Toric G_2 and $Spin(7)$ holonomy spaces from gravitational instantons and other examples

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Abstract

Non-compact G_2 holonomy metrics that arise from a T^2 bundle over a hyper-Kähler space are constructed. These are one parameter deformations of certain metrics studied by Gibbons, Lü, Pope and Stelle in [1]. Seven-dimensional spaces with G_2 holonomy fibered over the Taub-Nut and the Eguchi-Hanson gravitational instantons are found, together with other examples. By using the Apostolov-Salamon theorem [2], we construct a new example that, still being a T^2 bundle over hyper-Kähler, represents a non trivial two parameter deformation of the metrics studied in [1]. We then review the $Spin(7)$ metrics arising from a T^3 bundle over a hyper-Kähler and we find two parameter deformation of such spaces as well. We show that if the hyper-Kähler base satisfies certain properties, a non trivial three parameter deformations is also possible. The relation between these spaces with half-flat and almost G_2 holonomy structures is briefly discussed.

1. Introduction

The spaces of special holonomy have received remarkable attention within the context of high energy physics; being the spaces of G_2 holonomy one of the most important examples. This is due to the fact that these geometries, in the presence of singularities, give rise to a natural framework for reducing eleven-dimensional M-theory to $\mathcal{N} = 1$ four-dimensional realistic models [3, 4, 5]; see also [6, 7]. Actually, it is a well known fact that the geometries of this sort admit at least one globally defined covariantly constant spinor [8] and, when singularities are present, non-abelian gauge fields and chiral matter can also appear. All these features make evident that the study of spaces of G_2 holonomy deserved the attention of theoretical physicists. However, this is actually a hard tool to be studied, mainly because the explicit examples of compact spaces of G_2 holonomy have so far eluded discovery. In fact, even though such spaces are known to exist [21], the metrics which are explicitly known [8]-[20] turn out to be non-compact. Nevertheless, it was shown that the M-theory dynamics near the singularity is not strongly dependent on the global properties of the space [3]. On the other hand, applications for which the presence of singularities plays no role also exist. Furthermore, still in the case of non-compact spaces, it turns out that finding and classifying metrics of holonomy G_2 represents a highly non-trivial problem. In this work we discuss non-compact examples of toric geometries of such special holonomy, which are based on four-dimensional hyper-Kähler gravitational instantons.

The problem of classifying spaces of G_2 holonomy possessing one isometry whose Killing vector orbits form a Kähler six-dimensional space was analyzed both by physicist and mathematicians. In Ref. [23], it was concluded that such geometries are described by a sort of holomorphic monopole equation together with a condition related to the integrability of the complex structure. Such condition turns out to be stronger than the one required by supersymmetry. On the other hand, Apostolov and Salamon have proven in [2] that the Kähler condition yields the existence of a new Killing vector that commutes with the first, so that these metrics are toric. Besides, it was shown that such a G_2 metric yields a four-dimensional manifold equipped with a complex symplectic structure and a one-parameter family of functions and 2-forms linked by second order equations (henceforth called Apostolov-Salamon equation). The inverse problem, i.e. the one of constructing a torsion-free G_2 structure starting from such a four-dimensional space was also discussed in [2]. Then, a natural question arises as to whether both description of this classification problem are equivalent. In Ref [24], it was argued that this is indeed the case. Moreover, in Ref. [1, 2, 24] such a construction was employed to generate new G_2-metrics. In the present work, the solution generating technique will be analyzed in detail and a wider family of G_2-metrics will be written down. In particular, the Eguchi-Hanson and the Taub-Nut metrics will be dimensionally extended to new examples of G_2 holonomy following the construction proposed in [1]. In section 2 we discuss the Apostolov-Salamon theorem [2], which formalizes a method for systematically constructing spaces with special holonomy G_2 by starting with a hyper-Kähler space in four-dimensions. This construction is actually the one previously employed by Gibbons, Lü, Pope and Stelle in Ref. [1] and here we discuss it within the framework of [2]. Then, we describe some explicit examples in order to illustrate the procedure. In particular, we show how some of the G_2 metrics obtained in such way are one-parameter deformations of those examples discussed in the literature. In section 3, we present the examples that are based on non-trivial Gibbons-Hawking solutions. We discuss the G_2 spaces obtained by starting with the four-dimensional gravitational instantons and we write down the corresponding metrics explicitly. Other examples are also discussed. In particular, we find a two parameter deformations of the T^3 bundles with $Spin(7)$ holonomy over hyper-Kähler studied in [1]. Besides, in the strictly almost Kähler case, we obtain $Spin(7)$ metrics which correspond to non trivial T^3 bundles over hyper-Kähler metrics. To our knowledge, such metrics were not considered before in the literature. We show that all the presented metric spaces are foliated by equidistant hypersurfaces. In the G_2 holonomy case, these surfaces are half-flat manifolds, while for the $Spin(7)$ metrics they are almost G_2 holonomy spaces. So that we implicitly find a family of half-flat T^2 bundles and a family of almost G_2 holonomy T^3 bundles over hyper-Kähler spaces.

2. The general setup

Let us begin by explaining how the Apostolov-Salamon scheme works; detailed proofs can be found in [2]. Let us consider a four dimensional complex manifold M with a metric $g_4 = \delta_{ab}e^a \otimes e^b$. It is also assumed that this metric is a function of certain parameter μ, i.e. $g_4 = g_4(\mu)$. This parameter should not be confused with a coordinate of g_4. The property of M as being "complex" means the following: Consider the manifold M, for which there exists a $(1,1)$-tensor J_1 with the property $J_1 \cdot J_1 = -I$, such that $g_4(J_1\cdot,\cdot) = -g_4(\cdot,J_1\cdot)$. A metric for which the last property holds is called hermitian with respect to J_1, and it follows that $\widetilde{J}_1 = g_4(J_1\cdot,\cdot)$ is an antisymmetric tensor, so that it is a μ-dependent 2-form. In the cases in which there is no dependence on μ we will denote this tensor as $\overline{J}_1$. The tensor J_1 is called an almost complex structure and is not uniquely defined. Indeed, any $SO(4)$ rotation of the frame e^a induces a new almost complex structure for which the metric turns out to be again hermitian. If at least one element of such family of almost complex structures is covariantly constant with respect to the Levi-Civita connection Γ of g_4 (that is, $\nabla_X J_1 = 0$, X being an arbitrary element of TM) then the tensor J_1 will be called a complex structure, and then the manifold M will be complex. Conversely, for any complex manifold it is possible to find at least one complex structure. The condition $\nabla_X J_1 = 0$ is equivalent to the condition $d\overline{J}_1 = 0$ (or $d_M \widetilde{J}_1 = 0$ if there is a dependence on μ, being d_M the differentiation over M which does not involve derivatives with respect to μ) together with the integrability of J_1; that is, the vanishing of the Nijenhuis tensor associated to J_1.

It will be assumed also that the metric g_4 admits a complex symplectic form $\Omega = \overline{J}_2 + i\overline{J}_3$, where being "symplectic" means that it is closed, $d\Omega = 0$. On the other hand, being "complex" implies that

$$\overline{J}_2 \wedge \overline{J}_2 = \overline{J}_3 \wedge \overline{J}_3, \qquad \overline{J}_2 \wedge \overline{J}_3 = 0,$$

and that

$$\overline{J}_2(J_1 \cdot, \cdot) = \overline{J}_3(\cdot, \cdot). \tag{2.1}$$

Now let us introduce a function u depending on the coordinates of M and on the parameter μ, and satisfying

$$2\mu \widetilde{J}_1(\mu) \wedge \widetilde{J}_1(\mu) = u\Omega \wedge \overline{\Omega}. \tag{2.2}$$

This function always exists because the wedge products appearing in (2.2) are proportional to the volume form of M. With the help of the quantities defined above a seven-dimensional metric [2, 1]

$$g_7 = \frac{(d\alpha + H_2)^2}{\mu^2} + \mu \left(u\, d\mu^2 + \frac{(d\beta + H_1)^2}{u} + g_4(\mu) \right), \tag{2.3}$$

can be constructed. Here β and α are two new coordinates while H_1 and H_2 are certain 1-forms independent on β and α. The parameter μ of g_4 is now a coordinate of g_7 and the metric tensor g_7 is also independent on the coordinates α and β. This means that the vectors ∂_α and ∂_β are obviously Killing and commuting and therefore the metrics (2.3) are to be called "toric". The Apostolov-Salamon construction states that if the quantities appearing in (2.3) are related by the evolution equation

$$\frac{\partial^2 \widetilde{J}_1}{\partial^2 \mu} = -d_M d_M^c u, \tag{2.4}$$

and the forms H_1 and H_2 are defined on $M \times \mathbf{R}_\mu$ and M respectively by the equations

$$dH_1 = (d_M^c u) \wedge d\mu + \frac{\partial \widetilde{J}_1}{\partial \mu}, \qquad dH_2 = -\overline{J}_2, \tag{2.5}$$

with $d_M^c = J_1 d_M$, then the seven-dimensional metric (2.3) has holonomy in G_2. Moreover, the six-dimensional metric

$$g_6 = u\, d\mu^2 + \frac{(d\beta + H_1)^2}{u} + g_4(\mu), \tag{2.6}$$

is Kähler with Kähler form

$$K = (d\beta + H_1) \wedge d\mu + \widetilde{J}_1. \tag{2.7}$$

This condition is usually referred as the "strong supersymmetry condition" in the physical literature [23]. The calibration 3-form corresponding to the metrics (2.3) is

$$\Phi = \widetilde{J}_1(\mu) \wedge (d\alpha + H_2) + d\mu \wedge (d\beta + H_1) \wedge (d\alpha + H_2)$$

$$+\mu \ \Big(\, \overline{J}_2 \wedge (d\beta + H_1) + u\overline{J}_3 \wedge d\mu \, \Big) , \tag{2.8}$$

and by means of (2.5), (2.4) and (2.2) it can be seen directly seen that $d\Phi = d * \Phi = 0$. This is an standard feature of the reduction of the holonomy from $SO(7)$ to G_2 [8]. Moreover, the Killing field ∂_α preserves Φ and $*\Phi$.

The converse of all these statements are also true. That is, for given a G_2 holonomy manifold Y with a metric g_7 possessing a Killing vector that preserves the calibration forms Φ and $*\Phi$ and such that the six-dimensional metric g_6 obtained from the orbits of the Killing vector is Kähler, then there exists a coordinate system in which g_7 takes the form (2.3) being $g_4(\mu)$ a one-parameter four-dimensional metric admitting a complex symplectic structure Ω and a complex structure J_1, being the quantities appearing in this expression related by (2.1) and the conditions (2.5), (2.4) and (2.2). Details of these assertions can be found in the original reference [2].

2.1 G_2 holonomy metrics fibered over hyper-Kähler manifolds

It is worth mentioning that the integrability conditions

$$d\Big((d^c_M u) \wedge d\mu + \frac{\partial \widetilde{J}_1}{\partial \mu} \Big) = 0, \qquad d\overline{J}_2 = 0$$

for (2.5) are identically satisfied. The second is the closure of $\overline{J}_2$ while the first is a direct consequence of (2.4) together with the closure of $\widetilde{J}_1$. Equations (2.5) are considerably simplified when $d^c_M u = 0$. In this case, the first equation in (2.5) and the condition $d^2 H_1 = 0$ imply that $\partial \widetilde{J}_1 / \partial \mu$ should be μ-independent and closed. If this is so then (2.4) is trivially satisfied. One possible solution is to choose

$$\widetilde{J}_1(\mu) = (M + Q\mu)\overline{J}_1,$$

being $\overline{J}_1$ μ-independent and closed, and where M and Q certain real parameters. The metric g_4 for which $\widetilde{J}_1$ is a Kähler form is

$$g_4(\mu) = (M + Q\mu)\widetilde{g}_4, \tag{2.9}$$

being $\widetilde{g}_4$ a μ-independent metric and $\overline{J}_1$ its Kähler form. Relation (2.2) gives a simple algebraic equation for u, yielding

$$u = \mu(M + Q\mu)^2.$$

In conclusion, the metric $\widetilde{g}_4$ is quaternion hermitian with respect to the tensors J_i, and from (2.1) it follows that $J_i \cdot J_j = -\delta_{ij} + \epsilon_{ijk} J_k$. Besides, the two forms $\overline{J}_1$, $\overline{J}_2$ and $\overline{J}_3$ are closed. In four dimensions the closure of the hyper-Kähler triplet implies that the J_i are integrable. This means that $\widetilde{g}_4$ is hyper-Kähler. The G_2 holonomy metric corresponding to this case is then given by

$$g_7 = \frac{(d\beta + QH_1)^2}{(M + Q\mu)^2} + \frac{(d\alpha + H_2)^2}{\mu^2} + \mu^2 \ (M + Q\mu)^2 \ d\mu^2 + \mu \ (M + Q\mu) \ \widetilde{g}_4, \tag{2.10}$$

and the 1-forms H_1 and H_2 obey $dH_1 = \overline{J}_1$ and $dH_2 = -\overline{J}_2$, being $\overline{J}_1$ and $\overline{J}_2$ any pair of the three Kähler forms on M. It is important to remark that (2.10) gives three G_2 holonomy metrics for a given hyper-Kähler metric. This is because one pair of 2-forms, selected among those that form the hyper-Kähler triplet $\overline{J}_i^*$, is necessary. I.e. although there are six possible choices, the order of the selection has no relevance and there are essentially three pairs. If the parameter M is set to zero, then the resulting metrics correspond to those appearing in section 6.2 of reference [1]. Although (2.10) are just a subfamily of the Apostolov-Salamon metrics, the extension (2.10) gives rise to a powerful method to construct new G_2 examples. In principle, these have to be distinguished from other G_2 metrics presented in the literature [9]-[13], which are of the Bryant-Salamon type [8]. Regarding this, let us briefly comment on the amount of effective parameters appearing in both constructions. For Apostolov-Salamon metrics (2.10) there appear two parameters M and Q but only one of them is an effective one. It is not difficult to see that if $M \neq 0$ then it can be set 1 by simply rescaling as

$$\widetilde{g}_4 \to M\widetilde{g}_4, \qquad H_1 \to MH_1, \qquad H_2 \to MH_2$$
$$\alpha \to M\alpha, \qquad \beta \to M^2\beta, \qquad Q \to Q' = \frac{Q}{M}$$

Then, the number of effective parameters will be $1+n$ being n the numbers of those belonging to the base 4-space. On the other hand, if $M = 0$, then we can also make the following redefinitions

$$\widetilde{g}_4 \to Q\widetilde{g}_4, \qquad H_1 \to QH_1, \qquad H_2 \to QH_2$$
$$\alpha \to Q\alpha, \qquad \beta \to Q^2\beta,$$

and set $Q = 1$. In this case, the only parameters appearing in the metric will be those n of the hyper-Kähler base.

There is another way to check that g_7 is given by (2.10). Let us define the tetrad 1-forms

$$e^5 = \mu\,(M + Q\mu)\,d\mu, \qquad e^6 = \frac{d\alpha + H_2}{\mu}, \qquad e^7 = \frac{d\beta + QH_1}{M + Q\mu}. \tag{2.11}$$

Then calibration form (2.8) is then expressed as

$$\Phi = \mu\,(M + Q\mu)\,\overline{J}_1 \wedge e^6 + e^5 \wedge e^6 \wedge e^7 + \mu\,(M + Q\mu)\,\Big(\,\overline{J}_2 \wedge e^7 + \overline{J}_3 \wedge e^5\,\Big). \tag{2.12}$$

It is convenient to choose a tetrad basis $\widetilde{e}^i$ for which the hyper-Kähler metric is diagonal, i.e. $g_4 = \delta_{ab}\widetilde{e}^a \otimes \widetilde{e}^b$, and for which the hyper-Kähler triplet takes the form

$$\overline{J}_1 = \widetilde{e}^1 \wedge \widetilde{e}^2 + \widetilde{e}^3 \wedge \widetilde{e}^4, \qquad \overline{J}_2 = \widetilde{e}^1 \wedge \widetilde{e}^3 + \widetilde{e}^4 \wedge \widetilde{e}^2, \qquad \overline{J}_3 = \widetilde{e}^1 \wedge \widetilde{e}^4 + \widetilde{e}^2 \wedge \widetilde{e}^3. \tag{2.13}$$

Then, by making the redefinition

$$e^a = \mu^{1/2}\,(M + Q\mu)^{1/2}\widetilde{e}^a$$

we see that (2.12) takes the familiar octonionic form

$$\Phi = c_{abc}\; e^a \wedge e^b \wedge e^c,$$

where c_{abc} is the octonion constants. The form Φ is G_2 invariant as a consequence of the fact that G_2 is the automorphism group of the octonion algebra. The G_2 holonomy metric corresponding to Φ is simply $g_7 = \delta_{ab}e^a \otimes e^b$ and it can be checked that g_7 is indeed given by (2.10). This follows from the expressions for e^a given above. It is convenient to remark that, in (2.13), we were assuming that the hyper-Kähler triplet is positive oriented. In the negative oriented case it can be analogously shown that the holonomy will be also G_2.

2.2 The simplest example

An interesting example to illustrate this construction comes from considering the simplest hyper-Kähler 4-manifold, namely $\mathbf{R}^4$ provided with its flat metric $\mathbf{E}^4 = g_4 = dx^2+dy^2+dz^2+dt^2$. A closed hyper-Kähler triplet for $\mathbf{R}^4$ is given by

$$\overline{J}_1 = dt\wedge dy - dz\wedge dx, \qquad \overline{J}_2 = dt\wedge dx - dy\wedge dz, \qquad \overline{J}_3 = dt\wedge dz - dx\wedge dy. \tag{2.14}$$

Now, let us extend this example to a seven-dimensional metric by means of (2.10). This innocent looking case is indeed rather rich and instructive. As it will be explained below, it gives rise to metric with holonomy exactly G_2. This example was already discussed in Ref. [9] and we will extend it here to less simple cases.

As it has been mentioned, there are three possible G_2 metrics that can be constructed, depending on which pair of forms we select from (2.14). But in the flat case, this choice just corresponds to a permutation of coordinates and the resulting metrics will be actually the same. By selecting the first two $\overline{J}_i$ among those in (2.14) we obtain the potential forms

$$H_1 = -xdz - ydt,$$

$$H_2 = -ydz - xdt.$$

Hence, the corresponding G_2 holonomy metrics read

$$g_7 = \frac{(\ d\beta - Q(xdz+ydt)\ )^2}{(M+Q\mu)^2} + \frac{(d\alpha - ydz - xdt)^2}{\mu^2} + \mu^2\ (M+Q\mu)^2\ d\mu^2 \tag{2.15}$$

$$+\mu\ (M+Q\mu)(\ dx^2+dy^2+dz^2+dt^2\ ).$$

If we select $M=0$ and $Q=1$ the metric tensors (2.15) reduces to

$$g_7 = \frac{(d\beta - xdz + ydt)^2}{\mu^2} + \frac{(d\alpha - ydz - xdt)^2}{\mu^2} + \mu^4\ d\mu^2 + \mu^2\ (\ dx^2+dy^2+dz^2+dt^2\ ). \tag{2.16}$$

Actually, metrics (2.16) have been already obtained in the literature [1]. They have been constructed within the context of eleven-dimensional supergravity, by starting with a domain wall solution of the form

$$g_5 = H^{4/3}(\ dx^2+dy^2+dz^2+dt^2\ ) + H^{16/3}d\mu^2,$$

being $a=1,..,4$ and H a warp function (see [1] for the details). By making use of the often called oxidation rules, and by starting from the above space in five-dimensions, it is feasible to get a eleven-dimensional background of the form $g_{11} = g_{(3,1)} + g_7$ with the seven-dimensional metric being

$$g_7 = \frac{(d\beta - xdz + ydt)^2}{H^2} + \frac{(d\alpha - ydz - xdt)^2}{H^2} + H^4\ d\mu^2 + H^2\ (\ dx^2+dy^2+dz^2+dt^2\ ). \tag{2.17}$$

By selecting the homothetic case $H=\mu$ the last metric reduces to (2.16). It can be shown by explicit calculation of the curvature tensor that (2.17) is irreducible and has holonomy exactly G_2, and this happens even in the particular case $H=\mu$, i.e. the one in (2.16), cf. [1]. The isometry corresponding to (2.17) and (2.15) is the same $SU(2)$ group that acts linearly on the coordinates (x,y,z,t) on M and which simultaneously preserves the two forms $\overline{J}_1$ and $\overline{J}_2$. The

$SU(2)$ action on M supplies H_1 and H_2 with total differential terms that can be absorbed by a redefinition of the coordinates α and β. For instance, a general translation of the form

$$x \to x + \alpha_1, \qquad y \to y + \alpha_2, \qquad z \to z + \alpha_3, \qquad t \to t + \alpha_4, \tag{2.18}$$

does preserve $\overline{J}_1$ and $\overline{J}_2$ but does not preserve the 1-forms H_1 and H_2. Nevertheless, the effect of the translations (2.18) can be compensated by a coordinate transformation of the form

$$\beta \to \beta + \alpha_5 + \alpha_1 z - \alpha_2 t, \qquad \alpha \to \alpha + \alpha_6 + \alpha_2 z + \alpha_1 t. \tag{2.19}$$

Besides, more general $SU(2)$ transformations on M preserving $\overline{J}_1$ and $\overline{J}_2$ can be also absorbed by a redefinition of the coordinates α and β. For the metric (2.16) we also have the scale invariance under transformation

$$x \to \lambda x, \qquad y \to \lambda y, \qquad z \to \lambda z, \qquad t \to \lambda t,$$

$$\alpha \to \lambda^4 \alpha, \qquad \beta \to \lambda^4 \beta, \qquad \mu \to \lambda^2 \mu,$$

for a real parameter λ, which is generated by the homothetic Killing vector

$$D = 2x\partial_x + 2y\partial_y + 2z\partial_z + 2t\partial_t + \mu\partial_\mu + 4\alpha\partial_\alpha + 4\beta\partial_\beta. \tag{2.20}$$

Our task now is to construct more elaborated examples by means of similar procedure. Let us move into this.

3. G_2 metrics with three commuting Killing vectors

By construction, metrics (2.10) posses at least two Killing vectors. If a larger isometry group is desired, an inspection of the formula (2.10) shows that the hyper-Kähler basis $\widetilde{g}_4$ should already posses Killing vectors and that the action of the isometry group on H_1 and H_2 should be induced by gauge transforming $H_1 \to H_1 + df_1$ and $H_2 \to H_2 + df_2$. The effect of this transformation can be compensated by a redefinition of the coordinates $\alpha \to \alpha + f_1$ and $\beta \to \beta + f_2$, so that the local form of the metric would result unaltered. Considering $dH_1 \sim \overline{J}_1$ and $dH_2 \sim \overline{J}_2$ we see that $\overline{J}_2$ and $\overline{J}_3$ will be actually preserved by the isometry group. An obvious example is an hyperkahler metric with a the Killing vector which is tri-holomorphic, namely, one satisfying

$$\mathcal{L}_K \overline{J}_1 = \mathcal{L}_K \overline{J}_2 = \mathcal{L}_K \overline{J}_3 = 0.$$

For any metric admitting a tri-holomorphic Killing vector ∂_t there exists a coordinate system in which it takes generically the Gibbons-Hawking form [32]

$$g = V^{-1}(dt + A)^2 + V dx_i dx_j \delta^{ij}, \tag{3.21}$$

with a 1-form A and a function V satisfying the linear system of equations

$$\nabla V = \nabla \times A. \tag{3.22}$$

These metrics are hyper-Kähler with respect to the hyper-Kähler triplet

$$\overline{J}_1 = (dt + A) \wedge dx - V dy \wedge dz$$

$$\overline{J}_2 = (dt + A) \wedge dy - V dz \wedge dx \qquad (3.23)$$
$$\overline{J}_3 = (dt + A) \wedge dz - V dx \wedge dy$$

which is actually t-independent. The isometry group of the total G_2 space will be then enlarged to T^3.

Actually, one could naively suggest another possibility; namely, to choose a Killing vector which is not actually tri-holomorphic, but still preserves two of the three Kähler forms. However, as it will be shown below, an isometry preserving $\overline{J}_1$ and $\overline{J}_2$ is necessarily tri-holomorphic. If, on the other hand, a hyper-Kähler metric possesses an isometry that is not tri-holomorphic, then there always exists a coordinate system (x, y, z, t) for which the distance element takes the form [31]

$$g_h = u_z \ \Big(e^u(dx^2 + dy^2) + dz^2\Big) + u_z^{-1} \ \ (dt + (u_x dy - u_y dx))^2 \,, \qquad (3.24)$$

with u a function of (x, y, z) satisfying the $SU(\infty)$ Toda equation; namely

$$(e^u)_{zz} + u_{yy} + u_{xx} = 0; \qquad (3.25)$$

where we are denoting $f_{x^i} = \partial_{x^i} f$. It is evident that the vector field ∂_t is a Killing vector of (3.24). Metric (3.24) is thus hyper-Kähler with respect to the t-dependent hyper-Kähler triplet

$$\overline{J}_1 \ = \ e^u \ u_z dx \wedge dy + dz \wedge (dt + (u_x dy - u_y dx)) \,, \qquad (3.26)$$
$$\overline{J}_2 \ = \ e^{u/2} \ \cos(t/2)\tilde{\overline{J}}_2 + e^{u/2} \ \sin(t/2)\tilde{\overline{J}}_3, \qquad (3.27)$$
$$\overline{J}_3 \ = \ e^{u/2} \ \sin(t/2)\tilde{\overline{J}}_2 - e^{u/2} \ \cos(t/2)\tilde{\overline{J}}_3; \qquad (3.28)$$

where we defined the 2-forms

$$\tilde{\overline{J}}_2 = -u_z dz \wedge dy + (dt + u_y dx) \wedge dy, \qquad \tilde{\overline{J}}_3 = u_z dz \wedge dx + (dt + u_x dy) \wedge dx.$$

These should not be confused with the hyper-Kähler forms $\overline{J}_i$ in (3.28). From (3.28) it is clear that ∂_t preserve $\overline{J}_1$, but the other two Kähler forms turns out to be dependent on t. It means that in the non-tri-holomorphic case it is impossible to preserve two of the three $\overline{J}_i$ without preserving the third one. The local form (3.24) is general enough, and therefore, in four dimensions, a $U(1)$ isometry that preserves two of the closed Kähler forms of an hyper-Kähler metric is automatically tri-holomorphic.[1]

Actually, there exists a shorter argument in order to find the same conclusion. All the complex structures form a two-sphere, since $u_1 J_1 + u_2 J_2 + u_3 J_3$ is a complex structure so long as $u_1^2 + u_2^2 + u_3^2 = 1$. Now any U(1) action on a two sphere either is trivial and keeps all the complex structures inert or keeps only two points (i.e. only one complex structure and its conjugate) on the sphere fixed.[2]

3.1 On G_2 metrics and asymptotic behavior of ALG spaces

Let us consider now the Gibbons-Hawking metrics (3.21) for which the potential V and a vector potential A are independent on certain coordinate, say x. It is simple to check that the equations (3.22) reduce, up to a gauge transformation, to the Cauchy-Riemann equations

$$A_y = A_z = 0, \qquad \partial_y A_x = -\partial_z V, \qquad \partial_z A_x = \partial_y V.$$

[1] This discussion concerns only $U(1)^n$ isometry groups, in other case it does not apply.

[2] We thank S. Cherkis for this argument and for other important suggestions.

This means that we can write $A_x + iV = \tau_1 + i\tau_2 = \tau(w)$ with $w = z + iy$; namely, A_x and V turn out to be the real and the imaginary part of an holomorphic function τ of one complex variable w. The corresponding hyper-Kähler metric can be written as

$$g_4 = \frac{|dt + \tau dx|^2}{\tau_2} + \tau_2 dw d\overline{w}. \tag{3.29}$$

The "local aspect" of the isometry group corresponds to $\mathbf{R}^2$, $\mathbf{R} \times U(1)$ or $T^2 = U(1) \times U(1)$, depending on the range of the values of the coordinates x and t. Notice that the Cauchy-Riemann equations imply that the 1-forms

$$B_1 = \tau_1 dz - \tau_2 dy, \qquad B_2 = \tau_1 dy + \tau_2 dz \tag{3.30}$$

are closed. Therefore, two functions $\varrho(x, y)$ and $\omega(x, y)$ such that $d\varrho = B_1$ and $d\omega = B_2$ can be "locally" defined. It is easy to see that the B_i are the real and imaginary parts of the complex 1-form

$$B_1 + iB_2 = \tau(w)dw = dT(w),$$

being $T(w)$ the primitive of the function τ. Therefore $\varrho + i\omega = T(w)$. With the help of this functions the hyper-Kähler triplet for the family (3.29) is expressed as

$$\overline{J}_1^* = \Re\left(dt \wedge dx + \frac{1}{2} dw \wedge d\overline{T}\right)$$

$$\overline{J}_2^* = \Im\left(\Upsilon\right), \qquad \overline{J}_3^* = \Re\left(\Upsilon\right), \tag{3.31}$$

where we introduced the complex two 2-form

$$\Upsilon = dt \wedge dw + dx \wedge dT,$$

and where the symbols $\Re$ and $\Im$ denote the real and the imaginary part of the quantity between parenthesis. It is clear that both ∂_t and ∂_x preserve (3.31). Therefore ∂_x is also a tri-holomorphic Killing vector, which clearly commutes with ∂_t. The 1-forms H_i^* satisfying $dH_i^* = \overline{J}_i^*$ are easily found from (3.31), the result is

$$H_1^* = -\Re\left(xdt + \frac{1}{2}\overline{T}dw\right), \qquad H_3^* + iH_2^* = -wdt - Tdx,$$

up to total differential terms. By selecting $H_1 = H_1^*$ and $H_2 = H_2^*$ in (2.10) the following G_2 holonomy metric is obtained

$$g_7 = \frac{\left(d\beta - Q\,\Re(xdt + \frac{1}{2}\overline{T}dw)\,\right)^2}{(1 + Q\mu)^2} + \frac{(\,d\alpha - \Im(wdt + Tdx)\,)^2}{\mu^2} + \mu^2\,(1 + Q\mu)^2\,d\mu^2$$

$$+\mu\,(1 + Q\mu)\left(\frac{|dt + \tau dx|^2}{\tau_2} + \tau_2 dw d\overline{w}\right), \tag{3.32}$$

Metrics (3.32) constitute a family of G_2 holonomy metrics constructed essentially from a single holomorphic function τ and its primitive T. There are two more G_2 holonomy metrics obtained by selecting $H_1 = H_1^*$ and $H_2 = H_3^*$, and also $H_1 = H_2^*$ and $H_2 = H_3^*$ in (3.32). For conciseness, we will not write them explicitly here, but the procedure to find them follows straightforwardly. In all the cases there will be four commuting Killing vectors namely, ∂_α, ∂_β, ∂_t and ∂_x.

Metrics of the form (3.29) have been considered in the physical literature. For instance, the asymptotic form of any "ALG" instanton is (3.29) if the two coordinates x and t are periodically identified. Several examples of such geometries have been considered in [36, 37]. Let us recall that the term ALG usually stands for a complete elliptically fibered hyper-Kähler manifold. The metric (3.29) is not an ALG metric and in general is not complete; however, it is what an ALG metric approaches at infinity. Another context in which (3.29) have been considered in the context of stringy cosmic strings [38]. Besides, a particular case of such class of metrics were shown to describe the single matter hypermultiplet target space for type IIA superstrings compactified on a Calabi-Yau threefold when supergravity and D-instanton effects are switched off [39]. In this case we have $\tau = \log(w)$ with $T(w) = w(\log(w) - 1)$, which, from (3.32), it yields the following explicit G_2 holonomy metric

$$g_7 = \frac{(\,d\alpha - \Im\,(wdt + w(\log(w) - 1\,)dx))^2}{\mu^2} + \frac{\Big(\,d\beta - Q\,\Re(xdt + \frac{w}{2}(\,\log(w) - 1)dw\,)\,\Big)^2}{(1+Q\mu)^2}$$

$$+\mu^2\,(1+Q\mu)^2\,d\mu^2 + \,\mu\,(1+Q\mu)\,\left(\frac{|\,dt + \log(w)dx\,|^2}{\log|w|} + \log|w|\,dwd\overline{w}\right). \qquad (3.33)$$

The base hyper-Kähler metric possesses a rotational Killing vector $\partial_{\arg(w)}$. However, such vectors do not preserve the hyper-Kähler triplet (3.31) and, for this reason, they will not be a Killing vector of the seven-metric (3.33).

It is interesting to analyze the transformation properties of the generic metric (3.29) under the $SL(2,\mathbf{R})$ action

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + c}, \qquad t \to at - bx, \qquad x \to dx - ct,$$

where the parameters a, b, c, d satisfy $ad - bc = 1$. Then, we obtain that

$$\tau_2 \longrightarrow \frac{ad\tau_2}{|c\tau + d|^2},$$

and the metric (3.29) transforms as

$$g_4 = \frac{|dt + \tau dx|^2}{\tau_2} + \tau_2 \frac{dwd\overline{w}}{|c\tau + d|^2}.$$

By defining new complex coordinates ξ, by

$$d\xi = \frac{dw}{c\tau + d},$$

the metric becomes

$$g_4 = \frac{|dt + \tau dx|^2}{\tau_2} + \tau_2 d\xi d\overline{\xi},$$

which is of the form (3.29) and therefore hyper-Kähler. However, notice the transformed metric is different than the former one, because τ will be a different function after the coordinate transformation $w \longrightarrow \xi$. Therefore, in principle, the $SL(2,\mathbf{R})$ action is not a symmetry of (3.29), but instead it maps any element of the family of toric hyper-Kähler metric (3.29) into another one; namely, it is closed among such family and can be regarded as a subgroup of the asymptotic ALG symmetries.

If the coordinates x and t are periodic, the transverse space to the w-plane will be $T^2 = U(1) \times U(1)$. However, again, let us emphasize that only one of the $U(1)$ isometries of those which would correspond to the referred T^2 is globally defined for the metric (3.29). In this case τ can be interpreted as the modulus of the tori by restricting it to the fundamental $SL(2, \mathbf{Z})$ domain. Under this domain we have the action of the modular transformations $\tau \to \tau + 1$ and $\tau \to -1/\tau$. Function τ will have certain magnetic source singularities around the w-plane. By going around such singularities one comes up against a jump $\tau \to \tau+1$, so that the singularities behave like $2\pi\tau \sim -i \log(w - w_i)$. By changing coordinate in the complex plane the metric can be expressed as

$$g_4 = \frac{|dt + \tau dx|^2}{\tau_2} + \tau_2 |h(\xi)|^2 d\xi d\overline{\xi},$$

being $h(\xi)$ an holomorphic function. If we require this metric to be modular invariant we obtain certain restrictions over τ and $h(\xi)$ [38]. This requirement implies that τ is given by the rational function

$$j(\tau) = \frac{P(\xi)}{Q(\xi)},$$

being P and Q polynomials in the complex variable ξ. The function h is related to τ by

$$h(w) = \eta(\tau)\overline{\eta}(\tau) \prod_{k=1}^{N} (\xi - \xi_i)^{-1/12}$$

where $\eta(\tau)$ is the Dedekind function

$$\eta(\tau) = q^{1/24} \prod_n (1 - q^n), \qquad q = \exp(2\pi i \tau),$$

and N is the number of singularities ξ_i of the function τ. This corresponds to the seven brane solution in [38]. The corresponding 4-metric g_4 is hyper-Kähler with respect to the triplet

$$\begin{gathered} \overline{J}_1^* = \Re\left(dt \wedge dx + \frac{1}{2}\tau |h(\xi)|^2 d\xi \wedge d\overline{\xi}\right) \\ \overline{J}_2^* = \Im\left(\Upsilon\right), \qquad \overline{J}_3^* = \Re\left(\Upsilon\right), \\ \Upsilon = h(\xi) dt \wedge d\xi + dx \wedge dT. \end{gathered} \tag{3.34}$$

We see that only the first 2-form in (3.34) is modular invariant, therefore when modular invariant hyper-Kähler metrics are extended to a G_2 holonomy one, the modular invariance is generically lost through the extension.

4. G_2 holonomy metrics fibered over gravitational instantons

4.1 The Taub-Nut case

The family of G_2 metrics presented in the previous subsection was constructed with a toric hyper-Kähler basis whose corresponding Killing vectors were tri-holomorphic, i.e. they preserved the hyper-Kähler triplet (3.23). For this reason, they turned out to be isometries of the full G_2 metric. The following examples we consider do deal with toric hyper-Kähler metrics, but possessing only one tri-holomorphic Killing vector. Therefore only this vector will extend the isometry of the resulting seven-dimensional metric, and the isometry group will be $U(1)\times \mathbf{R}^2$. As the first example of a less simple family, let us consider the Gibbons-Hawking metrics (3.21), and select V as the potential for the electric field of certain configuration of charges. Then it follows from (3.22) that A will be a Wu-Yang potential describing a configuration of Dirac monopoles located at the same position where the electric charges are. For a single monopole located at the origin the potentials will take the form

$$V=1+\frac{a}{r},\qquad A=\frac{a(ydx-xdy)}{r(r+z)}\qquad z>0,\qquad \widetilde{A}=\frac{a(ydx-xdy)}{r(r-z)}\qquad z\leq 0; \tag{4.35}$$

where we have defined here the radius $r^2=x^2+y^2+z^2$. The vector potential is not globally defined in $\mathbf{R}^3$ due to the string singularities in the z axis. In the overlapping region, the potential A and potential $\widetilde{A}$ differ one to each other by a gauge transformation of the form $\widetilde{A}=A-2a\;d\arctan(y/x)$. Besides, if a further gauge transformation $A\to A+a\;d\arctan(y/x)$ and $\widetilde{A}\to\widetilde{A}-a\;d\arctan(y/x)$ is performed, the vector potential will be given by a single expression, namely

$$A=\frac{az}{r}\;d\arctan(y/x); \tag{4.36}$$

nevertheless, the potential (4.36) is clearly discontinuous at the origin, as it can be seen by evaluating the limit $\delta A=\lim_{z\to 0^+}A-\lim_{z\to 0^-}A=2a\;d\arctan(y/x)$. As usual, we are assuming that this limit is taken by crossing the origin along the z axis. The upper and lower limits are different but related by a gauge transformation, as expected. The Gibbons-Hawking metric corresponding to this single monopole configuration is the Taub-Nut self-dual instanton. The local form of its metric, written in Cartesian coordinates, reads

$$g=\left(\frac{r}{r+a}\right)\left(d\tau+\frac{az}{r}d\arctan(y/x)\right)^2+\left(\frac{r+a}{r}\right)\left(dx^2+dy^2+dz^2\right). \tag{4.37}$$

Although the function z/r is discontinuous, the metric tensors corresponding to the regions $z>0$ and $z\leq 0$ can be joined to form a globally defined regular metric by defining a new variable $\widetilde{\tau}=\tau+2a\arctan(y/x)$ in the region $z\leq 0$. This implies that the variable τ turns out to be periodic. To see this clearly it is convenient to introduce cylindrical coordinates

$$x=\rho\cos\varphi,\qquad y=\rho\sin\varphi,\qquad z=\eta,\qquad d\arctan(y/x)=d\varphi;$$

and then it follows that $\widetilde{\tau}=\tau+2a\varphi$. The angle φ is periodic with period 2π and therefore the coordinate τ should be actually periodic with period $4a\pi$. Then τ can be interpreted as an angular coordinate if the parameter a is an integer number, $a\in\mathbf{Z}_{\neq 0}$.

The explicit form for the hyper-Kähler triplet for the Taub-Nut metric (4.37) is given by

$$\begin{aligned}\overline{J}_1&=\left(d\tau+\frac{az}{r}d\arctan(y/x)\right)\wedge dx-\left(1+\frac{a}{r}\right)\;dy\wedge dz,\\ \overline{J}_2&=\left(d\tau-\frac{az}{r}d\arctan(y/x)\right)\wedge dy-\left(1+\frac{a}{r}\right)\;dz\wedge dx,\\ \overline{J}_3&=\left(d\tau+\frac{az}{r}d\arctan(y/x)\right)\wedge dz-\left(1+\frac{a}{r}\right)\;dx\wedge dy.\end{aligned} \tag{4.38}$$

Then it is elementary to find the integral forms H_i. These are

$$H_1=-x\;d\tau+(\;a\;\log(r+z)+z\;)\,dy-a\;x\;d\arctan(y/x),$$

$$H_2 = -y\, d\tau - (a\ \log(r+z) + z)\ dx - a\, y\, d\arctan(y/x), \tag{4.39}$$
$$H_3 = -z\ d\tau - a\, r\, d\arctan(y/x) - x\, dy,$$

up to a total differential term.

Let us show that the one monopole solution actually corresponds to the Taub-Nut hyper-Kähler metric. In spherical coordinates

$$x = r\sin\theta\cos\varphi, \qquad y = r\sin\theta\sin\varphi, \qquad z = r\cos\theta, \qquad d\arctan(y/x) = d\varphi,$$

the metric tensor acquires the form

$$g = \left(\frac{r}{r+a}\right)(d\tau + a\cos\theta d\varphi)^2 + \left(\frac{r+a}{r}\right)(dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)). \tag{4.40}$$

Next, by defining a new radial coordinate $R = 2r + a$, it becomes

$$g = \left(\frac{R-a}{R+a}\right)(d\tau + a\cos\theta d\varphi)^2 + \frac{1}{4}\left(\frac{R+a}{R-a}\right)dR^2 + \frac{1}{4}(R^2 - a^2)(d\theta^2 + \sin^2\theta d\varphi^2), \tag{4.41}$$

which turns out to be the most familiar expression for the Taub-Nut instanton. Nevertheless, this metric is defined only in the domain $R > a$ and therefore it has no well defined flat limit $a^2 \to \infty$.

With the help of equation (4.39), it is not difficult to find the G_2 holonomy metrics (2.10) fibered over the Taub-Nut instanton. The first one is thus obtained by selecting $H_1^* = H_1$ and $H_2^* = H_2$, and the result is

$$\begin{aligned} g_7 = \mu\ (M + Q\mu)\ \left(\frac{r}{r+a}\right)\ (d\tau + A(x,y,z))^2 + \mu\ (M+Q\mu)\ \left(\frac{r+a}{r}\right)\ (\ dx^2 + dy^2 + dz^2\ ) + \\ + \mu^{-2}\left(\ d\alpha + y\ G(x,y) + F(r,z)\ dx\right)^2 + \mu^2 (M+Q\mu)^2 d\mu^2 + \\ + \frac{1}{(M+Q\mu)^2}\left(\ d\beta - Qx\ d\tau + Q\ F(r,z)\ dy - Qa\ x\ d\arctan(y/x)\right)^2, \end{aligned} \tag{4.42}$$

where the function F and the 1-forms G and A are given by

$$F(r,z) = a\ \log(r+z) + z, \qquad G(x,y) = d\tau + a\, d\arctan(y/x) \qquad A(x,y,z) = \frac{az}{r}\, d\arctan(y/x),$$

and where Q and M are two positive real parameters. Again, for $M > 0$, it can be set to 1 without loss of generality. The case $M = 0$, on the other hand, is the one considered in Ref. [1], cf. Eq. (126) there. Besides, a second G_2 holonomy metric is obtained by selecting $H_1^* = H_3$ and $H_2^* = H_2$, leading to the form

$$\begin{aligned} g_7 = \mu\ (M + Q\mu)\ \left(\frac{r}{r+a}\right)\ (d\tau + A(x,y,z))^2 + \mu\ (M+Q\mu)\ \left(\frac{r+a}{r}\right)\ (\ dx^2 + dy^2 + dz^2\ ) + \\ + \mu^{-2}\ \left(d\alpha + y\ G(x,y) + F(r,z)\ dx\right)^2 + \mu^2 (M+Q\mu)^2 d\mu^2 \\ + \frac{1}{(M+Q\mu)^2}\ \left(d\beta - Qz\ d\tau - Q\ a\ r\ d\arctan(y/x) - Q\ x\ dy\ \right)^2. \end{aligned} \tag{4.43}$$

Notice that the metric holding for the case $H_1^* = H_2$ and $H_2^* = H_3$ just corresponds to a permutation of the coordinates in (4.43) and then gives no new geometry. The Killing vectors corresponding to both metrics above are ∂_τ, ∂_α and ∂_β. We have actually worked out these metrics in other coordinate systems, but the corresponding expressions are too cumbersome to write here. The curvature tensor corresponding to both cases is irreducible and the holonomy is exactly G_2.

4.2 The Eguchi-Hanson case

Now, let us discuss the case of two monopoles on the z axis. Without losing generality, it can be considered that the monopoles are located in the positions $(0,0,\pm c)$. The potentials for this configurations are

$$V=\frac{1}{r_+}+\frac{1}{r_-},\qquad A=A_++A_-=\left(\frac{z_+}{r_+}+\frac{z_-}{r_-}\right)d\arctan(y/x),\qquad r_\pm^2=x^2+y^2+(z\pm c)^2.$$

This case corresponds to the Eguchi-Hanson instanton, whose metric, in Cartesian coordinates, reads

$$g=\left(\frac{1}{r_+}+\frac{1}{r_-}\right)^{-1}\left(d\tau+\left(\frac{z_+}{r_+}+\frac{z_-}{r_-}\right)\,d\arctan(y/x)\right)^2+\left(\frac{1}{r_+}+\frac{1}{r_-}\right)(\,dx^2+dy^2+dz^2\,),\tag{4.44}$$

where $z_\pm=z\pm c$. In order to recognize the Eguchi-Hanson metric in its standard form it is convenient to introduce a new parameter $a^2=8c$, and the elliptic coordinates defined by [44]

$$x=\frac{r^2}{8}\sqrt{1-(a/r)^4}\sin\varphi\cos\theta,\quad y=\frac{r^2}{8}\sqrt{1-(a/r)^4}\sin\varphi\sin\theta,\quad z=\frac{r^2}{8}\cos\varphi.$$

In this coordinate system it can be checked that

$$r_\pm=\frac{r^2}{8}\left(1\pm(a/r)^2\cos\varphi\right),\qquad z_\pm=\frac{r^2}{8}\left(\cos\varphi\pm(a/r)^2\right),\qquad V=\frac{16}{r^2}\left(1-(a/r)^4\cos^2\varphi\right)^{-1},$$

$$A=2\,\left(1-(a/r)^4\cos^2\varphi\right)^{-1}\,\left(1-(a/r)^4\right)\,\cos\varphi\,d\theta,$$

and, with the help of these expressions, it is found

$$g=\frac{r^2}{4}\left(\,1-(a/r)^4\,\right)\,(\,d\theta+\cos\varphi d\tau\,)^2+\left(\,1-(a/r)^4\,\right)^{-1}\,dr^2+\frac{r^2}{4}\,(\,d\varphi^2+\sin^2\varphi d\tau\,)\tag{4.45}$$

This is actually a more familiar expression for the Eguchi-Hanson instanton, indeed. Its isometry group is $U(2)=U(1)\times SU(2)/\mathbf{Z}_2$, the same as the Taub-Nut one. Actually, the Eguchi-Hanson is a limit form of the Taub-Nut instanton [45]. The holomorphic Killing vector is ∂_τ. This space is asymptotically locally Euclidean (ALE), which means that it asymptotically approaches the Euclidean metric; and therefore the boundary at infinity is locally S^3. However, the situation is rather different in what regards its global properties. This can be seen by defining the new coordinate

$$u^2=r^2\left(1-(a/r)^4\right)$$

for which the metric is rewritten as

$$g=\frac{u^2}{4}\,(\,d\theta+\cos\varphi d\tau\,)^2+\left(\,1+(a/r)^4\,\right)^{-2}\,du^2+\frac{r^2}{4}\,(\,d\varphi^2+\sin^2\varphi d\tau\,).\tag{4.46}$$

The apparent singularity at $r=a$ has been moved now to $u=0$. Near the singularity, the metric looks like

$$g\simeq\frac{u^2}{4}\,(\,d\theta+\cos\varphi d\tau\,)^2+\frac{1}{4}du^2+\frac{a^2}{4}\,(\,d\varphi^2+\sin^2\varphi d\tau\,),$$

and, at fixed τ and φ, it becomes

$$g \simeq \frac{u^2}{4}\, d\theta^2 + \frac{1}{4} du^2.$$

This expression "locally" looks like the removable singularity of $\mathbf{R}^2$ that appears in polar coordinates. However, for actual polar coordinates, the range of θ covers from 0 to 2π, while in spherical coordinates in $\mathbf{R}^3$, $0 \leq \theta < \pi$. This means that the opposite points on the geometry turn out to be identified and thus the boundary at infinite is the lens space $S^3/\mathbf{Z}_2$. For an arbitrary ALE space, the boundary will be S^3/Γ, being Γ a finite subgroup that induces the identifications. In general, the multi Taub-Nut metrics, corresponding to V with no constant term, will be ALE spaces [30]. In particular, any ALE space admits an unique self-dual metric [47].

The expressions for the integral forms corresponding to the Eguchi-Hanson metrics is a bit longer than those of the Taub-Nut case; yielding

$$H_1 = -x\, d\tau + (\log(r_+ + z_+) + \log(r_- + z_-))\, dy - 2a\, x\, d\arctan(y/x), \tag{4.47}$$

$$H_2 = +y\, d\tau\ + (\log(r_+ + z_+) + \log(r_- + z_-))\ dx + 2a\, y\, d\arctan(y/x), \tag{4.48}$$

$$H_3 = -z d\tau - a\ (\ r_+ + r_-\ )\ d\arctan(y/x). \tag{4.49}$$

By using the formulas (4.48) and (2.10) the following G_2 holonomy metric is found

$$g_7 = \mu\ (M + Q\mu)\ \left(\ \mu\ (M + Q\mu)\ d\mu^2 + \left(\frac{1}{r_+} + \frac{1}{r_-}\right)^{-1} \left(\ d\tau + \left(\frac{z_+}{r_+} + \frac{z_-}{r_-}\right)\ d\arctan(y/x)\ \right)^2\ \right)$$

$$+\mu(M + Q\mu)\left(\frac{1}{r_+} + \frac{1}{r_-}\right)(\ dx^2 + dy^2 + dz^2\ ) + \frac{(\ d\beta + Q\ H_1^*)^2}{(\ M + Q\mu\ )^2} + \frac{(\ d\alpha + H_2^*\ )^2}{\mu^2}. \tag{4.50}$$

As before, H_1^* and H_2^* are any pair of 1-forms selected among those in (4.47)-(4.49), and there are essentially two different metrics coming from these possible choices. The Killing vectors of (4.50) turn out to be the same as those for the Taub-Nut case; namely, ∂_t, ∂_α and ∂_β.

4.3 Relation to the Ward metrics

In the previous subsections we considered an array of one and two monopoles along one axis. In this one we consider an arbitrary array along one axis. This case also corresponds to axially symmetric hyper-Kähler metrics, such as the Eguchi-Hanson and Taub-Nut instantons. Thus, it is convenient to write the flat three dimensional metric in cylindrical coordinates

$$dx^2 + dy^2 + dz^2 = d\rho^2 + d\eta^2 + \rho^2 d\varphi^2.$$

Then the potentials V and A, and consequently the hyper-Kähler metrics corresponding to such an array, will be φ-independent. This means that ∂_φ is a Killing vector but, unlike ∂_t, it is not tri-holomorphic. For this reason, ∂_φ will not be necessarily a Killing vector of the full G_2 metric. In this sense, there is no advantage in considering these toric 4-metrics if one wants to enlarge the isometry group of the 7-metrics. One can consider a configuration with at least three monopoles that are not aligned [34, 35], the resulting metrics will not be toric, but the isometry group of the G_2 holonomy space will be the same; that is, ∂_t, ∂_α and ∂_β.

The interest in considering the toric hyper-Kähler examples is mainly that they already encode many well known hyper-Kähler examples. Such spaces corresponding to aligned monopoles are known as the Ward spaces [33]. The solution of the Gibbons-Hawking equation in this case reads $A = \rho U_\rho d\varphi$ and $V = U_\eta$, being U a solution of the Ward monopole equation $(\rho U_\rho)_\rho + (\rho U_\eta)_\eta = 0$. The local form for such metrics, when written in cylindrical coordinates, is

$$g = \frac{(dt + \rho U_\rho d\varphi)^2}{U_\eta} + U_\eta(d\rho^2 + d\eta^2 + \rho^2 d\varphi^2). \tag{4.51}$$

For instance, the Taub-Nut metric (4.37) would look like

$$g = \frac{\sqrt{\rho^2+\eta^2}}{a+\sqrt{\rho^2+\eta^2}}\left(dt + \frac{\eta}{\sqrt{\rho^2+\eta^2}}d\varphi\right)^2 + \frac{a+\sqrt{\rho^2+\eta^2}}{\sqrt{\rho^2+\eta^2}}(d\rho^2 + d\eta^2 + \rho^2 d\varphi^2), \tag{4.52}$$

where, for instance, we recognize that the expression (4.52) is actually of the Ward form (4.51) for the function $U = \eta + a\log(\eta + \sqrt{\eta^2+\rho^2})$, which can be checked to satisfy $(\rho U_\rho)_\rho + (\rho U_\eta)_\eta = 0$. The form (4.51) is characteristic of a hyper-Kähler metric with two commuting isometries, one of which is self-dual while the other is not. Just for completeness, let us point out that the integral forms (4.39) for the Taub-Nut metric would be expressed in cylindrical coordinates as

$$\begin{aligned} H_1 &= -\rho\cos\varphi\,(d\tau + a\,d\varphi) + U(\sin\varphi d\rho + \rho\cos\varphi d\varphi), \\ H_2 &= +\rho\sin\varphi\,(d\tau + a\,d\varphi) + U\,(\cos\varphi d\rho - \rho\sin\varphi d\varphi), \\ H_3 &= -\eta\,d\tau\, - a\,\sqrt{\rho^2+\eta^2}\,d\varphi - \rho\cos\varphi\,(\sin\varphi d\rho + \rho\cos\varphi d\varphi). \end{aligned} \tag{4.53}$$

On the other hand, the Eguchi-Hanson solution corresponds to

$$U = \log\left(\eta - c + \sqrt{(\eta-c)^2+\rho^2}\right) + \log\left(\eta + c + \sqrt{(\eta+c)^2+\rho^2}\right)$$

with $c^2 > 0$. That is, we have two point sources on the η-axis (z-axis). Also, the case $c^2 < 0$ corresponds to the potential for an axially symmetric circle of charge, called Eguchi-Hanson metric of the type I, and which is always incomplete.

Further, let us consider the fundamental solution of the Ward equation, namely

$$U_i = a_i \log(\eta - \eta_i + \sqrt{(\eta-\eta_i)^2+\rho^2}).$$

This corresponds to a monopole of charge a_i located in the position η_i. It is not difficult to find the explicit expressions for the forms H_i in analogous way to that in the case analyzed before. If we consider an array of aligned monopoles, the forms H_i will be given by

$$\begin{aligned} H_1 &= -\rho\cos\varphi\,(\,dt + Ad\varphi\,) + U(\sin\varphi d\rho + \rho\cos\varphi d\varphi_i) + b\eta(\sin\varphi d\rho + \rho\cos\varphi d\varphi), \\ H_2 &= -\rho\sin\varphi\,(\,dt + Ad\varphi\,) - U\,(\cos\varphi d\rho - \rho\sin\varphi d\varphi) + b\eta(\cos\varphi d\rho - \rho\sin\varphi d\varphi), \\ H_3 &= -\eta\,d\tau - \sum_i a_i\,\sqrt{\rho^2+(\eta-\eta_i)^2}\,d\varphi - b\rho\cos\varphi\,(\sin\varphi d\rho + \rho\cos\varphi d\varphi). \end{aligned} \tag{4.54}$$

where now U would be given by $U = \sum_i U_i$ and $A = \sum_i a_i$. The parameter b adds a constant to V, for the Eguchi-Hanson metric we have $b = 0$ and for Taub-Nut $b = 1$. The case of infinite array of monopoles periodically distributed over the axis has been also considered in the literature; and this was within the context of D-instantons in type IIA superstring theory

[39]. All these examples concern the Ward-type spaces in four-dimensions. From (2.10), one can find the corresponding G_2 holonomy metrics, which again take the form

$$g_7 = \frac{\mu\,(M+Q\mu)}{U_\eta}\ (dt+\rho U_\rho d\varphi)^2 + \mu\,(M+Q\mu)\,U_\eta(d\rho^2+d\eta^2+\rho^2 d\varphi^2)$$

$$+\frac{(d\beta+QH_1^*)^2}{(M+Q\mu)^2}+\frac{(d\alpha+H_2^*)^2}{\mu^2}+\mu^2\,(M+Q\mu)^2\,d\mu^2, \tag{4.55}$$

being H_1^* and H_2^* any pair of forms selected among those in the hyper-Kähler triplet. Again, notice that expression (4.55) gives essentially a pair of G_2 metrics for a given Ward space (4.51).

5. Non trivial T^2 bundle over hyper-Kähler

All the G_2 metrics considered in the previous sections are solutions of the Apostolov-Salamon system (2.4), (2.5) and (2.2) together with the condition $d^c_M u = 0$. As we have seen, the equation (2.5) together with the integrability condition for H_1 implies that $\partial\widetilde{J}_1/\partial\mu$ should be μ-independent and closed. We have selected the solution $\widetilde{J}_1=(M+Q\mu)\overline{J}_1$, being $\overline{J}_1$ a closed two-form. The resulting base space was found to be hyper-Kähler with respect to certain triplet of 2-forms $\overline{J}_i$. Nevertheless, there exist in the literature examples of hyper-Kähler structures $(g_4,\,J_i,\,\overline{J}_i)$ which also admit an strictly almost Kähler structure $(g_4,\,J_0,\,\overline{J}_0)$ compatible with the opposite orientation defined by $\overline{J}_1$, $\overline{J}_2$, $\overline{J}_3$ [40]-[43]. This means that $\overline{J}_i\wedge\overline{J}_i=-\overline{J}_0\wedge\overline{J}_0$. Being "strictly almost hyper-Kähler" means that the 2-form $\overline{J}_0$ is closed though the corresponding almost complex structure J_0 defined by $\overline{J}_0=g_4(J_0\cdot,\cdot)$ is not integrable. Thus J_0 is not a complex structure. If this is the case, then we can consider the 2-form

$$\widetilde{J}_1=(M+Q\mu)\overline{J}_1+(M'+Q'\mu)\overline{J}_0, \tag{5.56}$$

for which the integrability condition $d^2H_1=0$ is also satisfied. From (2.2) it is obtained an algebraic equation for u with solution

$$u=\mu\ \Big(\,(M+Q\mu)^2-(M'+Q'\mu)^2\,\Big), \tag{5.57}$$

being M' and Q' two additional parameters. This solution again defines a G_2 holonomy space, which is in principle different from the one with $Q'=M'=0$ of the previous sections. The task of finding the corresponding G_2 holonomy metric is a little bit more complicated because the four dimensional base metric $\widetilde{g}_4(\mu)$ will not be simply given by a μ-dependent scaling of the hyper-Kähler metric g_4, as it was before. It is better to illustrate how to construct the G_2 metric with an example. Let us consider the distance element

$$g_4=x(dx^2+dy^2+dz^2)+\frac{1}{x}(dt+\frac{1}{2}zdy-\frac{1}{2}ydz)^2. \tag{5.58}$$

It is not hard to see that such metric tensor is of the Gibbons-Hawking type (3.21) and is therefore hyper-Kähler. Let us define the positive and negative oriented triplets

$$\overline{J}_1^{\pm}=(dt+\frac{1}{2}zdy)\wedge dz\pm xdx\wedge dy,$$

$$\overline{J}_2^{\pm}=(dt+\frac{1}{2}zdy-\frac{1}{2}ydz)\wedge dx\pm xdy\wedge dz, \tag{5.59}$$

$$\overline{J}_3^{\pm} = (dt - \frac{1}{2}ydz) \wedge dy \pm xdz \wedge dx.$$

Being "negative oriented" means that the square of such forms is minus the volume form of M. The metric (5.58) is hyper-Kähler with respect to the positive oriented triplet. But it is easy to see that also $d\overline{J}_1^- = d\overline{J}_3^- = 0$. One can also consider any rotated 2-form

$$\overline{J}_\theta^- = \cos\theta \overline{J}_1^- - \sin\theta \overline{J}_3^-$$

where θ runs from 0 to 2π. Such forms will be also closed and we have a whole circle of negative oriented symplectic forms $\overline{J}_\theta^-$. Nevertheless the almost complex structures J_θ^- associated to $\overline{J}_\theta^-$ are not integrable, that is, their Nijenhuis tensor is not zero. Thus they are not truly complex structures. This means that the metric (5.58) admits a circle bundle of negative oriented almost Kähler structures which are not Kähler [40]-[43]. This is the situation that we were talking about. The structures of this kind are known as strictly almost Kähler, for obvious reasons.

Now, let us select $\theta = 0$ for simplicity, and take $\overline{J}_1^-$ as $\overline{J}_0$ in (5.56). Then we have

$$\widetilde{J}_1 = (M + Q\mu)\overline{J}_1^+ + (M' + Q'\mu)\overline{J}_1^-. \tag{5.60}$$

It is convenient to introduce the basis

$$\widetilde{e}^1 = \frac{(dt + \frac{1}{2}zdy - \frac{1}{2}ydz)}{\sqrt{x}}, \qquad \widetilde{e}^2 = \sqrt{x}dz, \qquad \widetilde{e}^3 = \sqrt{x}dx, \qquad \widetilde{e}^4 = \sqrt{x}dy, \tag{5.61}$$

for the metric (5.58). Then the positive and negative oriented triplet will be written as

$$\overline{J}_1^{\pm} = \widetilde{e}^1 \wedge \widetilde{e}^2 \pm \widetilde{e}^3 \wedge \widetilde{e}^4, \qquad \overline{J}_2^{\pm} = \widetilde{e}^1 \wedge \widetilde{e}^3 \pm \widetilde{e}^4 \wedge \widetilde{e}^2, \qquad \overline{J}_3^{\pm} = \widetilde{e}^1 \wedge \widetilde{e}^4 \pm \widetilde{e}^2 \wedge \widetilde{e}^3, \tag{5.62}$$

and, combining (5.62) with (5.60), we obtain that

$$\widetilde{J}_1 = (\delta_+ M + \delta_+ Q\mu)\widetilde{e}^1 \wedge \widetilde{e}^2 + (\delta_- M + \delta_- Q\mu)\widetilde{e}^3 \wedge \widetilde{e}^4. \tag{5.63}$$

Here we have denoted $\delta_\pm M = M \pm M'$ and $\delta_\pm Q = Q \pm Q'$. By making the redefinitions

$$\begin{aligned} e^1 &= (\delta_+ M + \delta_+ Q\mu)^{1/2}\widetilde{e}^1, \qquad & e^2 &= (\delta_+ M + \delta_+ Q\mu)^{1/2}\widetilde{e}^2 \\ e^3 &= (\delta_- M + \delta_- Q\mu)^{1/2}\widetilde{e}^3, \qquad & e^4 &= (\delta_- M + \delta_- Q\mu)^{1/2}\widetilde{e}^4, \end{aligned} \tag{5.64}$$

we see that the 2-form (5.63) becomes

$$\widetilde{J}_1 = e^1 \wedge e^2 + e^3 \wedge e^4.$$

Thus $\widetilde{J}_1$ is the Kähler form of the metric $\widetilde{g}_4(\mu) = \delta_{ab} e^a \otimes e^b$. The explicit expression for $g_4(\mu)$ can be obtained from (5.64) and (5.61), the result is

$$g_4(\mu) = (\delta_- M + \delta_- Q\mu)\; x\; (\; dx^2 + dy^2\;) + (\delta_+ M + \delta_+ Q\mu)\; \Big(\; xdz^2 + \frac{1}{x}(\; dt + \frac{1}{2}zdy - \frac{1}{2}ydz\;)^2\;\Big). \tag{5.65}$$

We immediately observe that $\widetilde{g}_4(\mu)$ is not proportional to the hyper-Kähler metric (5.58) except if $Q' = M' = 0$, which corresponds to the cases considered in the previous sections.

Now, we can construct a G_2 holonomy metric for (5.65) by means of (2.3), the result is

$$g_7 = \frac{(d\beta + H_1)^2}{(M + Q\mu)^2 - (M' + Q'\mu)^2} + \left(\frac{d\alpha + H_2}{\mu}\right)^2 + \mu^2\; \Big((M + Q\mu)^2 - (M' + Q'\mu)^2\Big)\; d\mu^2$$

$$+ \mu \, (\delta_- M + \delta_- Q\mu) \, x \, (\, dx^2 + dy^2 \,) + \, \mu \, (\delta_+ M + \delta_+ Q\mu) \, \Big(\, xdz^2 + \frac{1}{x}(\, dt + \frac{1}{2}zdy - \frac{1}{2}ydz \,)^2 \, \Big), \tag{5.66}$$

where the forms H_1 and H_2 are defined by

$$dH_1 = \partial_\mu \widetilde{J}_1 = Q\overline{J}_1^+ + Q'\overline{J}_1^-, \qquad dH_2 = -\overline{J}_2^+. \tag{5.67}$$

It is really not difficult to obtain the explicit expressions of H_i from (5.59). Therefore the expression (5.66) is explicit. As before, more metrics can be obtained by selecting another elements of the hyper-Kähler triplet in order to solve (5.67).

There is another way to check that the expression (5.65) for the G_2 metric is correct. Let us consider the calibration form (2.8). Then from (5.57), (5.56) and (2.11) we find that

$$\Phi = \Big((M + Q\mu)\overline{J}_1^+ + (M' + Q'\mu)\overline{J}_1^-\Big) \wedge e^6 + e^5 \wedge e^6 \wedge e^7 \tag{5.68}$$

$$+\mu \, \sqrt{(M + Q\mu)^2 - (M' + Q'\mu)^2} \, \Big(\, \overline{J}_2 \wedge e^7 + \overline{J}_3 \wedge e^5 \, \Big).$$

By using (5.62) together with the redefinitions (5.64) it can be checked again that Φ takes the octonionic form $\Phi = c_{abc} e^a \wedge e^b \wedge e^c$. The corresponding G_2 metric is $g_7 = \delta_{ab} e^a \otimes e^b$ and after some calculation it is found the expression (5.66), which is what we wanted to show.

Although in principle the metric (5.66) contains four parameters (Q, M, Q', M'), only two of them are effective ones. In fact, by a convenient scaling in (5.64) we can select $\delta_+ M$ and $\delta_- M$ equal to one, which means that $M = 1$ and $M' = 0$. Therefore the G_2 extension presented in this subsection add two parameters to the 4-dimensional base space, unlike the extensions considered in previous sections, which did add only one. Nevertheless here we are imposing much stronger conditions on the 4-base metric. It should be not only hyper-Kähler, but also should possess a bundle of opposite oriented strictly almost Kähler structures. Only few of such spaces are known in the literature, and they are usually too simple. For instance, the 4-metric that we have presented in this subsection is one of the simplest Gibbons-Hawking ones, and contains no parameters. The resulting G_2 metric possesses two effective parameters, the same than the G_2 metrics presented in the previous sections. Investigating the existence of less trivial examples of this kind does deserve attention.

6. Half-flat associated metrics

Now, let us revisit the discussion of Ref. [2]-[9] about the half-flat metrics that are related to the G_2 spaces discussed here. For all the G_2 holonomy spaces presented so far, we can consider the six-dimensional hyper-surfaces corresponding to the foliation $\mu = const$. Then it follows from (2.10) that the metrics that have the form

$$g_6 = c_1(d\beta + QH_1)^2 + c_2(d\alpha + H_2)^2 + \widetilde{g}_4, \tag{6.69}$$

are then defined over such hyper-surfaces. Here, c_1 and c_2 are simply constants. As we will show below, metrics (6.69) are half-flat spaces [2]. These spaces are of interest in physics, specially in heterotic string compactifications [48]-[51]

It is not difficult to see that there exists a coordinate system for which metrics (2.10) take the simple form

$$g_7 = d\tau^2 + g_6(\tau), \tag{6.70}$$

being $g_6(\tau)$ a six-dimensional metric depending on τ as an evolution parameter. In fact, by introducing the new variable τ, defined by

$$\mu^2\,(M+Q\mu)^2\,d\mu^2 = d\tau^2, \tag{6.71}$$

it can be seen that (2.10) takes the desired form. Therefore these G_2 holonomy metrics are a wrapped product $Y = I_\tau \times N'$ being I a real interval. The coordinate τ is just a function of μ and is given by

$$\tau - \tau_0 = \int \mu\,(M+Q\mu)\,d\mu = \frac{M\mu^2}{2} + \frac{Q\mu^3}{3}. \tag{6.72}$$

An aspect to be emphasized is that $g_6(\tau)$ is a half-flat metric on any hypersurface Y_τ for which τ takes a constant value. Indeed, the G_2 structure (2.8) can be decomposed as

$$\Phi = \widehat{J}\wedge d\tau + \widehat{\psi}_3, \tag{6.73}$$

$$*\Phi = \widehat{\psi}_3'\wedge d\tau + \frac{1}{2}\widehat{J}\wedge\widehat{J}, \tag{6.74}$$

where we have defined

$$\widehat{J} = z^{1/2}\,\overline{J}_3 + z^{-1/2}(d\beta + A_1)\wedge(d\alpha + A_2), \tag{6.75}$$

$$\widehat{\psi}_3 = z^{-1/2}\,\widetilde{J}_1\wedge(d\alpha + A_2) + \mu\overline{J}_2\wedge(d\beta + A_1), \tag{6.76}$$

$$\widehat{\psi}_3' = \mu\,z^{-1/2}\overline{J}_2\wedge(d\alpha + A_2) - \mu^2 z^{1/2}\widetilde{J}_1\wedge(d\beta + A_1), \tag{6.77}$$

and $z = \mu^2\,(M+Q\mu)^2$. Then the G_2 holonomy conditions $d\Phi = d*\Phi = 0$ for (2.8) yield

$$d\Phi = d\widehat{\psi}_3 + (d\widehat{J} - \frac{\partial\widehat{\psi}_3}{\partial\tau})\wedge d\tau = 0,$$

$$d*\Phi = \widehat{J}\wedge d\widehat{J} + (d\widehat{\psi}_3 + \widehat{J}\wedge\frac{\partial\widehat{J}}{\partial\tau})\wedge d\tau = 0.$$

The last equations are satisfied if and only if

$$d\widehat{\psi}_3 = \widehat{J}\wedge d\widehat{J} = 0 \tag{6.78}$$

for any fixed value of τ, and

$$\frac{\partial\widehat{\psi}_3}{\partial\tau} = d\widehat{J}, \qquad \widehat{J}\wedge\frac{\partial\widehat{J}}{\partial\tau} = -d\widehat{\psi}_3. \tag{6.79}$$

These flow equations were considered by Hitchin in a rather different context, concerning certain Hamiltonian system whose details are not important here; see [26] and [2]. Equations (6.78) imply that, for every constant value τ, the metric g_6, together with $\widehat{J}$, $\widehat{\psi}_3$ and $\widehat{\psi}_3'$, form a half-flat or half-integrable structure [25]. A constant value for τ implies a constant value for μ, and the generic form of such half-flat metric corresponds to (6.69). The reasoning presented above can be also applied to the metric (5.66) by defining τ by

$$\tau - \tau_0 = \int \mu\,\sqrt{(1+Q\mu)^2 - Q'^2\mu^2}\,d\mu.$$

As an example, let us consider the G_2 metric (2.15). From (6.69), the following half-flat space is obtained

$$g_6 = c_1(\ dv - Q \ x \ dz - Q \ y \ dt \ )^2 + c_2(d\chi - y \ dz - x \ dt)^2 + (\ dx^2 + dy^2 + dz^2 + dt^2 \ ). \tag{6.80}$$

There are no technical difficulties in finding the half-flat metrics corresponding to each G_2 holonomy metric described along this work. Therefore a family of half-flat metrics for the stringy cosmic string, the Eguchi-Hanson, the Taub-Nut, the almost Kähler and the Ward cases have been found through this procedure.

7. Toric $Spin(7)$ holonomy metrics

7.1 $Spin(7)$ metrics that are T^3 bundle over hyper-Kähler

Now, we will dedicate the last section to try extend the construction described here to the case of eight-dimensional spaces with special holonomy in $Spin(7)$. In reference [1] a construction of eight-dimensional $Spin(7)$ metrics as T^3 bundles over hyper-Kähler metrics was presented. This construction is actually analogous to (2.10) for the G_2 holonomy case and leads to the following $Spin(7)$ metric

$$g_8 = \frac{(d\alpha + H_1)^2}{\mu^2} + \frac{(d\beta + H_2)^2}{\mu^2} + \frac{(d\gamma + H_3)^2}{\mu^2} + \mu^6 \ d\mu^2 + \mu^3 \ g_4, \tag{7.81}$$

where, as before, the metric g_4 is hyper-Kähler and the 1-forms H_i are given by $dH_i = \overline{J}_i$. There is not major difficulties in proving that the holonomy of this metric is in $Spin(7)$. Indeed, by defining the tetrad basis

$$e_0 = \mu^3 d\mu, \qquad e_1 = \frac{d\alpha + H_1}{\mu}, \qquad e_2 = \frac{d\beta + H_2}{\mu} \qquad e_3 = \frac{d\gamma + H_3}{\mu}, \qquad e_i = \mu^{3/2}\overline{e}_i,$$

where $\overline{e}_i$ is a tetrad for the hyper-Kähler basis and where the indices run over $i = 1, 2, 3, 4$ and $a = 1, 2, 3$. It follows that the 4-form defined by the dual octonion constants c_{abcd}

$$\Phi_4 = c_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d = e^0 \wedge e^1 \wedge e^2 \wedge e^3 + \frac{\mu^6}{6} \overline{J}^a \wedge \overline{J}^a + \mu^3 (e^0 \wedge e^a + \frac{\epsilon_{abc}}{2} e^a \wedge e^b) \wedge \overline{J}^a,$$

turns out to be closed. The presence of such closed form is characteristic from the reduction from $SO(8)$ from $Spin(7)$.

It is possible to deform this metric in order to get a new one, which will be again a T^3 bundle over an hyper-Kähler base, but now containing two more effective parameters. The natural deformation *ansatz* from (7.81) would be

$$g_8 = \mu^2 \ (M_1 + Q_1\mu)^2 \ (M_2 + Q_2\mu)^2 \ d\mu^2 + \mu \ (M_1 + Q_1\mu) \ (M_2 + Q_2\mu) \ g_4 + \frac{(d\alpha + Q_1 \ H_1)^2}{(M_1 + Q_1\mu)^2} + \frac{(d\beta + Q_2 \ H_2)^2}{(M_2 + Q_2\mu)^2} + \frac{(d\gamma + H_3)^2}{\mu^2}, \tag{7.82}$$

where M_i and Q_i are four real parameters. By defining the tetrad basis

$$e_1 = \frac{d\alpha + Q_1 \ H_1}{M_1 + Q_1\mu}, \qquad e_2 = \frac{d\beta + Q_2 \ H_2}{M_2 + Q_2\mu},$$

$$e_3 = \frac{d\gamma + H_3}{\mu}, \qquad e_i = \sqrt{\mu\ (M_1 + Q_1\mu)\ (M_2 + Q_2\mu)}\ \overline{e}_i,$$

it can be checked that also the 4-form

$$\Phi_4 = c_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d = e^0 \wedge e^1 \wedge e^2 \wedge e^3 + \frac{\mu^2\ (M_1 + Q_1\mu)^2\ (M_2 + Q_2\mu)^2}{6} \overline{J}^a \wedge \overline{J}^a$$

$$+\mu\ (M_1 + Q_1\mu)\ (M_2 + Q_2\mu)(e^0 \wedge e^a + \frac{\epsilon_{abc}}{2} e^a \wedge e^b) \wedge \overline{J}^a,$$

is closed. Therefore, the deformation (7.82) also defines an $Spin(7)$ holonomy metric. Although there are four parameters in the expression (7.82), only two of them are effective. It is easy to see that, when M_1 and M_2 are nonzero, we can set $M_1 = M_2 = 1$ by rescaling

$$\widetilde{g}_4 \to M_1\ M_2\ \widetilde{g}_4 \qquad \Rightarrow \qquad H_1 \to M_1\ M_2\ H_1, \qquad H_2 \to M_1\ M_2\ H_2 \qquad H_3 \to M_1\ M_2\ H_3$$

$$\alpha \to M_1^2\ M_2\ \alpha, \qquad \beta \to M_1\ M_2^2\ \beta, \qquad \gamma \to M_1\ M_2\ \gamma, \qquad Q_i \to \frac{Q_i}{M_i}.$$

It is actually feasible to extend any of the hyper-Kähler basis considered along this work to the case of metrics of $Spin(7)$ holonomy. By constructions, the resulting metrics will possess four commuting Killing vectors at least. For instance, for the hyper-Kähler metrics (3.29) the resulting $Spin(7)$ metrics will be given by

$$g_8 = \frac{\left(d\beta - Q_1\ \Re(xdt + \frac{1}{2}\overline{T}dw)\ \right)^2}{(1 + Q_1\mu)^2} + \frac{(\ d\alpha - Q_2\Re(wdt + Tdx)\ )^2}{(1 + Q_2\mu)^2} + \frac{(\ d\alpha - \Im(wdt + Tdx)\ )^2}{\mu^2}$$

$$+\mu^2\ (1 + Q_1\mu)^2\ (1 + Q_2\mu)^2\ d\mu^2 + \mu\ (1 + Q_1\mu)\ (1 + Q_2\mu)\ \left(\frac{|dt + \tau dx|^2}{\tau_2} + \tau_2 dw d\overline{w}\right), \quad (7.83)$$

being τ an holomorphic function on the variable w and T its primitive. This metric possesses five commuting Killing vectors ∂_α, ∂_β, ∂_γ, ∂_t and ∂_x. This procedure can be extended to the Eguchi-Hanson, Taub-Nut and Ward cases straightforwardly.

7.2 Non trivial T^3 bundle over hyper-Kähler

As a second example, let us comment on a case which is a non trivial example of T^3 bundle over hyper-Kähler. We can generalize our discussion of the section 5 in order to find metrics that are not of the Gibbons-Lü-Pope-Stelle type. As before, let us consider an hyper-Kähler structure $(g_4,\ J_i,\ \overline{J}_i)$ which also admits an strictly almost Kähler structure $(g_4,\ J_0,\ \overline{J}_0)$ compatible with the opposite orientation defined by $\overline{J}_1$, $\overline{J}_2$, $\overline{J}_3$. Let us note that the 4-form Φ_4 of an eight-dimensional metric $g_8 = \delta_{ab} e^a \otimes e^b$ can be expressed as

$$\Phi_4 = e^0 \wedge e^1 \wedge e^2 \wedge e^3 + \frac{\widetilde{J}^a \wedge \widetilde{J}^a}{6} + (e^0 \wedge e^a + \frac{\epsilon_{abc}}{2} e^b \wedge e^c) \wedge \widetilde{J}^a$$

where

$$\widetilde{J}^1 = e^5 \wedge e^6 + e^7 \wedge e^8, \qquad \widetilde{J}^2 = e^5 \wedge e^7 + e^8 \wedge e^6, \qquad \widetilde{J}^3 = e^5 \wedge e^8 + e^6 \wedge e^7. \qquad (7.84)$$

For the deformed Gibbons-Lü-Pope-Stelle metrics we have that

$$\widetilde{J}^i = \mu\,(M_1 + Q_1\mu)\,(M_2 + Q_2\mu)\overline{J}^i. \tag{7.85}$$

being $\overline{J}^i$ the hyper-Kähler triplet of the hyper-Kähler base metric. But if also $(g_4,\, J_0,\, \overline{J}_0)$ defines an strictly almost Kähler structure then equation (5.56) suggests us to modify the definition (7.85) and consider

$$\begin{aligned}
\widetilde{J}^1 &= \mu\,(M_2 + Q_2\mu)\,\Big(\,(M_1 + Q_1\mu)\,\overline{J}^1 + (M_1' + Q_1'\mu)\,\overline{J}^0\Big),\\
\widetilde{J}^2 &= \mu\,(M_2 + Q_2\mu)\,\sqrt{(M_1 + Q_1\mu)^2 - (M_1' + Q_1'\mu)^2}\,\overline{J}^2,\\
\widetilde{J}^3 &= \mu\,(M_2 + Q_2\mu)\,\sqrt{(M_1 + Q_1\mu)^2 - (M_1' + Q_1'\mu)^2}\,\overline{J}^3,
\end{aligned} \tag{7.86}$$

where Q_1' and M_1' are new parameters. There always exists an einbein $\overline{e}_i$ for which the hyper-Kähler triplet is written as

$$\begin{aligned}
\overline{J}_0 &= \overline{e}^1\wedge\overline{e}^2 - \overline{e}^3\wedge\overline{e}^4, \qquad & \overline{J}_1 &= \overline{e}^1\wedge\overline{e}^2 + \overline{e}^3\wedge\overline{e}^4,\\
\overline{J}_2 &= \overline{e}^1\wedge\overline{e}^3 + \overline{e}^4\wedge\overline{e}^2, \qquad & \overline{J}_3 &= \overline{e}^1\wedge\overline{e}^4 + \overline{e}^2\wedge\overline{e}^3,
\end{aligned} \tag{7.87}$$

Then the natural generalization of (5.64) for an einbein which "diagonalize" the tensors $\widetilde{J}^i$ is given by

$$\begin{aligned}
e^1 &= \mu^{1/2}\,(M_2 + Q_2\mu)^{1/2}(\delta_+M_1 + \delta_+Q_1\mu)^{1/2}\overline{e}^1,\\
e^2 &= \mu^{1/2}\,(M_2 + Q_2\mu)^{1/2}(\delta_+M_1 + \delta_+Q_1\mu)^{1/2}\overline{e}^2\\
e^3 &= \mu^{1/2}\,(M_2 + Q_2\mu)^{1/2}(\delta_-M_1 + \delta_-Q_1\mu)^{1/2}\overline{e}^3,\\
e^4 &= \mu^{1/2}\,(M_2 + Q_2\mu)^{1/2}(\delta_-M_1 + \delta_-Q_1\mu)^{1/2}\overline{e}^4,
\end{aligned} \tag{7.88}$$

where $\delta_\pm M_1 = M_1 \pm M_1'$ and $\delta_\pm Q_1 = Q_1 \pm Q_1'$. In terms of this basis it is easy to check by using (7.87) and (7.88) that the expressions (7.86) for $\widetilde{J}_i$ take the diagonal form (7.84), which is what we need. Also, by analogy with the cases discussed in the previous sections, we define the 1-forms

$$e_1 = \frac{d\alpha + H_1}{\sqrt{(M_1 + Q_1\mu)^2 - (M_1' + Q_1'\mu)^2}}, \qquad e_2 = \frac{d\beta + H_2}{M_2 + Q_2\mu}$$

$$e_3 = \frac{d\gamma + H_3}{\mu}, \qquad e_0 = \mu\,(M_2 + Q_2\mu)\,\sqrt{(M_1 + Q_1\mu)^2 - (M_1' + Q_1'\mu)^2}\,d\mu, \tag{7.89}$$

where the forms H_i will be given now by the equations

$$dH_1 = Q_1\overline{J}_1 + Q_1'\overline{J}_0, \qquad dH_2 = Q_3\overline{J}_2, \qquad dH_3 = \overline{J}_3. \tag{7.90}$$

The form Φ_4 corresponding to (7.89) is

$$\Phi_4 = d\mu\wedge(d\alpha + H_1)\wedge(d\beta + H_2)\wedge(d\gamma + H_3) + \frac{\mu^2\,(M_2 + Q_2\mu)^2}{6}\Big(\,f^2 - f'^2\,\Big)\overline{J}^a\wedge\overline{J}^a$$

$$+(d\beta + H_2)\wedge(d\gamma + H_3)\wedge\Big(\,f\,\overline{J}^1 + f'\,\overline{J}^0\,\Big) + (M_2 + Q_2\mu)(d\gamma + H_3)\wedge(d\alpha + H_1)\wedge\overline{J}^2$$

$$+\mu\ (d\alpha+H_1)\wedge(d\beta+H_2)\wedge\overline{J}^3+\mu^2\ (M_2+Q_2\mu)^2 d\mu\wedge(d\alpha+H_1)\wedge\Big(\ f\ \overline{J}^1+f'\ \overline{J}^0\ \Big)$$

$$+\mu\ (M_2+Q_2\mu)\ (f^2-f'^2)\ \Big(\mu\ d\mu\wedge(d\alpha+H_2)\wedge\overline{J}^2+(M_2+Q_2\mu)\ d\mu\wedge(d\gamma+H_3)\wedge\overline{J}^3\Big)$$

where we have defined $f = M_1 + Q_1\mu$ and $f' = M_1' + Q_1'\mu$. By virtue of (7.90) it follows that $d\Phi_4 = 0$, therefore Φ_4 defines an $Spin(7)$ holonomy metric. The expression for the metric $g_8 = \delta_{ab}e^a \otimes e^b$ is given by

$$g_8=\frac{(d\alpha+H_1)^2}{(M_1+Q_1\mu)^2-(M_1'+Q_1'\mu)^2}+\Big(\frac{d\beta+H_2}{M_2+Q_2\mu}\Big)^2+\Big(\frac{d\gamma+H_3}{\mu}\Big)^2+\mu\ (M_2+Q_2\mu)\ g_4(\mu), \quad (7.91)$$

where we have defined the one parameter depending four-dimensional metric

$$g_4(\mu)=(\delta_+M_1+\delta_+Q_1\mu)(\overline{e}^1\otimes\overline{e}^1+\overline{e}^2\otimes\overline{e}^2)+(\delta_-M_1+\delta_-Q_1\mu)(\overline{e}^3\otimes\overline{e}^3+\overline{e}^4\otimes\overline{e}^4).$$

Therefore, if we deal with an hyper-Kähler basis which is also strictly almost Kähler with a Kähler form with opposite orientation to the one defined by the hyper-Kähler triplet, then the expression (7.91) gives an $Spin(7)$ metric if the equations (7.90) are satisfied. This result can be applied for instance to the example (5.58). It is not difficult to check that the number of effective parameters appearing in this expression is three, we can select $M_1 = M_2 = 1$ and $M_1' = 0$ by an appropriate rescaling of coordinates. Metric (7.91) is constructed with an einbein $\overline{e}_i$ of an hyper-Kähler metric, and is a non trivial example of T^3 bundle over hyper-Kähler.

7.3 Almost G_2 holonomy hypersurfaces living inside $Spin(7)$ metrics

All the $Spin(7)$ metrics obtained in the previous subsections are of the form

$$g_8 = d\tau + g_7(\tau),$$

being τ certain coordinate. This means that all these spaces are foliated by equidistant hypersurfaces and the coordinate τ is the distance to a fixed hypersurface M. For instance for the metrics (7.81) the coordinate τ is defined by

$$\tau-\tau_0=\frac{\mu^4}{4}.$$

In such cases we have that $e^0 = d\tau$ and that the eight-space over which the metric is defined is decomposed as $M_8 = I_\tau \times M_7$ being I_τ a real interval. The closed 4-form

$$\Phi_4=e^0\wedge e^1\wedge e^2\wedge e^3+\frac{\widetilde{J}^a\wedge\widetilde{J}^a}{6}+(e^0\wedge e^a+\frac{\epsilon_{abc}}{2}e^b\wedge e^c)\wedge\widetilde{J}^a$$

can be expressed in this coordinates as

$$\Phi_4 = d\tau\wedge\omega + *_7\omega$$

where $*_7$ is the Hodge star operation defined on M_7. The closure of Φ_4 implies that

$$d(*_7\omega)=0, \quad \partial_\tau(*_7\omega)=d(\omega).$$

These equations were considered in [26]. The second is known as the gradient flow equation. The first implies that the 3-form ω is co-closed. The seven-spaces with this property are called almost G_2 holonomy spaces.

Technically, there are no difficulties to find the almost G_2 holonomy metrics for the examples presented in this section. The surfaces τ is constant are those for which μ is constant. The expression for these almost G_2 metrics

$$g_7 = c_1(d\alpha + H_1)^2 + c_2(d\beta + H_2)^2 + c_3(d\gamma + H_3)^2 + \widetilde{g}_4, \tag{7.92}$$

with c_i being constants, $\widetilde{g}_4$ being an hyper-Kähler metric and where H_i refers to the usual one-forms satisfying $dH_i = \overline{J}_i$. In fact, by making use of (7.92), any of the hyper-Kähler metrics presented along this work can be extended to an almost G_2 holonomy metric straightforwardly.

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