

**Surveying the three-dimensional fixed points of  $T$  duality**Eloy Ayón-Beato<sup>1,2,\*</sup> and Gastón Giribet<sup>3,†</sup><sup>1</sup>*Centro de Estudios Científicos (CECS), Casilla 1469, Valdivia, Chile.*<sup>2</sup>*Departamento de Física, CINVESTAV-IPN, Apdo. Postal 14-740, 07000, México D.F., México.*<sup>3</sup>*Departamento de Física, Universidad de Buenos Aires, Ciudad Universitaria, Pabellon I (1428), Buenos Aires, Argentina.*

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We explore the family of fixed points of  $T$ -duality transformations in three dimensions. For the simplest nontrivial self-duality conditions it is possible to show that, in addition to the spacelike isometry in which the  $T$ -duality transformation is performed, these backgrounds must be necessarily stationary. This allows us to prove that, for nontrivial string coupling, the low energy bosonic string backgrounds, which are additionally self- $T$ -dual along an isometry direction generated by a constant norm Killing vector, are uniquely described by a two-parametric class, including only three nonsingular cases: the charged black string, the exact gravitational wave propagating along the extremal black string, and the flat space with a linear dilaton. Besides, for constant string coupling, the only self- $T$ -dual lower energy string background under the same assumptions corresponds to the Coussaert-Henneaux spacetime. Thus, we identify minimum criteria that yield a classification of these quoted examples and only these. All these  $T$ -dual fixed points describe exact backgrounds of string theory.

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**I. INTRODUCTION**

One of the most fascinating aspects of string theory is the existence of the duality symmetries, which turn out to relate different regimes of the theory that, *a priori*, would seem to be substantially distinct. The catalog of duality symmetries playing an important role in string theory includes the quoted  $T$ -duality, which relates different spacetime configurations, suggesting that a more satisfactory picture is achieved when one considers those spacetimes as describing diverse sectors of the same *string model*. For instance, duality transformations may map flat backgrounds in curved ones, with the cost of generating a nontrivial antisymmetric tensor. Moreover, even the topological aspects of the space may abdicate in the presence of  $T$ -duality transformations. Consequently, such duality symmetry can be thought of as manifesting a deeper relation existing between all the fields arising in the formulation of the theory. In general, the understanding of such a duality between two backgrounds is that both are, actually, two spacetime interpretations of the same string solution. A particular realization of this idea is the fact that, even though a given low energy string solution and its dual can correspond to a pair of distinct background configurations, the conformal field theory description of both models is indeed the same [1] and, thus, both are equivalent from the string theory point of view [2].

The interpretation of  $T$ -duality as a symmetry manifesting such a deep aspect of string theory can be found with particular conciseness in Refs. [3,4]. This perspective establishes that the spacetime metric provides just a crude description of what string geometry actually is. Then, the

metric would not be a proper characterization of the *string geometry* since it is not enough to fully realize the symmetries of the string theory. To be precise, fundamental properties of the metric as its curvature, its causal structure, or its asymptotic behavior are not, in general, invariant under duality transformations [4]. Furthermore, it is currently suggested that the existence of  $T$ -duality could be seen as a manifestation of the fact that the spacetime itself is merely an emergent notion rather than a fundamental element in string theory [5]. Then, this point can be regarded as one of the motivations to study the properties remaining invariant in the space of solutions under duality and, in particular, to study the fixed points of it. The physics at the fixed points of  $T$ -duality manifests important properties of the string theory that, in other regimes, could remain hidden. Such aspects include the well known enhancement of the symmetries of the theory, with which we are familiarized due to the well known case of string theory on flat space compactified on the self-dual radius  $R = \sqrt{2\alpha'}$ . Besides, the thermodynamics of the self-dual solutions presenting event horizons also exhibits particular properties [6], and it can be also thought of as an additional motivation to study it within the context of string theory in curved spacetime. More importantly, self- $T$ -dual backgrounds are believed to play an important role in the search for string theory vacua [7].

Here we will analyze  $T$ -duality within the context of the three-dimensional low energy effective theory coming from the bosonic string theory. A particular property of three dimensions is that all the low energy solutions with constant string coupling, i.e. constant dilaton, turn out to be exact solutions to the string theory beyond the field theory approximation. This is related to the fact that, in three dimensions, the Weyl tensor is identically zero and thus the Riemann tensor is completely determined by the Ricci

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tensor through  $R_{\mu\nu\rho\sigma} = g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\rho} - \frac{1}{2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})R$ . Besides, in three dimensions the second order Gauss-Bonnet term vanishes,  $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + R^2 - 4R_{\mu\nu}R^{\mu\nu} = 0$ , and the antisymmetric Kalb-Ramond field strength  $H_{\mu\nu\rho}$  turns out to be proportional to the volume threeform  $\eta_{\mu\nu\rho}$ . All these conditions enable one to make use of the field redefinition freedom and eventually show that a picture in which the  $D = 3$  effective action has all  $\alpha'$  corrections depending only on the derivatives of the dilaton exists. [4]. Then, those solutions of the three-dimensional low energy effective theory for which the dilaton results to be constant remain exact to all orders. Moreover, all the solutions to Einstein theory with negative cosmological constant,  $\Lambda < 0$ , can be actually embedded in the three-dimensional string theory. In fact, the leading order solution with constant string coupling, i.e. constant dilaton, in three dimensions turns out to be the three-dimensional anti-de Sitter space, or its cosets over discrete subgroups. This includes, of course, the case of the Bañados-Teitelboim-Zanelli black hole [8,9] and the Coussaert-Henneaux [10] geometry (see also [11]). The last case is specially remarkable by its self- $T$ -dual character [10]. For the case of nonconstant dilaton, solutions which do not receive corrections do exist as well; the charged black string is perhaps the most celebrated example of this [12]. Thus, the catalog of solutions of three-dimensional string theory is rich enough to consider it as an interesting toy model to study the physical content of the theory.

In the next section we review the  $T$ -duality transformations and analyze the simplest properties that a nontrivial string background must satisfy in order to be a fixed point of these transformations; the possibility of having more general self-dual backgrounds is discussed in Appendix C. In Sec. III we establish a Birkhoff's theorem for such spacetimes of the simplest nontrivial class satisfying the low energy string equations. In doing this, we find two stationary branches depending on whether a nontrivial string coupling is allowed or not. For a trivial string coupling, we show that the only self- $T$ -dual background with a constant dilaton is precisely the previously quoted Coussaert-Henneaux spacetime [10]. This is actually related to the fact that the leading order three-dimensional solutions to the effective equations with constant dilaton must be constant negative curvature spacetimes as was mentioned before, i.e. they are locally  $\text{AdS}_3$ . In the other branch, corresponding to backgrounds exhibiting a non-constant dilaton, it is proven that the simplest nontrivial self- $T$ -duality configurations are uniquely described by a two-parametric family. This class is analyzed separately in Sec. IV using as guide the existence of hypersurface-orthogonal Killing fields. The above allows a classification depending on the values of the integration constants and containing just three self- $T$ -dual backgrounds which are regular or at least contain no naked singularities. For

generic values of the integration constants the backgrounds correspond to the charged black string [13]; for an extremal case they become an exact gravitational wave propagating along the extremal black string [14], and for vanishing constants we recover flat space with a linear dilaton. Some relevant information is included as Appendices. The first Appendix contains the explicit form of the lower energy string equations for the self- $T$ -dual configurations under study. A second one is devoted to justifying the gauge elections made on the dilaton in Sec. III. As it was mentioned, a third Appendix explores the possibility of constructing more general self-dual backgrounds besides the ones analyzed in the main part of the paper.

## II. ON SELF-T-DUALITY

The low energy effective action, describing the particle limit of the bosonic string dynamics in  $2 + 1$  dimensions, is given by

$$S = \int d^3x \sqrt{-g} e^{-2\Phi} \left( \frac{4}{k} + R + 4\nabla_\alpha \Phi \nabla^\alpha \Phi - \frac{1}{12} H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} \right), \quad (1)$$

where  $\Phi$  is the dilaton field,  $H_{\alpha\beta\gamma} = \partial_{[\alpha} B_{\beta\gamma]}$  is the field strength of the Kalb-Ramond field  $B_{\alpha\beta}$ , and the constant  $k$  turns out to be related with the central charge of the sigma model. For those cases where a given background has been recognized as an exact conformal one, the value of  $k$  typically corresponds to the level of the WZNW model involved. The equations of motion yielding from the action above read

$$R_{\alpha\beta} + 2\nabla_\alpha \nabla_\beta \Phi - \frac{1}{4} H_{\alpha\mu\nu} H_\beta{}^{\mu\nu} = 0, \quad (2a)$$

$$\nabla_\mu (e^{-2\Phi} H^{\mu\alpha\beta}) = 0, \quad (2b)$$

$$\square \Phi - 2\nabla_\alpha \Phi \nabla^\alpha \Phi + \frac{2}{k} + \frac{1}{12} H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} = 0. \quad (2c)$$

A very interesting property of string theory is the fact that the above low energy string equations turn out to be invariant under the often called  $T$ -duality transformations. This is a symmetry of the theory that maps any given solution  $(g_{\mu\nu}, B_{\mu\nu}, \Phi)$  with a translational symmetry, lets say along the direction  $\partial_x$ , to another solution  $(\tilde{g}_{\mu\nu}, \tilde{B}_{\mu\nu}, \tilde{\Phi})$  given by the Buscher rules [15]

$$\begin{aligned} \tilde{g}_{xx} &= 1/g_{xx}, & \tilde{g}_{xb} &= B_{xb}/g_{xx}, \\ \tilde{g}_{bc} &= g_{bc} - (g_{xb}g_{xc} - B_{xb}B_{xc})/g_{xx}, & \tilde{B}_{xb} &= g_{xb}/g_{xx}, \\ \tilde{B}_{bc} &= B_{bc} - 2g_{x[b}B_{c]x}/g_{xx}, & \tilde{\Phi} &= \Phi - \frac{1}{2} \ln g_{xx}, \end{aligned} \quad (3)$$

where the subindexes  $b, c$  label the components along the other directions, different from  $x$ .

Within this context, a natural question arises as to whether the previous system allows the existence of self- $T$ -dual configurations, i.e. the existence of string backgrounds, which result to be fixed points of the above  $T$ -duality transformations. In fact, it is well known that this is actually the case, as the study of the Coussaert-Henneaux spacetime shows [10], and what we want to analyze here is the existence of other relevant examples. These self-dual configurations are to be defined as those having a special symmetric direction such that they coincide with the corresponding  $T$ -dual modulo diffeomorphisms and gauge transformations.

The simplest way to obtain a self-dual configuration is to fix  $g_{xx} = 1$  and additionally to impose  $g_{xb} = 0$  and  $\mathbf{B} = 0$ . Under these conditions the background is manifestly self- $T$ -dual. However, we are interested in less trivial examples. That is why here we will focus our attention on all the particular cases of self- $T$ -dual configurations that present a Killing vector  $\mathbf{m} = \partial_\varphi$  with constant norm  $\mathbf{g}(\mathbf{m}, \mathbf{m}) = a^2 = \text{const}$ ; in this way the special direction is given by an appropriated normalization  $\partial_x = a^{-1}\partial_\varphi$ . Having  $g_{xx} = 1$  leaves invariant the dilaton and the component  $g_{xx}$  itself in the transformations (3). A remaining condition in order to obtain self- $T$ -duality would be then  $B_{xb} = g_{xb}$ . Notice that this last condition has to be understood as holding up to gauge transformations and coordinate transformations preserving the Killing direction  $\partial_x$ . Further, we will use this condition as a gauge fixing actually. Summarizing, demanding the existence of a constant norm Killing field is the simplest way to obtain nontrivial self-dual backgrounds. This is not a necessary requirement, which is why we study in Appendix C a possibility of constructing backgrounds without such property. Nevertheless, straightforward attempts to find an explicit example do not succeed.

The above considerations motivate the study of those string backgrounds for which the metric exhibits the generic form

$$\mathbf{g} = \mathbf{g}^{(2)} + a^2[d\varphi + W_b(x^c)dx^b]^2, \quad (4)$$

where  $\mathbf{g}^{(2)} = g_{bc}^{(2)}(x^b)dx^b dx^c$  ( $b, c = 0, 1$ ) is the metric of a two-dimensional spacetime. In the case of stationary configurations we have an additional timelike Killing field  $\mathbf{k} = \partial_t$ , and the two-dimensional metric can be written as

$$\mathbf{g}^{(2)} = -N(r)^2 F(r) dt^2 + \frac{dr^2}{F(r)}, \quad (5)$$

where a gauge election still remains to be fixed.

A particular example of the previous class is the metric of the Coussaert-Henneaux spacetime, which is given by [10,16]

$$F(r) = 1, \quad (6a)$$

$$N(r) = \cosh(2r/l), \quad (6b)$$

$$W_b(r) = \frac{1}{a} \sinh(2r/l) \delta_b^t, \quad (6c)$$

and where  $\varphi$  is periodically identified with  $\varphi + 2\pi$ . This example describes a spacetime with constant negative curvature  $R = -6/l^2$  and isometry group  $SO(2) \times SO(2, 1)$ . This background is also a solution to low energy string theory Eqs. (2) with  $\Phi = \text{const}$  and  $H_{\alpha\beta\gamma}$  given by the volume threeform modulo a constant [10]. This is because, for a constant dilaton, the low energy string action (1) becomes the Einstein-Hilbert action with an effective cosmological constant  $\Lambda = -1/l^2 = -1/k$ . This is what allows embedding of any three-dimensional constant negative curvature spacetime within low energy string theory, as it has been explicitly shown in Refs. [10,17] for the BTZ black hole [8,9] and the Coussaert-Henneaux spacetime [10], respectively; see also [4,18]. These constant curvature solutions could seem to be uninteresting from the string theory point of view since they solve the equations of motions in a regime where string theory just mimics general relativity. Nevertheless, it is worth noticing that such a regime can be, in some cases, related to a nontrivial one by  $T$ -duality, e.g. the BTZ black hole turns out to be  $T$ -dual to the charged black string solution, which presents a nontrivial dilaton configuration [17].

### III. BIRKHOFF'S THEOREM FOR A CLASS OF SELF-T-DUAL BACKGROUNDS

In this section we present a Birkhoff's theorem for the class of three-dimensional string backgrounds which remain invariant under  $T$ -duality transformations generated by constant norm directions. In fact, we prove that in addition to the constant norm spacelike Killing field  $\mathbf{m}$  realizing  $T$ -duality, there exists another Killing field  $\mathbf{k}$  which turns out to be timelike, so that the resulting three-dimensional self- $T$ -dual configurations must be necessarily stationary. This will allow us to identify an important set of quoted self- $T$ -dual examples within the class considered here.

In order to prove this result we will make use of the following explicit form for the metric

$$\mathbf{g} = -N(t, r)^2 F(t, r) dt^2 + \frac{dr^2}{F(t, r)} + a^2(d\varphi + W(t, r)dt)^2, \quad (7)$$

where we have already used part of the diffeomorphism invariance in order to fix the components  $g_{tr} = 0 = g_{\varphi r}$ . The low energy string equations (2), once evaluated for the ansatz (7), assuming that the dilaton and the Kalb-Ramond field are also  $\varphi$ -independent, are explicitly written down in Appendix A. Their integration depends on a remaining gauge election. Such gauge election and the resulting local

solutions depend on the nature of the surfaces  $e^{-\Phi} = \text{const}$ , defined by the string coupling; these surfaces can be timelike, spacelike or null, or they can be not defined at all if the string coupling turns out to be a constant. Hence, the corresponding open neighborhoods can be classified according to the three possible signs of the following norm:

$$\nu^2 \equiv \nabla_\mu e^{-\Phi} \nabla^\mu e^{-\Phi} = F e^{-2\Phi} \left( (\Phi')^2 - \frac{(\dot{\Phi})^2}{N^2 F^2} \right). \quad (8)$$

For each sign we make a separate analysis employing a different choice of coordinates, which also make easier the integration of the field equations. Notice that it is sufficient to analyze the case of positive  $F$  only, since changing the sign of  $F$  just corresponds to changing the sign of  $\nu^2$ .

### A. Case $\nu^2 > 0$ : regions $r < r_-$ or $r > r_+$

For open neighborhoods where  $\nabla_\mu e^{-\Phi} \nabla^\mu e^{-\Phi} > 0$  the surfaces  $e^{-\Phi} = \text{const}$  are timelike and we can fix the gauge such that

$$e^{-\Phi(t,r)} = \frac{r}{a}, \quad (9)$$

i.e. in this coordinate the string coupling  $e^\Phi$  vanishes when  $r$  goes to infinity. We provide a complete justification for this election in Appendix B. With this choice, the low energy string equations (see Appendix A) can be easily integrated, yielding the following metric functions:

$$F(t, r) = \frac{r^2}{k} - M + \frac{J^2}{4r^2}, \quad (10a)$$

$$N(t, r) = \frac{a}{r} N_0(t), \quad (10b)$$

$$W(t, r) = -\frac{J}{2r^2} N_0(t) + W_0(t), \quad (10c)$$

where  $N_0$  and  $W_0$  are arbitrary functions of  $t$ , and where  $M$  and  $J$  are two integration constants. Additionally, we obtain the self-duality condition

$$c^2 = \frac{J^2}{a^4}, \quad (11)$$

where the constant  $c$  is related to the axion charge per unit length, see Eq. (A1) in the Appendix A.

Moreover, functions  $N_0$  and  $W_0$  above can be eliminated by the following coordinate transformation

$$(t, r, \varphi) \mapsto \left( \int N_0(t) dt, r, \varphi + \int W_0(t) dt \right), \quad (12)$$

which makes evident that there is no time dependence.

Next, we have to integrate the Kalb-Ramond field from  $\mathbf{H} = d\mathbf{B}$  and Eq. (A1), considering that the antisymmetric field can be determined only up to gauge transformations  $\mathbf{B} \rightarrow \mathbf{B} + d\mathbf{A}$ , where  $\mathbf{A}$  is an arbitrary vector field. Then, we end up with the following stationary configuration:

$$\mathbf{g} = -\frac{a^2}{r^2} \left( \frac{r^2}{k} - M + \frac{J^2}{4r^2} \right) dt^2 + \left( \frac{r^2}{k} - M + \frac{J^2}{4r^2} \right)^{-1} dr^2 + a^2 \left( d\varphi - \frac{J}{2r^2} dt \right)^2, \quad (13a)$$

$$\Phi = -\ln\left(\frac{r}{a}\right), \quad (13b)$$

$$\mathbf{B} = \frac{Ja^2}{2r^2} dt \wedge d\varphi. \quad (13c)$$

According to our gauge election, this solution is valid only in the regions where  $F > 0$ . For values of the integration constants such that  $|J| \leq M\sqrt{k}$  the function  $F$  turns out to be positive for  $r < r_-$  or  $r > r_+$ , where  $r_\pm$  are the positive roots of the equation  $F = 0$ , and these are defined by

$$M = \frac{r_+^2 + r_-^2}{k}, \quad |J| = \frac{2r_+ r_-}{\sqrt{k}}. \quad (14)$$

On the other hand, for values of the integration constants obeying  $|J| > M\sqrt{k}$  the function  $F$  turns out to be positive everywhere. Accordingly, the coordinate  $r$  is restricted to be defined within the region  $\{r < r_-\} \cup \{r > r_+\}$  for  $|J| \leq M\sqrt{k}$  and has no restriction at all when  $|J| > M\sqrt{k}$ .

It is easy to check that in the special direction  $\partial_x = a^{-1} \partial_\varphi$  not only  $g_{xx} = 1$  but additionally

$$B_{xb} = g_{xb} = -\frac{Ja}{2r^2} \delta'_b, \quad (15)$$

and, as was mentioned at the beginning, this guarantees the self- $T$ -duality of the previous string background.

### B. Case $\nu^2 < 0$ : region $r_- < r < r_+$

Now, let us move to the open neighborhoods where  $\nu^2$  takes negative values. In these regions the surfaces  $e^{-\Phi} = \text{const}$  are spacelike and the time coordinate can be identified as (see Appendix B),

$$e^{-\Phi(t,r)} = \frac{t}{a}. \quad (16)$$

In this gauge the string coupling does decay in time. Then, as it can be deduced from the equations in Appendix A, this fact implies the following form for the metric functions,

$$F(t, r) = \frac{t^2}{a^2 N_0(r)^2 f(t)}, \quad (17a)$$

$$N(t, r) = -\frac{a}{t} N_0(r), \quad (17b)$$

$$W(t, r) = -\frac{J \int N_0(r) dr}{t^3} + W_0(t), \quad (17c)$$

where  $N_0$  and  $W_0$  are arbitrary functions of  $r$  and  $t$ , respectively, and where

$$f(t) = -\frac{t^2}{k} + M - \frac{J^2}{4t^2}, \quad (18)$$

with  $M$  and  $J$  being two integration constants. In this case the self-duality condition (11) is again fulfilled.

In order to preserve the condition  $F > 0$ , the function  $f$  must be positive as well. Hence, the above solution turns out to be valid only for  $|J| \leq M\sqrt{k}$  within the range  $t_- < t < t_+$ , where  $t_{\pm}$  are the positive roots of the equation  $f(t) = 0$ .

Once again, the functions  $N_0$  and  $W_0$  can be eliminated by rescaling the coordinate  $r$  and shifting the coordinate  $\varphi$  appropriately. However, making the following coordinate transformation:

$$(t, r, \varphi) \mapsto \left( \int N_0(r) dr, t, \varphi + \int W_0(t) dt + \frac{J}{2t^2} \int N_0(r) dr \right), \quad (19)$$

which additionally interchanges the resulting coordinates  $t$  and  $r$ , the solution takes the same form as the background (13), but now restricted to satisfy the condition  $r^2/k - M + J^2/4r^2 < 0$ . This is obeyed only when  $|J| \leq M\sqrt{k}$  for  $r_- < r < r_+$ . The open regions analyzed here for  $|J| \leq M\sqrt{k}$  are naturally glued with the corresponding ones of the previous section at the points  $r_{\pm}$ , where  $\nu^2$  changes its sign due to the vanishing of the function  $F$ , see definition (8).

Taking into account the discussion of this subsection and that of the previous one, we can conclude that the union of the different cases studied up to now gives rise to all the different patches of the background (13) for any value of its integration constants  $M$  and  $J$ . It remains to study now the case for which  $\nu^2$  vanishes in an open region.

### C. Case $\nu^2 = 0$ : Coussaert-Henneaux spacetime

Then, let us close this section discussing the case for which  $\nu$  vanishes in an open neighborhood. The fact that the surfaces  $e^{-\Phi} = \text{const.}$  are null surfaces implies

$$\dot{\Phi} = FN\Phi'. \quad (20)$$

Inserting this condition in the dilaton Eq. (A2g) we obtain

$$c^2 e^{4\Phi} = \frac{4}{k}, \quad (21)$$

i.e. the dilaton must take a constant value  $\Phi = \Phi_0$ . This implies that we must take a different gauge election in this case. Then, we choose one for which the coordinate  $r$  is a proper distance:  $F(t, r) = 1$ . The remaining metric functions are easily determined from the nontrivial field equations (see Appendix A) leading to the expressions

$$N(t, r) = N_0(t) \cosh\left(\frac{2r}{\sqrt{k}} + H_0(t)\right), \quad (22a)$$

$$W(t, r) = \frac{N_0(t)}{a} \sinh\left(\frac{2r}{\sqrt{k}} + H_0(t)\right) + W_0(t), \quad (22b)$$

where  $N_0$ ,  $W_0$ , and  $H_0$  are time dependent integration functions.

The functions  $N_0$  and  $W_0$  can be again eliminated by an appropriate rescaling of the time  $t$  and a shifting of the coordinate  $\varphi$ , similarly as the ones used in the transformation (12). This is equivalent to choose  $N_0 = 1$  and  $W_0 = 0$ . The conditions for making the function  $H_0$  to vanish turns out to be more subtle. In Ref. [16] a Birkhoff's theorem was proven for the class of metrics (7) described by gravity in presence of a negative cosmological constant. In particular, it was proven that for a metric determined by the functions above there exists a coordinate system in which  $H_0 = 0$ . The involved transformation is highly nontrivial, however, it can be found for any nontrivial function  $H_0$ , see the last Appendix of Ref. [16]. In terms of these coordinates the final configuration reads

$$g = -\cosh^2\left(\frac{2r}{\sqrt{k}}\right) dt^2 + dr^2 + a^2 \left[ d\varphi + \frac{1}{a} \sinh\left(\frac{2r}{\sqrt{k}}\right) dt \right]^2, \quad (23a)$$

$$\Phi = \Phi_0, \quad (23b)$$

$$B = -a \sinh\left(\frac{2r}{\sqrt{k}}\right) dt \wedge d\varphi, \quad (23c)$$

where the metric (23a) corresponds to the one of the Coussaert-Henneaux spacetime with constant negative curvature  $R = -6/k$  [10], if one imposes the identification  $\varphi = \varphi + 2\pi$ . The connection between the results of this subsection and those of Ref. [16] is not a mere coincidence, it is due to the fact that, for a constant dilaton, the three-dimensional low energy string effective action becomes the Einstein-Hilbert term plus a negative cosmological constant. Again, in the special direction  $\partial_x = a^{-1}\partial_\varphi$  the above background satisfies  $g_{xx} = 1$  and

$$B_{xb} = g_{xb} = \sinh\left(\frac{2r}{\sqrt{k}}\right) \delta_b^t, \quad (24)$$

which manifestly shows its self- $T$ -dual character.

## IV. CLASSIFYING BACKGROUNDS WITH NONTRIVIAL STRING COUPLING

In the previous section it was shown that there are two distinguishable branches of self- $T$ -dual backgrounds with constant norm self-dual direction, depending on whether the string coupling is constant or is not. For a constant dilaton, the Coussaert-Henneaux spacetime (23) was identified as being the only possibility. On the other hand, for a nonconstant dilaton, we obtained the general two-parameter solutions (13). In this section we will study these last solutions in order to completely identify the different string backgrounds contained within this class.

First, we notice that for  $J = 0$ ,  $M > 0$ , the configuration (13) reduces to the uncharged black string [13],

$$\mathbf{g} = -\left(1 - \frac{\mathcal{M}}{\hat{r}}\right)d\hat{t}^2 + \left(1 - \frac{\mathcal{M}}{\hat{r}}\right)^{-1} \frac{k d\hat{r}^2}{4\hat{r}^2} + d\mathbf{x}^2, \quad (25a)$$

$$\Phi = -\frac{1}{2} \ln\left(\frac{\sqrt{k}\hat{r}}{a^2}\right), \quad (25b)$$

with  $\mathbf{B} = 0$  and mass (per unit length)  $\mathcal{M} = r_+^2/\sqrt{k}$ , where we used the coordinate change

$$(t, r, \varphi) \mapsto (\bar{t} = at/\sqrt{k}, \hat{r} = r^2/\sqrt{k}, x = a\varphi). \quad (26)$$

It is known that this background corresponds to the direct product of the two-dimensional Witten black hole [19,20] and the line, which is evident making the redefinition  $\hat{r} = \mathcal{M} \cosh^2(\rho/\sqrt{k})$ ,

$$\mathbf{g} = -\tanh^2\left(\frac{\rho}{\sqrt{k}}\right)d\bar{t}^2 + d\rho^2 + d\mathbf{x}^2, \quad (27a)$$

$$\Phi = -\ln\left[\frac{\sqrt{k}^{1/2}\mathcal{M}}{a} \cosh\left(\frac{\rho}{\sqrt{k}}\right)\right]. \quad (27b)$$

This background is also known to be  $T$ -dual to the static BTZ black hole [17], which is an orbifold of the  $SL(2, \mathbb{R})_k$  WZNW model.

We will argue here that a similar situation occurs for  $J \neq 0$ , i.e. we will show that the configuration (13) coincides with the charged black string [13], except for special values of the integration constants. However, one may be puzzled about the fact that both the uncharged black string solution described above and the charged one turns out to be static spacetimes, while the background (13) does not seem to be so. In order to clarify this point, we will make a classification of the different spacetimes contained within the class (13) in terms of their properties concerning the existence of hypersurface-orthogonal Killing fields  $\mathbf{k}^s$ . This is equivalent to demand the fulfillment of the Frobenius integrability condition

$$\mathbf{k}^s \wedge d\mathbf{k}^s = 0, \quad (28)$$

for a given combination  $\mathbf{k}^s$  of the Killing fields.

On the one hand, for values such that  $|J| \leq M\sqrt{k}$  we observe that when  $r_+ > r_-$  the only combinations of the Killing fields which are hypersurface-orthogonal and time-like in the exterior region  $r > r_+$  must be proportional to

$$\mathbf{k}^s = \partial_t + \frac{r_-}{r_+\sqrt{k}} \partial_\varphi. \quad (29)$$

For the extremal case  $r_+ = r_- \neq 0$  the hypersurface-orthogonal Killing fields are also given by the previous expression, but they become null in this limit. Finally, for  $r_+ = 0 = r_-$ , any Killing combination is hypersurface-orthogonal.

On the other hand, for  $|J| \geq M\sqrt{k}$  there are no hypersurface-orthogonal Killing fields if  $J \neq 0$ , and the case of vanishing  $J$  allows both Killing fields to be hypersurface-orthogonal. We will analyze each of these cases separately in the following subsections, and we will

show that they actually describe spacetimes with different properties.

### A. Case $r_+ > r_-$ : The charged black string

For  $r_+ > r_-$  the existence of the stationary and hypersurface-orthogonal Killing field (29) in the exterior region  $r > r_+$  guarantees that this region is actually static, i.e. the off-diagonal terms in metric (13a) are just an artifact of the gauge that has been chosen. Then, we find it convenient to change to a new coordinate system adapted to  $\mathbf{k}^s$  and where the staticity turns out to be explicit; namely

$$(t, r, \varphi) \mapsto \left( \hat{t} = \frac{a}{\sqrt{k}} \frac{(r_+ t - r_- \sqrt{k} \varphi)}{(r_+^2 - r_-^2)^{1/2}}, \right. \\ \hat{r} = \frac{r^2}{\sqrt{k}}, \\ \hat{x} = \frac{a}{\sqrt{k}} \frac{(r_+ \sqrt{k} \varphi - r_- t)}{(r_+^2 - r_-^2)^{1/2}} \Big). \quad (30)$$

In these coordinates the string background (13) takes the form

$$\mathbf{g} = -\left(1 - \frac{\mathcal{M}}{\hat{r}}\right)d\hat{t}^2 + \left(1 - \frac{\mathcal{Q}^2}{\mathcal{M}\hat{r}}\right)d\hat{x}^2 + \left(1 - \frac{\mathcal{M}}{\hat{r}}\right)^{-1} \\ \times \left(1 - \frac{\mathcal{Q}^2}{\mathcal{M}\hat{r}}\right)^{-1} \frac{k d\hat{r}^2}{4\hat{r}^2}, \quad (31a)$$

$$\Phi = -\frac{1}{2} \ln\left(\frac{\sqrt{k}\hat{r}}{a^2}\right), \quad (31b)$$

$$\mathbf{B} = \frac{\mathcal{Q}}{\hat{r}} d\hat{t} \wedge d\hat{x}, \quad (31c)$$

where  $\mathcal{M} = r_+^2/\sqrt{k}$  and  $|\mathcal{Q}| = r_+ r_-/\sqrt{k}$ . This corresponds to the three-dimensional charged black string [13] with mass (per unit length)  $\mathcal{M}$  and axion charge (per unit length)  $\mathcal{Q}$ . This nonlinear sigma model corresponds to the WZNW model formulated on  $SL(2, \mathbb{R}) \times \mathbb{R}/\mathbb{R}$ .

The  $T$ -dual properties of the charged black string were explored in Refs. [12,17]. Its dualization along  $\hat{x}$  gives a boosted uncharged black string [12]. Dualizing along a more general spacelike direction both the BTZ black hole and another charged black strings can be obtained [17]. As it has been seen here, the charged black string is also a fixed point of the  $T$ -duality transformation where the self- $T$ -dual direction is given by

$$\partial_x = \frac{1}{(\mathcal{M}^2 - \mathcal{Q}^2)^{1/2}} (\mathcal{M}\partial_{\hat{x}} - |\mathcal{Q}|\partial_{\hat{t}}). \quad (32)$$

Next, let us move to consider the extremal case.

### B. Case $r_+ = r_- \neq 0$ : Gravitational wave propagating along the extremal black string

For the extremal case  $r_+ = r_- \neq 0$  the hypersurface-orthogonal Killing field (29) turns out to be null. Such kind of null symmetries are usually associated to the existence

of gravitational waves, and we will explicitly show that this is actually the case. Using the following coordinates adapted to the null vector,

$$(t, r, \varphi) \mapsto \left( v = \frac{a}{2\sqrt{k}}(t + \sqrt{k}\varphi), \right. \\ \hat{r} = \frac{r^2}{\sqrt{k}}, \\ \left. u = \frac{a}{\sqrt{k}}(t - \sqrt{k}\varphi) \right), \quad (33)$$

it is possible to express the configuration (13) as

$$g = -\left(1 - \frac{\mathcal{M}}{\hat{r}}\right)2dudv + \left(1 - \frac{\mathcal{M}}{\hat{r}}\right)^{-2} \frac{k d\hat{r}^2}{4\hat{r}^2} + \frac{\mathcal{M}}{\hat{r}} du^2, \quad (34a)$$

$$\Phi = -\frac{1}{2} \ln\left(\frac{\sqrt{k}\hat{r}}{a^2}\right), \quad (34b)$$

$$B = \frac{\mathcal{M}}{\hat{r}} d\mathbf{v} \wedge du, \quad (34c)$$

where, again,  $\mathcal{M} = r_+^2/\sqrt{k}$ . The first two terms of metric (34a) describe the extremal black string,  $Q^2 = \mathcal{M}^2$ , in null coordinates  $v = (\hat{t} + \hat{x})/2$ ,  $u = \hat{t} - \hat{x}$ . In fact, using an appropriate parametrization for  $k^s$  the above metric allows the following representation:

$$g_{\mu\nu} = g_{\mu\nu}^e + \frac{\mathcal{M}}{\hat{r}} \left(1 - \frac{\mathcal{M}}{\hat{r}}\right)^{-2} k_\mu^s k_\nu^s, \quad (35)$$

where  $g^e$  is the metric of the extremal black string. Since  $k^s$  is a null Killing field it is also geodesic, hence the above expression represents a generalized Kerr-Schild transformation of the extremal black string. In other words, the string background (34) describes an exact gravitational wave propagating along the extremal black string [14].

In Ref. [17] it was shown that the extremal BTZ black hole is  $T$ -dual to this gravitational wave (see also [21,22]). Here we have made the self- $T$ -duality of this wavelike solution explicit, and showed that this is realized along the direction

$$\partial_x = \frac{1}{2} \partial_v - \partial_u. \quad (36)$$

The next case that requires to be studied would be that for which all integration constants vanish.

### C. Case $r_+ = 0 = r_-$ : Flat space with linear dilaton

This case turns out to be the simplest one. By using the following coordinates,

$$(t, r, \varphi) \mapsto (\bar{t} = at/\sqrt{k}, \bar{r} = \sqrt{k} \ln(r/a), x = a\varphi), \quad (37)$$

the background (13) simply becomes flat space with a linear dilaton and vanishing axion; namely

$$g = -d\bar{t}^2 + d\bar{r}^2 + dx^2, \quad (38a)$$

$$\Phi = -\frac{\bar{r}}{\sqrt{k}}. \quad (38b)$$

In this case the self- $T$ -duality along the direction  $\partial_x$  is explicitly manifest. It is interesting to remark that this background is the asymptotic geometry of the two previously studied cases when their coordinate  $\hat{r}$  goes to infinity.

Finally, let us briefly discuss the cases where  $|J| > M\sqrt{k}$ .

### D. Case $|J| > M\sqrt{k}$

So far, we have examined those cases for which the horizon radius  $r_\pm$  turns out to be defined, and this actually occurs for values such that  $|J| \leq M\sqrt{k}$ . The reason for this is that for  $|J| > M\sqrt{k}$  the geometry (31a) presents a naked singularity at  $r = 0$ . This is similar to the case of the BTZ geometry. However, even though we focus our attention on those geometries where no such singularities exist, we find illustrative to discuss one particular case of that sort here. As it was previously pointed out, for these values of the integration constants the case with  $J = 0$  is special since it allows the existence of hypersurface-orthogonal Killing fields in contrast to the case  $J \neq 0$  where the existence of these fields is forbidden. Let us describe this case in some detail. For vanishing  $J$  the condition  $|J| > M\sqrt{k}$  implies that  $M$  is negative. Then, by choosing the following coordinates:

$$(t, r, \varphi) \mapsto (\bar{t} = at/\sqrt{k}, \rho = \sqrt{k} \operatorname{arcsinh}(r/\sqrt{-Mk}), x = a\varphi), \quad (39)$$

the background (13) takes the form

$$g = -\coth^2\left(\frac{\rho}{\sqrt{k}}\right) d\bar{t}^2 + d\rho^2 + dx^2, \quad (40a)$$

$$\Phi = -\ln\left[\frac{\sqrt{-Mk}}{a} \sinh\left(\frac{\rho}{\sqrt{k}}\right)\right], \quad (40b)$$

with a vanishing Kalb-Ramond field. This geometry turns out to be dual to the Witten 2D black hole times the real line (27), if the duality transformation is thought to be performed along the timelike direction  $\bar{t}$ . Timelike  $T$ -duality was discussed in Ref. [23] within the context of three-dimensional string theory, and it was shown to relate positive mass solutions to singular analogs of negative mass. Here, we are only considering standard spacelike duality transformations instead. Actually, applying  $T$ -duality in the spacelike  $x$ -direction one verifies that the background above remains invariant. It does represent a self-dual background describing a naked singularity.

Let us also notice here that a double Wick rotation can be performed, namely  $\bar{t} \rightarrow i\theta$ ,  $x \rightarrow i\tau$ , and then used to show that the metric takes the form

$$\tilde{g}^t = -d\tau^2 + d\rho^2 + \coth^2\left(\frac{\rho}{\sqrt{k}}\right)d\theta^2. \quad (41)$$

This geometry is also a solution of the low energy string equations, and corresponds to the product between the time direction  $\tau$  and the often called *trumpet geometry*, which turns out to be the  $T$ -dual to the Witten cigar [2], i.e. this is dual of the Euclidean version of the two-dimensional black hole. In fact, by performing  $T$ -duality in the  $\theta$ -direction one gets

$$\tilde{g}^t = -d\tau^2 + d\rho^2 + \tanh^2\left(\frac{\rho}{\sqrt{k}}\right)d\theta^2, \quad (42)$$

which is the product between the time direction  $\tau$  and the Witten cigar. Since the function  $\tanh^2(\rho/\sqrt{k})$  vanishes at  $\rho = 0$ , the  $\theta$ -direction finds a fixed point at the origin, where its  $T$ -duality transformation is not actually defined; this explains the divergence that its dual (41) develops at  $\rho = 0$ . Notice that this is analog to the quoted example of the three-dimensional Minkowski space  $\boldsymbol{\eta}$  expressed in polar coordinates  $(\tau, \rho, \theta)$ , which, once dualized along the  $\theta$ -direction, yields a dual  $\tilde{\boldsymbol{\eta}}$  that develops a singularity at the origin  $\rho = 0$  [4]. It is clear from Eq. (42) that performing the double Wick rotation backwards one ends up with the Witten black hole times the line (27). The trumpet geometry (41) is also  $T$ -dual to  $\text{AdS}_3$  space, as it was studied in [17] as a particular case of the BTZ geometry; and more important for us is that (40) results self-dual as well. All this *cascade of dualities* turns out to be interesting and, indeed, suggestive. First, we should point out that, from the viewpoint of the CFT world sheet formulation of string theory, both backgrounds (41) and (42) are completely equivalent. The CFT involved turns out to be the  $SL(2, \mathbb{R})_k/U(1)$  WZNW model, which presents a  $SL(2, \mathbb{R})_k \times SL(2, \mathbb{R})_k$  symmetry group, and the duality transformation that translates one target-space picture into the other simply corresponds to changing the sign in one of the currents  $J^a$  that generate one of the two  $SL(2, \mathbb{R})$ 's factors (let us say the right-handed factor). This also resembles the symmetry under dualizing the radius as  $R \rightarrow R^{-1}$  in the compactification of the free boson, which represents the prototypical example to discuss the  $T$ -duality. Regarding the CFT description of the  $T$ -duality, it would be certainly interesting to fully understand how the duality symmetry connecting both (41) and (42), and the fact that  $T$ -duality (though in a different direction) also connects the trumpet geometry to the  $\text{AdS}_3$  space. Actually, the CFT description of  $\text{AdS}_3$  strings is closely related to that of the theory formulated on the manifold (42) which, as mentioned, is the product of time and the Euclidean 2D black hole. Algebraically, such relation regards a natural realization of the symmetries that the worldsheet theory presents; and having a geometrical picture of it provides a way for working out the details of the connection existing between both realizations.

## V. CONCLUSIONS

In this paper, we studied the  $T$ -duality in three-dimensional bosonic string theory from the viewpoint of the low energy effective action. Such duality symmetry is known to manifest that different spacetime configurations can be interpreted as two different regimes of the same string background. Within the framework of the CFT description of the theory, two models that are connected by  $T$ -duality are indeed completely equivalent and consequently the two target-space interpretations are equally valid. More generally, if the isometry with respect to which one performs a given  $T$ -duality transformation corresponds to a spacelike compact direction, then the original solution and its dual correspond to the same conformal field theory [1]. More concisely, the CFT description of both sides of the duality map are equivalent and, in terms of the stringy description, this is typically realized by the interchange between winding and Kaluza-Klein momenta in the compact direction. According to this picture, those configurations that result to be self-dual are such that this interchange of quantum numbers can be realized on the same spacetime. These configurations are thought of as gathering important properties of the string theory, in particular in what respects to its symmetries.

Here, we investigated the simplest case of nontrivial self- $T$ -dual configurations and identified minimum criteria that yield a classification of previously known exact solutions of three-dimensional string theory. This amounts in imposing stringent self-duality conditions, so that the duality transformation was thought to be performed along an isometry direction generated by a Killing vector with constant norm. It is highly remarkable that the pdes system, resulting from the low energy string equations, become fully integrable in this case. The first consequence of this fact is that the resulting solutions are necessarily stationary, i.e. a Birkhoff' theorem of the sort of the one proven in [16] for pure gravity is obeyed by these configurations. An unusual fact we found through the computations is that, in order to make integrability manifest, it was necessary to take one of the gauge elections as imposed on the string coupling: the dilaton. In spite of the fact that the above is a nonstandard procedure it results fully justifiable at the local level (as we show in Appendix B).

Our main results can be stated as follows: for the case of nontrivial string coupling, the lower energy string backgrounds with a constant norm Killing field that are additionally self- $T$ -dual are uniquely described by a two-parametric class, including only three nonsingular cases: the charged black string, the exact gravitational wave propagating along the extremal black string, and flat space with a linear dilaton. Besides, for a constant string coupling, the only self- $T$ -dual lower energy string background under the same assumptions corresponds to the Coussaert-Henneaux spacetime. We also discussed other cases and, along this work, we went through the bestiary of three-

dimensional string backgrounds. Actually, we presented a survey of fixed points of  $T$ -duality transformations in three-dimensional low energy effective bosonic string theory. We did this by means of standard techniques for solving the equations of motion of three-dimensional gravity models, and the fact of having worked out a classification for the described self-dual solutions in such a simple way provides an example of how the techniques developed in Ref. [16] were suitable to be used within a more general context.

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### APPENDIX A: LOW ENERGY STRING EQUATIONS FOR SELF-T-DUAL BACKGROUNDS

A low energy string background is determined by Eqs. (2). In three dimensions the Kalb-Ramond strength must be proportional to the volume threeform  $\eta_{\alpha\beta\gamma} = \sqrt{-g}\epsilon_{\alpha\beta\gamma}$  ( $\epsilon_{tr\varphi} = +1$ ) and such proportionality is straightforwardly fixed by Eq. (2b) as being

$$H_{\alpha\beta\gamma} = ce^{2\Phi}\eta_{\alpha\beta\gamma}, \quad (\text{A1})$$

where  $c$  is an integration constant related to the axion charge per unit length. Using the above expression, the independent Einstein equations for a geometric background of the form (7) take the following form

$$-\frac{NF}{2}(E_t{}^t - E_r{}^r - WE_\varphi{}^t) = \left(\frac{\dot{\Phi}}{N}\right) + N^2F^2\left(\frac{\Phi'}{N}\right)' = 0, \quad (\text{A2a})$$

$$E_r{}^r + E_\varphi{}^\varphi + WE_\varphi{}^t = \frac{e^{2\Phi}}{2N}\left(\frac{e^{-2\Phi}(F^{-1})}{N}\right)' - \frac{e^{-2\Phi}}{2N}\left(\frac{e^{6\Phi}(e^{-4\Phi}N^2F)'}{N}\right)' - 4F(\Phi')^2 + c^2e^{4\Phi} = 0, \quad (\text{A2b})$$

$$-E_r{}^t = \frac{(F\Phi)'}{N^2F^2} + \left(\frac{\dot{\Phi}}{N^2F}\right)' = 0, \quad (\text{A2c})$$

$$\frac{2Ne^{-2\Phi}}{a^2}E_\varphi{}^t = \left(\frac{e^{-2\Phi}W'}{N}\right)' = 0, \quad (\text{A2d})$$

$$-\frac{2Ne^{-2\Phi}}{a^2}E_\varphi{}^r = \left(\frac{e^{-2\Phi}W'}{N}\right)' = 0, \quad (\text{A2e})$$

$$-\frac{2e^{-4\Phi}}{a^2}(E_\varphi{}^\varphi + WE_\varphi{}^t) = \left(\frac{e^{-2\Phi}W'}{N}\right)^2 - \frac{c^2}{a^2} = 0, \quad (\text{A2f})$$

where  $(\dot{\phantom{x}})$  and  $(\dots)'$  denote derivatives with respect to the coordinates  $t$  and  $r$ , respectively, and  $E_{\alpha\beta} = 0$  are the components of the Einstein Eqs. (2a). The remaining dilaton Eq. (2c) is given by

$$-\frac{e^{2\Phi}}{N}\left(\frac{e^{-2\Phi}\dot{\Phi}}{NF}\right)' + \frac{e^{2\Phi}}{N}(NF e^{-2\Phi}\Phi')' - \frac{c^2}{2}e^{4\Phi} + \frac{2}{k} = 0. \quad (\text{A2g})$$

From Eqs. (A2d) and (A2e) it is clear that the quantity

$$J = \frac{a^3 e^{-2\Phi} W'}{N}, \quad (\text{A3})$$

is an integration constant. The remaining equations determine the form of  $F(t, r)$ ,  $N(t, r)$ , and  $W(t, r)$ , while  $\Phi(t, r)$

is fixed by choosing an appropriate coordinate, as it is justified in the Appendix B.

### APPENDIX B: JUSTIFYING GAUGE ELECTIONS

Something that can be found puzzling is the fact that the last gauge election on the cases  $\nu^2 \neq 0$  of Sec. III were taken on the dilaton and not on the metric functions as usual. However, this choice is fully consistent with the previous ones which allow to write the metric as in Eq. (7). In order to avoid any confusion we dedicate this appendix to justify this item. The key point here is that after fixing  $g_{tr} = 0 = g_{\varphi r}$ , metric (7) presents a residual symmetry, it is form-invariant under the coordinate transformation

$$(t, r, \varphi) \mapsto (\check{t} = f_1(t, r), \check{r} = f_2(t, r), \check{\varphi} = \varphi + f_3(t, r)), \quad (\text{B1})$$

together with the redefinitions

$$\check{F} = F \left( (f_2')^2 - \frac{(\dot{f}_2)^2}{N^2 F^2} \right), \quad (\text{B2})$$

$$\check{N} = N \frac{f_2'}{\dot{f}_1} \left( (f_2')^2 - \frac{(\dot{f}_2)^2}{N^2 F^2} \right)^{-1}, \quad (\text{B3})$$

$$\check{W} = \frac{W - \dot{f}_3}{\dot{f}_1}, \quad (\text{B4})$$

where the functions  $f_i$ ,  $i = 1, 2, 3$ , obey the two conditions

$$f_1' f_2' - \frac{\dot{f}_1 \dot{f}_2}{N^2 F^2} = 0, \quad (\text{B5})$$

$$\dot{f}_1 f_3' - f_1' (\dot{f}_3 - W) = 0, \quad (\text{B6})$$

related to the fact that the transformation (B1) respects the gauge election, i.e.  $g_{\check{t}\check{r}} = 0 = g_{\check{\varphi}\check{r}}$ . So, the remaining gauge choice just corresponds to fix one of the above functions. Hence, in order to recover, for example, the gauge election (9) we just need to make a transformation of the type discussed above, with  $f_2 = a e^{-\Phi}$ , and where the functions  $f_1$  and  $f_3$  are obtained from the linear first order pdes

$$\left( \partial_t - \frac{\Phi' N^2 F^2}{\dot{\Phi}} \partial_r \right) f_1 = 0, \quad (\text{B7})$$

$$\left( \partial_t - \frac{\Phi' N^2 F^2}{\dot{\Phi}} \partial_r \right) f_3 = W. \quad (\text{B8})$$

On the other hand, the gauge election (16) is obtained by means of a similar transformation, where this time  $f_1 = a e^{-\Phi}$  and where the functions  $f_2$  and  $f_3$  come from solving the linear first order pdes

$$\left( \partial_t - \frac{\Phi' N^2 F^2}{\dot{\Phi}} \partial_r \right) f_2 = 0, \quad (\text{B9})$$

$$\left( \partial_t - \frac{\dot{\Phi}}{\Phi'} \partial_r \right) f_3 = W. \quad (\text{B10})$$

Finally, for the gauge choice of Subsec. III C we substitute  $\check{F} = 1$  in Eq. (B2), which provides a third condition on the functions  $f_i$ ,  $i = 1, 2, 3$ , additional to the conditions (B5) and (B6), and additionally determines the corresponding transformation. The solutions of all the previous first order pdes can be found integrating their corresponding characteristic ordinary systems. This guarantees the existence of the related coordinate systems.

### APPENDIX C: ON SELF-T-DUAL DIRECTIONS WITH NONCONSTANT NORM

Now, let us comment on the existence of more general examples, while still satisfying the requirements for self- $T$ -duality, but do not necessarily present a constant norm Killing vector along the isometry direction where the  $T$ -duality is being performed. We explore this possibility within a simple set-up according to which the Kalb-Ramond field  $B_{\mu\nu}$  is set to zero and the metric and dilaton acquire the following form

$$g = e^{2\xi(t,y)} (-dt^2 + dy^2) + e^{2\Psi(t,y)} dx^2, \quad (\text{C1a})$$

$$\Phi = \Phi(t, y), \quad (\text{C1b})$$

for a pair of differentiable functions  $\xi$  and  $\Psi$  which, like the dilaton  $\Phi$ , are assumed to depend only on the coordinates  $t$  and  $y$ .

Thus, the idea is to find a configuration that could still be self- $T$ -dual along the direction  $\partial_x$  even though the function  $\frac{1}{2} \ln g_{xx} = \Psi(t, y)$  is nonconstant. This would be possible due to the fact that a self- $T$ -dual solution is one that, after performing the  $T$ -duality transformation, recovers its original form up to diffeomorphisms and gauge transformations. Hence, a given configuration (C1) is self-dual if there exists a diffeomorphism

$$* : (t, y) \mapsto (T(t, y), Y(t, y)), \quad (\text{C2})$$

that leaves the two-dimensional metric block

$$g^{(2)} = e^{2\xi(t,y)} (-dt^2 + dy^2),$$

invariant, and for which the conditions

$$\check{\Psi}(t, y) = -\Psi(t, y) = \Psi(T(t, y), Y(t, y)), \quad (\text{C3a})$$

$$\check{\Phi}(t, y) = \Phi(t, y) - \Psi(t, y) = \Phi(T(t, y), Y(t, y)), \quad (\text{C3b})$$

are obeyed. These functional conditions do not look a priori so restrictive; in particular, if the coordinate transformation satisfy to be an involution,  $*^2 = 1$ , then the first condition (C3a) is a consequence of the second one (C3b), meaning that it is simply requesting that the corresponding metric function is the noninvariant part of the dilaton under the action of  $*$ , so that  $\Psi(t, y)$  is odd with respect to it.

It is instructive to think in the concise example where  $\xi(t, y)$  turns out to be symmetric under parity transformation  $*: y \mapsto -y$  while the function  $\Psi(t, y)$  is assumed to agree with the odd part of the dilaton, namely  $\Psi(t, y) = \Phi(t, y) - \Phi(t, -y)$ . Notice that such an ansatz would lead to a self- $T$ -dual configuration even for a nonconstant  $\Psi(t, y)$  satisfying these requirements. In fact, by applying  $T$ -duality along the direction  $x$ , one would obtain

$$\check{g} = e^{2\xi(t,y)} (-dt^2 + dy^2) + e^{-2\Psi(t,y)} dx^2,$$

$$\check{\Phi} = \Phi(t, y) - \Psi(t, y),$$

which means that  $\Psi(t, y)$  is odd under reversing the sign of  $y$ . Now, the requirements mentioned above would yield the

following dual configurations:

$$\tilde{\mathbf{g}} = e^{2\xi(t,-y)}(-dt^2 + (-1)^2 dy^2) + e^{2\Psi(t,-y)} d\mathbf{x}^2,$$

$$\tilde{\Phi} = \Phi(t, -y).$$

Hence, by simply renaming coordinates as  $y \mapsto Y = -y$ , we would certainly reobtain the original configuration. This can be straightforwardly extended to the generic form  $*$ :  $(t, y) \mapsto (T(t, y), Y(t, y))$  we were discussing above.

This kind of construction would provide concrete examples of self-dual backgrounds only if the conditions

(C3) are compatible with the low energy string equations. Nevertheless, the straightforward efforts to find explicit nontrivial examples do not prove to succeed. In fact, the stationary branch related to the configuration (C1), when there is no dependence on time, turns out to be fully determined from the low energy string equations by using a different gauge election. The resulting stationary configuration is only self-dual if  $\Psi(y) = 0$ , i.e. for a constant norm self-dual direction.

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