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Note on $\mathbb{Z}_2$ symmetries of the Knizhnik-Zamolodchikov equation

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We continue the study of hidden $\mathbb{Z}_2$ symmetries of the four-point $\hat{sl(2)}_k$ Knizhnik-Zamolodchikov equation initiated by Giribet [Phys. Lett. B **628**, 148 (2005)]. Here, we focus our attention on the four-point correlation function in those cases where one spectral flowed state of the sector $\omega=1$ is involved. We give a formula that shows how this observable can be expressed in terms of the four-point function of non spectral flowed states. This means that the formula holding for the winding violating four-string scattering processes in AdS_3 has a simple expression in terms of the one for the conservative case, generalizing what is known for the case of three-point functions, where the violating and the nonviolating structure constants turn out to be connected one to each other in a similar way. What makes this connection particularly simple is the fact that, unlike what one would naively expect, it is not necessary to explicitly solve the five-point function containing a single spectral flow operator to this end. Instead, nondiagonal functional relations between different solutions of the Knizhnik-Zamolodchikov equation turn out to be the key point for this short path to exist. Considering such functional relation is necessary but it is not sufficient; besides, the formula also follows from the relation existing between correlators in both Wess-Zumino-Novikov-Witten (WZNW) and Liouville conformal theories. © *2007 American Institute of Physics.*
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I. INTRODUCTION

The $SL(2,\mathbb{R})_k$ Wess-Zumino-Novikov-Witten (WZNW) model plays an important role within the context of string theory. This conformal model enters in the worldsheet description of the theory formulated on exact noncompact curved backgrounds, being the most celebrated examples: the string theory on AdS_3 and on the two-dimensional (2D) black hole (the last, by means of its gauged $SL(2,\mathbb{R})_k/U(1)$ construction). Consequently, among the main motivations, this topic received particular attention due to its relevance for black hole physics and its relation with the anti-de Sitter space (AdS)/conformal field theory (CFT) correspondence. Actually, this conformal field theory raises the hope to work out the details of the correspondence beyond the particle limit approximation.[2–4] Here, we continue the study of the observables in this theory, focusing out attention on the four-point function.

A. String amplitudes in AdS_3

Although the structure of the WZNW model on $SL(2,\mathbb{R})$ (corresponding to strings in Lorentzian AdS_3) is not yet well understood, the observables of this theory are assumed to be well defined in terms of the analytic continuation of the correlation functions in the Euclidean case, i.e.,

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in the $SL(2,\mathbb{C})_k/SU(2)$ WZNW gauged model. Hence, the string scattering amplitudes in Lorentzian AdS_3 are obtained by integrating over the worldsheet insertions of vertex operators in the model on $SL(2,\mathbb{C})/SU(2)$ and then extending the range of the indices of continuous representations in order to include the discrete representations of $SL(2,\mathbb{R})$ as well. The correlation functions defined in such a way typically develop poles in the space of representation indices and these poles are then interpreted as state conditions of bounded states (called "short strings" in Ref. 5) and yield selection rules[6,7] for the string scattering amplitudes. Besides, the continuous representations (the "long strings" in Ref. 5) have a clear interpretation as asymptotic states and define the S matrix in the Lorentzian target space.

In the last decade, we gained important information on the explicit functional form of the WZNW correlation functions, and this enabled us to study the string theory on AdS_3 beyond the supergravity approximation, both at the level of the spectrum[8] and at the level of its interactions (see Refs. 9 and 5 and references therein). Originally, some particular cases of the two- and three-point functions were explicitly computed in Refs 10 and 11, and Teschner was who presented the general result in Ref. 12 and carefully described its formal aspects in Refs. 12–14, by employing the bootstrap approach and the minisuperspace approximation. Subsequently, other approaches, such as the path integral techniques[15] and the free field representation,[16,17] were employed to rederive such correlation functions. The operator product expansion was studied in Refs. 6 and 7 in order to investigate the fusion rules, and the crossing symmetry of the correlation functions was eventually proven in Ref. 18 by making use of its analogy with the Liouville field theory[19,20] (see also Ref. 21). The study of the correlation functions was also shown to be useful for consistency checks of the conjectured dualities between this and other conformal models.[22–24]

After these objectives were achieved, the study of correlation functions in the $SL(2,\mathbb{R})_k$ WZNW model acquired a new dimension since Maldacena and Ooguri pointed out the existence of new representations of $SL(2,\mathbb{R})_k$ contributing to the spectrum of the string theory in AdS_3.[5] Then, the correlation functions involving these new states had to be analyzed as well. These new representations, obtained from the standard ones by acting with the spectral flow transformation, are semiclassically related to the possibility of the AdS_3 strings to have a nonzero winding number. This interpretation in terms of "winding numbers" does not regard a topological winding, but it turns out to be a consequence of the presence of a nontrivial NS-NS $B_{\mu\nu}$ background field. Then, from the beginning, this winding number, as a nontopological degree of freedom, was assumed to be possibly violated when the interactions were to take place. This violation was actually first suggested in a unpublished work by Fateev *et al.* for the case of the 2D black hole.[25] A free field computation of three-point function of such winding states, including the violating winding case, appeared in Ref. 9, and a similar free field realization was studied in more detail in Refs. 16 and 26. An impressive analysis of the correlation functions in the $SL(2,\mathbb{R})_k$ WZNW model was then presented in the paper by Maldacena and Ooguri.[5] There, the pole structure of two-, three- and four-point functions was discussed in the framework of the semiclassical interpretation and the AdS/CFT correspondence. The exact expressions of two- and three-point functions, including the violating winding three-point function, were fully analyzed. Besides, the string theory interpretation of such observables, as representing scattering amplitudes, was given with precision. The four-point function was also studied in Ref. 5, and, even though there is still no closed expression for the generic case available, our understanding of its analytic structure was substantially increased due to the results of that work. By making use of the factorization ansatz given by Teschner in Ref. 14, Maldacena and Ooguri proposed an analytic extension of the expression for the $SL(2,\mathbb{C})_k/SU(2)$ conformal blocks. Thus, they gave a precise prescription to integrate the monodromy invariant expression over the space of $SL(2,\mathbb{R})_k$ representations and to pick up the pole contribution of the discrete states. Perhaps, the two main observations made in Ref. 5 regarding the four-point functions are the existence of additional poles in the middle of the moduli space, and the fact that the factorization of the four-point function only permits the usual interpretation in terms of physical intermediate states for particular incoming and outgoing kinematic configurations. The analytic structure of particular four-point functions, those leading to logarithmic singularities, was studied in Ref. 27, showing that this example fits the standard structure of four-point functions in the AdS/CFT correspondence.

Despite all this information we get about four-point functions, it is worth noticing that the mentioned cases only took into account nonflowed representations (representing nonwinding string states) as those describing the incoming and outgoing states. For instance, even though the winding strings of the sector $\omega=-1$ were shown to arise in the intermediate channels of four-point functions,[5] this was shown by analyzing the processes that only involve incoming and outgoing states of the sector $\omega=0$. Then, it would be interesting to extend the study to the case of four-point functions that involve external states of the sectors $\omega\neq 0$. Here is where our result enters in the game since it actually permits to get information of the four-point winding violating functions from all what is already known about the conservative case. In fact, in this note we will show how to connect the correlation functions involving one flowed state (winding string state) of the sector $\omega=-1$ (with winding number $\omega=-1$) to the analogous quantity that merely involves nonflowed states (just strings with winding number $\omega=0$). We will also argue that this is actually analogous to the relation existing between the violating winding three-point function and the conservative one. The way of showing such connection takes into account a recent result that presents a new map between WZNW and Liouville correlators. This map, different from that employed in Ref. 14 to prove the crossing symmetry in $SL(2,\mathbb{C})/SU(2)$, was discovered by Stoyanovsky some years ago,[28] and it was further developed by Ribault and Teschner[29] and Ribault.[30,31] In Refs. 32–35 this map between both CFTs (henceforth denominated Stoyanovsky-Ribault-Teschner map, and denoted SRT map) was analyzed in the context of the implications it has in string theory (see also Ref. 36 for recent works on the relation between Liouville and WZNW correlators). In Refs. 37 and 38, a free field realization of the SRT map was given, and it was shown to reproduce the correct three-point function for the case where one spectral flowed state (string state with winding $\omega=1$) is considered. Such observable had been also computed in Ref. 1 through "the other" connection to Liouville theory, discovered by Fateev and Zamolodchikov in Ref. 19 and extended in Refs. 14, 20, and 21 to the noncompact case (henceforth denominated Fateev-Zamolodchikov map, and denoted FZ map). We will be more precise in the following paragraph.

B. Overview

Recently, a new possibility to study the four-point function involving winding states has raised because of a discovery made by Ribault[30] and Fateev,[39] stating that correlators involving winding states in WZNW can be written in terms of correlators in Liouville field theory. In the case of four-point function with one state in the sector $\omega=-1$, on which we are interested here, this turns out to correspond to the Liouville five-point function. In principle, this does not seem to imply an actual simplification since five-point function in Liouville theory cannot be simply solved either. However, we noticed that the SRT map is not the only way of mapping "the same" Liouville five-point function to a (different) four-point function of the WZNW theory. Indeed, we can also do this by employing the noncompact generalization of the FZ map.[19] If this "second map" is used, the four-point function reached in the WZNW side is one that contains four nonwinding states, enabling us to relate winding violating processes in AdS_3 with their conservative analogs. However, this is not the whole story since, unlike the SRT map,[28–30] the FZ map involves a nondiagonal correspondence between the quantum numbers of both Liouville and WZNW sides. Hence, besides the appropriate combination of the SRT and the FZ maps, a sort of a "diagonalization procedure" is also required in order to present the result in a clear form. This diagonalization is eventually achieved by using the FZ map itself and the reflection symmetry of Liouville theory. By doing something similar to that in Ref. 1, we will first indicate how such diagonalization is realized due to identities holding between different exact solutions of the KZ equation. In fact, this paper can be regarded as an addendum to Ref. 1, being the second part of our study of hidden symmetries in the four-point $\widehat{sl(2)}_k$ KZ equation. In Ref. 1, a set of $\mathbb{Z}_2$ symmetry transformations of the KZ equation was studied. Such involutions were realized by means of the action on the four indices of $SL(2,\mathbb{R})$ representations and led to prove identities between different exact solutions of the KZ equation. The main tool for working out such identities was the FZ map, mapping four-point functions of the WZNW theory to a particular subset of five-point functions of the Liouville field theory. Following this line, here we explore the implications of new symmetry transformations on the solutions of KZ equation. The plan of the paper goes as it follows: In the next section we will review the connections between four-point functions in WZNW theory and five-point function in Liouville field theory. The fact that there is no unique map of this kind [or, more precisely, the fact that the connection between the different maps which are known turns out to be non trivial, (see Ref. 29)] is the reason for a nontrivial relation between correlators of winding and nonwinding states to exist. In Sec. III, as a preliminary result, we first prove a new identity between exact solutions of the KZ equation, complementing the catalog presented in Refs. 1 and 27. This identity is again realized by a nondiagonal $\mathbb{Z}_2$ transformation of the class studied in Ref. 1, acting on the four indices of the representations of $SL(2,\mathbb{R})$. Then, this leads us to show how the four-point correlation function involving one spectral flowed state in the winding sector $\omega=-1$ can be written in terms of the correlation function of nonflowed (non winding) states.

II. CONFORMAL FIELD THEORY

A. The WZNW model

We are interested in the WZNW model formulated on the $SL(2,\mathbb{R})$ group manifold. Its action corresponds to the non-linear σ model of strings in Lorentzian AdS_3 space. On the other hand, its Euclidean version is similarly given by the gauged $SL(2,\mathbb{C})/SU(2)$ model. The states of the Euclidean model are characterized by normalizable operators on the Poincaré hyper-half-plane $H_3^+=SL(2,\mathbb{C})/SU(2)$. These operators can be conveniently written as

$$\Phi_j(x|z) = \frac{1-2j}{\pi}(u^{-1}+u|\gamma-x|^2)^{-2j}, \tag{1}$$

where the variables $\gamma,\bar{\gamma}$, and u are associated with the AdS_3 Euclidean metric in Poincaré coordinates, namely,

$$\mathrm{d}s^2 = k(u^{-2}\mathrm{d}u^2 + u^2\mathrm{d}\gamma\mathrm{d}\bar{\gamma}),$$

while the complex coordinates x and $\bar{x}$ represent auxiliary variables that expand the $SL(2,\mathbb{C})$ representations as it follows

$$J^a(z_1)\Phi_j(x|z_2) \sim \frac{1}{z_1-z_2}D_x^a\Phi_j(x|z_2) + \cdots,$$

for $a=\{3,+,-\}$. The dots "…" stand for "regular terms″ in the OPE, while the differential operators D_x^a correspond to the realization

$$D_x^3 = x\partial_x + j, \quad D_x^+ = \partial_x, \quad D_x^- = x^2\partial_x + 2jx.$$

Besides, the operators $J^a(z)$ are the local Kac-Moody currents, whose Fourier modes are defined by $J^a(z)=\Sigma_{n\in\mathbb{Z}}J_n^a z^{-1-n}$ and satisfy the $\widehat{sl(2)}_k$ affine Kac-Moody algebra of level k. The value of k is related to the string length and the AdS radius through $k=l_{\mathrm{AdS}}^2/l_s^2$. This keeps track of the conformal invariance of the WZNW theory. The Sugawara construction yields the stress tensor that can be used to compute the central charge of this theory, being

$$c = 3 + \frac{6}{k}. \tag{2}$$

The formula for the conformal dimension of operators (1) reads

$$h_j = \frac{j(1-j)}{k-2}, \tag{3}$$

where the indices take the values $j=\frac{1}{2}+i\mathbb{R}_{>0}$ for $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$. In order to propose a similar algebraic realization for the Lorentzian string theory, it is usually assumed that the observables of the Euclidean theory admit an analytic continuation in the variable j, now parametrizing both continuous and discrete representations of $\mathrm{SL}(2,\mathbb{R})$. It is worth noticing that the transformation $j\to 1-j$ is a symmetry of formula (3); this corresponds to the so-called Weyl reflection symmetry and will be important for us in Sec. III. Operators (1) represent the vertex operators of the string theory in AdS_3 and define the correlation functions. Then, after integrating over the variables z_μ, the functional form of the N-point correlators will still depend on the auxiliary variables x_μ, with $\mu=\{1,2,\ldots,N\}$. Within the context of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ interpretation, those auxiliary variables are interpreted as the coordinates of the dual conformal field theory in the boundary of AdS_3, and then acquire a geometrical meaning. This picture, employing the variable x to label the representations, is usually referred as the x basis. On the other hand, there exists a different picture that is also convenient, and in which the quantum numbers labeling the representations permits to define string states with well defined momenta in the bulk. This is the often called m basis, and employs the standard way of parametrizing representations of the $\mathrm{SL}(2,\mathbb{R})$ group, by using a pair of indices j,m. In this frame, the string states in AdS_3 are given by vectors $|j,m\rangle\otimes|j,\bar{m}\rangle$ of the $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ representations. Furthermore, as it was mentioned in the Introduction, in Ref. 5 it was shown that an additional quantum number should be included in order to fully characterize the space of states in AdS_3. This quantum number, denoted ω, is associated with the winding number of strings in AdS_3, at least in what respects to the states with a suitable asymptotic description (the long strings). From an algebraic point of view, ω labels the spectral flow transformation that generates new representations of the theory. This is the reason because we will refer to the "spectral flowed states" as "winding states," indistinctly. The vertex operators representing winding states (states with $\omega\neq 0$) in the m basis are to be denoted $\Phi^{\omega}_{j,m,\bar{m}}(z)$ and define the correlation functions in this basis, which we will denote as $\langle\Phi_{j_1,m_1,\bar{m}_1}(z_1)\Phi_{j_2,m_2,\bar{m}_2}(z_2),\ldots,\Phi_{j_N,m_N,\bar{m}_N}(z_N)\rangle$. The conformal dimension of the states in the m basis depends both on m and ω through the expression

$$h_{j,m,\omega} = \frac{j(1-j)}{k-2} - m\omega - \frac{k}{4}\omega^2.$$

In the case $\omega=0$, operators $\Phi^{\omega=0}_{j,m,\bar{m}}(z)$ are related to those of the x basis through the Fourier transform

$$\Phi^{\omega=0}_{j,m,\bar{m}}(z) = \int d^2x\,\Phi_j(x|z)x^{m-j}\bar{x}^{\bar{m}-j}. \tag{4}$$

On the other hand, the definition of string states with $\omega\neq 0$ in the x basis was studied in Ref. 5 and 40; however, these do not have a simple expression. Now, let us discuss the correlation functions in more detail.

B. The four-point KZ equation

As it was commented, the two and three-point functions in the WZNW model are known,[11] and the four-point function in the sector $\omega=0$ was studied in detail in Ref. 14, where a consistent ansatz was proposed based in the analogy with other CFTs; we detail such proposal below.

The four-point correlation functions on the zero-genus topology are determined by conformal invariance up to a factor f, which is a function of the cross ratio z and the variables j_i, x_i, and $\bar{x}_i$ that label the representations, namely,

$$\langle \Phi_{j_1}(x_1|z_1)\Phi_{j_2}(x_2|z_2)\Phi_{j_3}(x_3|z_3)\Phi_{j_4}(x_4|z_4)\rangle = \prod_{a<b}^{4} |x_a - x_b|^{2J_{ab}} \prod_{a<b}^{4} |z_a - z_b|^{2h_{ab}} |f_{j_1,j_2,j_3,j_4}(x,z)|^2, \tag{5}$$

being

$$x = \frac{(x_2-x_1)(x_3-x_4)}{(x_4-x_1)(x_3-x_2)}, \quad z = \frac{(z_2-z_1)(z_3-z_4)}{(z_4-z_1)(z_3-z_2)},$$

and where $h_{34}=h_1+h_2-h_3-h_4$, $h_{14}=-2h_1$, $h_{24}=h_1-h_2+h_3-h_4$, and $h_{23}=h_4-h_1-h_2-h_3$ and $J_{34}=j_1+j_2+j_3-j_4$, $J_{14}=-2j_1$, $J_{24}=j_1-j_2+j_3-j_4$, and $J_{13}=-j_1-j_2-j_3+j_4$.

The function $f=f_{j_1,j_2,j_3,j_4}(x,z)$ is then given by certain linear combination of solutions to the Knizhnik-Zamolodchikov (KZ) partial differential equation; i.e., that combination which turns out to be monodromy invariant. The KZ equation in the case of the $\mathrm{SL}(2,\mathbb{R})_k$ WZNW model takes the form

$$(k-2)z(z-1)\frac{\partial}{\partial z} f_{j_1,j_2,j_3,j_4}(x,z) = ((z-1)\mathcal{D}_1 + z\mathcal{D}_0) f_{j_1,j_2,j_3,j_4}(x,z), \tag{6}$$

where the differential operators are

$$\mathcal{D}_1 = x^2(x-1)\frac{\partial^2}{\partial x^2} - ((j_4-j_3-j_2-j_1-1)x^2 + 2j_2x + 2j_1x(1-x))\frac{\partial}{\partial x} + 2(j_1+j_2+j_3-j_4)j_1x - 2j_1j_2,$$

$$\mathcal{D}_0 = -(1-x)^2x\frac{\partial^2}{\partial x^2} + ((-j_1-j_2-j_3+j_4+1)(1-x) - 2j_3 - 2j_1x)(x-1)\frac{\partial}{\partial x}$$
$$+ 2(j_1+j_2+j_3-j_4)j_1(1-x) - 2j_1j_3.$$

With Ref. 14, we can consider the following ansatz for the solution:

$$f_{j_1,j_2,j_3,j_4}(x,z) = \int_{\mathcal{C}} \mathrm{d}j \frac{C(j_1,j_2,j)C(j,j_3,j_4)}{B(j)} \mathcal{G}_{j_1,j_2,j,j_3,j_4}(x|z) \times \overline{\mathcal{G}}_{j_1,j_2,j,j_3,j_4}(\bar{x}|\bar{z}), \tag{7}$$

where the functions $C(j_1,j_2,j_3)$ and $B(j_1)$ are given by the structure constants and the reflection coefficient of the $\mathrm{SL}(2,\mathbb{R})_k$ WZNW model, respectively, and where the contour of integration is defined as covering the curve $\mathcal{C}=\frac{1}{2}+i\mathbb{R}$. The integration along $\mathcal{C}$ turns out to be redundant for a monodromy invariant solution since such particular linear combination is invariant under Weyl reflection $j\to 1-j$, for which the contour transforms as the complex conjugation $\mathcal{C}\to\overline{\mathcal{C}}=-\mathcal{C}$. Here, we are interested in the relation between different solutions to Eq. (6).

C. The Liouville field theory

Now, let us briefly review the Liouville field theory, which is the other CFT in which we are interested here. Its action reads

$$S = \frac{1}{4\pi}\int \mathrm{d}^2z(-\partial\varphi\bar{\partial}\varphi + QR\varphi + \mu e^{\sqrt{2}b\varphi}), \tag{8}$$

where the background charge is given by $Q=b+b^{-1}$, and the exponential self-interaction thus corresponds to a marginal deformation. We will set the Liouville cosmological constant μ to have an appropriate value in order to make the correlation functions acquire a simple form. This is achieved by properly rescaling the zero mode of $\varphi(z)$. The Liouville field theory is reviewed with impressive detail in Ref. 41 (see also Refs. 43 and 44 and the recent in Ref. 45). The central charge of the theory is given by

$$c = 1 + 6Q^2 > 1,$$

and the vertex operators have the exponential form[46]

$$V_\alpha(z) = e^{\sqrt{2}\alpha\phi(z)}, \tag{9}$$

whose conformal dimension is given by

$$\Delta_\alpha = \alpha(Q-\alpha). \tag{10}$$

Vertex operators [Eq. (9)] define the Liouville N-point correlation functions, which are to be denoted by $\langle V_{a_1}(z_1)V_{a_2}(z_2),\ldots,V_{a_N}(z_N)\rangle$. Notice that formula (10) remains invariant under $\alpha \to Q-\alpha$, henceforth called Liouville reflection. This symmetry induces the identification between both fields $V_\alpha(z)$ and $V_{Q-\alpha}(z)$, yielding the operator valued relation

$$\langle V_{a_1}(z_1)V_{a_2}(z_2),\ldots,V_{a_N}(z_N)\rangle = R_b(\alpha_1)\langle V_{Q-a_1}(z_1)V_{a_2}(z_2),\ldots,V_{a_N}(z_N)\rangle, \tag{11}$$

which is valid for any vertex $i=\{1,2,\ldots,N\}$ (though we exemplified it here for $i=1$), and where $R_b(\alpha_1)$ represents the Liouville reflection coefficient,

$$R_b(\alpha) = \left(\pi\mu\frac{\Gamma(1-b^2)}{\Gamma(1+b^2)}\right)^{(2/b)\alpha-1-b^{-2}}\frac{\Gamma(2b\alpha-b^2)\Gamma(2b^{-1}\alpha-b^{-2})}{\Gamma(2-2b\alpha+b^2)\Gamma(2-2b^{-1}\alpha+b^{-2})}. \tag{12}$$

Another important feature of the Liouville correlation functions is the fact that those involving states with momentum $\alpha=-1/2b$ satisfy a well known partial differential equation, called the Belavin-Polyakov-Zamolodchikov (BPZ) equation. This is similar to that of the minimal models and actually holds for a wider family of correlators, namely, all of those involving certain state with momentum $\alpha_{m,n}=[(1-m)/2]b+[(1-n)/2]b^{-1}$ for any pair $m,n\in\mathbb{Z}_{>0}$. In particular, here we are interested in Liouville five-point correlators of the form

$$\langle V_{a_1}(z_1)V_{a_2}(z_2)V_{a_3}(z_3)V_{a_4}(z_4)V_{a_{1,2}=-1/2b}(z_5)\rangle.$$

Now, once both WZNW and Liouville theories were introduced, we move to the following ingredient in our discussion: the close relation existing between correlation functions of each of these two conformal theories. Such relation has multiple aspects indeed; we will discuss two of them Secs. II D and II E.

D. The Fateev-Zamolodchikov identity

The often called FZ map is a dictionary that connects four-point correlation functions in WZNW models to five-point correlation in Liouville field theory. This result was developed in Ref. 18 (see also Ref. 21) and, among other ingredients involved in its derivation, is based on the relation existing between solution of the KZ and the BPZ differential equations. Such relation was originally noticed by Fateev and Zamolodchikov in Ref. 19 for the case of the compact $SU(2)_k$ WZNW case and the minimal models, and it basically states that, starting from any given solution of the four-point KZ equation, a systematical way of constructing a solution of the five-point BPZ equation exists. The FZ map, or strictly speaking its adaptation to the noncompact WZNW model, was employed to investigate several properties of the $SL(2,\mathbb{R})_k$ conformal theory, becoming one of the most fruitful tools to this end. In particular, Teschner gave to it its closed form and used it to prove the crossing symmetry of the WZNW model by assuming that a similar relation holds for the conformal blocks. Besides, Andreev rederived the fusion rules of admissible representations of the $\widehat{sl(2)}_k$ affine algebra by similar means,[20] and Ponsot discussed the monodromy of the theory with such techniques.[21] In Ref. 1, the FZ map was shown to be useful to prove several identities between exact solutions of the KZ equation, and we will extend such result in the next section here. But first, let us briefly review the FZ statement: The observation made in Ref. 19 is that the KZ equation satisfied by the four-point functions in the WZNW model agrees with the BPZ equation satisfied by a particular set of five-point functions in Liouville field theory. More specifically, Fateev and Zamolodchikov showed that it is possible to get a solution of the KZ equation by starting with one of the BPZ system. This yields the relation

$$\langle \Phi_{j_2}(0|0)\Phi_{j_1}(x|z)\Phi_{j_3}(1|1)\Phi_{j_4}(\infty|\infty)\rangle = F_k(j_1,j_2,j_3,j_4)X_k(j_1,j_2,j_3,j_4|x,z) \times \langle V_{\alpha_2}(0)V_{\alpha_1}(z)V_{\alpha_3}(1)V_{-1/2b}(x)V_{\alpha_4}(\infty)\rangle, \tag{13}$$

where

$$X_k(j_1,j_2,j_3,j_4|x,z) = |z|^{-4b^2 j_1 j_2 + \alpha_1\alpha_2}|1-z|^{-4b^2 j_1 j_3 + \alpha_1\alpha_3}|x-z|^{-2b^{-1}\alpha_1}|x|^{-2b^{-1}\alpha_2}|1-x|^{-2b^{-1}\alpha_3},$$

$$F_k(j_1,j_2,j_3,j_4) = c_b\left(\lambda \pi b^{2b^2}\frac{\Gamma(1-b^2)}{\Gamma(1+b^2)}\right)^{1-3j_1-j_2-j_3-j_4}(\lambda\mu)^{2j_1}\prod_{\mu=1}^{4}\frac{\Upsilon_b(2j_\mu+1)}{\Upsilon_b(2\alpha_\mu)}, \tag{14}$$

and where the function $\Upsilon_b(x)$ is the one introduced in Ref. 42 when presenting the explicit form of the Liouville three-point function (see the Appendix for the definition and a survey of functional properties of the Υ_b function). The coefficient c_b is a b-dependent factor, independent of the quantum numbers j_μ and α_μ, whose explicit value can be found in Ref. 18 and references therein, though its explicit value is not important for our purpose here. The relation between the indices j_μ, that label the $\mathrm{SL}(2,\mathbb{R})_k$ representations, and the Liouville momenta α_μ involves a nondiagonal invertible transformation defined through

$$2\alpha_1 = b(j_1+j_2+j_3+j_4-1), \quad 2\alpha_i = b(j_1-j_2-j_3-j_4+2j_i)+Q, \quad i=\{2,3,4\}. \tag{15}$$

On the other hand, the relation between the WZNW level k and the Liouville parameter b is given by

$$b^{-2} = k-2 \in \mathbb{R}_{>0}.$$

Now, let us make a remark on the KPZ scaling in Eq. (13): The parameter λ is a free parameter of the WZNW theory and it can be regarded as the mass of the two-dimensional black hole when the gauged $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)$ is being considered. This is just a parameter and is free of physical significance since it can be set to any positive value by means of an appropriate rescaling of the zero mode of the dilaton. Thus, we could absorb the factor $\pi b^{2b^2}\Gamma(1-b^2)/\Gamma(1+b^2)$ in Eq. (14) by simply shifting λ. This freedom can be used to simplify the functional form of the correlators.[23]

E. The Stoyanovsky-Ribault-Teschner identity

Analogously as to the case of the FZ map, the SRT map, discovered by Stoyanovsky in Ref. 28, translates solutions of the KZ equation into solutions of the BPZ equation. But this is, in some sense, more general. Unlike the FZ map, the SRT map presents two advantages on its partner: the first is that it involves a diagonal transformation between the quantum numbers j_μ and α_μ; the second advantage, and more important indeed, is that it holds for the case of the N-point KZ equation for an arbitrary N. One of the original versions of the SRT map states the correspondence between N-point functions in the WZNW theory and the $2N-2$-point function of the Liouville theory. Furthermore, this was generalized by Ribault in order to connect any N-point functions in WZNW to M-point functions in Liouville CFT with $N \leqslant M \leqslant 2N-2$. Let us describe such generalized form below.

The Ribault formula reads

$$\left\langle \prod_{i=1}^{N} \Phi^{\omega_i}_{j_i,m_i,\bar{m}_i}(z_i) \right\rangle = \mathcal{N}_k(j_1, \ldots, j_N; m_1, \ldots, m_N) \times \prod_{r=1}^{M} \int d^2 w_r \mathcal{F}_k(j_1, \ldots, j_N; m_1, \ldots, m_N | z_1, \ldots, z_N; w_1, \ldots, w_M) \times \left\langle \prod_{t=1}^{N} V_{\alpha_t}(z_t) \prod_{r=1}^{M} V_{-1/2b}(w_r) \right\rangle \delta\left(\sum_{\mu=1}^{N} m_\mu - \frac{k}{2} M \right) \delta\left(\sum_{\mu=1}^{N} \bar{m}_\mu - \frac{k}{2} M \right), \tag{16}$$

where the normalization factor is given by

$$\mathcal{N}_k(j_1, \ldots, j_N; m_1, \ldots, m_N) = \frac{2\pi^{3-2N} b}{M! c_k^{M+2}} \prod_{i=1}^{N} \frac{c_k \Gamma(-m_i + j_i)}{\Gamma(1 - j_i + \bar{m}_i)},$$

while the z-dependent function is

$$\mathcal{F}_k(j_1, \ldots, j_N; m_1, \ldots, m_N | z_1, \ldots, z_N; w_1, \ldots, w_M) = \frac{\prod_{1 \le r < l}^{N} |z_r - z_l|^{k-2(m_r + m_l + \omega_r \omega_l k/2 + \omega_l m_r + \omega_r m_l)}}{\prod_{1 < r < l}^{M} |w_r - w_l|^{-k} \prod_{t=1}^{N} \prod_{r=1}^{M} |w_r - z_t|^{k - 2m_t}} \times \frac{\prod_{1 \le r < l}^{N} (\bar{z}_r - \bar{z}_l)^{m_r + m_l - \bar{m}_r - \bar{m}_l + \omega_l(m_r - \bar{m}_r) + \omega_r(m_l - \bar{m}_l)}}{\prod_{1 < r < l}^{M} (\bar{w}_r - \bar{z}_t)^{m_t - \bar{m}_t}}. \tag{17}$$

Here, the Liouville cosmological constant μ has been set to the value $\mu = b^2 \pi^{-2}$ for convenience; this is to make the KPZ scaling of both sides of Eq. (16) match. It is also important to keep in mind the presence of the free parameter λ of the WZNW theory, and the fact that this can still be set to an appropriate value in order to absorb the powers of $\Gamma(b^2)/\Gamma(1+b^2)$ in the KPZ overall factor of WZNW correlators.[23] For our purpose, we do not need to focus the attention on the specific value of the j-independent normalization factor, which we wrote above just for completeness, being equal to $2\pi^{3-2N} b / M! c_k^{M+2}$ where c_k represents a k-dependent factor whose exact value is discussed in Ref. 30 but, again, is not relevant for us.

As in the case of the FZ map, the relation between the level k of the WZNW theory and the parameter b of the Liouville theory is given by

$$b^{-2} = k - 2 \in \mathbb{R}_{>0},$$

while the quantum numbers labeling the states of both theories are related through

$$\alpha_i = -b j_i + b + b^{-1}/2 = b(k/2 - j_i), \quad i = \{1, 2, \ldots, N\}. \tag{18}$$

We also have

$$s = -b^{-1} \sum_{i=1}^{N} \alpha_i + b^{-2} \frac{M}{2} + 1 + b^{-2},$$

and, then, the total winding number is given by

$$\sum_{i=1}^{N}\omega_i = M + 2 - N. \tag{19}$$

This manifestly shows that scattering processes leading to the violation of the total winding number conservation can occur in principle. In particular, Ribault formula states that the four-point function involving one flowed state of the sector $\omega=-1$ obeys

$$\begin{aligned}
&\langle \Phi^{\omega_2=0}_{J_2,m_2,\bar{m}_2}(0)\Phi^{\omega_1=-1}_{J_1,m_1,\bar{m}_1}(z)\Phi^{\omega_3=0}_{J_3,m_3,\bar{m}_3}(1)\Phi^{\omega_4=0}_{J_4,m_4,\bar{m}_4}(\infty)\rangle \\
&\quad = \hat{c}_b \prod_{\mu=1}^{N} \frac{\Gamma(-m_\mu + J_\mu)}{\Gamma(1 - J_\mu + \bar{m}_\mu)} \delta\left(m_1 + m_2 + m_3 - \frac{k}{2}\right)\delta\left(\bar{m}_1 + \bar{m}_2 + \bar{m}_3 - \frac{k}{2}\right)(z)^{k/2-m_1} \\
&\qquad \times (1-z)^{k/2-m_1}(\bar{z})^{k/2-\bar{m}_1}(1-\bar{z})^{k/2-\bar{m}_1}\int d^2x (x)^{m_2-k/2}(\bar{x})^{\bar{m}_2-k/2}(1-x)^{m_3-k/2} \\
&\qquad \times (1-\bar{x})^{\bar{m}_3-k/2}(x-z)^{m_1-k/2}(\bar{x}-\bar{z})^{\bar{m}_1-k/2}\langle V_{b(k/2-J_2)}(0)V_{b(k/2-J_1)}(z) \\
&\qquad \times V_{b(k/2-J_3)}(1)V_{-1/2b}(x)V_{b(k/2-J_3)}(\infty)\rangle,
\end{aligned} \tag{20}$$

where $\hat{c}_b$ is, again, a J-independent factor whose specific value, $\hat{c}_b = 2\pi^{-5} b c_k^{N-2-M}$, is known in terms of c_k, though not important for us. Notice that we have changed the notation here, where we replaced j_μ by J_μ; this will be convenient later. Regarding the relation between J_μ and α_μ, let us make some remarks here: Notice that, according to Eq. (18), the Liouville reflection $\alpha_\mu \to Q - \alpha_\mu$ translates in terms of the WZNW correlators in doing the Weyl reflection $j \to 1-j$. This is a simple comment, but is also important. In particular, because of Eq. (11), this implies that the correlation function involving the field $\Phi_j(x|z)$ and the one involving the field $\Phi_{1-j}(x|z)$ are connected one to each other by the overall factor $R_b(b(k/2-j))$. It is not difficult to see that this simple affirmation leads directly to the exact expression for the WZNW two-point function, for instance. This comment will become important below.

III. THE FOUR-POINT FUNCTION OF WINDING STRINGS

A. The idea

The idea is to use Eqs. (13) and (20) in order to relate the four-point function $\langle \Phi^{\omega_1=-1}_{J_1,m_1,\bar{m}_1}(z_1)\Pi_{i=2}^{4}\Phi^{\omega_i=0}_{J_i,m_i,\bar{m}_i}(z_i)\rangle$ with the four-point function $\langle \Pi_{\mu=1}^{4}\Phi_{j_\mu}(x_\mu|z_\mu)\rangle$. The nexus will be the Liouville five-point function $\langle \Pi_{\mu=1}^{4} V_{\alpha_\mu}(z_\mu)V_{-1/2b}(x)\rangle$, appearing in the right hand side of both expressions: this is connected to the first through the SRT map while it is connected to the second through the FZ map. After doing this, the second step would be to "diagonalize" the relation between indices j_μ and indices J_μ. We will do this by using the preliminary result of Sec. III B. Notice that, according to Eqs. (13) and (20), the relation between momenta J_μ and j_μ is given by Eq. (29) below. The third step would be to perform a Fourier transform in order to translate the operators $\Phi_{j_\mu}(x_\mu|z_\mu)$ of the x basis into operators $\Phi_{j_\mu,m_\mu,\bar{m}_\mu}(z_\mu)$ of the m basis.

B. Nondiagonal symmetry of the KZ equation

As mentioned, the purpose here is to prove a functional relation obeyed by two solutions of the KZ equation that will be useful further. Specifically, we will see that four-point correlation functions with momenta $j_1, \ldots, j_4$ have a simple expression in terms of the analogous quantity with momenta $\tilde{j}_1, \ldots, \tilde{j}_4$, being related through

$$2\tilde{j}_\mu = \sum_{\nu=1}^{4} j_\nu - 2j_\mu .$$

To see this, let us begin by observing that, according to Eq. (15), the fact of performing the change $\tilde{j}_\mu \to j_\mu$ for all the indices $\mu=\{1,2,3,4\}$ translates in terms of the Liouville momenta α_μ in doing the change $\alpha_i \to Q-\alpha_i$ for $i=\{2,3,4\}$, while keeping α_1 unchanged. Then, by taking into account the Liouville reflection symmetry [Eq. (11)] and the dictionary [Eq. (15)], we get (I thank Garraffo for discussions on this formula.)

$$\langle \Phi_{j_1}(x|z)\Phi_{j_2}(0|0)\Phi_{j_3}(1|1)\Phi_{j_4}(\infty|\infty)\rangle = \frac{F_k(j_1,j_2,j_3,j_4)X_k(j_1,j_2,j_3,j_4|x,z)}{F_k(\tilde{j}_1,\tilde{j}_2,\tilde{j}_3,\tilde{j}_4)X_k(\tilde{j}_1,\tilde{j}_2,\tilde{j}_3,\tilde{j}_4|x,z)}\prod_{i=2}^{4} R_b(\alpha_i)$$
$$\times \langle \Phi_{\tilde{j}_1}(x|z)\Phi_{\tilde{j}_2}(0|0)\Phi_{\tilde{j}_3}(1|1)\Phi_{\tilde{j}_4}(\infty|\infty)\rangle. \quad (21)$$

The factor $F_k(j_1,j_2,j_3,j_4)/F_k(\tilde{j}_1,\tilde{j}_2,\tilde{j}_3,\tilde{j}_4)$ involves a quotient of products of Υ_b functions that, by making use of the formulas in the Appendix [see Eq. (35) below], can be substantially simplified once one takes into account the presence of the three factors $R_b(\alpha_i)$ $(i=2,3,4)$. Then, the results reads

$$\langle \Phi_{j_1}(x|z)\Phi_{j_2}(0|0)\Phi_{j_3}(1|1)\Phi_{j_4}(\infty|\infty)\rangle$$
$$= b^{2\mathcal{P}(j)}|x|^{2b^{-1}(Q-2\alpha_2)}|1-x|^{2b^{-1}(Q-2\alpha_3)}|z|^{b^2(j_1+j_2)^2-b^2(j_3+j_4)^2}$$
$$\times|1-z|^{b^2(j_1+j_3)^2-b^2(j_2+j_4)^2}\prod_{\mu=1}^{4}\frac{\Upsilon_b(2j_\mu b-b)}{\Upsilon_b\left(b\sum_{\nu=1}^{4}j_\nu-2j_\mu b-b\right)}$$
$$\times\langle \Phi_{\tilde{j}}(x|z)\Phi_{\tilde{j}_2}(0|0)\Phi_{\tilde{j}_3}(1|1)\Phi_{\tilde{j}_4}(\infty|\infty)\rangle, \quad (22)$$

where $\mathcal{P}(j)$ is a polynomial in the indices j_μ, namely, $\mathcal{P}(j)=-3j_1+j_2+j_3+j_4$, though it is not actually relevant here. Since $j_1+j_2=\tilde{j}_3+\tilde{j}_4$ and $j_1+j_3=\tilde{j}_2+\tilde{j}_4$, one immediately notices that the $\mathbb{Z}_2$-invariant form under the involution $j_\mu \to \tilde{j}_\mu$ is given by

$$\mathcal{I}_k^{\pm}(x,z) = Z_k^{\pm}(j_1,j_2,j_3,j_4|x,z)\langle \Phi_{j_1}(x|z)\Phi_{j_2}(0|0)\Phi_{j_3}(1|1)\Phi_{j_4}(\infty|\infty)\rangle,$$

where

$$Z_k^{+}(j_1,j_2,j_3,j_4|x,z) = b^{2(j_1-j_2-j_3-j_4)}|x|^{j_1+j_2-j_3-j_4+k-1}|1-x|^{j_1-j_2+j_3-j_4+k-1}$$
$$\times|z|^{+b^2(j_3+j_4)^2}|1-z|^{+b^2(j_2+j_4)^2}\prod_{\mu=1}^{4}\Upsilon_b^{-1}(2j_\mu b-b),$$

$$Z_k^{-}(j_1,j_2,j_3,j_4|x,z) = b^{2(j_1-j_2-j_3-j_4)}|x|^{j_1+j_2-j_3-j_4+k-1}|1-x|^{j_1-j_2+j_3-j_4+k-1}$$
$$\times|z|^{-b^2(j_1+j_2)^2}|1-z|^{-b^2(j_1+j_3)^2}\prod_{\mu=1}^{4}\Upsilon_b^{-1}(2j_\mu b-b).$$

Formula (22) will be useful in the next section. Similar functional relations were studied in Refs. 1, 20, and 47. Now, let us move to the case of winding strings.

C. The four-point function of winding strings

In this section, as an application of our formula [Eq. (22)], we will employ it to show that, by making use of both FZ and SRT maps, it is feasible to write down a formula that expresses the winding violating four-point functions in terms of the zero-winding four-point function. To do this, let us begin by considering the FZ identity

$$\langle \Phi_{j_2}(0|0)\Phi_{1-j_1}(x|z)\Phi_{j_3}(1|1)\Phi_{j_4}(\infty|\infty)\rangle = F_k(1-j_1,j_2,j_3,j_4)X_k(1-j_1,j_2,j_3,j_4|x,z) \times \langle V_{\hat{\alpha}_2}(0)V_{\hat{\alpha}_1}(z)V_{\hat{\alpha}_3}(1)V_{-1/2b}(x)V_{\hat{\alpha}_4}(\infty)\rangle, \tag{23}$$

where the quantum numbers $\hat{\alpha}_\mu$ and j_μ are then related as it follows

$$\hat{\alpha}_1 = \frac{b}{2}(-j_1+j_2+j_3+j_4), \quad \hat{\alpha}_2 = \frac{b}{2}(-j_1+j_2-j_3-j_4+k),$$

$$\hat{\alpha}_3 = \frac{b}{2}(-j_1-j_2+j_3-j_4+k), \quad \hat{\alpha}_4 = \frac{b}{2}(-j_1-j_2-j_3+j_4+k).$$

Now, let us also define quantum numbers J_μ as

$$b(k/2-J_\mu) = \hat{\alpha}_\mu, \tag{24}$$

for $\mu=\{1,2,3,4\}$. Then, by taking into account Eq. (11), we have

$$\langle V_{\hat{\alpha}_2}(0)V_{\hat{\alpha}_1}(z)V_{\hat{\alpha}_3}(1)V_{-1/2b}(x)V_{\hat{\alpha}_4}(\infty)\rangle = \tilde{A}_k(J_1,J_2,J_3,J_4|x,z)\langle \Phi_{j_2}(0|0)\Phi_{1-j_1}(x|z)\Phi_{j_3}(1|1)\Phi_{j_4}(\infty|\infty)\rangle. \tag{25}$$

with

$$\tilde{A}_k(J_1,J_2,J_3,J_4|x,z) = F_k^{-1}(1-j_1,j_2,j_3,j_4)X_k^{-1}(1-j_1,j_2,j_3,j_4|x,z). \tag{26}$$

Consequently, we can write the following SRT identity:

$$\begin{aligned}\langle \Phi^{\omega_2=0}_{J_2,m_2,\bar{m}_2}(0)\Phi^{\omega_1=-1}_{J_1,m_1,\bar{m}_1}(z)\Phi^{\omega_3=0}_{J_3,m_3,\bar{m}_3}(1)\Phi^{\omega_4=0}_{J_4,m_4,\bar{m}_4}(\infty)\rangle \\ = \hat{c}_b\prod_{\mu=1}^{N}\frac{\Gamma(-m_\mu+J_\mu)}{\Gamma(1-J_\mu+\bar{m}_\mu)}\delta(m_1+m_2+m_3-k/2)\delta(\bar{m}_1+\bar{m}_2+\bar{m}_3-k/2)(z)^{k/2-m_1} \\ \times(\bar{z})^{k/2-\bar{m}_1}(1-z)^{k/2-m_1}(1-\bar{z})^{k/2-\bar{m}_1}\int d^2x(x)^{m_2-k/2}(1-x)^{m_3-k/2}(\bar{x})^{\bar{m}_2-k/2} \\ \times(1-\bar{x})^{\bar{m}_3-k/2}(x-z)^{m_1-k/2}(\bar{x}-\bar{z})^{\bar{m}_1-k/2}\langle V_{\hat{\alpha}_2}(0)V_{\hat{\alpha}_1}(z)V_{\hat{\alpha}_3}(1)V_{-1/2b}(x)V_{\hat{\alpha}_4}(\infty)\rangle.\end{aligned} \tag{27}$$

Hence, plugging Eq. (25) into Eq. (27), we find

$$\begin{aligned}\langle \Phi^{\omega_1=-1}_{J_1,m_1,,\bar{m}_1}(z)\Phi^{\omega_1=0}_{J_2,m_2,,\bar{m}_2}(0)\Phi^{\omega_1=0}_{J_3,m_3,\bar{m}_3}(1)\Phi^{\omega_4=0}_{J_4,m_4,\bar{m}_4}(\infty)\rangle \\ = \prod_{\mu=1}^{4}\frac{\Gamma(J_\mu-m_\mu)}{\Gamma(1-J_\mu+\bar{m}_\mu)}\delta(m_1+m_2+m_3-k/2)\delta(\bar{m}_1+\bar{m}_2+\bar{m}_3-k/2)(z)^{-m_1}(\bar{z})^{-\bar{m}_1}(1-z)^{-m_1} \\ \times(1-\bar{z})^{-\bar{m}_1}\int d^2x(x-z)^{m_1}(\bar{x}-\bar{z})^{\bar{m}_1}(x)^{m_2}(\bar{x})^{\bar{m}_2}(1-x)^{m_3}(1-\bar{x})^{\bar{m}_3} \\ \times A_k(J_1,J_2,J_3,J_4|x,z)\langle \Phi_{1-j_1}(x|z)\Phi_{j_2}(0|0)\Phi_{j_3}(1|1)\Phi_{j_4}(\infty|\infty)\rangle,\end{aligned} \tag{28}$$

where we simply defined

$$A_k(J_1,J_2,J_3,J_4|x,z) = \left|\frac{z(1-z)}{x(1-x)(z-x)}\right|^k \tilde{A}_k(J_1,J_2,J_3,J_4|x,z),$$

and where, according to the definitions above, the indices J_μ and j_μ turn out to be related by

$$2J_1 = k + j_1 - j_2 - j_3 - j_4, \quad 2J_i = j_1 + j_2 + j_3 + j_4 - 2j_i, \tag{29}$$

for $i=\{2,3,4\}$. Then, it is feasible to show that

$$\begin{aligned} A_k(J_1,J_2,J_3,J_4|x,z) \sim & \frac{\Upsilon_b(2b(k/2-J_1))}{\Upsilon_b(b(k/2+1-J_1-J_2-J_3-J_4))}|x|^{-2J_2}|1-x|^{-2J_3}|x-z|^{-2J_1} \\ & \times \prod_{i=2}^{4} \frac{\Upsilon_b(b(2J_i-1))}{\Upsilon_b(b(2J_i-J_1-J_2-J_3-J_4+k/2-1))} \\ & \times |z|^{k+4b^2(1-j_1)j_2-4\hat{\alpha}_1\hat{\alpha}_2}|1-z|^{k+4b^2(1-j_1)j_3-4\hat{\alpha}_1\hat{\alpha}_3}. \end{aligned} \tag{30}$$

where the symbol "$\sim$" stands just because we are not writing the k-dependent overall factor and the precise KPZ scaling here [this can be directly read from Eq. (26) if necessary]. In short, in the last equation we have employed the notation

$$2\hat{\alpha}_1 = b(k/2-J_1) = b(-j_1+j_2+j_3+j_4), \quad 2\hat{\alpha}_i = b(k/2-J_i) = b(k+2j_i-j_1-j_2-j_3-j_4), \tag{31}$$

and one could also find it convenient to define

$$2\alpha_1 = b(j_1+j_2+j_3+j_4-1), \quad 2\alpha_i = b(j_1-j_2-j_3-j_4+2j_i+k-1). \tag{32}$$

Identity (28) does already express the winding violating correlators in terms of the conservative ones. However, such expression is still not fully satisfactory since the right hand side of Eq. (28) contains a four-point correlator that, instead of involving states with momenta J_μ, involves states with momenta j_μ, $\mu=\{1,2,3,4\}$. Here is where our result of Sec. III B becomes useful, since the following step would be to relate the correlator in the j_μ basis with the one that is diagonal in the J_μ basis. This can be simply achieved by making use of Equation (22) and by taking into account Eq. (29), yielding

$$\begin{aligned} \langle \Phi_{j_1}(x|z)\Phi_{j_2}(0|0)\Phi_{j_3}(1|1)\Phi_{j_4}(\infty|\infty)\rangle = & B_k(J_1,J_2,J_3,J_4|x,z) \\ & \times \langle \Phi_{(k/2)-J_1}(x|z)\Phi_{J_2}(0|0)\Phi_{J_3}(1|1)\Phi_{J_4}(\infty|\infty)\rangle, \end{aligned} \tag{33}$$

with the normalization factor being

$$\begin{aligned} B_k(J_1,J_2,J_3,J_4|x,z) = & b^{2\hat{\mathcal{P}}(j)}|z|^{b^2(J_3+J_4)^2-b^2(k/2-J_1+J_2)^2}|1-z|^{b^2(J_2+J_4)^2-b^2(k/2-J_1+J_3)^2} \\ & \times \frac{\Upsilon_b(b(J_1+J_2+J_3+J_4-k/2-1))}{\Upsilon_b(b(k-2J_1-1))}\prod_{i=2}^{4}\frac{\Upsilon_b(b(2J_i-J_1-J_2-J_3-J_4+k/2-1))}{\Upsilon_b(b(2J_i-1))} \\ & \times |x|^{k-2(J_1-J_2+J_3+J_4)}|1-x|^{k-2(J_1+J_2-J_3+J_4)}. \end{aligned} \tag{34}$$

Here, $\hat{\mathcal{P}}(j)$ is again a polynomial in the indices j_μ that is related to the $\mathcal{P}(j)$ in Eq. (22) through the replacement $j_i \to J_i$, $i=\{2,3,4\}$, $j_1 \to k/2-J_1$. In writing down Eq. (34) we used the fact that, as in Sec. III B, the combination of the Υ_b functions and the Γ functions in $R_b(\alpha_i)$ leads to a very simple expression. In this case, it comes from

$$\frac{\Upsilon_b(b(j_1-j_2-j_3-j_4+2j_i))}{\Upsilon_b(b(-j_1+j_2+j_3+j_4-2j_i))R_b((Q/2)-(b/2)(j_1-j_2-j_3-j_4+2j_i))}$$
$$=\left(\pi b^{2(b^2-1)}\frac{\Gamma(1-b^2)}{\Gamma(1+b^2)}\right)^{J_1-J_2-J_3-J_4+2J_i-k/2}. \tag{35}$$

Hence, up to the Weyl reflection $j_1\to 1-j_1$ that we discuss below, Eqs. (33) and (28) represent the result we wanted to prove: the four-point function involving one winding state of the sector $\omega=-1$ admits to be expressed in terms of the four-point function of nonwinding states. Let us discuss the final form of this result in the following paragraphs [see Eq. (37) below], where we address the comparison with the case of the three-point function. We will also comment on possible applications and conclude with some remarks in the following section.

D. Analogy with the case of the three-point function

The three-point function that includes one spectral flowed state of the sector $\omega=-1$ can be written in terms of the structure constant of three nonflowed states.[5,25] In Sec. III C above we proved a similar relation at the level of the four-point functions. Then, the natural question we want to address now is whether both relations are connected in some way. As we will see below, these are indeed analogous. Actually, expressions (28) and (33), once considered together, correspond to the generalization of the formula that holds for the three-point structure constants. First, one of the aspects one notices in the formulas above is that the first operator in the right hand side of Eq. (33) represents the state of momentum $(k/2)-J_1$, instead that of momentum J_1. Actually, this should not be a surprise since it precisely resembles what happens at the level of the three-point function, where the violating winding correlator

$$\langle \Phi^{\omega_1=-1}_{J_1,m_1,,\bar{m}_1}(0)\Phi^{\omega_2=0}_{J_2,m_2,,\bar{m}_2}(1)\Phi^{\omega_3=0}_{J_3,m_3,\bar{m}_3}(\infty)\rangle$$

turns out to be proportional to the integral of the conservative structure constant that corresponds to

$$\sim B^{-1}(k/2-J_1)\langle \Phi_{k/2-J_1}(0|0)\Phi_{J_2}(1|1)\Phi_{J_3}(\infty|\infty)\rangle.$$

The same association between momenta J_1 and $k/2-J_1$ occurs for the four-point functions here. Actually, such relation between states with quantum numbers $\omega=0$, J and those with $\omega=\pm 1$, $k/2-J$ is understood from the algebraic point of view: This concerns the identification between the discrete series of the SL $(2,\mathbb{R})_k$ representations by means of the spectral flow automorphism. This is related to the fact that both states with $\omega=0$ and $|\omega|=1$ satisfy similar KZ equations. Besides, this is related to the fact that the OPE between the vertex operator $\Phi_J(z)$ and the spectral flow operator $\Phi_{k/2}(w)$, which is necessary to provide the winding $\omega_1=1$ to the first, yields the single string contribution $\Phi_{k/2-J}(z)$. The involution $J\to k/2-J$, as a symmetry of the KZ equation, was studied in the first part of our work.[1] As it was detailed in Ref. 5, when presenting the original derivation of Ref. 25, the proportionality factor connecting both violating winding and nonviolating winding three-point correlators is basically given by the reflection coefficient $B(J_1)\sim B^{-1}(k/2-J_1)$ of the SL $(2,\mathbb{R})_k$ WZNW model (see the formula above). This quantity has the form

$$B(J)=\frac{1}{\pi b^2}\left(\lambda\pi\frac{\Gamma(1-b^2)}{\Gamma(1+b^2)}\right)^{1-2J}\frac{\Gamma(1+b^2-2Jb^2)}{\Gamma(2Jb^2-b^2)}, \tag{36}$$

and therefore

$$B(J) = \frac{1}{\pi^2 b^4}\left(\frac{1}{\lambda\pi}\frac{\Gamma(1+b^2)}{\Gamma(1-b^2)}\right)^{b^{-2}} B^{-1}(k/2-J).$$

This permits to show that the picture for the three-point functions turns out to be similar to the one we are obtaining here for the four-point functions. Actually, this is not only manifested in the shifting $J_1 \to k/2 - J_1$ but also in the fact that, when combining both Eqs. (28) and (33), a factor $\Upsilon_b(2b(k/2-J_1))/\Upsilon_b(2b(k/2-J_1)-b)$ also arises in the product between $A_k(J_1,J_2,J_3,J_4|x,z)$ and $B_k(J_1,J_2,J_3,J_4|x,z)$. This factor is precisely proportional to $(2J-1)B(J)$; thus, this is actually in equal footing as to how the WZNW reflection factor stands in the relation between violating and nonviolating three-point correlators.

On the other hand, some nice cancellations occur when combining expressions (28) and (33). First, we observed that, instead of the operator $\Phi_{j_1}(x|z)$, the correlator on the right hand side of Eq. (28) involves its Weyl reflected operator $\Phi_{1-j_1}(x|z)$, and, as we commented at the end of Sec. II, this implies that a factor $R_b^{-1}(b(k/2-j_1))$ has to be included as well in order to plug Eq. (28) into Eq. (33). Then, such reflection coefficient is eventually simplified due to another contribution standing when multiplying $A_k(J_1,J_2,J_3,J_4|x,z)$ and $B_k(J_1,J_2,J_3,J_4|x,z)$, namely,

$$\frac{\Upsilon_b(b(J_1+J_2+J_3+J_4-k/2-1))}{\Upsilon_b(-b(J_1+J_2+J_3+J_4-k/2-1))} = \frac{\Upsilon_b(2j_1b-b)}{\Upsilon_b(b-2j_1b)} \sim R_b(b(k/2-j_1)).$$

Moreover, these are not all the cancellations that take place. Let us also observe that three factors of the form $\Upsilon_b(b(2J_i-1))/\Upsilon_b(b(2J_i-J_1-J_2-J_3-J_4+k/2-1))$ (with $i=\{2,3,4\}$) and their respective inverses are mutually canceled as well. These come from the second line of Eq. (30) and the second line of Eq. (34), respectively. Finally, Eqs. (28) and (33), considered together, lead to the following expression:

$$\begin{aligned}
&\langle \Phi^{\omega_1=-1}_{J_1,m_1,\bar{m}_1}(z)\Phi^{\omega_2=0}_{J_2,m_2,\bar{m}_2}(0)\Phi^{\omega_3=0}_{J_3,m_3,\bar{m}_3}(1)\Phi^{\omega_4=0}_{J_4,m_4,\bar{m}_4}(\infty)\rangle\\
&\sim B(J_1)\prod_{\mu=1}^{4}\frac{\Gamma(J_\mu-m_\mu)}{\Gamma(1-J_\mu+\bar{m}_\mu)}\int d^2x(x-z)^{m_1+\Delta_1}(\bar{x}-\bar{z})^{\bar{m}_1+\Delta_1}\\
&\times\prod_{i=1}^{2}(z-z_i)^{-m_i+\Delta_i}(\bar{z}-\bar{z_i})^{-\bar{m}_i+\Delta_i}(x-x_i)^{m_i+\tilde{\Delta}_i}(\bar{x}\\
&-\bar{x}_i)^{\bar{m}_i+\tilde{\Delta}_i}\langle\Phi_{(k/2)-J_1}(x|z)\Phi_{J_2}(0|0)\Phi_{J_3}(1|1)\Phi_{J_4}(\infty|\infty)\rangle\\
&\times\delta(m_1+m_2+m_3-k/2)\delta(\bar{m}_1+\bar{m}_2+\bar{m}_3-k/2),
\end{aligned} \tag{37}$$

where $B(J_1)$ is the WZNW reflection coefficient [Eq. (36)], while the symbol $\sim$ stands for some k-dependent overall factor. For short, in this expression we denoted $x_2=z_2=0$ and $x_3=z_3=1$, and exponents $\Delta_{1,2,3}$ and $\tilde{\Delta}_{2,3}$ refer to J-dependent (m-independent) linear combinations that are directly given by the exponents arising in Eqs. (28) and (33). Expression (37) represents the main result here, and it turns out to be actually analogous to the formula connecting winding violating three-point functions to those involving merely three nonwinding states. Namely, we showed that the SL $(2,R)_k$ WZNW four-point function,

$$\langle \Phi^{\omega_1=-1}_{J_1,m_1,\bar{m}_1}(z)\Phi^{\omega_2=0}_{J_2,m_2,\bar{m}_2}(0)\Phi^{\omega_3=0}_{J_3,m_3,\bar{m}_3}(1)\Phi^{\omega_4=0}_{J_4,m_4,\bar{m}_4}(\infty)\rangle,$$

while involving one state of the spectral flowed sector $\omega_1=-1$, can be expressed in terms of the integral of the four-point function

$$\sim B^{-1}(k/2 - J_1)\langle \Phi_{(k/2)-J_1}(x|z)\Phi_{J_2}(0|0)\Phi_{J_3}(1|1)\Phi_{J_4}(\infty|\infty)\rangle,$$

defined in terms of merely nonspectral flowed states $\omega_\mu=0$, $\mu=\{1,2,3,4\}$. The integration over the complex variable x clearly stands for the requirement of the Fourier transform when changing to the m basis. Furthermore, it is worth pointing out that the factor

$$\prod_{\mu=1}^{4} \frac{\Gamma(J_\mu - m_\mu)}{\Gamma(1 - J_\mu + \bar{m}_\mu)}$$

is also present here, completing the analogy with the three-point function case. The fact that a similar pattern is found at the level of the four-point correlators opens a window that would permit to gain information about the violating winding four-point functions by making use of all that is known about the WZNW conservative amplitudes.

E. Further applications

So far, we presented a concise application of our results of Ref. 1, by showing that the symmetries of the KZ equation, that were inferred by means of its relation to the BPZ equation in Liouville theory, lead to a relation between violating and conserving winding amplitudes in AdS_3 string theory. Besides, there exist some further application of formula (28) we obtained here: One of these feasible applications concerns the integral representation of the four-point conformal blocks and the factorization ansatz. Actually, let us return to Eq. (7) and expand the chiral conformal blocks as it follows

$$\mathcal{G}_{j_1,j_2,j,j_3,j_4}(x|z) = z^{h_j-h_{j_1}-h_{j_2}}x^{j-j_1-j_2}\sum_{n=0}^{\infty} \mathcal{G}^{(n)}_{j_1,j_2,j,j_3,j_4}(x)z^n, \tag{38}$$

where, now, j acts as an internal index that labels different solutions to the differential equation. In the *stringy* interpretation it is feasible to assign physical meaning to such index: this is the one parametrizing the intermediate states interchanged in a given four-point scattering process. Then, the integration over the internal index j stands in order to include all the contributions of the conformal blocks [Eq. (7)] to the four-point amplitude [Eq. (5)]. By replacing [Eq. (38)] into the KZ equation one finds that the leading term contribution in the z-power expansion obeys the hypergeometric differential equation. For instance, this permits to analyze the monodromy properties at $x=z$ for leading orders in the large $1/x$ limit. Studying this regime turns out to be important to fully understand the analytic structure of the four-point function.[5] For such purpose, it is useful to study the solution in the vicinity of the point $x=z$. By assuming the extension of the expressions above to the SL $(2,\mathbb{R})$ case and by deforming the contour of integration as $\mathcal{C} \to (k/2)-\mathcal{C}=(k-1)/2-i\mathbb{R}$, one can see that, once the integral over j is performed, the solution is actually monodromy invariant at $z=x$. Then, the leading contribution of the solution can be written as it follows[5]

$$|f_{j_1,j_2,j_3,j_4}(x,z)|^2 = \frac{1}{2}\int_{(k-1)/2+i\mathbb{R}} \mathrm{d}j |x|^{2(h_j-h_{j_1}-h_{j_2}-j+j_1+j_2)}|zx^{-1}|^{2(h_j-h_{j_1}-h_{j_2})}\frac{C(j_1,j_2,j)C(j,j_3,j_4)}{B(j)}$$
$$\times |F(j_1+j_2-j,j_3+j_4-j,k-2j,zx^{-1})|^2(1+\mathcal{O}(z^{-1}x)) + 2\pi i \sum_{\{x_i\}} \mathrm{Re}\, s_{(x=x_i)},$$

where $F(a,b,c;d)$ is the hypergeometric function, and where $\{x_i\}$ refers to the set of poles located in the region $1<\mathrm{Re}(2j)<k-1$. These poles take the form $j-j_1-j_2\in\mathbb{N}$ if the constraint $\Sigma_{i=1}^4 j_i < k$ is assumed (see Ref. 5 for the details of the construction and the issue of the integration over the complex variables z and x). Taking all this into account, one could try to plug the expression for the conformal blocks [Eq. (7)] in the equation we wrote in Eq. (28). This would lead to an integral realization of the winding violating four-point function. Moreover, such a realization could be then evaluated in a particular case $J_4=k/2$ and, by means of the prescription of Ref. 25, be used to compute the three-point function $\langle \Phi^{\omega_1=-1}_{J_1,m_1,\bar{m}_1} \Phi^{\omega_2=0}_{J_2,m_2,\bar{m}_2} \Phi^{\omega_3=+1}_{J_3,m_3,\bar{m}_3} \rangle$, with two winding states of the sectors $\omega=+1$ and $\omega=-1$. This idea does deserve to be explored in future work (I thank Minces for pointing out this possible application.).

Other possible analysis that can be done regard the Coulomb gaslike integral representation recently presented in Refs. 37 and 38. There, it was proven that the SRT map can be thought of as a free field representation of the SL $(2,\mathbb{R})_k$ model in terms of the action

$$S = \frac{1}{4\pi} \int d^2z(-\partial\varphi\bar{\partial}\varphi + QR\varphi + \mu e^{\sqrt{2}b\varphi}) + S_M, \tag{39}$$

where the specific model representing the "matter sector" S_M corresponds to a $c<1$ conformal field theory defined by the action

$$S_M = \frac{1}{4\pi} \int d^2z(\partial X^0\bar{\partial}X^0 - \partial X^1\bar{\partial}X^1 - i\sqrt{k}RX^1).$$

This free field representation leads to an integral expression for the four-point function when the winding number conservation is being maximally violated. On the other hand, for the cases where the winding is conserved, the action S_M has to be perturbed by introducing a new term of the form

$$\mathcal{O} = \int d^2z e^{-\sqrt{(k-2)/2}\varphi(z)+i\sqrt{k/2}X^1(z)}. \tag{40}$$

This is a perturbation, represented by a primary operator of the matter sector and properly dressed with the coupling to the Liouville field in order to turn it into a marginal deformation. It could be interesting to check formulas (28)–(33) in terms of this representation.

IV. CONCLUDING REMARKS

As an application of our Ref. 1, in this brief note we proved a new relation existing between the four-point function of winding strings in AdS_3 [spectral flowed states of the SL $(2,\mathbb{R})_k$ WZNW model] and the four-point function of nonwinding strings [nonflowed SL $(2,\mathbb{R})_k$ states]. We showed that the former admits a simple expression in terms of the last, which was studied in more detail in the literature. The fact that such a simple expression exits is mainly due to two facts: first, the nondiagonal $\mathbb{Z}_2$ symmetry of the KZ equation realized by Eq. (22); second, to the fact that correlators of the WZNW model are connected to those of the Liouville theory in more than one way.

To be more precise, here we have shown the relation that connects the four-point WZNW correlation function involving one $\omega=-1$ spectral flowed state to the four-point function of non-spectral flowed states. This consequently relates the scattering amplitude involving one winding string state in AdS_3 with the analogous observable for nonwinding strings. Such relation is realized by Eq. (37), and is reminiscent of the relation obeyed by the SL $(2,\mathbb{R})_k$ structure constants. On the other hand, it is likely that the four-point function involving one state in the sector $\omega=-1$ corresponds to a limiting procedure of a WZNW five-point function involving one spectral flow operator of momentum $J_5=k/2$. Consequently, it is plausible that the connection manifested by Eqs. (28) and (33) could then be obtained in an alternative way; for instance, by directly studying the operator product expansion of operators $\Phi_{J_1}(x_1|z_1)\Phi_{k/2}(x_2|z_2)$ in the coincidence limit in the five-point conformal blocks. In such case, our result would be seen from a different perspective since, as far as our derivation of Eqs. (28) and (33) is invertible, this would lead to the possibility of deriving the FZ map from a particular case of the SRT map. This was, indeed, one of the main motivations we had for studying this, though the operator product expansion involving an additional spectral flow operator (a fifth operator) turns out to be complicated enough and thus we leave it for the future. Conversely, as far as the relation between both maps is not trivial, as it is emphasized in Ref. 29, our way of proving the connection between violating winding correlators and conserving winding correlators turns out to be an ingenious trick. The conciseness of such a deduction becomes particularly evident once one gets familiarized with the complicated definition of the action of spectral flow operator in the x basis. Moreover, even in the m basis the computation turns out to be simplified by our method because no explicit reference to the decoupling equation of the degenerate state $J=k/2$ was needed at all, since it was already encoded in the Ribault formula [Eq. (20)]. Our hope is that the result of this brief paper might be useful in working out the details of the SL $(2,\mathbb{R})_k$ WZNW four-point functions, which is our main challenge within this line of research.

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APPENDIX: THE FUNCTION $\Upsilon_b(x)$

The function $\Upsilon_b(x)$ was introduced by Zamolodchikov and Zamoldchikov in Ref. 42, and is defined as it follows

$$\log \Upsilon_b(x) = \frac{1}{4}\int_{\mathbb{R}_{>0}} \frac{\mathrm{d}\tau}{\tau}\left((Q-2x)^2 \mathrm{e}^{-\tau} - \frac{\sinh^2((\tau/4)(Q-2x))}{\sinh(b\tau/2)\sinh(\tau/2b)}\right), \tag{A1}$$

where $Q=b+b^{-1}$, being $b \in \mathbb{R}_{>0}$. It also admits a definition in terms of a limiting procedure involving the double Barnes Γ_2 function, though we did not find such a relation necessary here. This function has its zeros at the points

$$x = mb + nb^{-1},$$

$$x = -(m+1)b - (n+1)b^{-1},$$

for any pair of positive integers $m,n \in \mathbb{Z}_{>0}$. From Eq. (A1), it turns out to be evident that this function is symmetric under the inversion of the parameter $b \to 1/b$, namely,

$$\Upsilon_b(x) = \Upsilon_{1/b}(x). \tag{A2}$$

This is the first of a list of nice functional properties of this function. The second identity we find useful is the reflection property

$$\Upsilon_b(x) = \Upsilon_b(Q-x), \tag{A3}$$

which is keeps track of the reflection symmetry of Liouville structure constants when these are written in terms of $\Upsilon_b(x)$. Besides, Eq. (A1) also presents the following properties under fixed translations:

$$\Upsilon_b(x+b^{\pm 1}) = \Upsilon_b(x)\frac{\Gamma(b^{\pm 1}x)}{\Gamma(1-b^{\pm 1}x)} b^{\pm 1 \mp 2b^{\pm 1}x}. \tag{A4}$$

The above identities are then gathered in the equation

$$\Upsilon_b(Q \mp x) = \pm \Upsilon_b(x) \frac{\Gamma(bx)\Gamma(b^{-1}x)}{\Gamma(\pm bx)\Gamma(\pm b^{-1}x)} b^{2x(b^{\pm 1}-b)}, \tag{A5}$$

which, in particular, includes Eq. (A3). The relations above also permit to prove that the following relation holds:

$$\Upsilon_b(x) = \Upsilon_b(-x) b^{2x(b-b^{-1})} \frac{\Gamma(-bx)\Gamma(-b^{-1}x)}{\Gamma(bx)\Gamma(b^{-1}x)}. \tag{A6}$$

All these relations were employed through our computation.

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