

# Boundedness of the fractional maximal operator on variable exponent Lebesgue spaces: a short proof

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## Abstract

We give a simple proof of the boundedness of the fractional maximal operator providing in this way an alternative approach to the one given by C. Capone, D. Cruz Uribe and A. Fiorenza in [CCUF].

## 1 Introduction and main results

Given an open set  $\Omega \subset \mathbb{R}^n$  and a measurable function  $p : \Omega \rightarrow [1, +\infty)$ , the variable exponent Lebesgue space  $L^{p(\cdot)}$  consists of all measurable functions  $f$  on  $\Omega$  such that for some  $\lambda > 0$ ,  $\int_{\Omega} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty$  equipped with the norm

$$\|f\|_{p(\cdot),\Omega} = \inf\{\lambda > 0 : \int_{\Omega} (|f(x)|/\lambda)^{p(x)} dx \leq 1\}.$$

Given  $\alpha$ ,  $0 < \alpha < n$ , the fractional maximal operator  $M_{\alpha}$  is defined by

$$M_{\alpha}f(x) = \sup_{B \ni x} \frac{1}{|B|^{1-\alpha/n}} \int_{B \cap \Omega} |f(y)| dy,$$

where the supremum is taken over all balls  $B$  which contain  $x$ . When  $\alpha = 0$ ,  $M_0 = M$  is the Hardy-Littlewood maximal operator.

In [CCUF], C. Capone, D. Cruz-Uribe and A. Fiorenza gave an extension of the classical  $L^p - L^q$  boundedness result for  $M_{\alpha}$  for  $1 < p < n/\alpha$  and  $1/q = 1/p - \alpha/n$  on the variable Lebesgue context. The authors proved the following two interesting pointwise inequalities relating both operators  $M$  and  $M_{\alpha}$  which turn out to be crucial in proving the boundedness result. Nevertheless, their proofs are not trivial at all and they require additional lemmas. Moreover the hypotheses of log-Hölder continuity on the exponent is needed.

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(1.1) **Proposition:** ([CCUF]) Given an open set  $\Omega \subset \mathbb{R}^n$  and  $0 < \alpha < n$ , let  $p : \Omega \rightarrow [1, \infty)$  be such that  $1 < \inf_{x \in \Omega} p(x) \leq p(x) \leq \sup_{x \in \Omega} p(x) < n/\alpha$  and such that

$$(1.2) \quad |p(x) - p(y)| \leq \frac{C}{-\log|x-y|}, \quad x, y \in \Omega, \quad |x-y| < 1/2.$$

Let  $q$  be such that  $1/q(x) = 1/p(x) - \alpha/n$ . Then, for all  $f \in L^{p(\cdot)}(\Omega)$  such that  $\|f\|_{p(\cdot), \Omega} \leq 1$  and such that  $f(x) \geq 1$  or  $f(x) = 0$ ,  $x \in \Omega$ ,

$$(1.3) \quad M_\alpha f(x) \leq CM f(x)^{p(x)/q(x)}.$$

(1.4) **Proposition:** ([CCUF]) Given an open set  $\Omega \subset \mathbb{R}^n$  and  $0 < \alpha < n$ , let  $p : \Omega \rightarrow [1, \infty)$  be such that  $1 < \inf_{x \in \Omega} p(x) \leq p(x) \leq \sup_{x \in \Omega} p(x) < n/\alpha$  and such that

$$(1.5) \quad |p(x) - p(y)| \leq \frac{C}{\log(e+|x|)}, \quad x, y \in \Omega, \quad |y| \geq |x|.$$

Let  $q$  be such that  $1/q(x) = 1/p(x) - \alpha/n$ . Then, for all  $f \in L^{p(\cdot)}(\Omega)$  such that  $\|f\|_{p(\cdot), \Omega} \leq 1$  and such that  $0 \leq f(x) < 1$ ,  $x \in \Omega$ ,

$$M_\alpha f(x) \leq CM f(x)^{p(x)/I_q(x)},$$

where  $I_q(x) = \sup_{|y| \geq |x|} q(y)$

Thus to obtain the boundedness result the authors have to split the function  $f$  into  $f_1$  and  $f_2$  properly and then apply the propositions above to  $M_\alpha(f_1)$  and  $M_\alpha(f_2)$  respectively. Then the final result follows by applying the continuity of  $M$  proved in [CUFN].

When  $p$  is a constant exponent, an inequality in the spirit of (1.3) was proved in [H].

The following elementary lemma is a successful substitute of the propositions above. It should be also noticed that no conditions of continuity on the exponent  $p$  are required.

(1.6) **Lemma:** Let  $0 < \alpha < n$  and  $p$  be a function such that  $1 < \inf_{x \in \Omega} p(x) \leq p(x) \leq \sup_{x \in \Omega} p(x) < n/\alpha$ . Let  $q$  be defined by  $1/q(x) = 1/p(x) - \alpha/n$ . Then the following inequality

$$M_\alpha(f)(x) \leq \left( M \left( |f|^{\frac{p(\cdot)}{q(\cdot)} \frac{n}{n-\alpha}} \right) (x) \right)^{1-\alpha/n} \left( \int_\Omega |f(y)|^{p(y)} dy \right)^{\alpha/n}$$

holds for every function  $f$ .

*Proof :* Let  $Q \subset \mathbb{R}^n$  be a cube containing  $x$ . Taking into account that  $p(y)/q(y) + \alpha p(y)/n = 1$ , by applying Hölder's inequality we get

$$\begin{aligned} \frac{1}{\mu(Q)^{1-\alpha/n}} \int_{Q \cap \Omega} |f(y)| dy &= \frac{1}{\mu(Q)^{1-\alpha/n}} \int_{Q \cap \Omega} |f(y)|^{p(y)/q(y)} |f(y)|^{\alpha p(y)/n} dy \\ &\leq \left( \frac{1}{|Q|} \int_{Q \cap \Omega} |f(y)|^{\frac{p(y)}{q(y)} \frac{n}{n-\alpha}} dy \right)^{1-\alpha/n} \left( \int_\Omega |f(y)|^{p(y)} dy \right)^{\alpha/n} \\ &\leq \left( M \left( |f|^{\frac{p(\cdot)}{q(\cdot)} \frac{n}{n-\alpha}} \right) (x) \right)^{1-\alpha/n} \left( \int_\Omega |f(y)|^{p(y)} dy \right)^{\alpha/n}. \end{aligned}$$

Thus the desired inequality follows immediately.  $\square$

A straightforward application of lemma 1.6 allows us to obtain the boundedness of  $M_\alpha$  in the variable context. In fact, since  $f \in L^{p(x)}$  implies that  $|f|^{\frac{p(x)}{q(x)} \frac{n}{n-\alpha}} \in L^{q(x)(1-\alpha/n)}$  the result follows from the boundedness of  $M$ . Particularly if  $p$  satisfies (1.2) and (1.5) this was proved in [CUFN].

(1.7) **Remark:** A weighted pointwise inequality in a more general context was proved in [GPS]. In that paper the authors took advantage of this result to obtain weighted results for the boundedness of the fractional maximal operator in the variable context with a non-necessary doubling measure.

## References

- [CCUF] Capone, C., Cruz Uribe, D. and Fiorenza, A.: *The fractional maximal operator and fractional integrals on variable  $L^p$  spaces*, Rev. Mat. Iberoam. 23 (2007), no. 3, 743–770.
- [CUFN] Cruz Uribe, D., Fiorenza, A. and Neugebauer C.J.: *The maximal function on variable  $L^p$  spaces* Ann. Acad. Sci. Fenn 28 (2003), 223-238.
- [GPS] Gorosito, O., Pradolini, G. and Salinas, O.: *Boundedness of fractional operators in weighted variable exponent spaces with non doubling measures*, preprint.
- [H] Hedberg, L.: *On Certain Convolution Inequalities*, Proc. of Amer. Math. Soc., Vol. 36, No. 2 (1972), 505-510.