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Abstract

In this work we obtain boundedness on L^p, for $1<p<\infty$, of commutators $T_b f = bTf - T(bf)$ where T is any of the Riesz transforms or their conjugates associated to the Schrödinger operator $-\Delta+V$ with V satisfying an appropriate reverse Hölder inequality. The class where b belongs is larger than the usual BMO. We also obtain a substitute result for $p=\infty$, under a slightly stronger condition on b.

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Commutators of Riesz Transforms Related to Schrödinger Operators

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Abstract In this work we obtain boundedness on L^p, for $1 < p < \infty$, of commutators $T_b f = bTf - T(bf)$ where T is any of the Riesz transforms or their conjugates associated to the Schrödinger operator $-\Delta + V$ with V satisfying an appropriate reverse Hölder inequality. The class where b belongs is larger than the usual BMO. We also obtain a substitute result for $p = \infty$, under a slightly stronger condition on b.

Keywords Schrödinger operator · BMO · Commutators · Riesz transforms

Mathematics Subject Classification (2000) Primary 42B35 · Secondary 35J10

1 Introduction

Let us consider the Schrödinger operator

$$\mathfrak{L} = -\Delta + V$$

in $\mathbb{R}^d$, $d \geq 3$. The function V is non-negative, $V \neq 0$, and belongs to a reverse-Hölder class RH_q for some exponent $q > d/2$, i.e. there exists a constant C such

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that

$$\left(\frac{1}{|B|}\int_B V(y)^q\,dy\right)^{1/q} \leq \frac{C}{|B|}\int_B V(y)\,dy, \tag{1}$$

for every ball $B \subset \mathbb{R}^d$.

We associate to the differential operator $\mathfrak{L}$ the vector valued Riesz Transform

$$\mathcal{R} = \nabla(-\Delta + V)^{-1/2}.$$

This operator has been considered in [11], where the author shows that it is bounded on $L^p(\mathbb{R}^d)$ for $1 < p < p_0$, with p_0 depending on q in a way that if $V \in RH_q$ with $q \geq d$, it results $p_0 = \infty$. Moreover, Shen shows that in that case $\mathcal{R}$ and its adjoint $\mathcal{R}^*$ are in fact Calderón-Zygmund operators (see [11]).

As in [11], we will use the auxiliary function ρ defined for $x \in \mathbb{R}^d$ as

$$\rho(x) = \sup\left\{r > 0 : \frac{1}{r^{d-2}}\int_{B(x,r)} V \leq 1\right\}. \tag{2}$$

Under the above conditions on V, we have $0 < \rho(x) < \infty$.

For $\theta > 0$, we define the class $BMO_\theta(\rho)$ of locally integrable functions b such that

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |b(y) - b_B|\,dy \leq C\left(1 + \frac{r}{\rho(x)}\right)^\theta, \tag{3}$$

for all $x \in \mathbb{R}^d$ and $r > 0$, where $b_B = \frac{1}{B}\int_B b$. A norm for $b \in BMO_\theta(\rho)$, denoted by $[b]_\theta$, is given by the infimum of the constants satisfying (3), after identifying functions that differ upon a constant. Notice that if we let $\theta = 0$ in (3) we obtain the John-Nirenberg space BMO.

Now, with the above definition in mind, we define $BMO_\infty(\rho) = \bigcup_{\theta>0} BMO_\theta(\rho)$. Clearly $BMO \subset BMO_\theta(\rho) \subset BMO_{\theta'}(\rho)$ for $0 < \theta \leq \theta'$, and hence $BMO \subset BMO_\infty(\rho)$. Moreover, it is in general a larger class. As an example, when ρ is constant (which corresponds to V a positive constant) the functions $b_j(x) = |x_j|$, $1 \leq j \leq d$, belong to $BMO_\infty(\rho)$ but not to BMO. Also, when $V(x) = |x|^2$ and $\mathfrak{L}$ becomes the Hermite operator, we obtain $\rho(x) \simeq \frac{1}{1+|x|}$ and we may take $b(x) = |x_j|^2$.

We denote by T either $\mathcal{R}$ or $\mathcal{R}^*$. For $b \in BMO_\infty(\rho)$ we will consider the commutator operator

$$T_b f(x) = T(bf)(x) - b(x)Tf(x), \quad x \in \mathbb{R}^d. \tag{4}$$

Before stating the main theorems we introduce the definition of the *reverse Hölder index* of V as $q_0 = \sup\{q : V \in RH_q\}$. It is known that $V \in RH_q$ implies $V \in RH_{q+\epsilon}$ for some $\epsilon > 0$ (see [5]). Therefore, under the assumption $V \in RH_{d/2}$ we may conclude $q_0 > d/2$.

Finally recall that $V \in RH_q$ for some $q > 1$ implies that V satisfies the doubling condition, i.e., there exist constants $\mu \geq 1$ and C such that

$$\int_{tB} V \leq Ct^{d\mu}\int_B V, \tag{5}$$

holds for every ball B and $t > 1$.

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Now, we are in position to state our first result.

Theorem 1 *Let $V \in RH_{d/2}$, $b \in BMO_\infty(\rho)$ and p_0 such that $1/p_0 = (1/q_0 - 1/d)^+$, where q_0 is the reverse Hölder index of V.*

(i) *If $1 < p < p_0$, then*

$$\|\mathcal{R}_b f\|_p \le C_b \|f\|_p,$$

for all $f \in L^p$.

(ii) *If $p_0' < p < \infty$, then*

$$\|\mathcal{R}_b^* f\|_p \le C_b \|f\|_p,$$

for all $f \in L^p$.

Moreover, $C_b \lesssim [b]_\theta$ whenever $b \in BMO_\theta(\rho)$.

In order to present our result concerning the behavior of commutators for $p = \infty$ we need the following definition.

The space $BMO_{\mathfrak{L}}$ is defined as the set of functions f in L^1_{loc} satisfying that there exists a constant C such that for every ball $B = B(x,r)$,

$$\int_B |f - f_B| \le C|B|,$$

if $r < \rho(x)$, and

$$\int_B |f| \le C|B|,$$

if $r \ge \rho(x)$.

This space was introduced in [4] as the appropriate substitute of BMO in the study of the boundedness of operators associated to $\mathfrak{L}$.

Regarding the Riesz transforms, it was shown in [1] that $\mathcal{R}^*$ preserves $BMO_{\mathfrak{L}}$ when $q_0 > d/2$, and the same occurs with $\mathcal{R}$ under the stronger assumption $q_0 > d$. Since L^∞ is continuously embedded in $BMO_{\mathfrak{L}}$, these results imply the $L^\infty - BMO_{\mathfrak{L}}$ continuity of $\mathcal{R}$ and $\mathcal{R}^*$, under the stated hypothesis on q_0. We point out that even when $\mathcal{R}$ and $\mathcal{R}^*$ are Calderón-Zygmund these results are sharper than those derived from Calderón-Zygmund theory since $BMO_{\mathfrak{L}} \subset BMO$.

It is a natural question to ask for the class of functions b such that $\mathcal{R}_b$ and $\mathcal{R}_b^*$ are also bounded operators from L^∞ into $BMO_{\mathfrak{L}}$. For this purpose we introduce the following definition.

For $\theta > 0$, we denote by $BMO_\theta^{\log}(\rho)$ the set of functions b such that

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |b - b_B| \le C \frac{(1 + r/\rho(x))^\theta}{1 + \log^+(\rho(x)/r)},$$

for all $x \in \mathbb{R}^d$ and $r > 0$. Correspondingly, we define $BMO_\infty^{\log}(\rho) = \bigcup_{\theta>0} BMO_\theta^{\log}(\rho)$.

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Our second result can be stated as follows.

Theorem 2 *Let $V \in RH_{d/2}$ and $b \in BMO_\infty(\rho)$, then*

(i) $\mathcal{R}_b^*: L^\infty \mapsto BMO_{\mathfrak{L}}$ *if and only if* $b \in BMO_\infty^{\log}(\rho)$.
(ii) *If $V \in RH_d$, the above result is also true for $\mathcal{R}_b$.*

The contents of Theorem 1 were already known for functions b in BMO. In the case $q_0 > d$, since $\mathcal{R}$ and $\mathcal{R}^*$ are Calderón-Zygmund operators, the boundedness of commutators follows from the general theory (see [2] and [9] for instance). The result for $\mathcal{R}_b^*$ when $d/2 < q_0 < d$ was recently proved in [6]. The novelty of Theorem 1 relies on the extension of the L^p-boundedness for b belonging to the larger class $BMO_\infty(\rho)$. Theorem 2 is completely new for this kind of Riesz transforms. However, there is a result in that direction for the classical case $\mathfrak{L} = -\Delta$ in [7]. There, the authors show that commutators of the Hilbert transform are never bounded from L^∞ into BMO except for the trivial case when b is constant.

Our approach to handle commutators is the Strömberg technique that was also used in [6]. That involves to obtain a point-wise majorization of the sharp maximal function of the commutators. In this article we reduce the problem to estimate a more appropriate and smaller sharp maximal function which takes into account only local balls, namely those contained in a critical ball. In order to do so we prove a suitable Fefferman-Stein inequality (see Lemma 2).

The clue that allows us to enlarge the class of functions b with respect to the classical case, relies on the stronger decay of the kernels and their modulus of continuity outside critical balls, contained in Lemmas 3 and 4.

The paper is organized as follows. In the next section we present some properties of the space $BMO_\infty(\rho)$ and a Fefferman-Stein type inequality. In Sect. 3 we collect some useful estimates of the kernels of $\mathcal{R}$ and $\mathcal{R}^*$. Section 4 is devoted to prove some estimates of averages and oscillations related to commutators that will be used in the last section to prove Theorem 1 as well as Theorem 2.

2 Preliminary Lemmas and Propositions

Proposition 1 [11] *Let $V \in RH_{d/2}$. For the associated function ρ there exist C and $k_0 \geq 1$ such that*

$$C^{-1}\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{-k_0} \leq \rho(y) \leq C\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{k_0}{k_0+1}} \tag{6}$$

for all $x, y \in \mathbb{R}^d$.

A ball $B(x, \rho(x))$ is called *critical*.

Proposition 2 [3] *There exists a sequence of points $\{x_k\}_{k=1}^\infty$ in $\mathbb{R}^d$, so that the family of critical balls $Q_k = B(x_k, \rho(x_k))$, $k \geq 1$, satisfies*

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(i) $\bigcup_k Q_k = \mathbb{R}^d$.
(ii) *There exists* N *such that for every* $k \in \mathbb{N}$, $\operatorname{card}\{j: 4Q_j \cap 4Q_k \neq \emptyset\} \leq N$.

Inequality (6) implies that if $x, y \in Q$, and Q is a critical ball, then

$$\rho(x) \leq C_0 \rho(y) \tag{7}$$

where the constant C_0 depends on the constants C and k_0 in (6).

Proposition 3 *Let* $\theta > 0$ *and* $1 \leq s < \infty$. *If* $b \in BMO_\theta(\rho)$, *then*

$$\left(\frac{1}{|B|}\int_B |b - b_B|^s\right)^{1/s} \lesssim [b]_\theta \left(1 + \frac{r}{\rho(x)}\right)^{\theta'}, \tag{8}$$

for all $B = B(x, r)$, *with* $x \in \mathbb{R}^d$ *and* $r > 0$, *where* $\theta' = (k_0 + 1)\theta$ *and* k_0 *the constant appearing in* (6).

Proof From the standard John-Nirenberg inequality (see [8]), given a ball B_0 and a function $g \in BMO(B_0)$ we have, for each $1 \leq s < \infty$,

$$\left(\frac{1}{|B|}\int_B |g - g_B|^s\right)^{1/s} \leq C\|g\|_{BMO(B_0)}, \tag{9}$$

for every ball $B \subset B_0$, where the constant C does not depend on the ball B_0.

Therefore, to prove (8) we only need to show the claim: if $R \geq 1$ and Q is a critical ball, then we have $b \in BMO(RQ)$ and

$$\|b\|_{BMO(RQ)} \lesssim [b]_\theta (1 + R)^{(k_0+1)\theta}.$$

If this is true, an application of (9), gives that for any ball $B \subset RQ$,

$$\left(\frac{1}{|B|}\int_B |b - b_B|^s\right)^{1/s} \lesssim [b]_\theta (1 + R)^{(k_0+1)\theta}. \tag{10}$$

Now, let $B = B(x, r)$ and $Q = B(x, \rho(x))$, with $x \in \mathbb{R}^d$ and $r > 0$. If $r \leq \rho(x)$, we choose $R = 1$, and we may apply (10) to get (8). In the case $r > \rho(x)$, we notice that $B = \frac{r}{\rho(x)} Q$. Then we apply (10) with $R = \frac{r}{\rho(x)}$ which yields (8).

It remains to prove the claim. Let $B = B(z, r) \subset RQ$, with $z \in \mathbb{R}^d$ and $r > 0$. Due to (6), we have

$$\rho(x)(1 + R)^{-k_0} \lesssim \rho(z),$$

then, since $r < R\rho(x)$,

$$\frac{r}{\rho(z)} \lesssim (1 + R)^{(k_0+1)}.$$

Using that $b \in BMO_\theta(\rho)$, it leads to

$$\frac{1}{|B|}\int_B |b - b_B| \lesssim [b]_\theta (1 + R)^{(k_0+1)\theta}.$$ □

Lemma 1 *Let $b \in BMO_\theta(\rho)$, $B = B(x_0, r)$ and $s \geq 1$, then*

$$\left(\frac{1}{|2^k B|}\int_{2^k B}|b - b_B|^s\right)^{1/s} \lesssim [b]_\theta k\left(1 + \frac{2^k r}{\rho(x_0)}\right)^{\theta'},$$

for all $k \in \mathbb{N}$, with θ' as in (8).

Proof Following standard arguments and Proposition 3, we have

$$\begin{aligned}
&\left(\frac{1}{|2^k B|}\int_{2^k B}|b - b_B|^s\right)^{1/s} \\
&\quad\lesssim \left(\frac{1}{|2^k B|}\int_{2^k B}|b - b_{2^k B}|^s\right)^{1/s} + \sum_{j=1}^{k}|b_{2^j B} - b_{2^{j-1}B}| \\
&\quad\lesssim [b]_\theta \sum_{j=1}^{k}\left(1 + \frac{2^j r}{\rho(x_0)}\right)^{\theta'} \\
&\quad\lesssim [b]_\theta k\left(1 + \frac{2^k r}{\rho(x_0)}\right)^{\theta'}.
\end{aligned}$$

□

Given $\alpha > 0$ we define the following maximal functions for $g \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,

$$M_{\rho,\alpha} g(x) = \sup_{x\in B\in\mathcal{B}_{\rho,\alpha}} \frac{1}{|B|}\int_B |g|,$$

$$M^\sharp_{\rho,\alpha} g(x) = \sup_{x\in B\in\mathcal{B}_{\rho,\alpha}} \frac{1}{|B|}\int_B |g - g_B|,$$

where $\mathcal{B}_{\rho,\alpha} = \{B(y,r):\ y \in \mathbb{R}^d, \text{ and } r \leq \alpha\rho(y)\}$.

Also, given a ball $Q \subset \mathbb{R}^d$, for $g \in L^1_{\text{loc}}(Q)$ and $x \in Q$, we define

$$M_Q g(x) = \sup_{x\in B\in\mathcal{F}(Q)} \frac{1}{|B\cap Q|}\int_{B\cap Q}|g|, \tag{11}$$

and

$$M^\sharp_Q g(x) = \sup_{x\in B\in\mathcal{F}(Q)} \frac{1}{|B\cap Q|}\int_{B\cap Q}|g - g_{B\cap Q}|, \tag{12}$$

where $\mathcal{F}(Q) = \{B(y,r):\ y \in Q,\ r > 0\}$.

Let us note that if g is supported in Q, operators (11) and (12) coincide with the standard definitions of Hardy-Littlewood and sharp maximal functions defined in Q viewed as a space of homogeneous type with the Euclidean metric and the Lebesgue measure restricted to Q.

Lemma 2 (Fefferman–Stein type inequality) *For $1 < p < \infty$, there exist β and γ such that if $\{Q_k\}_{k=1}^{\infty}$ is a sequence of balls as in Proposition 2, then*

$$\int_{\mathbb{R}^d} |M_{\rho,\beta}(g)|^p \lesssim \int_{\mathbb{R}^d} |M^{\sharp}_{\rho,\gamma}(g)|^p + \sum_k |Q_k| \left(\frac{1}{|Q_k|} \int_{2Q_k} |g| \right)^p,$$

for all $g \in L^1_{\text{loc}}(\mathbb{R}^d)$.

Proof The main tool to prove this lemma is the Fefferman-Stein inequality in the setting of spaces of homogeneous type with finite measure given by Proposition 3.4 in [10]. We point out that in this case the finiteness of the L^p norm of the maximal function is not needed (in fact that assumption is only used to prove that the left hand side of inequality (3.14) there is finite, but this follows immediately from the finiteness of the measure of the space).

If Q is a critical ball and $x \in Q$, it is not difficult to see that

$$M_{\rho,\beta}g(x) \le M_{2Q}(g\chi_{2Q})(x), \tag{13}$$

with $\beta = \frac{1}{2C_0^2}$ (where C_0 is the constant appearing in (7)), and for $x \in 2Q$,

$$M^{\sharp}_{2Q}(g\chi_{2Q})(x) \lesssim M^{\sharp}_{\rho,2}g(x). \tag{14}$$

We give an outline of the proof of the last inequality since (13) is even easier. In fact, given a ball $B = B(y,r) \in \mathcal{F}(2Q)$, we divide the argument according to r greater or less than $3^{-\frac{k_0}{k_0+1}} \frac{\rho(x_0)}{C}$ where C and k_0 are the constants appearing in (6). In the first case $B \cap 2Q$ has measure comparable to $2Q$ which belongs to $\mathcal{B}_{\rho,2}$. In the other case we just use that $B \in \mathcal{B}_{\rho,1} \subset \mathcal{B}_{\rho,2}$ and that $|B \cap 2Q|$ is comparable with $|B|$.

Now we use the decomposition of $\mathbb{R}^d$ given by Proposition 2, the mentioned Proposition 3.4 in [10], and inequalities (13) and (14), to obtain

$$\begin{aligned}
\int_{\mathbb{R}^d} |M_{\rho,\beta}(g)|^p &\le \sum_k \int_{Q_k} |M_{\rho,\beta}(g)|^p \\
&\le \sum_k \int_{Q_k} |M_{2Q_k}(g\chi_{2Q_k})|^p \\
&\lesssim \sum_k \int_{2Q_k} |M^{\sharp}_{2Q_k}(g\chi_{2Q_k})|^p + \sum_k |2Q_k| \left(\frac{1}{|2Q_k|} \int_{2Q_k} |g| \right)^p \\
&\lesssim \sum_k \int_{2Q_k} |M^{\sharp}_{\rho,4}(g)|^p + \sum_k |Q_k| \left(\frac{1}{|Q_k|} \int_{2Q_k} |g| \right)^p \\
&\lesssim \int_{\mathbb{R}^d} |M^{\sharp}_{\rho,4}(g)|^p + \sum_k |Q_k| \left(\frac{1}{|Q_k|} \int_{2Q_k} |g| \right)^p,
\end{aligned}$$

where in the last inequality we have used the finite overlapping property given by Proposition 2. □

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3 Estimates for the Kernels of $\mathcal{R}$ and $\mathcal{R}^*$

Let $\mathcal{K}$ and $\mathcal{K}^*$ be the vector valued kernels of $\mathcal{R}$ and $\mathcal{R}^*$ respectively.

Lemma 3 *If $V \in RH_{d/2}$, then we have:*

(i) *For every N there exists a constant C such that*

$$|\mathcal{K}^*(x,z)| \leq \frac{C(1+\frac{|x-z|}{\rho(x)})^{-N}}{|x-z|^{d-1}}\left(\int_{B(z,|x-z|/4)} \frac{V(u)}{|u-z|^{d-1}}du + \frac{1}{|x-z|}\right). \tag{15}$$

Moreover, the last inequality also holds with $\rho(x)$ replaced by $\rho(z)$.

(ii) *For every N and $0<\delta<\min\{1, 2-d/q_0\}$ there exists a constant C such that*

$$\begin{aligned} &|\mathcal{K}^*(x,z) - \mathcal{K}^*(y,z)| \\ &\quad \leq \frac{C|x-y|^\delta(1+\frac{|x-z|}{\rho(x)})^{-N}}{|x-z|^{d-1+\delta}}\left(\int_{B(z,|x-z|/4)} \frac{V(u)}{|u-z|^{d-1}}du + \frac{1}{|x-z|}\right) \end{aligned} \tag{16}$$

whenever $|x-y| < \frac{2}{3}|x-z|$. Moreover, the last inequality also holds with $\rho(x)$ replaced by $\rho(z)$.

(iii) *If $\mathbf{K}^*$ denotes the $\mathbb{R}^d$ vector valued kernel of the adjoint of the classical Riesz operator, then for every $0<\sigma<2-d/q_0$,*

$$\begin{aligned} &|\mathcal{K}^*(x,z) - \mathbf{K}^*(x,z)| \\ &\quad \leq \frac{C}{|x-z|^{d-1}}\left(\int_{B(z,|x-z|/4)} \frac{V(u)}{|u-z|^{d-1}}du + \frac{1}{|x-z|}\left(\frac{|x-z|}{\rho(x)}\right)^\sigma\right), \end{aligned} \tag{17}$$

whenever $|x-y| < \rho(x)$.

(iv) *When $q_0 > d$, the term involving V can be dropped from inequalities (15), (16) and (17).*

Proof Inequalities (15) and (17) are basically contained in [11], and (16) can be found in [6]. Statement (iv) for (17) is a consequence of Lemma 1 in [1] since it gives the boundedness of the first term by the second one. The remaining inequalities follow from the same lemma, applying (15) and (16) with perhaps a different N. □

Lemma 4 *If $V \in RH_d$, then we have:*

(i) *For every N there exists a constant C such that*

$$|\mathcal{K}(x,z)| \leq \frac{C(1+\frac{|x-z|}{\rho(x)})^{-N}}{|x-z|^{d}}. \tag{18}$$

(ii) *For every N and $0<\delta<\min\{1, 1-d/q_0\}$ there exists a constant C such that*

$$|\mathcal{K}(x,z) - \mathcal{K}(y,z)| \leq \frac{C|x-y|^\delta(1+\frac{|x-z|}{\rho(x)})^{-N}}{|x-z|^{d+\delta}} \tag{19}$$

whenever $|x-y| < \frac{2}{3}|x-z|$.

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(iii) *If* $\mathbf{K}$ *denotes the* $\mathbb{R}^d$ *vector valued kernel of the classical Riesz operator, for every* $0 < \sigma < 2 - d/q_0$, *we have*

$$|\mathcal{K}(x,z) - \mathbf{K}(x,z)| \le \frac{C}{|x-z|^d}\left(\frac{|x-z|}{\rho(z)}\right)^{\sigma}. \tag{20}$$

Proof Estimate (18) can be found in [11, inequality (6.5)]. Estimates (19) and (20) are also basically contained in [11]. Details for (20) are given in [1]. As for (19) in [11] it is proved for $N = 0$. Nevertheless, the same argument can be applied to any positive N. □

Remark 1 Let us observe that when $V \in RH_d$, (18) and (19) together with (16) and Lemma 3(iv) imply that $\mathcal{K}$ and $\mathcal{K}^*$ are Calderón-Zygmund kernels.

4 Technical Lemmas

As usual we denote by M the Hardy-Littlewood maximal function and, for $s > 1$, by M_s the operator defined as $M_s f = (M(f^s))^{1/s}$.

Lemma 5 *Let* $V \in RH_{d/2}$, $1/p_0 = (1/q_0 - 1/d)^+$, *and* $b \in BMO_\theta(\rho)$. *Then, for any* $s > p_0'$ *there exists a constant* C *such that*

$$\frac{1}{|Q|}\int_Q |\mathcal{R}_b^* f| \le C[b]_\theta \inf_{y\in Q} M_s f(y),$$

for all $f \in L^s_{\text{loc}}(\mathbb{R}^d)$ *and every ball* $Q = B(x_0, \rho(x_0))$. *Additionally, if* $q_0 > d$, *the above estimate also holds for* $\mathcal{R}$ *instead of* $\mathcal{R}^*$.

Proof Let $f \in L^p(\mathbb{R}^d)$ and $Q = B(x_0, \rho(x_0))$. We first observe

$$\mathcal{R}_b^* f = (b - b_Q)\mathcal{R}^* f - \mathcal{R}^*\big(f(b - b_Q)\big), \tag{21}$$

and so we have to deal with the average on Q of each term.

By Hölder's inequality with $s > p_0'$ and Lemma 1,

$$\frac{1}{|Q|}\int_Q |(b-b_Q)\mathcal{R}^* f| \le \left(\frac{1}{|Q|}\int_Q |b - b_Q|^{s'}\right)^{1/s'}\left(\frac{1}{|Q|}\int_{\mathbb{R}^d} |\mathcal{R}^* f|^s\right)^{1/s}$$
$$\lesssim [b]_\theta \left(\frac{1}{|Q|}\int_Q |\mathcal{R}^* f|^s\right)^{1/s}.$$

If we write $f = f_1 + f_2$ with $f_1 = f\chi_{2Q}$ then, using that $\mathcal{R}^*$ is bounded on $L^s(\mathbb{R}^d)$ with $s > p_0'$,

$$\left(\frac{1}{|Q|}\int_Q |\mathcal{R}^* f_1|^s\right)^{1/s} \lesssim \left(\frac{1}{|Q|}\int_{2Q} |f|^s\right)^{1/s}$$
$$\lesssim \inf_{y\in Q} M_s f(y). \tag{22}$$

Now, for $x \in Q$ and using (15) in Lemma 3, we have

$$\begin{aligned} |\mathcal{R}^* f_2(x)| &= \left| \int_{|x_0-z|>2\rho(x_0)} \mathcal{K}^*(x,z) f(z)\, dz \right| \\ &\lesssim I_1(x) + I_2(x), \end{aligned}$$

where

$$I_1(x) = \int_{|x_0-z|>2\rho(x_0)} \frac{|f(z)|}{|x-z|^d (1+\frac{|x-z|}{\rho(x)})^N}\, dz$$

and

$$I_2(x) = \int_{|x_0-z|>2\rho(x_0)} \frac{|f(z)|}{|x-z|^{d-1} (1+\frac{|x-z|}{\rho(x)})^N} \int_{B(z,|x-z|/4)} \frac{V(u)}{|u-z|^{d-1}}\, du\, dz.$$

To deal with $I_1(x)$, using that in our situation $\rho(x) \simeq \rho(x_0)$ and $|x-z| \simeq |x_0-z|$, we split into annuli to obtain

$$\begin{aligned} I_1(x) &\lesssim \sum_{k\geq 1} \frac{2^{-Nk}}{(2^k \rho(x_0))^d} \int_{|x_0-z|<2^k\rho(x_0)} |f(z)|\, dz \\ &\lesssim \inf_{y\in Q} Mf(y). \end{aligned} \tag{23}$$

To take care of $I_2(x)$, having in mind Lemma 3(iv) we may assume $d/2 < q_0 < d$. Then, since $x \in Q$,

$$\begin{aligned} I_2(x) &\lesssim \int_{|x_0-z|>2\rho(x_0)} \frac{|f(z)|}{|x_0-z|^{d-1} (1+\frac{|x_0-z|}{\rho(x_0)})^N} \int_{B(x_0,4|x_0-z|)} \frac{V(u)}{|u-z|^{d-1}}\, du\, dz \\ &\lesssim \sum_{k\geq 1} \frac{2^{-Nk}}{(2^k\rho(x_0))^{d-1}} \int_{|x_0-z|<2^{k+1}\rho(x_0)} |f(z)| \int_{B(x_0,2^{k+3}\rho(x_0))} \frac{V(u)}{|u-z|^{d-1}}\, du\, dz \\ &\lesssim \sum_{k\geq 1} \frac{2^{-Nk}}{(2^k\rho(x_0))^{d-1}} \int_{|x_0-z|<2^k\rho(x_0)} |f| \mathcal{I}_1(V\chi_{B(x_0,2^k\rho(x_0))}). \end{aligned}$$

Let $p_0' < s < d$ (this is always possible because $q_0 > 1$, and also sufficient since $M_s f$ increases with s). Using first Hölder's inequality and the boundedness of the fractional integral $\mathcal{I}_1 : L^s \mapsto L^q$ with $1/q = 1/s' + 1/d$, we obtain

$$\begin{aligned} &\int_{|x_0-z|<2^k\rho(x_0)} |f| \mathcal{I}_1(V\chi_{B(x_0,2^k\rho(x_0))}) \\ &\quad \leq \|f\chi_{B(x_0,2^k\rho(x_0))}\|_s \|\mathcal{I}_1(V\chi_{B(x_0,2^k\rho(x_0))})\|_{s'} \\ &\quad \lesssim \|f\chi_{B(x_0,2^k\rho(x_0))}\|_s \|V\chi_{B(x_0,2^k\rho(x_0))}\|_q. \end{aligned}$$

Since $V \in RH_q$, from our assumptions on s, we obtain

$$\begin{aligned}\|V\chi_{B(x_0,2^k\rho(x_0))}\|_q &\lesssim \left(2^k\rho(x_0)\right)^{-d/q'}\int_{B(x_0,2^k\rho(x_0))} V\\ &\lesssim 2^{k(d\mu-d/q')}\rho(x_0)^{-d/q'}\int_{B(x_0,\rho(x_0))} V\\ &\lesssim 2^{k(d\mu-d/q')}\rho(x_0)^{-d/q'+d-2},\end{aligned} \tag{24}$$

where in the last two inequalities we have used (5) and the definition of ρ respectively.

Therefore,

$$I_2(x) \lesssim \rho(x_0)^{-d/q'-1}\sum_{k\geq 1} 2^{k(-N+1-d+d\mu-d/q')}\|f\chi_{B(x_0,2^k\rho(x_0))}\|_s. \tag{25}$$

Finally, observing that

$$\|f\chi_{B(x_0,2^k\rho(x_0))}\|_s \lesssim \left(2^k\rho(x_0)\right)^{d/s}\inf_{y\in Q} M_s f(y)$$

and using that $d/s - d/q' = 1$, we have

$$I_2(x) \lesssim \inf_{y\in Q} M_s f(y)\sum_{k\geq 1} 2^{k(-N+d\mu-d+2)}, \tag{26}$$

since N can be chosen large enough the last series converges.

To deal with the second term of (21), we split again $f = f_1 + f_2$. Choosing $p_0' < \tilde{s} < s$ and denoting $\nu = \frac{\tilde{s}s}{s-\tilde{s}}$, using the boundedness of $\mathcal{R}^*$ on $L^{\tilde{s}}(\mathbb{R}^d)$ (see [11]) and applying Hölder's inequality,

$$\begin{aligned}\frac{1}{|Q|}\int_Q |\mathcal{R}^* f_1(b-b_Q)| &\leq \left(\frac{1}{|Q|}\int_Q |\mathcal{R}^* f_1(b-b_Q)|^{\tilde{s}}\right)^{1/\tilde{s}}\\ &\lesssim \left(\frac{1}{|Q|}\int_{2Q} |f(b-b_Q)|^{\tilde{s}}\right)^{1/\tilde{s}}\\ &\lesssim \left(\frac{1}{|Q|}\int_{2Q} |f|^s\right)^{1/s}\left(\frac{1}{|Q|}\int_{2Q} |(b-b_Q)|^{\nu}\right)^{1/\nu}\\ &\lesssim [b]_\theta \inf_{y\in Q} M_s f(y),\end{aligned}$$

where in the last inequality we have used Proposition 3.

For the remaining term we have to deal with

$$\tilde{I}_1(x) = \int_{|x-z|>2\rho(x_0)} \frac{|f(z)(b-b_Q)|}{|x-z|^d(1+\frac{|x-z|}{\rho(x)})^N}\,dz$$

and

$$\tilde{I}_2(x) = \int_{|x-z|>2\rho(x_0)} \frac{|f(z)(b-b_Q)|}{|x-z|^{d-1}(1+\frac{|x-z|}{\rho(x)})^N}\int_{B(z,|x-z|/4)} \frac{V(u)}{|u-z|^{d-1}}\,du\,dz.$$

We start by observing that for $1 \le \tilde{s} < s$ and $\nu = \frac{\tilde{s}s}{s-\tilde{s}}$, using Lemma 1, we obtain

$$\begin{aligned}
&\| f(b-b_Q)\chi_{B(x_0,2^k\rho(x_0))}\|_{\tilde{s}} \\
&\quad \le \| f\chi_{B(x_0,2^k\rho(x_0))}\|_s \, \|(b-b_Q)\chi_{B(x_0,2^k\rho(x_0))}\|_\nu \\
&\quad \lesssim \left(2^k\rho(x_0)\right)^{d/\tilde{s}} \inf_{y\in Q} M_s f(y) k 2^{k\theta'}[b]_\theta .
\end{aligned} \tag{27}$$

For $\tilde{I}_1(x)$ we proceed as for $I_1(x)$, and using (27) with $\tilde{s}=1$, we arrive to

$$\begin{aligned}
\tilde{I}_1(x) &\lesssim \sum_{k\ge 1} \frac{2^{-Nk}}{(2^k\rho(x_0))^d} \int_{|x_0-z|<2^k\rho(x_0)} |b(z)-b_Q||f(z)|\,dz \\
&\lesssim [b]_\theta \inf_{y\in Q} M_s f(y) \sum_{k\ge 1} k 2^{k(-N+\theta')} \\
&\lesssim [b]_\theta \inf_{y\in Q} M_s f(y).
\end{aligned}$$

To deal with $\tilde{I}_2(x)$ we argue as in the estimate for $I_2(x)$ with $f(b-b_Q)$ instead of f and $\tilde{s}$ and $\tilde{q}$ instead of s and q, where $1/\tilde{q} = 1/\tilde{s}' + 1/d$. In this way, as in (25), using also (27), we have

$$\begin{aligned}
\tilde{I}_2(x) &\lesssim \rho(x_0)^{-1-d/\tilde{q}'} \sum_{k\ge 1} 2^{k(-N+1-d+d\mu-d/\tilde{q}')} \| f(b-b_Q)\chi_{B(x_0,2^k\rho(x_0))}\|_{\tilde{s}} \\
&\lesssim [b]_\theta \inf_{y\in Q} M_s f(y) \sum_{k\ge 1} k 2^{k(-N+\theta'+2-d+d\mu)} \\
&\lesssim [b]_\theta \inf_{y\in Q} M_s f(y),
\end{aligned} \tag{28}$$

choosing N large enough.

Finally, we notice that in the proof above, we only have used the size of $\mathcal{K}^*$ given by (15) in Lemma 3, therefore in the case $q_0 > d$ we also have the result for $\mathcal{R}$ in view of Lemma 4. □

Remark 2 It is easy to check that if the critical ball Q is replaced by $2Q$, last lemma also holds.

Lemma 6 *Let $V \in RH_{d/2}$ and $b \in BMO_\infty(\rho)$, then for any $s > p_0'$ and $\gamma \ge 1$, there exists a constant C such that*

$$\int_{(2B)^c} |\mathcal{K}^*(x,z) - \mathcal{K}^*(y,z)||b(z)-b_B||f(z)|\,dz \le C[b]_\theta \inf_{u\in B} M_s f(u), \tag{29}$$

for all f and $x, y \in B = B(x_0, r)$, with $r < \gamma\rho(x_0)$. Additionally, if $q_0 > d$, the above estimate also holds for $\mathcal{K}$ instead of $\mathcal{K}^*$.

Proof Denoting $Q = B(x_0, \gamma\rho(x_0))$, by (16), and since in our situation $\rho(x) \simeq \rho(x_0)$ and $|x-z| \simeq |x_0 - z|$, we need to bound four terms

$$I_1 = r^\delta \int_{Q\setminus 2B} \frac{|f(z)||b(z)-b_B|}{|x_0-z|^{d+\delta}}\,dz,$$

$$I_2 = r^\delta \rho(x_0)^N \int_{Q^c} \frac{|f(z)||b(z)-b_B|}{|x_0-z|^{d+\delta+N}}\,dz,$$

$$I_3 = r^\delta \int_{Q\setminus 2B} \frac{|f(z)||b(z)-b_B|}{|x_0-z|^{d-1+\delta}} \int_{B(x_0,4|x_0-z|)} \frac{V(u)}{|u-z|^{d-1}}\,du\,dz,$$

and

$$I_4 = r^\delta \rho(x_0)^N \int_{Q^c} \frac{|f(z)|,|b(z)-b_B|}{|x_0-z|^{d-1+\delta+N}} \int_{B(x_0,4|x_0-z|)} \frac{V(u)}{|u-z|^{d-1}}\,du\,dz.$$

Splitting into annuli, we have

$$I_1 \lesssim \frac{1}{r^d} \sum_{j=2}^{j_0} 2^{-j(d+\delta)} \int_{2^j B} |f||b-b_B|,$$

where j_0 is the least integer such that $2^{j_0} \geq \gamma\rho(x_0)/r$.

By Hölder's inequality and Lemma 1 we obtain for $j \leq j_0$,

$$\int_{2^j B} |f||b-b_B| \leq j[b]_\theta |2^j B| \inf_{y\in B} M_s f(y).$$

Then,

$$\begin{aligned} I_1 &\lesssim [b]_\theta \inf_{y\in B} M_s f(y) \sum_{j=2}^{\infty} j 2^{-j\delta} \\ &\lesssim [b]_\theta \inf_{y\in B} M_s f(y). \end{aligned}$$

To deal with I_2, splitting into annuli, using Lemma 1 and choosing $N > \theta'$, we have

$$\begin{aligned} I_2 &\lesssim \frac{\rho(x_0)^N}{r^{N+d}} \sum_{j=j_0-1}^{\infty} 2^{-j(d+\delta+N)} \int_{2^j B} |f||b-b_B| \\ &\lesssim [b]_\theta \inf_{y\in B} M_s f(y) \left(\frac{\rho(x_0)}{r}\right)^{N-\theta'} \sum_{j=j_0-1}^{\infty} j 2^{-j(\delta+N-\theta')} \\ &\lesssim [b]_\theta \inf_{y\in B} M_s f(y) \sum_{j=j_0-1}^{\infty} j 2^{-j\delta} \\ &\lesssim [b]_\theta \inf_{y\in B} M_s f(y). \end{aligned}$$

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To deal with I_3 and I_4, due to Lemma 3(iv) we may assume $d/2 < q_0 \le d$. Now,

$$I_3 \lesssim \frac{1}{r^{d-1}} \sum_{j=2}^{j_0} 2^{-j(d-1+\delta)} \int_{2^j B} |f(z)||b(z) - b_B| \mathcal{I}_1(V\chi_{2^{j+2}B})(z)\, dz.$$

If $p_0' < \tilde{s} < s$, $\nu = \frac{\tilde{s}s}{s-\tilde{s}}$ and q such that $1/q = 1/\tilde{s}' + 1/d$, then

$$\begin{aligned}\int_{2^j B} |f||b - b_B| \mathcal{I}_1(V\chi_{2^{j+2}B}) &\le \|f\chi_{2^j B}\|_s \|(b - b_B)\chi_{2^j B}\|_\nu \|\mathcal{I}_1(V\chi_{2^{j+2}B})\|_{\tilde{s}'} \\ &\lesssim j|2^j B|^{1/\tilde{s}} [b]_\theta \inf_{y\in B} M_s f(y) \|V\chi_{2^{j+2}B}\|_q, \end{aligned} \tag{30}$$

where in the last inequality we use Lemma 1 and that $j \le j_0$.

Since $V \in RH_q$, from our assumptions on $\tilde{s}$,

$$\begin{aligned}\|V\chi_{2^{j+2}B}\|_q &\lesssim \|V\chi_Q\|_q \\ &\lesssim \rho(x_0)^{-d/q'} \int_Q V \\ &\lesssim \rho(x_0)^{d/q-2}, \end{aligned}$$

for all $j \le j_0$. Therefore, since $d/\tilde{s} = d + 1 - d/q$ and $2 - d/q > 0$,

$$\begin{aligned} I_3 &\lesssim [b]_\theta \inf_{y\in B} M_s f(y) \frac{r^{d/\tilde{s}-d+1}}{\rho(x_0)^{2-d/q}} \sum_{j=2}^{j_0} j 2^{-j(d-1+\delta-d/\tilde{s})} \\ &\lesssim [b]_\theta \inf_{y\in B} M_s f(y) \left(\frac{r}{\rho(x_0)}\right)^{2-d/q} \sum_{j=2}^{j_0} j 2^{-j(d/q-2+\delta)} \\ &\lesssim [b]_\theta \inf_{y\in B} M_s f(y) \left(\frac{r}{\rho(x_0)}\right)^{2-d/q} 2^{j_0(2-d/q)} \sum_{j=2}^{j_0} j 2^{-j\delta} \\ &\lesssim [b]_\theta \inf_{y\in B} M_s f(y). \end{aligned}$$

Finally, for I_4 we have

$$I_4 \lesssim \frac{\rho(x_0)^N}{r^{d-1+N}} \sum_{j=j_0-1}^{\infty} 2^{-j(d-1+\delta+N)} \int_{2^j B} |f(z)||b(z) - b_B| \mathcal{I}_1(V\chi_{2^{j+2}B})\, dz.$$

Now we proceed as in (30) to obtain, for $j > j_0$,

$$\int_{2^j B} |f||b - b_B| \mathcal{I}_1(V\chi_{2^{j+2}B}) \lesssim [b]_\theta \inf_{y\in B} M_s f(y) j \frac{(2^j r)^{\theta'+d/\tilde{s}}}{\rho(x_0)^{\theta'}} \|V\chi_{2^{j+2}B}\|_q,$$

moreover,

$$\begin{aligned}\|V\chi_{2^{j+2}B}\|_q &\lesssim (2^j r)^{-d/q'}\int_{2^j B} V\\ &\lesssim 2^{j(d\mu-d/q')}\frac{r^{-d/q'+d\mu}}{\rho(x_0)^{d\mu}}\int_Q V\\ &\lesssim 2^{j(d\mu-d/q')}\frac{r^{-d/q'+d\mu}}{\rho(x_0)^{d\mu-d+2}}.\end{aligned}$$

With this estimate, choosing N large enough so that $d-2+N-\theta'-d\mu>0$, we have

$$\begin{aligned}I_4 &\lesssim [b]_\theta \inf_{y\in B} M_s f(y)\left(\frac{\rho(x_0)}{r}\right)^{d-2+N-\theta'-d\mu}\sum_{j=j_0-1}^{\infty} j2^{-j(d-2+N-\theta'-d\mu+\delta)}\\ &\lesssim [b]_\theta \inf_{y\in B} M_s f(y),\end{aligned}$$

and we have finished the proof (29).

Now, suppose $q_0>d$. To obtain the estimate for $\mathcal{K}$ we use (19) in Lemma 4 to get

$$\int_{(2B)^c}|\mathcal{K}(x,z)-\mathcal{K}(y,z)||b(z)-b_B||f(z)|\,dz\lesssim I_1+I_2,$$

completing the proof of the lemma. □

5 Proofs of the Main Results

Proof of Theorem 1 We will prove part (ii) and part (i) follows by duality. We start with a function $f\in L^p(\mathbb{R}^d)$ with $p_0'<p<\infty$, and we notice that due to Lemma 5 we have $\mathcal{R}_b^* f\in L^1_{\text{loc}}(\mathbb{R}^d)$.

By using Lemma 2, Lemma 5 with $p_0'<s<p$ and Remark 2, we have

$$\begin{aligned}\|\mathcal{R}_b^* f\|_p^p &\le \int_{\mathbb{R}^d}\left|M_{\rho,\beta}(\mathcal{R}_b^* f)\right|^p\\ &\lesssim \int_{\mathbb{R}^d}\left|M^\sharp_{\rho,\gamma}(\mathcal{R}_b^* f)\right|^p+\sum_k |Q_k|\left(\frac{1}{|Q_k|}\int_{2Q_k}|\mathcal{R}_b^* f|\right)^p\\ &\lesssim \int_{\mathbb{R}^d}\left|M^\sharp_{\rho,\gamma}(\mathcal{R}_b^* f)\right|^p+[b]_\theta^p\sum_k\int_{2Q_k}|M_s f|^p.\end{aligned}$$

By the finite overlapping property given by Proposition 2 and the boundedness of M_s in $L^p(\mathbb{R}^d)$ the second term is controlled by $[b]_\theta^p\|f\|_p^p$. Thus, we have to take care of the first term.

Our goal is to find a point-wise estimate of $M^{\sharp}_{\rho,\gamma}(\mathcal{R}^*_b f)$. Let $x \in \mathbb{R}^d$ and $B = B(x_0, r)$, with $r < \gamma\rho(x_0)$ such that $x \in B$. If $f = f_1 + f_2$, with $f_1 = f\chi_{2B}$, then we write

$$\mathcal{R}^*_b f = (b - b_B)\mathcal{R}^* f - \mathcal{R}^*\big(f_1(b - b_B)\big) - \mathcal{R}^*\big(f_2(b - b_B)\big). \tag{31}$$

Therefore, we need to control the mean oscillation on B of each term that we call $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$.

Let $s > p_0'$, an application of Hölder's inequality and Proposition 3 gives

$$\begin{aligned}
\mathcal{O}_1 &\le \frac{2}{|B|}\int_B |(b - b_B)\mathcal{R}^* f| \\
&\lesssim \left(\frac{1}{|B|}\int_B |b - b_B|^{s'}\right)^{1/s'}\left(\frac{1}{|B|}\int_B |\mathcal{R}^* f|^s\right)^{1/s} \\
&\lesssim [b]_\theta M_s \mathcal{R}^* f(x),
\end{aligned}$$

since $\frac{r}{\rho(x_0)} < \gamma$.

To estimate $\mathcal{O}_2$, let $p_0' < \tilde{s} < s$ and $\nu = \frac{\tilde{s}s}{s-\tilde{s}}$. Then,

$$\begin{aligned}
\mathcal{O}_2 &\le \frac{2}{|B|}\int_B \big|\mathcal{R}^*\big((b - b_B)f_1\big)\big| \\
&\lesssim \left(\frac{1}{|B|}\int_B \big|\mathcal{R}^*\big((b - b_B)f_1\big)\big|^{\tilde{s}}\right)^{1/\tilde{s}} \\
&\lesssim \left(\frac{1}{|B|}\int_{2B} |(b - b_B)f|^{\tilde{s}}\right)^{1/\tilde{s}} \\
&\lesssim \left(\frac{1}{|B|}\int_{2B} |b - b_B|^{\nu}\right)^{1/\nu}\left(\frac{1}{|B|}\int_{2B} |f|^s\right)^{1/s} \\
&\lesssim [b]_\theta M_s f(x).
\end{aligned} \tag{32}$$

For $\mathcal{O}_3$ we observe that

$$\mathcal{O}_3 \lesssim \frac{1}{|B|^2}\int_B\int_B \big|\mathcal{R}^*\big(f_2(b - b_B)\big)(u) - \mathcal{R}^*\big(f_2(b - b_B)\big)(z)\big|\, du\, dz$$

and the integral is clearly bounded by the left hand side of (29). Therefore, Lemma 6 asserts

$$\mathcal{O}_3 \lesssim [b]_\theta M_s f(x). \tag{33}$$

Therefore, we have proved that

$$\big|M^{\sharp}_{\rho,\gamma}\big(\mathcal{R}^*_b f\big)\big| \lesssim [b]_\theta\big(M_s\mathcal{R}^* f + M_s f\big).$$

Since $s < p$, we obtain the desired result. □

Proof of Theorem 2 We first assume $V \in RH_d$, and we denote T either $\mathcal{R}$ or $\mathcal{R}^*$ and G either $\mathcal{K}$ or $\mathcal{K}^*$.

Let $f \in L^\infty(\mathbb{R}^d)$ and $Q = B(x_0, \rho(x_0))$. In view of Proposition 2, it is not hard to see that it is enough to consider averages over critical balls (see [4]). Due to Lemma 5,

$$\frac{1}{|Q|}\int_Q |T_b f| \lesssim [b]_\theta \inf_{y\in Q} M_s f(y) \lesssim [b]_\theta \|f\|_\infty.$$

In order to deal with the oscillations, let $B = B(x_0, r)$ with $r < \rho(x_0)$. Notice that by Lemma 5 the function $T_b f$ belongs to $L^1_{\text{loc}}(\mathbb{R}^d)$.

We write as in (31)

$$T_b f = (b - b_B)Tf - T\big(f_1(b - b_B)\big) - T\big(f_2(b - b_B)\big),$$

and its mean oscillations on B as $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$.

The estimate for terms $\mathcal{O}_2$ and $\mathcal{O}_3$ are already done in (32) and (33) for $\mathcal{R}^*$. They also can be performed for $\mathcal{R}$ as long as $q_0 > d$, due to the boundedness of $\mathcal{R}$ in $L^{\tilde{s}}(\mathbb{R}^d)$ for $\tilde{s} > 1$ and Lemma 6. Thus, both terms are bounded by $[b]_\theta \|f\|_\infty$.

To deal with $\mathcal{O}_1$ we fixed $u \in B$ and write,

$$\begin{aligned}(b - b_B)Tf &= (b - b_B)Tf_1 + (b - b_B)\big(Tf_2 - Tf_2(u)\big)\\ &\quad + Tf_{21}(u)(b - b_B) + Tf_{22}(u)(b - b_B),\end{aligned} \tag{34}$$

where $f_2 = f_{21} + f_{22}$, with $f_{22} = f\chi_{4Q\setminus 2B}$ and $Q = B(x_0, \rho(x_0))$. We denote each oscillation $\mathcal{O}_{11}$, $\mathcal{O}_{12}$, $\mathcal{O}_{13}$ and $\mathcal{O}_{14}$.

We observe that $Tf_{21}(u)$ and $Tf_{22}(u)$ are finite for any $u \in B$, since $f \in L^\infty$ and

$$\int_{(2B)^c} |G(u,z)|\,dz < \infty. \tag{35}$$

We will see that $\mathcal{O}_{11}$, $\mathcal{O}_{12}$ and $\mathcal{O}_{13}$ are bounded under the condition $b \in BMO_\infty(\rho)$. For $\mathcal{O}_{11}$, choosing s so that T is bounded on $L^s(\mathbb{R}^d)$, we have

$$\begin{aligned}\mathcal{O}_{11} &\le \frac{2}{|B|}\int_B |(b - b_B)Tf_1|\\ &\lesssim \left(\frac{1}{|B|}\int_B |b - b_B|^{s'}\right)^{1/s'} \left(\frac{1}{|B|}\int_{\mathbb{R}^d} |Tf_1|^s\right)^{1/s}\\ &\lesssim \left(\frac{1}{|B|}\int_B |b - b_B|^{s'}\right)^{1/s'} \left(\frac{1}{|B|}\int_{2B} |f|^s\right)^{1/s}\\ &\lesssim [b]_\theta \|f\|_\infty.\end{aligned} \tag{36}$$

For $\mathcal{O}_{12}$ we claim

$$|Tf_2(x) - Tf_2(u)| \lesssim \|f\|_\infty,$$

for any x and u in B.

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First, observe that when $V \in RH_d$, the claim follows easily, since both kernels are Calderón-Zygmund. Therefore, for $V \in RH_q$ and $d/2 \le q < d$, and $T = \mathcal{R}^*$ due to (16) in Lemma 3, we only need to estimate

$$J_1 = r^\delta \int_{Q\setminus 2B} \frac{|f(z)|}{|x_0 - z|^{d-1+\delta}} \int_{B(x_0,4|x_0-z|)} \frac{V(u)}{|u-z|^{d-1}}\, du\, dz,$$

and

$$J_2 = r^\delta \rho(x_0)^N \int_{Q^c} \frac{|f(z)|}{|x_0 - z|^{d-1+\delta+N}} \int_{B(x_0,4|x_0-z|)} \frac{V(u)}{|u-z|^{d-1}}\, du\, dz.$$

Since the remaining term can be handled as in the Calderón-Zygmund case, we proceed as in Lemma 6 when estimating I_3 and I_4. In fact, splitting into annuli we have

$$\begin{aligned} J_1 &\lesssim \frac{\|f\|_\infty}{r^{d-1}} \sum_{j=2}^{j_0} 2^{-j(d-1+\delta)} \int_{2^j B} \int_{2^{j+2}B} \frac{V(u)}{|u-z|^{d-1}}\, du\, dz \\ &\lesssim \frac{\|f\|_\infty}{r^{d-2}} \int_{4Q} V \sum_{j=2}^{j_0} 2^{-j(d-2+\delta)} \\ &\lesssim \|f\|_\infty \left(\frac{\rho(x_0)}{r}\right)^{d-2} 2^{-j_0(d-2+\delta)} \\ &\lesssim \|f\|_\infty, \end{aligned}$$

and

$$\begin{aligned} J_2 &\lesssim \|f\|_\infty \frac{\rho(x_0)^N}{r^{d-1+N}} \sum_{j=j_0-1}^{\infty} 2^{-j(d-1+\delta+N)} \int_{2^j B} \int_{2^{j+2}B} \frac{V(u)}{|u-z|^{d-1}}\, du\, dz \\ &\lesssim \|f\|_\infty \frac{\rho(x_0)^{N-d\mu}}{r^{d-2+N-d\mu}} \sum_{j=j_0-1}^{\infty} 2^{-j(d-2+\delta+N-d\mu)} \int_Q V \\ &\lesssim \|f\|_\infty \left(\frac{\rho(x_0)}{r}\right)^{d-2+N-d\mu} 2^{-j_0(d-2+N-d\mu-\delta)} \\ &\lesssim \|f\|_\infty, \end{aligned}$$

thus the claim is proved.

Then,

$$\begin{aligned} \mathcal{O}_{12} &\le \frac{2}{|B|} \int_B |b(x) - b_B||Tf_2(x) - Tf_2(u)|\, dx \\ &\lesssim [b]_\theta \|f\|_\infty. \end{aligned}$$

That $\mathcal{O}_{13} \lesssim [b]_\theta \|f\|_\infty$, is a consequence of (35).

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Therefore, the theorem will follow if and only if there exists a constant C_b such that for any $B \in \mathcal{B}_{\rho,1}$ and $u \in B$,

$$\frac{1}{|B|}\left(\int_B |b(z)-b_B|\,dz\right)\left|\int_{4Q\setminus 2B} G(u,z)f(z)\,dz\right| \le C_b\|f\|_\infty. \tag{37}$$

But, adding and subtracting $K(u,z)$, the kernel of the classical Riesz Transform or its adjoint accordingly to the case, estimate (37) will hold if and only if

$$\frac{1}{|B|}\left(\int_B |b(z)-b_B|\,dz\right)\left|\int_{4Q\setminus 2B} K(u,z)f(z)\,dz\right| \le C_b\|f\|_\infty. \tag{38}$$

In fact, by using (17) or (20), it is easy to check that $\int_{4Q}|G(u,z)-K(u,z)|\,dz$ is bounded independently of the critical ball Q, more precisely

$$\int_{4Q}\frac{1}{|u-z|^d}\left(\frac{|u-z|}{\rho(u)}\right)^\sigma dz \lesssim \rho(x_0)^{-\sigma}\int_{4Q}\frac{1}{|x_0-z|^{d-\sigma}}\,dz \lesssim 1.$$

Due to the self-improvement of the reverse-Hölder inequality, we may assume $V \in RH_q$ for $d/2 < q < d$. Setting $1/s = 1/d + 1/q'$, we have

$$\begin{aligned}
&\int_{4Q}\frac{1}{|u-z|^{d-1}}\int_{B(z,|u-z|/4)}\frac{V(w)}{|w-z|^{d-1}}\,dw\,dz\\
&\quad\lesssim \left(\int_{4Q}\frac{dz}{|z-x_0|^{s(d-1)}}\right)^{1/s}\|I_1(V\chi_{4Q})\|_{s'}\\
&\quad\lesssim \rho(x_0)^{1-d/s'}\|V\chi_{4Q}\|_q \lesssim 1,
\end{aligned}$$

where in the last inequality we have used (24) for $k=2$.

Note that up to this point we only have used $b \in BMO_\infty(\rho)$.

Now, if we assume that b satisfies the stronger condition $b \in BMO_\infty^{\log}(\rho)$, since

$$\left|\int_{4Q\setminus 2B} K(u,z)f(z)\,dz\right| \le C_b\|f\|_\infty \log\bigl(\rho(x_0)/r\bigr), \tag{39}$$

we conclude that (38) holds proving the boundedness of T_b.

On the other hand if we suppose that T_b is bounded with $b \in BMO_\infty(\rho)$, then (38) must hold for each component K_i, $i = 1,\dots,d$, of K and for any f in L^∞. Choosing $f = \operatorname{sg}(u_i - z_i)$, and adding over i, inequality (38) implies

$$\frac{1}{|B|}\int_B |b(z)-b_B|\,dz\int_{4Q\setminus 2B}\frac{\sum_{i=1}^d|z_i-u_i|}{|z-u|^{d+1}}\,dz \le C_b$$

since $|z-u| \simeq |z-x_0|$, performing the integration, the inequality

$$\frac{1}{|B|}\int_B |b(z)-b_B|\,dz \le \frac{C_b}{1+\log(\rho(x_0)/r)},$$

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must hold for any $B \in \mathcal{B}_{\rho,1}$. Since we assume that $b \in BMO_\infty(\rho)$, we conclude that $b \in BMO_\infty^{\log}(\rho)$. □

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