*
(1.1.3) *E_α is bounded and $|E_\alpha| = \alpha^\tau |E_1|$, where τ is the trace of A.*
(1.1.4) *If $0 < \alpha < \beta$, then $E_\alpha \subsetneq E_\beta$.*
(1.1.5) *For each α positive and each $\varepsilon > 0$ only finitely many of the components of E_α meet the complement of the ball $B(0, \varepsilon)$.*

Sometimes, like in the elliptic case $\Gamma = \Gamma_e^{1/(n-2)}$, the sets Σ_α are compact in $\mathbb{R}^n - \{0\}$. This is not the generic situation. In fact for $\Gamma = \Gamma_p^{1/d}$ we see that the origin is a limit point of any Σ_α.

Regarding (1.1.5), let us notice that when Γ is allowed to vanish but Γ is nontrivial, the number of connected components of E_1 could be infinite. In fact, the function $\Gamma : \mathbb{R}^2 - \{0\} \to \mathbb{R}$ defined in the polar coordinates of $\mathbb{R}^2 - \{0\}$ around $\theta = 0$ by $\Gamma(re^{i\theta}) = \frac{\theta^2}{r}(1 + \sin\frac{1}{\theta})$ is homogeneous of degree -1 in the usual sense ($A = I$).

The main result of this note is contained in the next statement.

Theorem 1.2 *Let $\Gamma \in \mathscr{C}^1(\mathbb{R}^n - \{0\})$ be a nontrivial and nonnegative A-homogeneous function of degree -1. Then*

(1.2.a) *the following generalized Euler formula*

$$\Gamma(x) = -\nabla\Gamma(x) \cdot Ax$$

holds for every $x \neq 0$;
(1.2.b) *for each $r > 0$, Σ_r is locally $\mathscr{C}^1$ around each one of its points;*
(1.2.c) *for every $\mathscr{C}_0^\infty(\mathbb{R}^n - \{0\})$ function ψ we have that*

$$\frac{d}{dr}\int_{E_r} \psi\, dx = \int_{\Sigma_r} \psi\, d\mu_r, \tag{1.3}$$

with

$$\mu_r(E) = \frac{1}{r} \int_{E\cap\Sigma_r} Ay \cdot \vec{n}_r(y)\, d\sigma_r(y) \tag{1.4}$$

for every Borel subset E of $\mathbb{R}^n - \{0\}$, where $d\sigma_r$ is the surface area measure on Σ_r and $\vec{n}_r$ is the unit outward-pointing normal to Σ_r for E_r;
(1.2.d) *for each $r > 0$,*

$$d\mu_r = \frac{1}{r^2}\frac{1}{|\nabla\Gamma|} d\sigma_r;$$

(1.2.e) *for each* $\psi \in \mathscr{C}_0^\infty(\mathbb{R}^n - \{0\})$ *we have*

$$\int_{\mathbb{R}^n} \psi(x)\,dx = \int_0^\infty \left(\int_{\Sigma_r} \psi(y)\,d\mu_r(y) \right) dr;$$

(1.2.f) *for each* $R > 0$

$$|E_R| = \int_0^R \left(\int_{\Sigma_r} d\mu_r(y) \right) dr;$$

(1.2.g) *for each* $r > 0$, $\mu_r(\Sigma_r) = r^{\tau-1}\mu_1(\Sigma_1)$.

Let us notice that for the parabolic case, taking $n = d + 1$ we may regard $\mathbb{R}^n$ as $\{(x,t) : x \in \mathbb{R}^d; t \in \mathbb{R}\}$ and we have that $\Gamma_p(x,t)$ defined as in (1.1) is A-homogeneous of degree $-d$ with the matrix

$$A = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ & & 1 \\ 0 & & & 2 \end{pmatrix}.$$

If $u(x,t)$ is a solution of the heat equation $\frac{\partial u}{\partial t} = \Delta u = \sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2}$, then from (1.2), (1.2.e) and a simple smooth truncation argument around (x,t) we see that

$$u(x,t) = \frac{1}{4R^d} \int_0^R \left(\frac{1}{r} \int_{\partial E((x,t);r)} u(y,s) \frac{|x-y|^2}{(t-s)^2}(y,2s) \cdot \vec{n_r}(y,s)\,d\sigma_r(y,s) \right) dr.$$

By taking the derivative with respect to R in the above formula, we get

$$0 = -dR^{-1}u(x,t) + \frac{1}{4R^{d+1}} \int_{\partial E((x,t);R)} u(y,s) \frac{|x-y|^2}{(t-s)^2}(y,2s) \cdot \vec{n_R}(y,s)\,d\sigma_R(y,s).$$

Hence, for each $r > 0$

$$u(x,t) = -\frac{1}{4dr^d} \int_{\partial E((x,t);r)} u(y,s) \frac{|x-y|^2}{(t-s)^2}(y,2s) \cdot \vec{n_r}(y,s)\,d\sigma_r(y,s).$$

A final remark is in order, the scaling property for μ_r reflected in (1.2.g) is in consonance with the fact that the Hausdorff dimension of Σ_r with respect to the parabolic distance induced by A is precisely $\tau - 1$.

2 Proofs

We shall first prove Theorem 1.2 and then Lemma 1.1.

Proof of (1.2.a) From the homogeneity of Γ we have that $\lambda^{-1}\Gamma(x) = \Gamma(T_\lambda x)$ for every $\lambda > 0$. Taking the derivative with respect to λ for fixed x we have

$$-\lambda^{-2}\Gamma(x) = \frac{d}{d\lambda}\left(\Gamma(T_\lambda x)\right) = \frac{d}{d\lambda}\left(\Gamma(e^{A\log\lambda}x)\right) = \nabla\Gamma(e^{A\log\lambda}x)\cdot\left(\frac{1}{\lambda}Ae^{A\log\lambda}\right)x.$$

With $\lambda = 1$ we get the generalized Euler formula $-\Gamma(x) = \nabla\Gamma(x)\cdot Ax$ for $x \in \mathbb{R}^n - \{0\}$. □

Proof of (1.2.b) From the homogeneity of Γ we only have to see that the level set Σ_1 is a $\mathscr{C}^1$ surface contained in $\mathbb{R}^n - \{0\}$. From the implicit function theorem it suffices to prove that for each $x \in \Sigma_1$ the gradient of Γ at x does not vanish. Applying the generalized Euler formula we have that $\nabla\Gamma(x)\cdot Ax = -1$. Hence $\nabla\Gamma(x) \neq 0$. □

Proof of (1.2.c) For fixed positive r, the functional that to each $\psi \in \mathscr{C}_0^\infty(\mathbb{R}^n - \{0\})$ assigns the real number $\frac{d\eta_\psi}{d\alpha}(r)$ where $\eta_\psi(\alpha) = \int_{E_\alpha}\psi\,dx$ is well defined and linear. Let us start by computing the derivative of the function η_ψ at $r > 0$. Notice that from (1.1.2) and the semigroup property of T_α, we have that with $\nu > 0$,

$$\eta_\psi(\nu r) = \int_{E_{\nu r}}\psi(x)dx = \nu^\tau\int_{y\in E_r}\psi\left(T_\nu y\right)dy.$$

Hence

$$\begin{aligned}\frac{\eta_\psi(\nu r) - \eta_\psi(r)}{(\nu-1)r} &= \frac{1}{(\nu-1)r}\left(\int_{E_{r\nu}}\psi - \int_{E_r}\psi\right)\\ &= \frac{1}{r}\int_{E_r}\frac{\nu^\tau\psi\left(T_\nu y\right) - \psi(y)}{\nu-1}dy.\end{aligned}$$

Taking limit for $\nu \to 1$, we get

$$\frac{d\eta_\psi}{dr} = \frac{1}{r}\int_{E_r}\frac{d}{d\lambda}\left(\lambda^\tau\psi\left(T_\lambda y\right)\right)_{|\lambda=1}dy. \tag{2.1}$$

Since $\psi\left(T_\lambda y\right) = \psi(\lambda^{a_1}y_1,\ldots,\lambda^{a_n}y_n)$, its derivative with respect to λ is given by

$$\begin{aligned}\frac{d}{d\lambda}\psi\left(T_\lambda y\right) &= \nabla\psi\left(T_\lambda y\right)\cdot\left(a_1\lambda^{a_1-1}y_1,\ldots,a_n\lambda^{a_n-1}y_n\right)\\ &= \frac{1}{\lambda}\nabla\psi\left(T_\lambda y\right)\cdot A\left(T_\lambda y\right)\end{aligned} \tag{2.2}$$

With $\vec{F}(z) = \psi(z)Az$ we have that

$$\nabla\psi\left(T_\lambda y\right)\cdot A\left(T_\lambda y\right) = \operatorname{div}\vec{F}\left(T_\lambda y\right) - \tau\psi\left(T_\lambda y\right). \tag{2.3}$$

From (2.2) and (2.3) we obtain

$$\begin{aligned}\frac{d}{d\lambda}\left(\lambda^\tau\psi\left(T_\lambda y\right)\right) &= \tau\lambda^{\tau-1}\psi\left(T_\lambda y\right) + \lambda^\tau\frac{d}{d\lambda}\left(\psi\left(T_\lambda y\right)\right)\\ &= \tau\lambda^{\tau-1}\psi\left(T_\lambda y\right) + \frac{\lambda^\tau}{\lambda}\left[\operatorname{div}\vec{F}\left(T_\lambda y\right) - \tau\psi\left(T_\lambda y\right)\right]\\ &= \lambda^{\tau-1}\operatorname{div}\vec{F}\left(T_\lambda y\right),\end{aligned}$$

which for $\lambda = 1$ gives

$$\frac{d}{d\lambda}\left(\lambda^\tau\psi\left(T_\lambda y\right)\right)_{|\lambda=1} = \operatorname{div}\vec{F}(y).$$

By substitution in (2.1) we get

$$\frac{d\eta_\psi}{dr} = \frac{1}{r}\int\limits_{E_r}\operatorname{div}\vec{F}(y)\,dy.$$

Now we aim to apply the Divergence Gauss Theorem. *A priori*, the domain E_r has typically an infinite number of connected components which for general A are not even locally Lipschitz at the origin. Nevertheless the fact that the support of ψ does not contain the origin together with (1.1.5) and (1.2.b) gives the desired result

$$\frac{d\eta_\psi}{dr} = \frac{1}{r}\int\limits_{\Sigma_r}\psi(y)Ay\cdot\vec{n}_r(y)\,d\sigma_r(y).$$

□

Proof of (1.2.d) Let $r > 0$ be given. Since for each $y \in \Sigma_r$, $\nabla\Gamma(y)$ is perpendicular to Σ_r at y, we have that $\vec{n}_r(y) = -\frac{\nabla\Gamma(y)}{|\nabla\Gamma(y)|}$. Now, from (1.2.a) we get $Ay\cdot\vec{n}_r(y) = \frac{\Gamma(y)}{|\nabla\Gamma(y)|}$. Since $\Gamma(y) = r^{-1}$, then by substitution in (1.4) we get the expression $\left(r^2\left|\nabla\Gamma(y)\right|\right)^{-1}d\sigma_r$ for $d\mu_r$. □

Proof of (1.2.e) We only have to integrate both sides of (1.3) with respect to r. □

Proof of (1.2.f) Let us take a sequence ψ_n of $\mathscr{C}_0^\infty(\mathbb{R}^{n-\{0\}})$ functions such that $0 \leq \psi_n \leq \psi_{n+1} \leq \mathcal{X}_{E_R}$ and $\lim\psi_n(x) = \mathcal{X}_{E_R}(x)$ almost everywhere. Then (1.2.f) follows from (1.2.e) for ψ_n and Beppo-Levi convergence theorem. □

Proof of (1.2.g) From (1.1.3) we have that $\frac{d}{dr}\left|E_r\right| = \tau r^{\tau-1}\left|E_1\right|$. Hence $\tau r^{\tau-1}\left|E_1\right| = \mu_r(\Sigma_r)$ for every $r > 0$. In particular $\tau\left|E_1\right| = \mu_1(\Sigma_1)$. So that $r^{\tau-1}\mu_1(\Sigma_1) = \mu_r(\Sigma_r)$. □

Let us now prove Lemma 1.1.

Proof of (1.1.1) It is clear that the sets ∂_α, E_α and G_α are pairwise disjoint and cover the whole space $\mathbb{R}^n - \{0\}$. □

Proof of (1.1.2) From the homogeneity of Γ,

$$\begin{aligned} T_\alpha(\Sigma_1) &= \left\{T_\alpha x : x \in \mathbb{R}^n - \{0\}, \Gamma(x) = 1\right\} \\ &= \left\{y \in \mathbb{R}^n - \{0\} : \Gamma\left(T_{\alpha^{-1}}(y)\right) = 1\right\} \\ &= \left\{y \in \mathbb{R}^n - \{0\} : \alpha\Gamma(y) = 1\right\} \\ &= \Sigma_\alpha \end{aligned}$$

for $\alpha > 0$. The identities for E_α and G_α follow in a similar way. □

Proof of (1.1.3) To prove that each E_α is bounded, from (1.1.2) we only have to show that E_1 is bounded.

For $x \in E_1$, take $y = y(x) \in S^{n-1}$ and $\lambda = \lambda(x) > 0$ such that $x = T_\lambda y$. Applying the homogeneity of Γ we have that $\lambda^{-1}\Gamma(y) = \Gamma(x) > 1$. Since Γ is continuous on the compact set S^{n-1} we have that $\lambda < \Gamma(y) \leq \kappa$ for some $\kappa \geq 1$. Then

$$|x|^2 = \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} \lambda^{2a_i} y_i^2 \leq \kappa^{2a_n},$$

which proves the boundedness of E_1.

On the other hand, the identity $|E_\alpha| = \alpha^\tau |E_1|$ follows from the fact that $\det T_\alpha = \alpha^\tau$. □

Proof of (1.1.4) Notice first that for $\alpha < \beta$ the inclusion $E_\alpha \subset E_\beta$ holds. Let us show that E_α and E_β can not coincide. Since Γ in nontrivial, then for some point $\xi \neq 0$ we must have $\Gamma(\xi) > 0$. Hence $\Gamma(T_s(\xi)) = s^{-1}\Gamma(\xi)$ as a function of s is one to one and onto $\mathbb{R}^+$. To prove that $E_\alpha \neq E_\beta$ it is enough to take $s = \frac{\alpha+\beta}{2(\Gamma(\xi))^{-1}}$. □

Proof of (1.1.5) It is enough to prove (1.1.5) for $\alpha = 1$. Notice first that if Γ does not vanish, then $E_1 = \{\Gamma > 1\}$ is open and connected in $\mathbb{R}^n - \{0\}$. So that we shall keep assuming that Γ is nontrivial A-homogeneous of degree -1 and that Γ vanishes at some points of $\mathbb{R}^n - \{0\}$. Then $E_1 = \cup_{i\in I} C_i$ with C_i disjoint, open and connected and I a finite or countable index set. Let $\varepsilon > 0$ be given. Since E_1 is bounded it could happen that $E_1 \subset B(0, \varepsilon)$. In this case *(1.1.5)* is trivial. Hence we may consider that $\varepsilon > 0$ is small enough. Assume that there exists $J \subset I$ with $\#(J) = \aleph_0$ such that for each $j \in J$ we have $C_j \cap B^c(0, \varepsilon) \neq \varnothing$. Since Γ is A-homogeneous of negative degree we have that for each $j \in J$ there exists a point $y_j \in C_j$ with $\left|y_j\right| = \varepsilon$. Thus for some y^* with $|y^*| = \varepsilon$ and some subsequence y_{j_k} of y_j we have that $y_{j_k} \to y^*$ as $k \to \infty$.

Since given $k_1 \neq k_2$ the points $y_{j_{k_1}}$ and $y_{j_{k_2}}$ belong to different connected components, then in the arc in $\{|x| = \varepsilon\}$ joining $y_{j_{k_1}}$ and $y_{j_{k_2}}$ there exist points $z_{k_1k_2}$ with $\Gamma(z_{k_1k_2}) = 0$. So that this sequence $\{z_{k_1k_2}\}$ has also y^* as a limit point. But while $\Gamma(y_{j_k}) > 1$, $\Gamma(z_{k_1k_2}) = 0$ and Γ can not be continuous at y^*. □

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