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Completions in Subvarieties of BL-algebras

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In the present paper we extend the results of [4] by completely characterizing dual canonical subvarieties of BL-algebras. These are subvarieties of algebras that satisfy the equation $x^k = x^{k+1}$ for some integer $k \geq 1$. As a corollary we get a full description of subvarieties of BL-algebras that admit completions.

Keywords: Completions, BL-algebras, Canonical Extensions, Distributive lattices, Basic logic

1 INTRODUCTION

A lattice-based algebra **A** is called *complete* if its lattice reduct is complete, i.e., the join and the meet of any subset of the universe A of **A** exist. A variety $\mathcal{V}$ of lattice-based algebras is said to *admit completions* if for every algebra $\mathbf{A} \in \mathcal{V}$ there is a complete algebra $\mathbf{B} \in \mathcal{V}$ and an embedding $\phi : \mathbf{A} \to \mathbf{B}$. Completions of algebras have proved to be powerful tools to deal with completeness theorems in logic.

Basic Fuzzy Logic and their algebraic counterparts, BL-algebras, are presented by Hájek in [10]. BL-algebras have different equivalent definitions: they are commutative integral bounded residuated lattices (see [7]) satisfying prelinearity and divisibility, and they are bounded basic hoops (see [1]). They form a variety of lattice-based algebras, thus it is natural to study which subvarieties of BL-algebras admit completions.

Different methods to complete lattice-based algebras have been developed. One of the most investigated is the method of canonical extensions (see [9]). If A is a distributive lattice, the canonical extension A^σ (unique up to isomorphism) of A is a doubly algebraic distributive lattice that contains A as a separating and compact sublattice. Since a BL-algebra $\mathbf{A}$ is a bounded distributive lattice with two additional operations $\cdot$ and $\rightarrow$, to obtain the canonical extension of $\mathbf{A}$ we extend the base lattice A to the complete lattice A^σ and we need also to extend the extra operations $\cdot$ and $\rightarrow$ to A^σ. There are two natural ways to extend an operation f: one is the canonical extension f^σ and the other is the dual canonical extension f^π (see [8]). Then there are two possible candidates for the canonical extension of a BL-algebra $\mathbf{A}$, namely the canonical extension $\mathbf{A}^\sigma$ and the dual canonical extension $\mathbf{A}^\pi$. A class of algebras is called *canonical* or *dual canonical* if it is closed under canonical or dual canonical extensions respectively. In [4], we studied canonical extensions of subvarieties of BL-algebras and we proved that a subvariety of BL-algebras is canonical if and only if it is finitely generated. We also proved that there are some non-finitely generated subvarieties of BL-algebras that are dual canonical. Thus, in the case of BL-algebras, dual canonical extensions seem to be more interesting than canonical extensions. On the other hand, a crucial result from [11] shows the existence of BL-algebras that can not be embedded into any complete BL-algebra, hence there are subvarieties of BL-algebras that do not admit any completion.

Combining the results of [11] and [4], we characterize those subvarieties of BL-algebras that admit completions. Our main result is:

Main Theorem: Let $\mathcal{V}$ be a subvariety of BL-algebras. Then the following statements are equivalent:

(i) The equation $x^k = x^{k+1}$ is satisfied in $\mathcal{V}$ for some integer $k \geq 1$.

(ii) $\mathcal{V}$ is dual canonical.

(iii) $\mathcal{V}$ admits completions.

In the second section, we recall the definition of BL-algebra and we collect some results about BL-algebras needed to achieve our aim. We also introduce k-subvarieties and we offer a classification theorem for subvarieties of BL-algebras, which is crucial in the proof of the Main Theorem. In the third section, we study dual canonicity of k-subvarieties. Using the results of these two sections, we prove the Main Theorem in the last section.

2 CLASSIFICATION OF SUBVARIETIES OF BL-ALGEBRAS

A *hoop* is an algebra $\mathbf{A} = \langle A, \cdot, \rightarrow, 1\rangle$ of type $(2, 2, 0)$, such that $\langle A, \cdot, 1\rangle$ is a commutative monoid and for all $x, y, z \in A$:

1. $x \rightarrow x = 1$,

2. $x \cdot (x \to y) = y \cdot (y \to x)$,
3. $x \to (y \to z) = (x \cdot y) \to z$.

A natural order $\leq$ can be defined in any hoop by $x \leq y$ iff $x \to y = 1$. A hoop $\mathbf{A}$ is called *basic* if it is a subdirect product of totally ordered hoops. Therefore, the natural order in a basic hoop is a distributive lattice. More information about hoops can be found in [3].

A *BL-algebra* is a bounded basic hoop, i.e., an algebra $\mathbf{A} = \langle A, \cdot, \to, 0, 1\rangle$ of type $(2, 2, 0, 0)$ such that $\langle A, \cdot, \to, 1\rangle$ is a basic hoop, where 0 and 1 are the lower and upper bounds of the natural order of $\mathbf{A}$. The BL-algebra $\mathbf{A}$ with only one element, that is $0 = 1$, is called the *trivial* BL-algebra.

Among the subvarieties of BL-algebras we highlight: the subvariety $\mathcal{MV}$, of MV-algebras (see [5]), the subvariety $\mathcal{G}$ of Gödel algebras (or prelinear Heyting algebras), and the subvariety $\mathcal{PL}$ of product algebras (see [6]).

When the natural order of a basic hoop (BL-algebra) $\mathbf{A}$ is total, $\mathbf{A}$ is called a *totally ordered hoop* (*BL-chain*). Every subvariety $\mathcal{V}$ of BL-algebras is generated by a family of BL-chains (see [10]). In particular, the set of BL-chains in a subvariety of BL-algebras is a generating set. This is why in the next section we shall start our study of completions in subvarieties of BL-algebras by investigating completions of the BL-chains in the variety.

Let $(I, \leq)$ be a totally ordered set with lower bound $\perp$. For each $i \in I$ let $\mathbf{A}_i = \langle A_i, \cdot_i, \to_i, 1\rangle$ be a totally ordered hoop such that for every $i \neq j$, $A_i \cap A_j = \{1\}$. Then we can define the *ordinal sum* as the hoop $\bigoplus_{i\in I} \mathbf{A}_i = \langle \cup_{i\in I} A_i, \cdot, \to, 1\rangle$ where the operations $\cdot, \to$ are given by:

$$x \cdot y = \begin{cases} x \cdot_i y & \text{if } x, y \in A_i, \\ x & \text{if } x \in A_i \setminus \{1\}, y \in A_j \text{ and } i < j, \\ y & \text{if } y \in A_i \setminus \{1\}, x \in A_j \text{ and } i < j. \end{cases}$$

$$x \to y = \begin{cases} 1 & \text{if } x \in A_i \setminus \{1\}, y \in A_j \text{ and } i < j, \\ x \to_i y & \text{if } x, y \in A_i, \\ y & \text{if } y \in A_i, x \in A_j \text{ and } i < j. \end{cases}$$

If in addition $\mathbf{A}_\perp$ is a BL-chain, then $\bigoplus_{i\in I} \mathbf{A}_i$ is a BL-chain, whose lower bound is the lower bound of $\mathbf{A}_\perp$. For simplicity we denote by $\mathbf{A}_1 \oplus \mathbf{A}_2$ the ordinal sum of two hoops, where the order of the index set is $1 < 2$.

The natural order in an ordinal sum $\bigoplus_{i\in I} \mathbf{A}_i$ is given by:

$$x \leq y \qquad \text{iff} \qquad \begin{cases} x, y \in A_i \text{ and } x \leq_i y, \text{ or} \\ x \in A_i \setminus \{1\}, y \in A_j \text{ and } i < j. \end{cases}$$

A basic hoop is called *sum irreducible* if it is totally ordered and it cannot be written as the ordinal sum of two non-trivial totally ordered hoops. The following result is crucial for our investigations:

Theorem 2.1. *(see [1]) Each non-trivial BL-chain admits a unique decomposition as an ordinal sum of non-trivial sum irreducible hoops.*

In [1], it is proved that sum irreducible hoops are equivalent to totally ordered Wajsberg hoops, i.e., totally ordered hoops that satisfy the equation

$$(x \to y) \to y = (y \to x) \to x. \tag{1}$$

Bounded Wajsberg hoops are reducts of MV-algebras, in the following sense: if $\mathbf{W} = \langle W, \cdot, \to, 1\rangle$ is a Wajsberg hoop and 0 is the lower bound of W, then $\mathbf{W} = \langle W, \cdot, \to, 0, 1\rangle$ is the corresponding MV-algebra. Unbounded Wajsberg hoops coincide with cancellative totally ordered hoops, i.e., totally ordered hoops whose based monoids are cancellative (see [3, Section 1]). Summarizing, if $\mathbf{A}$ is a BL-chain and $\mathbf{A} = \bigoplus_{i\in I} \mathbf{W}_i$ is its decomposition into sum irreducible hoops, then $\mathbf{W}_\perp$ is an MV-chain and for each $i \in I \setminus \{\perp\}$, $\mathbf{W}_i$ is either a reduct of an MV-chain or a cancellative Wajsberg hoop.

For $n \geq 2$, let

$$L_n = \{\frac{0}{n-1}, \frac{1}{n-1}, \dots \frac{n-1}{n-1}\}.$$

The algebra $\mathbf{L}_n = \langle L_n, \cdot, \to, 0, 1\rangle$ with $x \cdot y = \max(0, x+y-1)$ and $x \to y = \min(1, 1-x+y)$, is the unique (up to isomorphism) n-element MV-chain (see [5, Corollary 3.5.4]). From our previous observations, each finite totally ordered Wasjberg hoop is isomorphic to the hoop reduct of $\mathbf{L}_n$ for some $n \geq 2$. With an abuse of notation we shall denote by $\mathbf{L}_n$ the MV-chain as well as its hoop reduct.

We collect some results about BL-algebras that will be needed throughout the paper.

Lemma 2.2. *(see [1]) Let $\mathbf{A}$ be a BL-chain and let $\mathbf{A} = \bigoplus_{i\in I} \mathbf{W}_i$ be its decomposition into sum irreducible hoops given by Theorem 2.1. The subhoops of $\mathbf{A}$ are totally ordered hoops of the form $\mathbf{B} = \bigoplus_{i\in J} \mathbf{V}_i$, such that $J \subseteq I$ and for each $i \in J$, $\mathbf{V}_i$ is a subhoop of $\mathbf{W}_i$. The subalgebras of $\mathbf{A}$ are obtained similarly, but we must require that $\perp \in J$ and that $\mathbf{V}_\perp$ is a subalgebra of $\mathbf{W}_\perp$.*

Theorem 2.3. *(see [4]) Let $\mathcal{V}$ be a variety of BL-algebras. If there exists an infinite set S of natural numbers such that*

$$\{\mathbf{L}_2 \bigoplus \mathbf{L}_t : t \in S\} \subseteq \mathcal{V},$$

then there exists an infinite totally ordered Wajsberg hoop $\mathbf{W}$, *such that* $\mathbf{L}_2 \oplus \mathbf{W} \in \mathcal{V}$.

Theorem 2.4. *(see [4]) Let* $\mathbf{W}$ *be an infinite totally ordered Wajsberg hoop and let* $\mathbf{A} = \mathbf{L}_2 \oplus \mathbf{W}$ *be a BL-chain. Then the variety of BL-algebras generated by* $\mathbf{A}$ *contains the variety* $\mathcal{PL}$ *of product algebras.*

Given a natural number $k \geq 1$, a subvariety $\mathcal{V}$ of BL-algebras is called a *k-subvariety* if every algebra $\mathbf{A} \in \mathcal{V}$ is *k-potent*, i.e., it satisfies the equation:

$$x^k = x^{k+1}, \tag{2}$$

where $x^1 = x$ and for every k, $x^{k+1} = x^k \cdot x$. Observe that the MV-chain (hoop) $\mathbf{L}_n$ satisfies (2) for every $n \leq k+1$. Moreover, a BL-chain $\mathbf{A}$ satisfies equation (2) if and only if $\mathbf{A} \cong \bigoplus_{i \in I} \mathbf{L}_{r_i}$ where $2 \leq r_i \leq k+1$ for each $i \in I$. We will call these BL-chains *k-chains*.

Theorem 2.5. *Let* $\mathcal{V}$ *be a subvariety of BL-algebras that it is not a k-subvariety for any* $k = 1, 2, \ldots$. *Hence there is a BL-chain* $\mathbf{B} \in \mathcal{V}$ *such that if* $\mathbf{B} = \bigoplus_{i \in I} \mathbf{W}_i$ *is the unique decomposition of* $\mathbf{B}$ *into sum irreducible hoops, then for some* $i \in I$, $\mathbf{W}_i$ *is an infinite totally ordered Wajsberg hoop.*

Proof. Assume on the contrary, that every BL-chain $\mathbf{B} \in \mathcal{V}$ can be uniquely decomposed as $\mathbf{B} = \bigoplus_{i \in I} \mathbf{L}_{r_i}$ for some lower bounded totally ordered set I and natural numbers $r_i \geq 2$, $i \in I$. Since $\mathcal{V}$ is not a k-subvariety and $\mathcal{V}$ is generated by its chains, for every $k = 1, 2, \ldots$, there are a BL-chain $\mathbf{B}_k \in \mathcal{V}$ and a natural number $r_k > k+1$ such that $\mathbf{L}_{r_k}$ is a subhoop of $\mathbf{B}_k$. Recalling that $\mathbf{L}_2$ is a subalgebra of $\mathbf{L}_n$ for every integer $n \geq 2$, from Lemma 2.2, we get that for each $k = 1, 2, \ldots$, there is $r_k > k+1$ such that either $\mathbf{L}_{r_k}$ is an MV-algebra in $\mathcal{V}$ or $\mathbf{L}_2 \oplus \mathbf{L}_{r_k}$ is a BL-chain in $\mathcal{V}$. Suppose first that

$$T = \{k \in \mathbb{N} \mid \mathbf{L}_{r_k} \text{ is an MV-algebra in } \mathcal{V}\}$$

is infinite. Then there is an infinite set of non-isomorphic finite MV-chains contained in $\mathcal{V}$. In [5, Proposition 8.1.2], it is proved that any infinite set of non-isomorphic finite MV-chains generates the variety $\mathcal{MV}$. Therefore $\mathcal{MV} \subseteq \mathcal{V}$ implies the existence of an infinite MV-chain in $\mathcal{V}$. Otherwise, if T is finite, then the set

$$S = \{k \in \mathbb{N} \mid \mathbf{L}_2 \oplus \mathbf{L}_{r_k} \in \mathcal{V}\}$$

is infinite. Because of Theorem 2.3, there is an infinite totally ordered Wajsberg hoop $\mathbf{W}$ such that $\mathbf{L}_2 \oplus \mathbf{W} \in \mathcal{V}$, also contradicting our original assumption. □

We present a classification of subvarieties of BL-algebras that will be necessary to give a complete description of subvarieties that admits completions.

Theorem 2.6. *Let $\mathcal{V}$ be a subvariety of BL-algebras. Then one and only one of the following happens:*

1. *There is $k \geq 1$ such that $\mathcal{V}$ is a k-subvariety.*
2. *$\mathcal{PL} \subseteq \mathcal{V}$ or there is a non-finitely generated subvariety $\mathcal{S}$ of MV-algebras such that $\mathcal{S} \subseteq \mathcal{V}$.*

Proof. Assume that $\mathcal{V}$ is a k-subvariety. Since no infinite MV-chain satisfies (2), $\mathcal{V}$ does not contain a non-finitely generated subvariety of $\mathcal{MV}$. The standard product algebra $\mathbf{P}$ defined over the real unit interval $[0, 1]$ (where $\cdot$ is taken as the usual product, see [10]) does not satisfy equation (2) for any k. Therefore $\mathcal{PL} \nsubseteq \mathcal{V}$.

Assume now that $\mathcal{V}$ is not a k-subvariety. From Theorem 2.5, there is a BL-chain $\mathbf{B} \in \mathcal{V}$ such that in its decomposition as ordinal sum of sum irreducible hoops, there is an infinite totally ordered Wajsberg hoop $\mathbf{W}_i$. If $i = \perp$, then Lemma 2.2 yields that $\mathbf{W}_i$ is an infinite MV-chain in $\mathcal{V}$. The subvariety of MV-algebras generated by $\mathbf{W}_i$ is a non-finitely generated subvariety of MV-algebras contained in $\mathcal{V}$. Otherwise, from Lemma 2.2, $\mathbf{L}_2 \oplus \mathbf{W}_i$ is a subalgebra on $\mathbf{B}$. From Theorem 2.4 we get $\mathcal{PL} \subseteq \mathcal{V}$. □

3 COMPLETIONS OF SUBVARIETIES OF BL-ALGEBRAS

The family of subvarieties that admit completions is constraint by the next result:

Theorem 3.1. *([11]) Let $\mathcal{V}$ be a subvariety of BL-algebras. If $\mathcal{PL} \subseteq \mathcal{V}$ or $\mathcal{V}$ contains a non-finitely generated subvariety of $\mathcal{MV}$, then $\mathcal{V}$ does not admit completions.*

From Theorem 2.6 it is left to study completions in k-subvarieties.

There are different methods to complete a lattice-based algebra. Most of them focus on completing the distributive lattice reduct first and extending the extra operations to the complete lattice then. We will concentrate on dual canonical extensions. The canonical extension of a distributive lattice L is a doubly algebraic distributive lattice L^σ containing L as a separating compact sublattice, i.e., satisfying:

Separation: If p and q are complete join irreducible elements in L^σ, $p \neq q$ then there exists $a \in L$ such that $a \leq p$ and $a \neq q$.

Compactness: For every $X, Y \subseteq L$ with $\bigwedge X \leq \bigvee Y$, there exist finite sets $F \subseteq X$, $I \subseteq Y$ such that $\bigwedge F \leq \bigvee I$.

An easy application of the definition canonical extension yields:

Lemma 3.2. *Let I be a totally ordered set and let I^σ be its canonical completion. For each finite subchain $i_1 < i_2 < \ldots < i_n$ of I^σ, there is a finite subchain $j_1 < j_2 < \ldots < j_n$ of I such that if $i_k \in I$, then $j_k = i_k$.*

Proof. We denote $J^\infty(I^\sigma)$ the set of complete join irreducible elements of I^σ. If $i_k \in I^\sigma \setminus I$, then

$$i_k = \bigvee\{c \in J^\infty(I^\sigma) \mid c \leq i_k\}.$$

Since I^σ is a totally ordered set, there exists $c \in J^\infty(I^\sigma)$ such that $i_{k-1} < c \leq i_k < i_{k+1}$. Similarly, there exists $d \in J^\infty(I^\sigma)$ with $i_{k-1} < c \leq i_k < d \leq i_{k+1}$. Since I is a separating sublattice of I^σ, there is $j_k \in I$ such that $i_{k-1} < c < j_k \leq d < i_{k+1}$. If $i_k \in I$ take $j_k = i_k$. Now the sequence $j_1 < j_2 < \ldots < j_n$ of I satisfies the desired result. □

As mentioned in the introduction, if an algebra **A** has a distributive lattice reduct there are two standard ways to extend each operation in **A** to the complete lattice A^σ. They are called the *canonical extension* and the *dual canonical extension* (see [8] or [9]). Since we are only going to deal with dual canonical extensions, we briefly recall its definition. For details see [9].

Given a BL-algebra $\mathbf{A} = \langle A, \cdot, \rightarrow, 0, 1\rangle$, the dual canonical extension is an algebra $\mathbf{A}^\pi = \langle A^\sigma, \cdot^\pi, \rightarrow^\pi, 0, 1\rangle$ of the same type. To define the extensions $\cdot^\pi$ and $\rightarrow^\pi$ of the operations $\cdot$ and $\rightarrow$, let us first recall that the set of open elements of A^σ is given by

$$O(A^\sigma) = \{x \in A^\sigma : x = \bigvee (x \downarrow \cap A)\}$$

where $x \downarrow$ is the set of elements of A^σ less or equal than x. Similarly the set of closed elements of A^σ is given by

$$K(A^\sigma) = \{y \in A^\sigma : y = \bigwedge (y \uparrow \cap A)\},$$

where $x \uparrow$ is the set of elements of A^σ greater or equal than x. Then $\cdot^\pi$, $\rightarrow^\pi$ are defined as follows:

If $x, y \in O(A^\sigma)$ and $z \in K(A^\sigma)$, then

$$x \cdot^\pi y = \bigvee\{a \cdot b\colon a \leq x, b \leq y, a, b \in A\}$$

and

$$z \rightarrow^\pi x = \bigvee\{a \rightarrow b\colon z \leq a, b \leq x, a, b \in A\}$$

In any other case,

$$c \cdot^\pi d = \{x \cdot^\pi y\colon c \leq x, d \leq y, x, y \in O(A^\sigma)\}$$

and

$$e \to^{\pi} f = \bigwedge\{z \to^{\pi} x \colon z \leq e, f \leq x, z \in K(A^{\sigma}), x \in O(A^{\sigma})\}.$$

A subvariety $\mathcal{V}$ of BL-algebras is called *dual canonical* if $\mathbf{A}^{\pi} \in \mathcal{V}$ for every $\mathbf{A} \in \mathcal{V}$.

The following result, that can be deduced from [9, Thm 4.3], allows to extend dual canonicity of a family of algebras to the variety that they generate.

Theorem 3.3. *Let $\mathcal{S}$ be a set of BL-algebras closed under ultraproducts and dual canonical extensions. Then the subvariety of BL-algebras generated by $\mathcal{S}$ is dual canonical.*

Next we describe the dual canonical extension of a k-chain.

Lemma 3.4. *([4, Thm. 4.4]) Let $\mathbf{A} = \bigoplus_{i \in I} \mathbf{L}_{r_i}$ be a BL-chain. Then*

$$\mathbf{A}^{\pi} \cong \bigoplus_{i \in I^{\sigma}} \mathbf{D}_i,$$

where

$$\mathbf{D}_i = \begin{cases} \mathbf{L}_{r_i} & \text{if } i \in I \\ \mathbf{L}_2 & \text{if } i \notin I \end{cases}$$

As a immediate consequence of this lemma we get:

Corollary 3.5. *The dual canonical extension of a k-chain is a k-chain.*

Lemma 3.6. *Let $\mathbf{A}$ be a k-chain for some $k \geq 2$. Then every finite subalgebra of $\mathbf{A}^{\pi}$ is isomorphic to a finite subalgebra of $\mathbf{A}$.*

Proof. Let $\mathbf{A} = \bigoplus_{i \in I} \mathbf{L}_{r_i}$ be the decomposition of $\mathbf{A}$ as ordinal sum of Wasjberg hoops. According to Lemmas 3.4 and 2.2, if $\mathbf{B}$ is a finite subalgebra of $\mathbf{A}^{\pi}$, there is a finite subset $J = \{\bot = j_0 < j_1 < \ldots < j_n\} \subseteq I^{\sigma}$, such that $\mathbf{B} = \bigoplus_{j \in J} \mathbf{L}_{t_j}$ and for each $j \in J$,

$$\mathbf{L}_{t_j} = \begin{cases} \text{is a subalgebra of } \mathbf{L}_{r_j} & \text{if } j \in I, \\ \mathbf{L}_2 & \text{if } j \in I^{\sigma} \setminus I. \end{cases}$$

From Lemma 3.2, we get a finite subset $K = \{\bot = k_0 < k_1 < \ldots < k_n\} \subseteq I$ such that if $j_w \in I$ then $k_w = j_w$. For each $k \in K$ we take

$$\mathbf{L}_{s_k} = \begin{cases} \mathbf{L}_{t_j} & \text{if } k = j \text{ for some } j \in J \\ \mathbf{L}_2 & \text{if } k \neq j \text{ for any } j \in J. \end{cases}$$

Observe that $\bigoplus_{k\in K} \mathbf{L}_{s_k} \cong \bigoplus_{j\in J} \mathbf{L}_{t_j} = \mathbf{B}$. Since $K \subseteq I$ and for each $k \in K \setminus J$, $\mathbf{L}_2$ is a subalgebra of $\mathbf{L}_{r_k}$, we can conclude that $\bigoplus_{k\in K} \mathbf{L}_{s_k}$ is a subalgebra of $\mathbf{A}$, and the lemma is proved. □

Theorem 3.7. *For any k-chain* $\mathbf{A}$*,* $\mathbf{A}^\pi$ *is in the subvariety of BL-algebras generated by* $\mathbf{A}$.

Proof. Assume that τ is an equation in the language of BL-algebras which is not satisfied by $\mathbf{A}^\pi$. Therefore τ is not satisfied by a finite subalgebra $\mathbf{B}$ of $\mathbf{A}^\pi$. By the previous Lemma there is a finite subalgebra $\mathbf{B}'$ of $\mathbf{A}$ isomorphic to $\mathbf{B}$ and clearly τ is not satisfied in $\mathbf{B}'$. Now, since τ is not satisfied by a finite subalgebra of $\mathbf{A}$, then τ is not satisfied by $\mathbf{A}$. We conclude that every equation satisfied by $\mathbf{A}$ is satisfied by $\mathbf{A}^\pi$ and the result of the theorem follows. □

Now we are ready to prove the main result of this section.

Theorem 3.8. *Let* $\mathcal{V}$ *be a k-subvariety. Then* $\mathcal{V}$ *is dual canonical.*

Proof. We denote by $\mathcal{S}$ the set of chains in $\mathcal{V}$. Clearly $\mathcal{V}$ is generated by $\mathcal{S}$. Since $\mathcal{V}$ is a k-subvariety, each element of $\mathcal{S}$ is a k-chain. By Theorem 3.7, $\mathcal{S}$ is closed under dual canonical extensions. An ultraproduct of BL-chains in $\mathcal{V}$ is a BL-chain in $\mathcal{V}$, hence $\mathcal{S}$ is closed under ultraproducts. Now an application of Theorem 3.3 proves that $\mathcal{V}$ is dual canonical. □

4 CONCLUSION

We have developed the tools to prove:

Main Theorem Let $\mathcal{V}$ be a subvariety of BL-algebras. Then the following statements are equivalent:

(i) The equation $x^k = x^{k+1}$ is satisfied in $\mathcal{V}$ for some integer $k \geq 1$.

(ii) $\mathcal{V}$ is dual canonical.

(iii) $\mathcal{V}$ admits completions.

Proof. (i)⇒(ii) Follows from Theorem 3.8. (ii)⇒(iii) Straightforward. (iii)⇒(i) It is a direct consequence of Theorem 3.1 and Theorem 2.6. This can also be deduced from the results in [2] (for more details see the Acknowledgments). □

The Main Theorem comes to close two circles of ideas: one started in [4], where we characterized subvarieties of BL-algebras that are canonical, and we observed that the class of subvarieties that are dual canonical is strictly bigger. Now we can offer a complete characterization of subvarieties of BL-algebras that are dual canonical. The second is that, based on the important results of [11], we provide a complete algebraic description of the subvarieties of BL-algebras that admit completions.

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