REGULARITY OF THE SCHRÖDINGER EQUATION FOR THE HARMONIC OSCILLATOR

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Abstract. We consider the Schrödinger equation for the harmonic oscillator $i\partial_t u = Hu$, where $H = -\Delta + |x|^2$, with initial data in the Hermite–Sobolev space $H^{-s/2}L^2(\mathbb{R}^n)$. We prove smoothing and maximal estimates and apply these to almost everywhere convergence and initial value problems.

1. Introduction

The solution to the free Schrödinger equation $i\partial_t u = -\Delta u$, with initial datum $u(\,\cdot\,,0) = f$, has been studied extensively. In particular, the question of whether

$$\lim_{t\to 0} e^{it\Delta} f = f(x), \qquad \text{a.e. } x \in \mathbb{R}^n,$$

for all $f \in H^s(\mathbb{R}^n)$, has been considered. Here $H^s(\mathbb{R}^n)$ denotes the inhomogeneous Sobolev space $(I-\Delta)^{-s/2}L^2(\mathbb{R}^n)$.

In one spatial dimension, the almost everywhere convergence was first proven by Carleson [6] for data in $H^s(\mathbb{R})$ with $s > 1/4$, and Dahlberg and Kenig [9] proved that the convergence is not guaranteed when $s < 1/4$. In two spatial dimensions, the best known result is due to Lee [17] who proved the convergence for data in $H^s(\mathbb{R}^2)$ with $s > 3/8$.

In higher dimensions, the best known result is independently due to Sjölin [23] and Vega [28, 29] who proved the convergence for $H^s(\mathbb{R}^n)$ with $s > 1/2$, and this follows from their *local smoothing* estimate

$$\|e^{it\Delta} f\|_{L^2(B_R\times[0,1])} \leqslant C_R \|f\|_{H^{-1/2}(\mathbb{R}^n)},$$

where B_R denotes a ball of radius R. This was also proven independently by Constantin and Saut [7]. These questions have subsequently received a lot of attention (see for example [10, 22]).

We consider the regularity of the Schrödinger equation $i\partial_t u = Hu$ with initial data $u(\,\cdot\,,0) = f$, where H is the Hermite operator defined by

$$H = -\Delta + |x|^2, \quad x \in \mathbb{R}^n. \tag{1}$$

This is an important model in quantum mechanics (see for example [11]).

The trigonometric polynomials are the eigenfunctions of Δ, and this is what makes the Fourier transform such an effective tool to attack the free equation. Similarly, this enables us to measure the smoothness of the initial data with fractional power Sobolev spaces defined via the Fourier transform.

The Schrödinger equation for the harmonic oscillator has been considered with respect to these fractional Sobolev spaces (see for example [33]). However, the

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eigenfunctions of H are the Hermite functions which are also dense in $L^2(\mathbb{R}^n)$, and so it is more efficient to decompose the initial data with these. Similarly, it seems in some sense more natural to measure the 'smoothness' of the initial data in the Hermite–Sobolev space $\mathcal{H}^s(\mathbb{R}^n) = H^{-s/2}L^2(\mathbb{R}^n)$.

Although the spectrum of H is discrete, recalling the free equation with periodic data (see for example [4]), the results will generally bear more resemblance to those for the nonperiodic free equation. In particular, we will see that there is 'smoothing' which is unavailable in the periodic case.

We prove almost everywhere convergence of the solution to the initial data, as time tends to zero for certain data.

Theorem 1. *Let $f \in \mathcal{H}^s(\mathbb{R}^n)$ with $s > 1/2$ if $n \geqslant 2$, or $s > 1/3$ if $n = 1$. Then*

$$\lim_{t\to 0} e^{-itH} f(x) = f(x) \qquad a.e. \quad x \in \mathbb{R}^n.$$

Cowling [8] proved this convergence for data in $\mathcal{H}^s(\mathbb{R}^n)$ with $s > 1$. In one spatial dimension, this was improved by Torrea and the first author [2] (see [3] for a Laguerre version) to include data in $\mathcal{H}^s(\mathbb{R})$ with $s > 1/2$.

By a theorem of Thangavelu [26], $f \in H^s(\mathbb{R}^n)$ with compact support is also contained in $\mathcal{H}^s(\mathbb{R}^n)$, thus we recover the almost everywhere convergence result of Yajima [33].

We also prove the following theorem, which is sharp with respect to the Sobolev index s. Note that s is negative when $p < \frac{2n}{n-2}$.

Theorem 2. *Let $n \geqslant 2$ and $p \in [\frac{2(n+3)}{n+1}, \frac{2n}{n-2}]$. Then*

$$\|e^{-itH} f\|_{L^p_x(\mathbb{R}^n,\, L^2_t[0,2\pi])} \leqslant C_s \|f\|_{\mathcal{H}^s(\mathbb{R}^n)}, \qquad s = \frac{n}{3}\Big(\frac{1}{2} - \frac{1}{p}\Big) - \frac{1}{3}.$$

Finally, we note an application for the initial value problem for the Schrödinger equation with potential of the form

$$\begin{cases} i\frac{du}{dt} + \Delta u = |x|^2 u + V(x)\, u \\ u(\,\cdot\,,0) = u_0. \end{cases}$$

By combining Theorem 2 with arguments of Ruiz–Vega [21], existence of a solution is guaranteed when $n \geqslant 2$ and $\|V\|_{L^{n/2}}$ is sufficiently small. For $n \geqslant 3$ this can be also be obtained via the Strichartz estimates and the arguments of Yajima [32].

Throughout, c and C will denote positive constants that may depend on the dimension n. Their values may change from line to line.

2. Preliminaries

In one dimension, the Hermite polynomials H_k are defined by

$$H_k(x) = (-1)^k\, e^{x^2} \frac{d^k}{dx^k}(e^{-x^2}), \quad x \in \mathbb{R},$$

and the Hermite functions h_k are defined by the normalization

$$h_k(x) = \frac{e^{-x^2/2} H_k(x)}{(\pi^{1/2} 2^k k!)^{1/2}}, \quad x \in \mathbb{R}.$$

In higher dimensions, for each multi-index $\mathbf{k} = (k_j)_{j=1}^n \in \mathbb{N}_0^n$, the Hermite functions $h_{\mathbf{k}}$ are defined by

$$h_{\mathbf{k}}(x) = \prod_{j=1}^{n} h_{k_j}(x_j), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

These are the eigenvectors of the Hermite operator defined in (1). In fact

$$H h_{\mathbf{k}} = (2|\mathbf{k}| + n)\, h_{\mathbf{k}},$$

where $|\mathbf{k}| = \sum_{j=1}^n k_j$.

We consider the space of finite linear combinations of Hermite functions $\mathfrak{F}(\mathbb{R}^n)$,

$$f = \sum_{\mathbf{k}\in\mathbb{N}_0^n :\, |\mathbf{k}|\leqslant N} a_{\mathbf{k}} h_{\mathbf{k}},$$

where $a_{\mathbf{k}}$ are the Fourier–Hermite coefficients

$$a_{\mathbf{k}} = \int_{\mathbb{R}^n} f(x) h_{\mathbf{k}}(x) dx.$$

These are dense in $L^2(\mathbb{R}^n)$, and so, by the orthonormality of the Hermite functions,

$$\|f\|_{L^2(\mathbb{R}^n)} = \left(\sum_{\mathbf{k}\in\mathbb{N}_0^n} |a_{\mathbf{k}}|^2 \right)^{1/2}, \tag{2}$$

and the Hermite–Sobolev norm is defined accordingly,

$$\|f\|_{\mathcal{H}^s(\mathbb{R}^n)} = \left(\sum_{\mathbf{k}\in\mathbb{N}_0^n} (2|\mathbf{k}| + n)^s |a_{\mathbf{k}}|^2 \right)^{1/2}.$$

For initial data $f \in \mathfrak{F}(\mathbb{R}^n)$, the solution to the Schrödinger equation for the harmonic oscillator can be written

$$e^{-itH} f = \sum_{\mathbf{k}\in\mathbb{N}_0^n :\, |\mathbf{k}|\leqslant N} e^{-it(2|\mathbf{k}|+n)} a_{\mathbf{k}} h_{\mathbf{k}}. \tag{3}$$

Note that the solution is periodic in time. Comparing (2) with (3) we see that

$$\|e^{-itH} f\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}, \tag{4}$$

for all time t, and so we can extend the operator e^{-itH} so that it is defined on $L^2(\mathbb{R}^n)$.

Finally, for $f \in \mathfrak{F}(\mathbb{R}^n)$, by the Mehler formula we also have the integral representation

$$e^{-itH} f(x) = \int_{\mathbb{R}^n} K_t(x, y)\, f(y)\, dy, \quad t \neq \frac{j\pi}{2},\ j \in \mathbb{Z}, \tag{5}$$

where

$$K_t(x, y) = \frac{1}{[2\pi i \sin(2t)]^{n/2}} \exp\left(\frac{i}{2} |x - y|^2 \cot(2t) - i x \cdot y \tan(t) \right).$$

3. Smoothing estimates

For the free equation, Kenig, Ponce and Vega [15] proved the sharp estimate

$$\sup_{x\in\mathbb{R}} \|e^{it\Delta}f(x)\|_{L^2_t(\mathbb{R})} \leqslant C\|f\|_{\dot{H}^{-1/2}(\mathbb{R})},$$

where $\dot{H}^s(\mathbb{R}^n)$ denotes the homogeneous Sobolev space $(-\Delta)^{-s/2}L^2(\mathbb{R}^n)$. The estimate is false when the homogeneous space is replaced by the inhomogeneous one. For the harmonic oscillator, we prove something similar. Note that the spectrum of H is bounded away from the origin, so there is no distinction between the homogeneous and inhomogeneous Hermite–Sobolev spaces.

In order to get a global bound in space with no decay, in the following estimate we loose some regularity with respect to the free equation. The relationship between the decay and the regularity is sharp however. To see this, consider $f = h_k$, so that the inequality in the proof can be reversed.

Theorem 3. *Let $1/6 \leqslant s \leqslant 1/2$. Then*

$$\sup_{x\in\mathbb{R}} (1+|x|)^{1/6-s}\|e^{-itH}f(x)\|_{L^2_t[0,2\pi]} \leqslant C_s\|f\|_{\mathcal{H}^{-s}(\mathbb{R})}.$$

Proof. As $\mathfrak{F}(\mathbb{R}^n) = H^{s/2}\mathfrak{F}(\mathbb{R}^n)$ is dense in $\mathcal{H}^{-s}(\mathbb{R}^n)$, it will suffice to consider $f \in \mathfrak{F}(\mathbb{R}^n)$ and we write $f = \sum_{k\in\mathbb{N}_0} a_k h_k$. Observe that by the orthogonality of the trigonometric polynomials,

$$\begin{aligned}\|e^{-itH}f(x)\|^2_{L^2_t[0,2\pi]} &= \int_0^{2\pi}\left(\sum_{j\in\mathbb{N}_0} e^{-it(2j+1)}a_j h_j(x)\right)\left(\sum_{k\in\mathbb{N}_0} e^{it(2k+1)}\overline{a}_k h_k(x)\right)dt\\ &= 2\pi\sum_{k\in\mathbb{N}_0}|a_k|^2 h_k^2(x).\end{aligned}$$

We use the following property of the Hermite functions which can be found in [24, Theorem 8.91.3]:

Let $0 \leqslant \alpha \leqslant 1/3$. Then there exists constants c_0 and k_0 such that

$$c_0^{-1}k^{-\alpha/2-1/12} \leqslant \sup_{x\in\mathbb{R}}(1+|x|)^{-\alpha}h_k(x) \leqslant c_0 k^{-\alpha/2-1/12}, \quad k \geqslant k_0. \tag{6}$$

Thus, interchanging the sum and the supremum,

$$\sup_{x\in\mathbb{R}}(1+|x|)^{-2\alpha}\|e^{-itH}f(x)\|^2_{L^2_t[0,2\pi]} \leqslant c_0^2\sum_{k\in\mathbb{N}_0}\frac{(2k_0+1)^{\alpha+1/6}}{(2k+1)^{\alpha+1/6}}|a_k|^2.$$

Finally, by writing $s = \alpha + 1/6$, and taking the square root,

$$\sup_{x\in\mathbb{R}}(1+|x|)^{1/6-s}\|e^{-itH}f(x)\|_{L^2_t[0,2\pi]} \leqslant C_s\left(\sum_{k\in\mathbb{N}_0}(2k+1)^{-s}|a_k|^2\right)^{1/2},$$

as desired. □

For the free equation, Vega [28] (see also [13, 19, 21]) proved that for $n \geqslant 2$ and $p \geqslant \frac{2(n+1)}{n-1}$,

$$\|e^{it\Delta}f\|_{L^p_x(\mathbb{R}^n,\, L^2_t(\mathbb{R}))} \leqslant C_s\|f\|_{\dot{H}^s(\mathbb{R}^n)}, \quad s = n\Big(\frac{1}{2}-\frac{1}{p}\Big)-1.$$

Note that s is negative in the range $p \in [\frac{2(n+1)}{n-1}, \frac{2n}{n-2})$. In the following theorem, we again loose some regularity with respect to the free equation, however we will see that it is sharp.

Theorem 4. *Let $n \geqslant 2$ and $p \geqslant 2$. Then*

$$\|e^{-itH} f\|_{L^p_x(\mathbb{R}^n,\, L^2_t[0,2\pi])} \leqslant C_s \|f\|_{\mathcal{H}^s(\mathbb{R}^n)},$$

where

$$s = \begin{cases} \frac{1}{p} - \frac{1}{2}, & 2 \leqslant p \leqslant \frac{2(n+3)}{n+1} \\ \frac{n}{3}\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{3}, & \frac{2(n+3)}{n+1} \leqslant p \leqslant \frac{2n}{n-2} \\ n\left(\frac{1}{2} - \frac{1}{p}\right) - 1, & \frac{2n}{n-2} \leqslant p \leqslant \infty. \end{cases}$$

Proof. By density we can write $f = \sum_{\mathbf{k}\in\mathbb{N}_0^n} a_\mathbf{k} h_\mathbf{k}$. As before,

$$\begin{aligned} \|e^{-itH} f\|^2_{L^2_t[0,2\pi]} &= \int_0^{2\pi} \left(\sum_{\mathbf{j}\in\mathbb{N}_0^n} e^{-it(2|\mathbf{j}|+n)} a_\mathbf{j} h_\mathbf{j} \right) \left(\sum_{\mathbf{k}\in\mathbb{N}_0^n} e^{it(2|\mathbf{k}|+n)} \overline{a}_\mathbf{k} h_\mathbf{k} \right) dt \\ &= 2\pi \left(\sum_{\mathbf{j},\mathbf{k}:\ 2|\mathbf{k}|+n=2|\mathbf{j}|+n} a_\mathbf{j}\, \overline{a}_\mathbf{k} h_\mathbf{j}\, h_\mathbf{k} \right) \\ &= 2\pi \left(\sum_{\lambda\in\mathbb{N}} \sum_{\mathbf{j}\,:\,2|\mathbf{j}|+n=\lambda} \sum_{\mathbf{k}\,:\,2|\mathbf{k}|+n=\lambda} a_\mathbf{j}\, \overline{a}_\mathbf{k} h_\mathbf{j}\, h_\mathbf{k} \right). \end{aligned}$$

We recall the spectral projection operators P_λ defined by

$$P_\lambda f(x) = \sum_{2|\mathbf{k}|+n=\lambda} a_\mathbf{k} h_\mathbf{k}(x).$$

We see that

$$\|e^{-itH} f\|_{L^2_t[0,2\pi]} = (2\pi)^{1/2} \left(\sum_{\lambda\in\mathbb{N}} P_\lambda f \overline{P_\lambda f} \right)^{1/2} = (2\pi)^{1/2} \left(\sum_{\lambda\in\mathbb{N}} |P_\lambda f|^2 \right)^{1/2},$$

and by Minkowski's inequality in $L^{p/2}_x$,

$$\|e^{-itH} f\|_{L^p_x(\mathbb{R}^n,\, L^2_t[0,2\pi])} \leqslant (2\pi)^{1/2} \left(\sum_{\lambda\in\mathbb{N}} \|P_\lambda f\|^2_{L^p(\mathbb{R}^n)} \right)^{1/2}. \tag{7}$$

Now combining the results of Karadzhov [12], Thangavelu [25] and Koch–Tataru [16], we have the sharp projection estimates

$$\|P_\lambda f\|^2_{L^p(\mathbb{R}^n)} \leqslant C\lambda^s \|P_\lambda f\|^2_{L^2(\mathbb{R}^n)}, \tag{8}$$

where s is as in the statement of the theorem. By orthogonality,

$$\|P_\lambda f\|^2_{L^2(\mathbb{R}^n)} = \sum_{\mathbf{k}\,:\,2|\mathbf{k}|+n=\lambda} |a_\mathbf{k}|^2,$$

so that using (8) we see that

$$\sum_{\lambda\in\mathbb{N}} \|P_\lambda f\|^2_{L^p(\mathbb{R}^n)} \leqslant \sum_{\lambda\in\mathbb{N}} \lambda^s \|P_\lambda f\|^2_{L^2(\mathbb{R}^n)} = \|f\|^2_{\mathcal{H}^s(\mathbb{R}^n)}. \tag{9}$$

The argument is completed by substituting (9) into (7). □

To see that these estimates are sharp we observe that $|e^{-itH}P_\lambda f| = |P_\lambda f|$ so that $\|e^{-itH}P_\lambda f\|_{L^2_t[0,2\pi]} = (2\pi)^{1/2}|P_\lambda f|$. Thus, an improvement of the previous estimate would yield improved estimates for the spectral projection operator, which is not possible (see [16]).

For the free equation, Kenig, Ponce and Vega [15] proved that for all $\alpha > 1$,

$$\left(\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} |e^{it\Delta}f(x)|^2 \frac{dxdt}{(1+|x|)^\alpha}\right)^{1/2} \leqslant C_\alpha \|f\|_{\dot{H}^{-1/2}(\mathbb{R}^n)}.$$

On the other hand, considering the inhomogeneous Sobolev space with $n \geqslant 2$, Ben–Artzi–Klainerman [1] and Kato–Yajima [13] proved that for all $\alpha > 2$,

$$\left(\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} |e^{it\Delta}f(x)|^2 \frac{dxdt}{(1+|x|)^\alpha}\right)^{1/2} \leqslant C_\alpha \|f\|_{H^{-1/2}(\mathbb{R}^n)}, \tag{10}$$

and this is false when $\alpha < 2$ (see [31]). In an involved argument, Yajima [33] proved that if one integrates over a compact integral of time, then (10) holds for $\alpha > 1$ with Δ replaced by a class of operators that includes both Δ and H. Considering $\mathcal{H}^{-1/2}(\mathbb{R}^n)$ instead of $H^{-1/2}(\mathbb{R}^n)$ enables the following simple proof more in the spirit of [15].

Theorem 5. *For all* $\alpha > 1$,

$$\left(\int_0^{2\pi}\int_{\mathbb{R}^n} |e^{-itH}f(x)|^2 \frac{dxdt}{(1+|x|)^\alpha}\right)^{1/2} \leqslant C_\alpha \|f\|_{\mathcal{H}^{-1/2}(\mathbb{R}^n)}.$$

Proof. By density we can write $f = \sum_{\mathbf{k}\in\mathbb{N}_0^n} a_\mathbf{k} h_\mathbf{k}$. Observe that by the orthogonality of the trigonometric polynomials,

$$\begin{aligned}\|e^{-itH}f\|^2_{L^2_t[0,2\pi]} &= \int_0^{2\pi}\left(\sum_{\mathbf{j}\in\mathbb{N}_0^n} e^{-it(2|\mathbf{j}|+n)}a_\mathbf{j}h_\mathbf{j}\right)\left(\sum_{\mathbf{k}\in\mathbb{N}_0^n} e^{it(2|\mathbf{k}|+n)}\overline{a}_\mathbf{k}h_\mathbf{k}\right)dt\\ &= 2\pi\left(\sum_{\mathbf{j},\mathbf{k}\,:\,j_1=k_1+|\overline{\mathbf{k}}|-|\overline{\mathbf{j}}|} a_\mathbf{j}\,\overline{a}_\mathbf{k} h_{j_1}h_{k_1}h_{\overline{\mathbf{j}}}h_{\overline{\mathbf{k}}}\right),\end{aligned}$$

where $\overline{\mathbf{j}} = (j_2,\ldots,j_n)$ and $\overline{\mathbf{k}} = (k_2,\ldots,k_n)$. By Fubini's theorem,

$$\begin{aligned}&\int_0^{2\pi}\int_{[-R,R]\times\mathbb{R}^{n-1}} |e^{-itH}f(x)|^2dxdt\\ =&2\pi\left(\sum_{\mathbf{j},\mathbf{k}\,:\,j_1=k_1+|\overline{\mathbf{k}}|-|\overline{\mathbf{j}}|} a_\mathbf{j}\,\overline{a}_\mathbf{k}\int_{-R}^R h_{j_1}(x_1)h_{k_1}(x_1)dx_1\int_{\mathbb{R}^{n-1}} h_{\overline{\mathbf{j}}}(\overline{x})h_{\overline{\mathbf{k}}}(\overline{x})d\overline{x}\right),\end{aligned}$$

so that, by the orthonormality of the Hermite functions in $n-1$ variables,

$$\int_0^{2\pi}\int_{-R}^R\int_{\mathbb{R}^{n-1}} |e^{-itH}f(x)|^2dxdt = 2\pi\sum_\mathbf{k}|a_\mathbf{k}|^2\int_{-R}^R h^2_{k_1}(x_1)dx_1. \tag{11}$$

Of course, we can repeat the argument for each variable, and so for $i = 1,\ldots,n$,

$$\int_0^{2\pi}\int_{[-R,R]^n} |e^{-itH}f(x)|^2dxdt \leqslant 2\pi\sum_\mathbf{k}|a_\mathbf{k}|^2\int_{-R}^R h^2_{k_i}(x_i)dx_i.$$

Now another well known property of the Hermite functions (see [25]) is the following: There exists a constant c_0 such that

$$h_k(x) \leqslant c_0 k^{-1/4}, \quad x \in [-R, R], \quad k \geqslant R^2. \tag{12}$$

This yields

$$\int_{-R}^{R} h_{k_i}^2(x_i)\, dx_i \leqslant C\frac{R}{k_i^{1/2}}.$$

Note that the inequality is trivial when $k_i^{1/2} \leqslant R$. Substituting into (11), we see that

$$\int_0^{2\pi}\int_{[-R,R]^n} |e^{-itH} f(x)|^2 dxdt \leqslant CR\sum_{\mathbf{k}} (2k_i+1)^{-1/2}|a_{\mathbf{k}}|^2. \tag{13}$$

Now we can decompose our function $f = \sum_{i=1}^n f_i$, where $f_i = \sum_{\mathbf{k}} a_{\mathbf{k}}^i h_{\mathbf{k}}$ and

$$a_{\mathbf{k}}^i = \begin{cases} a_{\mathbf{k}}, & k_i \geqslant k_j \text{ for all } j \neq i, \text{ and } k_i \neq k_j \text{ for all } j < i \\ 0, & \text{otherwise.} \end{cases}$$

By (13), we see that for $i = 1, \ldots, n$,

$$\begin{aligned}\int_0^{2\pi}\int_{[-R,R]^n} |e^{-itH} f_i(x)|^2 dxdt &\leqslant CR\sum_{\mathbf{k}} (2k_i+1)^{-1/2}|a_{\mathbf{k}}^i|^2 \\ &\leqslant Cn^{1/2}R\sum_{\mathbf{k}} (2|\mathbf{k}|+n)^{-1/2}|a_{\mathbf{k}}^i|^2,\end{aligned}$$

where we have used the fact that $nk_i \geqslant |\mathbf{k}|$ when $a_{\mathbf{k}}^i \neq 0$. By Minkowski's inequality followed by Cauchy–Schwarz,

$$\left(\int_0^{2\pi}\int_{[-R,R]^n} |e^{-itH} f(x)|^2 dxdt\right)^{1/2} \leqslant Cn^{3/4}R^{1/2}\left(\sum_{\mathbf{k}} (2|\mathbf{k}|+n)^{-1/2}|a_{\mathbf{k}}|^2\right)^{1/2},$$

and the result follows by summing dyadic pieces. □

It would be interesting to know what the sharp powers of R and n should be in the final inequality of the previous proof, however we do not know. To see that it is sharp with respect to the regularity, we require the following lemma.

Lemma 1. *Let I_m denote intervals of length $\frac{1}{\sqrt{k}}$ centred at $x_m = \frac{\sqrt{2}\pi m}{\sqrt{k}}$. Then there exist positive constants c_0, k_0 and μ such that*

$$c_0^{-1}k^{-1/4} \leqslant h_{4k}(x) \leqslant c_0 k^{-1/4}$$

for all $k \geqslant k_0$ when $x \in I_m$ and $m = \lfloor\sqrt{k}/\mu\rfloor, \ldots, \lfloor 2\sqrt{k}/\mu\rfloor$.

Proof. For k an even integer, there is an explicit formula for the Hermite functions given by

$$h_k(x) = \frac{2}{\pi^{3/4}}(-1)^{k/2}\frac{2^{k/2}}{\sqrt{k!}}e^{\frac{x^2}{2}}\int_0^\infty e^{-s^2}s^k\cos(2xs)ds \tag{14}$$

(see [24]). Note that by the formula for the Gamma function and a change of variables,

$$\int_0^\infty e^{-s^2}s^k\cos(2xs)ds \leqslant \int_0^\infty e^{-s^2}s^k ds = \frac{1}{2}\Gamma\left(\frac{k+1}{2}\right).$$

We will see later that this bound will suffice to provide the upper bound, so we concentrate on the lower bound.

Consider an interval I_m of length $\frac{1}{\sqrt{k}}$ with centre $x_m = \frac{\sqrt{2}\pi m}{\sqrt{k}}$, where

$$m = \lfloor \sqrt{k}/\mu \rfloor, \ldots, \lfloor 2\sqrt{k}/\mu \rfloor,$$

with μ to be choosen later. We split the integral

$$\int_0^\infty e^{-s^2} s^k \cos(2xs)ds = \int_0^{\sqrt{\frac{k}{2}}(1-\frac{1}{8m})} + \int_{\sqrt{\frac{k}{2}}(1-\frac{1}{8m})}^{\sqrt{\frac{k}{2}}} + \int_{\sqrt{\frac{k}{2}}}^{\sqrt{\frac{k}{2}}(1+\frac{1}{8m})} + \int_{\sqrt{\frac{k}{2}}(1+\frac{1}{8m})}^{\infty}$$
$$=: I_1 + I_2 + I_3 + I_4.$$

The function $e^{-s^2}s^k$ attains its maximum when $s = \sqrt{k/2}$, and so is monotone in $(0, \sqrt{k/2}(1-\frac{1}{8m}))$. By the second mean value theorem for integrals,

$$|I_1| \leqslant e^{-\frac{k}{2}(1-\frac{1}{8m})^2} \left(\frac{k}{2}\right)^{\frac{k}{2}} \left(1 - \frac{1}{8m}\right)^k \frac{1}{x}.$$

Squaring out and using the fact that $m \leqslant \lfloor 2\sqrt{k}/\mu \rfloor$ and $1/x < \mu$, for sufficiently large k,

$$|I_1| \leqslant \mu e^{-\frac{\mu^2}{256}} e^{\frac{k}{8m}} \left(1 - \frac{1}{8m}\right)^k e^{-\frac{k}{2}} \left(\frac{k}{2}\right)^{\frac{k}{2}}.$$

On the other hand, $\cos(2xs)$ is positive on the interval $(\sqrt{k/2}(1-\frac{1}{8m}), \sqrt{k/2})$ for $x \in I_m$, and strictly greater than $\cos(3/2)$ on $(\sqrt{k/2}(1-\frac{1}{16m}), \sqrt{k/2})$, so that

$$I_2 \geqslant c \int_{\sqrt{\frac{k}{2}}(1-\frac{1}{16m})}^{\sqrt{\frac{k}{2}}} e^{-s^2} s^k ds.$$

Now, we are integrating over an interval of length $\geqslant c\mu$, so considering the smallest value of the integrand,

$$I_2 \geqslant c\mu e^{-\frac{k}{2}(1-\frac{1}{16m})^2} \left(\frac{k}{2}\right)^{\frac{k}{2}} \left(1 - \frac{1}{16m}\right)^k.$$

Squaring out as before and using the fact that $m \geqslant \lfloor \sqrt{k}/\mu \rfloor$, we have

$$I_2 \geqslant c\mu e^{-\frac{\mu^2}{512}} e^{\frac{k}{16m}} \left(1 - \frac{1}{16m}\right)^k e^{-\frac{k}{2}} \left(\frac{k}{2}\right)^{\frac{k}{2}}.$$

Now, $e^{kx}(1-x)^k$ is a decreasing function on $[0,1]$, so we can also write

$$I_2 \geqslant c\mu e^{-\frac{\mu^2}{512}} e^{\frac{k}{8m}} \left(1 - \frac{1}{8m}\right)^k e^{-\frac{k}{2}} \left(\frac{k}{2}\right)^{\frac{k}{2}}.$$

Comparing with the upper bound for $|I_1|$, and choosing μ sufficiently large, this yields

$$I_1 + I_2 \geqslant c e^{\frac{k}{8m}} \left(1 - \frac{1}{8m}\right)^k e^{-\frac{k}{2}} \left(\frac{k}{2}\right)^{\frac{k}{2}},$$

and by a completely analogous argument we also have

$$I_3 + I_4 \geqslant c e^{-\frac{k}{8m}} \left(1 + \frac{1}{8m}\right)^k e^{-\frac{k}{2}} \left(\frac{k}{2}\right)^{\frac{k}{2}}.$$

Now as
$$e^{\frac{k}{8m}} \geqslant (1+\frac{1}{8m})^k \quad \text{and} \quad e^{-\frac{k}{8m}} \geqslant (1-\frac{1}{8m})^k,$$
we see that
$$c\Big(1-\frac{1}{64m^2}\Big)^k e^{-\frac{k}{2}}\left(\frac{k}{2}\right)^{\frac{k}{2}} \leqslant \int_0^\infty e^{-s^2}s^k\cos(2xs)ds \leqslant \frac{1}{2}\Gamma\left(\frac{k+1}{2}\right).$$

Finally, as $m^2 \approx k$ and
$$\Gamma\left(\frac{k+1}{2}\right) = 2\sqrt{\pi}\frac{k!}{2^k\left(\frac{k}{2}\right)!}, \tag{15}$$
(see [24]), by (14) we have
$$c_0\frac{2^{k/2}}{\sqrt{k!}}e^{-\frac{k}{2}}\left(\frac{k}{2}\right)^{\frac{k}{2}} \leqslant h_k(x) \leqslant c_1\frac{2^{k/2}}{\sqrt{k!}}\frac{k!}{2^k\left(\frac{k}{2}\right)!}$$
for $k/2$ even, and the proof is completed by Stirling's formula. □

We consider g_N defined by
$$g_N(x) = h_{4N}(x_1)h_0(x_2)\dots h_0(x_n).$$
Note that
$$\begin{aligned}\|e^{-itH}g_N\|^2_{L^2([0,2\pi]\times[0,1]^n)} &= 2\pi\left(\int_0^1 h^2_{4N}(x_1)dx_1\int_0^1 e^{-x_2^2}dx_2\dots\int_0^1 e^{-x_n^2}dx_n\right)\\ &= C\int_0^1 h^2_{4N}(x_1)dx_1.\end{aligned}$$

Now by Lemma 1, h_{4k} take values $\approx k^{-1/4}$ when x belongs to one of $\approx k^{1/2}$ subintervals of $[0,1]$ of length $k^{-1/2}$. Thus
$$\int_0^1 h^2_{4k}(x)dx \geqslant ck^{-1/2},$$
so that
$$\|e^{-itH}g_N\|_{L^2([0,1]^n\times[0,2\pi])} \geqslant CN^{-1/4}.$$
Now as $\|g_N\|_{\mathcal{H}^s(\mathbb{R}^n)} = (8N+n)^{s/2}$, letting N tend to infinity, we see that $s \geqslant -1/2$ is a necessary condition for the local smoothing estimate to hold.

4. Pointwise convergence

By Cauchy–Schwarz, functions $F:[0,2\pi]\to\mathbb{C}$ that satisfy
$$\Big\|\sum_{\lambda\in\mathbb{Z}}|\lambda|^\alpha\widehat{F}(\lambda)\,e^{-it\lambda}\Big\|_{L^2[0,2\pi]} < \infty, \quad \alpha > 1/2,$$
are in fact continuous, where $\widehat{F}$ denotes the Fourier transform of F. Writing
$$e^{-itH}f(x) = \sum_{\lambda\in\mathbb{N}}\left(\sum_{\mathbf{k}\,:\,2|\mathbf{k}|+n=\lambda}a_{\mathbf{k}}h_{\mathbf{k}}(x)\right)e^{-it\lambda} = \sum_{\lambda\in\mathbb{N}}P_\lambda f(x)\,e^{-it\lambda}, \tag{16}$$

by Theorem 5, we have

$$\Big\| \| \sum_{\lambda\in\mathbb{N}} |\lambda|^{\alpha} P_\lambda f\, e^{-it\lambda} \|_{L^2_t[0,2\pi]} \Big\|_{L^2_x(B_R)} \leqslant C_R \Bigg(\sum_{\mathbf{k}\in\mathbb{N}_0^n} (2|\mathbf{k}|+n)^{-1/2} |(2|\mathbf{k}|+n)^{\alpha} a_{\mathbf{k}}|^2 \Bigg)^{1/2}$$
$$\leqslant C_R \|f\|_{\mathcal{H}^{2\alpha-1/2}(\mathbb{R}^n)}.$$

Thus, when $f\in\mathcal{H}^s(\mathbb{R}^n)$ with $s>1/2$, we see that $e^{-itH}f(x)$ is a continuous function of t for almost every $x\in B_R$. Writing

$$\mathbb{R}^n = \bigcup_{j\in\mathbb{Z}} B_{2^j}\backslash B_{2^{j-1}},$$

we see that the set of divergence is null, which proves Theorem 1 for $n\geqslant 2$.

For the one dimensional improvement, we note that the the integral representation (5) can be combined with the machinery of Keel and Tao [14] so that

$$\|e^{-itH} f\|_{L^q_t([0,2\pi],L^p_x(\mathbb{R}^n))} \leqslant C_p \|f\|_{L^2(\mathbb{R}^n)} \tag{17}$$

when $q\geqslant 2$ and $\frac{n}{p}+\frac{2}{q}=\frac{n}{2}$, excluding the case $(p,q,n)\neq(\infty,2,2)$. Koch and Tataru [16] proved (17) for a more general class of operators that includes H, and also noted that there can be no such estimates for p outside of $[2,\frac{2n}{n-2}]$. Applying Hölder in the temporal integral yields (17) in the range $p\in[2,\frac{2n}{n-2}]$ when $\frac{n}{p}+\frac{2}{q}\geqslant\frac{n}{2}$, excluding the case $(p,q,n)\neq(\infty,2,2)$. We will see later that for $n\geqslant 3$ the estimate is completely sharp in the sense that (17) cannot hold when $\frac{n}{p}+\frac{2}{q}<\frac{n}{2}$.

Theorem 2 and (17) are the key ingredients in the proof of the following theorem. For the best known results in this direction for the free equation see [13, 15, 19–21].

Theorem 6. *Let* $p\in[\frac{2(n+2)}{n},\infty]$, $q\in[2,\infty)$ *and* $\frac{n}{p}+\frac{2}{q}\leqslant\frac{n}{2}$. *Then*

$$\|e^{-itH} f\|_{L^p_x(\mathbb{R}^n,\, L^q_t[0,2\pi])} \leqslant C_s \|f\|_{\mathcal{H}^s(\mathbb{R}^n)}, \qquad s = n\Big(\frac{1}{2}-\frac{1}{p}\Big)-\frac{2}{q}.$$

Proof. For $1<r<q<\infty$, we recall the following fractional Sobolev inequality (see [30]):

$$\Big\| \sum_{\substack{\lambda\in\mathbb{Z}\\ \lambda\neq 0}} |\lambda|^{-\alpha} \widehat{F}(\lambda)\, e^{-it\lambda} \Big\|_{L^q[0,2\pi]} \leqslant C\, \|F\|_{L^r[0,2\pi]}, \qquad \alpha = \frac{1}{r}-\frac{1}{q}.$$

In particular, by (16) we see that

$$\begin{aligned} \|e^{-itH} f(x)\|_{L^q_t[0,2\pi]} &\leqslant C \Big\| \sum_{\lambda\in\mathbb{N}} |\lambda|^{\alpha} P_\lambda f(x)\, e^{-it\lambda} \Big\|_{L^r_t[0,2\pi]} \\ &= C \Big\| \sum_{\mathbf{k}\in\mathbb{N}_0^n} (2|\mathbf{k}|+n)^{\alpha} a_{\mathbf{k}} h_{\mathbf{k}}(x)\, e^{-it(2|\mathbf{k}|+n)} \Big\|_{L^r_t[0,2\pi]}, \end{aligned} \tag{18}$$

where $f=\sum_{\mathbf{k}\in\mathbb{N}_0^n} a_{\mathbf{k}} h_{\mathbf{k}}$ is initially a member of $\mathfrak{F}(\mathbb{R}^n)$.

Taking $r=2$, in the range $p\in[\frac{2n}{n-2},\infty]$, by Theorem 2, we see that

$$\|e^{-itH}f\|_{L^p_x(\mathbb{R}^n,\,L^q_t[0,2\pi])}\leqslant C\left(\sum_{k\in\mathbb{N}_0^n}(2|\mathbf{k}|+n)^s(2|\mathbf{k}|+n)^{1-\frac{2}{q}}|a_{\mathbf{k}}|^2\right)^{1/2}$$
$$\leqslant C\,\|f\|_{\mathcal{H}^{s+1-\frac{2}{q}}(\mathbb{R}^n)},$$

where $s=n(\frac12-\frac1p)-1$. This yields the desired inequality.

In the range $p\in[\frac{2(n+2)}{n},\frac{2n}{n-2})$, by combining (17) and (18), we see that

$$\|e^{-itH}f\|_{L^p_x(\mathbb{R}^n,\,L^q_t[0,2\pi])}\leqslant C\left(\sum_{k\in\mathbb{N}_0^n}(2|\mathbf{k}|+n)^{\frac{2}{q_0}-\frac{2}{q}}|a_{\mathbf{k}}|^2\right)^{1/2}$$
$$\leqslant C\,\|f\|_{\mathcal{H}^{\frac{2}{q_0}-\frac{2}{q}}(\mathbb{R}^n)},$$

where $\frac{2}{q_0}=n\big(\frac12-\frac1p\big)$, and so we are done. □

To complete the proof of Corollary 1 we again appeal to [30]. There it was proven that functions

$$F(t)=\sum_{\lambda\in\mathbb{N}}\widehat{F}(\lambda)\,e^{-it\lambda}$$

which satisfy

$$\Big\|\sum_{\lambda\in\mathbb{N}}|\lambda|^\alpha\widehat{F}(\lambda)\,e^{-it\lambda}\Big\|_{L^q[0,2\pi]}<\infty,\qquad \alpha>1/q,$$

are also continuous. By Theorem 6 we see that for certain $q<\infty$,

$$\Big\|\sum_{\lambda\in\mathbb{N}}|\lambda|^\alpha P_\lambda f(x)\,e^{-it\lambda}\Big\|_{L^p_x(\mathbb{R}^n,\,L^q_t[0,2\pi])}\leqslant C\|f\|_{\mathcal{H}^s(\mathbb{R}^n)},\quad \alpha=\frac12\Big(s-n\big(\frac12-\frac1p\big)+\frac2q\Big).$$

In particular, taking $p=\frac{2(n+2)}{n}$ and $s>\frac{n}{n+2}$, we see that $\alpha>1/q$ so that $t\to e^{-itH}f(x)$ is continuous for almost every $x\in\mathbb{R}^n$.

Almost everywhere convergence results can also be obtained from maximal inequalities. By an appropriate dyadic decomposition, Theorem 6 implies that

$$\big\|\sup_{t\in\mathbb{R}}|e^{-itH}f|\,\big\|_{L^p_x(\mathbb{R}^n)}\leqslant C_s\,\|f\|_{\mathcal{H}^s(\mathbb{R}^n)},\quad s>n\big(\frac12-\frac1p\big),\quad p\geqslant\frac{2(n+2)}{n}.$$

Curiously, this is not trivial even when $p=\infty$. Indeed, for a dyadic piece $f_N=\sum_{N\leqslant|\mathbf{k}|\leqslant 2N}a_{\mathbf{k}}h_{\mathbf{k}}$, we can write

$$\sup_{x\in\mathbb{R}^n,\,t\in[0,2\pi]}|e^{-itH}f_N(x)|\leqslant\sup_{x\in\mathbb{R}^n}\left(\sum_{N\leqslant|\mathbf{k}|\leqslant 2N}|h_{\mathbf{k}}(x)|^2\right)^{1/2}\left(\sum_{N\leqslant|\mathbf{k}|\leqslant 2N}|a_{\mathbf{k}}|^2\right)^{1/2},$$

however, the property (6) only provides the estimate

$$\sup_{x\in\mathbb{R}^n}\left(\sum_{N\leqslant|k|\leqslant 2N}|h_{\mathbf{k}}(x)|^2\right)^{1/2}\leqslant CN^{\frac12\frac{5n}{6}}.$$

On the other hand, using the local property (12),

$$\sup_{x\in B_R}\left(\sum_{N\leqslant|k|\leqslant 2N}|h_{\mathbf{k}}(x)|^2\right)^{1/2}\leqslant CN^{\frac{1}{2}\frac{n}{2}},$$

and so we recover a local version of our estimate. Thangavelu [27] noted a similar phenomenon for the Bochner–Reisz problem for hermite expansions and concluded that the local Bochner–Reisz conjecture was the appropriate version. Here we see that global estimates are indeed possible even though this is not immediately apparent.

As we saw in the previous section, necessary conditions for the harmonic oscillator are harder to see than for the free equation. That Theorem 6 is sharp with respect to the regularity is a consequence of the following lemma.

Lemma 2. *There exist positive constants c_0 and c_1 such that*

$$h_{4k}(x)\geqslant c_0k^{-1/4}$$

for all $k\in\mathbb{N}$ when $|x|<c_1k^{-1/2}$.

Proof. For k an even integer, $|x|<\frac{1}{4\sqrt{k}}$ and $0<s<\sqrt{k}$, we have $\cos(2xs)>1/2$, so that

(19)

$$\begin{aligned}\left|\int_0^\infty e^{-s^2}s^k\cos(2xs)ds\right| &\geqslant \int_0^{\sqrt{k}} e^{-s^2}s^k\cos(2xs)ds-\left|\int_{\sqrt{k}}^\infty e^{-s^2}s^k\cos(2xs)ds\right|\\ &\geqslant\frac{1}{2}\int_0^{\sqrt{k}}e^{-s^2}s^kds-\int_{\sqrt{k}}^\infty e^{-s^2}s^kds\\ &=\frac{1}{2}\int_0^\infty e^{-s^2}s^kds-\frac{3}{2}\int_{\sqrt{k}}^\infty e^{-s^2}s^kds.\end{aligned}$$

Now, by the formula for the Gamma function and a change of variables,

$$\frac{1}{2}\int_0^\infty e^{-s^2}s^kds=\frac{1}{4}\Gamma\left(\frac{k+1}{2}\right). \tag{20}$$

On the other hand, making the change of variables $r=\frac{s}{\sqrt{2}}$,

$$\begin{aligned}\int_{\sqrt{k}}^\infty e^{-s^2}s^kds &\leqslant e^{-\frac{k}{2}}\int_{\sqrt{k}}^\infty e^{-\frac{s^2}{2}}s^kds\leqslant e^{-\frac{k}{2}}\int_0^\infty e^{-\frac{s^2}{2}}s^kds\\ &=\sqrt{2}\left(\frac{2}{e}\right)^{\frac{k}{2}}\int_0^\infty e^{-r^2}r^kdr=\frac{\sqrt{2}}{2}\left(\frac{2}{e}\right)^{\frac{k}{2}}\Gamma\left(\frac{k+1}{2}\right)\\ &\leqslant\frac{1}{16}\Gamma\left(\frac{k+1}{2}\right)\end{aligned} \tag{21}$$

for all $k\geqslant k_0=2\frac{\log(16)}{\log\frac{e}{2}}$ (since $h_k(0)>0$ when $k/2$ is even, it is sufficient to prove the assertion for $k\geqslant k_0$).

Substituting (20) and (21) into (19), we obtain

$$\left|\int_0^\infty e^{-s^2}s^k\cos(2xs)ds\right|\geqslant\frac{1}{8}\Gamma\left(\frac{k+1}{2}\right),$$

so that from (14), we see that

$$h_k(x) \geqslant \frac{1}{8\pi^{3/4}} \frac{2^{k/2}}{\sqrt{k!}} \Gamma\left(\frac{k+1}{2}\right)$$

for all $k \geqslant k_0$ when $|x| < \frac{1}{4\sqrt{k}}$ and $k/2$ is even.

Now, from (15), we have

$$h_k(x) \geqslant \frac{1}{4\pi^{1/4}} \frac{\sqrt{k!}}{2^{k/2}\left(\frac{k}{2}\right)!},$$

and the result follows by Stirling's formula as before. □

Consider g_N defined by

$$g_N = \sum_{\mathbf{k}\,:\,N \leqslant k_j < 2N} h_{4\mathbf{k}}.$$

When $|t| \leqslant \frac{1}{100nN}$ and $|\mathbf{k}| \leqslant 2nN$, we have

$$\left|\Re(e^{-it(8|\mathbf{k}|+n)} - 1)\right| = |\cos t(8|\mathbf{k}| + n) - 1| < 1/2,$$

so that

$$\begin{aligned} |e^{-itH} g_N| &\geqslant \left|\sum_{\mathbf{k}\,:\,N \leqslant k_j < 2N} h_{4\mathbf{k}}\right| - \left|\sum_{\mathbf{k}\,:\,N \leqslant k_j < 2N} [\cos(t(8|\mathbf{k}| + n)) - 1]\, h_{4\mathbf{k}}\right| \\ &\geqslant \left|\sum_{\mathbf{k}\,:\,N \leqslant k_j < 2N} h_{4\mathbf{k}}\right| - \frac{1}{2} \sum_{\mathbf{k}\,:\,N \leqslant k_j < 2N} |h_{4\mathbf{k}}|. \end{aligned}$$

Thus, by Lemma 2, if $|x_j| < \frac{c_1}{2N^{1/2}}$ for all $j = 1, \ldots, n$, then

$$|e^{-itH} g_N(x)| \geqslant \frac{1}{2} \sum_{\mathbf{k}\,:\,N \leqslant k_j < 2N} h_{4\mathbf{k}}(x) \geqslant cN^{n - \frac{n}{4}}.$$

Calculating, we see that

$$\|e^{-itH} g_N\|_{L^p_x(\mathbb{R}^n,\, L^q_t[0,2\pi])} \geqslant cN^{\frac{3n}{4} - \frac{n}{2p} - \frac{1}{q}}.$$

On the other hand, $\|g_N\|_{\mathcal{H}^s(\mathbb{R}^n)} \leqslant CN^{\frac{s}{2} + \frac{n}{2}}$, so that letting N tend to infinity, for (6) to hold, it is necessary that

$$s \geqslant \frac{n}{2} - \frac{n}{p} - \frac{2}{q}.$$

Finally we note that by the same calculation,

$$\|e^{-itH} g_N\|_{L^q_t([0,2\pi],\, L^p_x(\mathbb{R}^n))} \geqslant cN^{\frac{3n}{4} - \frac{n}{2p} - \frac{1}{q}},$$

so that, taking $s = 0$, we see that the Strichartz estimates (17) are also sharp.

5. The forced harmonic oscillator

We consider the Cauchy problem for the Schrödinger equation of the form

$$\text{(FHO)}\qquad \begin{cases} i\frac{du}{dt}+\Delta u=|x|^2u+V(x,t)\,u\\ u(\,\cdot\,,0)=u_0,\end{cases}$$

where V is periodic in time (a Floquet potential). In the following theorem, when $n\geqslant 3$ the hypothesis $\|V\|_{L^q_x(\mathbb{R}^n,\,L^\infty_t[0,2\pi])}$ sufficiently small can be changed to $\|V\|_{L^\infty_t([0,2\pi],L^q_x(\mathbb{R}^n))}$ sufficiently small, by using Theorem 6 instead of Theorem 4.

Theorem 7. *Let $n\geqslant 2$ and $\frac{2}{p}+\frac{1}{q}=1$, and suppose that $\|V\|_{L^q_x(\mathbb{R}^n,\,L^\infty_t[0,2\pi])}$ is sufficiently small, where $q\geqslant n/2$. Then there exists a unique global solution of (FHO) belonging to $C([0,\infty),L^2_x(\mathbb{R}^n))\cap L^p_x(\mathbb{R}^n,\,L^2_{loc}[0,\infty))$.*

Proof. We use the standard contraction mapping argument. A solution must satisfy Duhamel's formula

$$u(x,t)=e^{-itH}u_0+i\int_0^t e^{-i(t-\tau)H}V(\cdot,\tau)u(\cdot,\tau)(x)\,d\tau.$$

For $2\leqslant p\leqslant\frac{2n}{n-2}$, by Theorem 4, there exists a constant $C_0>1$ such that

$$\|e^{-itH}f\|_{L^p_x(\mathbb{R}^n,\,L^2_t[0,2\pi])}\leqslant C_0\|f\|_{L^2(\mathbb{R}^n)}, \tag{22}$$

and, by duality, this yields

$$\Big\|\int_0^t e^{i\tau H}V(\,\cdot\,,\tau)F(\,\cdot\,,\tau)\,d\tau\Big\|_{L^2_x(\mathbb{R}^n)}\leqslant C_0\,\|VF\|_{L^{p'}_x(\mathbb{R}^n,L^2_t[0,2\pi])},\quad t\in[0,2\pi]. \tag{23}$$

Thus, by writing

$$\int_0^t e^{-i(t-\tau)H}G(\cdot,\tau)(x)\,d\tau=e^{-itH}\int_0^t e^{i\tau H}G(\cdot,\tau)(x)\,d\tau,$$

we can apply first (22), then (23) to obtain

$$\Big\|\int_0^t e^{-i(t-\tau)H}V(\cdot,\tau)F(\cdot,\tau)(x)\,d\tau\Big\|_{L^p_x(\mathbb{R}^n,L^2_t[0,2\pi])}\leqslant C_0^2\|VF\|_{L^{p'}_x(\mathbb{R}^n,L^2_t[0,2\pi])}. \tag{24}$$

We define the Banach space $X=C([0,2\pi],L^2_x(\mathbb{R}^n))\cap L^p_x(\mathbb{R}^n,L^2_t[0,2\pi])$ via the norm

$$\|u\|_X=\sup_{t\in[0,2\pi]}\|u(\,\cdot\,,t)\|_{L^2_x(\mathbb{R}^n)}+\|u\|_{L^p_x(\mathbb{R}^n,\,L^2_t[0,2\pi])},$$

and the nonlinear map $\mathcal{L}:X\to X$ by

$$\mathcal{L}F=e^{-itH}u_0+i\int_0^t e^{-i(t-\tau)H}V(\cdot,\tau)F(\cdot,\tau)(x)\,d\tau.$$

By (22) and the conservation of the L^2 norm for the fundamental solution (4) we see that

$$\|e^{-itH}u_0\|_X\leqslant C_0^2\|u_0\|_{L^2(\mathbb{R}^n)},$$

and combining (23) and (24), we also have

$$\Big\|i\int_0^t e^{-i(t-\tau)H}V(\cdot,\tau)F(\cdot,\tau)(x)\,d\tau\Big\|_X\leqslant C_0^2\|V\|_{L^q_x(\mathbb{R}^n,L^\infty_t[0,2\pi])}\|F\|_X;$$

here we have used the fact that

$$\|VF\|_{L^{p'}_xL^2_t}\leqslant\|V\|_{L^q_xL^\infty_t}\|F\|_{L^p_xL^2_t},\quad \frac{2}{p}+\frac{1}{q}=1.$$

Thus we see that $\mathcal{L}$ maps $\{F : \|F\|_X \leqslant 2(C_0+1)\|u_0\|_{L^2(\mathbb{R}^n)}\}$ into itself when $\|V\|_{L^q_x(\mathbb{R}^n, L^\infty_t[0,2\pi])}$ is sufficiently small. We can also guarantee that

$$\|\mathcal{L}(F-G)\|_X \leqslant \frac{1}{2}\|F-G\|_X, \tag{25}$$

so that by the contraction mapping principle, there exists a solution. Iterating the process, replacing u_0 with $u(\cdot, 2k\pi)$, $k \in \mathbb{N}$, we obtain a global solution.

To see that the solution is unique in $L^p_x(\mathbb{R}^n, L^2_{loc}[0,\infty))$, suppose that u_1 and u_2 are solutions. Then by (24) as before, we see that

$$\|u_1-u_2\|_{L^p_x(\mathbb{R}^n,\, L^2_t[2k\pi, 2(k+1)\pi])} \leqslant \frac{1}{2}\|u_1-u_2\|_{L^p_x(\mathbb{R}^n,\, L^2_t[2k\pi, 2(k+1)\pi])}$$

for all $k \geqslant 0$, so they are in fact the same. □

6. Final remarks

We combine the Strichartz estimates with the orthogonality of the trigonometric polynomials to obtain some mysterious inequalities for the Hermite functions. Observe that for $f = \sum_{\mathbf{k}\in E} a_{\mathbf{k}} h_{\mathbf{k}}$, where $E \subset \mathbb{N}_0^n$,

$$\begin{aligned}\|e^{-itH}f\|^4_{L^4_t[0,2\pi]} &= \int_0^{2\pi} \left| \left(\sum_{\mathbf{j}\in E} e^{-it(2|\mathbf{j}|+n)} a_{\mathbf{j}} h_{\mathbf{j}}\right) \left(\sum_{\mathbf{k}\in E} e^{it(2|\mathbf{k}|+n)} \overline{a}_{\mathbf{k}} h_{\mathbf{k}}\right)\right|^2 dt \\ &= 2\pi \left(\sum_{\substack{\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}\in E \\ |\mathbf{i}|+|\mathbf{k}|=|\mathbf{j}|+|\mathbf{l}|}} a_{\mathbf{i}}\overline{a}_{\mathbf{j}} a_{\mathbf{k}}\overline{a}_{\mathbf{l}} h_{\mathbf{i}} h_{\mathbf{j}} h_{\mathbf{k}} h_{\mathbf{l}} \right).\end{aligned}$$

By (17), we have

$$\Big\| \|e^{-itH}f\|_{L^4_t[0,2\pi]} \Big\|_{L^4_x(\mathbb{R}^2)} \leqslant C\|f\|_2,$$

so that setting $a_{\mathbf{k}} = 1$, we have

$$\sum_{\substack{\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}\in E \\ |\mathbf{i}|+|\mathbf{k}|=|\mathbf{j}|+|\mathbf{l}|}} \int_{\mathbb{R}^2} h_{\mathbf{i}} h_{\mathbf{j}} h_{\mathbf{k}} h_{\mathbf{l}} \leqslant CN^2, \quad \#E = N.$$

In one spatial dimension, by the same procedure we obtain

$$\sum_{i,j,k,l,m\in E} \int_{\mathbb{R}} h_i h_j h_k h_l h_m h_{i+k+m-j-l} \leqslant CN^3, \quad \#E = N.$$

We see that there is cancelation. A better understanding of this cancelation would presumably yield improved results.

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