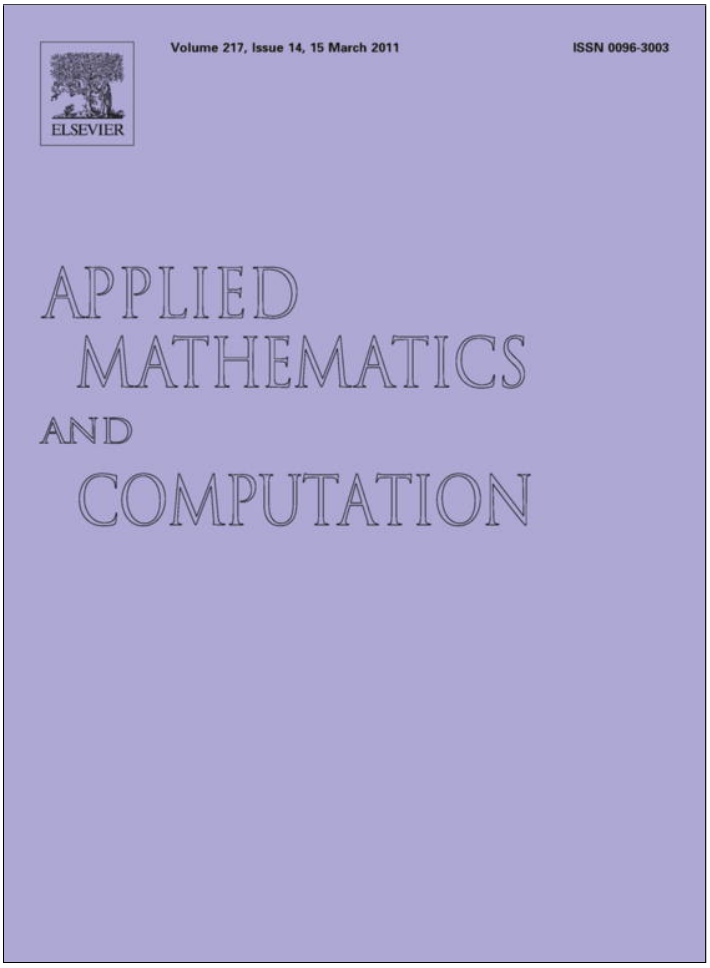
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A finite extensibility nonlinear oscillator

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ABSTRACT

The dynamics of a finite extensibility nonlinear oscillator (FENO) is studied analytically by means of two different approaches: a generalized decomposition method (GDM) and a linearized harmonic balance procedure (LHB). From both approaches, analytical approximations to the frequency of oscillation and periodic solutions are obtained, which are valid for a large range of amplitudes of oscillation. Within the generalized decomposition method, two different versions are presented, which provide different kinds of approximate analytical solutions. In the first version, it is shown that the truncation of the perturbation solution up to the third order provides a remarkable degree of accuracy for almost the whole range of amplitudes. The second version, which expands the nonlinear term in Taylor's series around the equilibrium point, exhibits a little lower degree of accuracy, but it supplies an infinite series as the approximate solution. On the other hand, a linearized harmonic balance method is also employed, and the comparison between the approximate period and the exact one (numerically calculated) is slightly better than that obtained by both versions of the GDM. In general, the agreement between the results obtained by the three methods and the exact solution (numerically integrated) for amplitudes (A) between $0 < A \leqslant 0.9$ is very good both for the period and the amplitude of oscillation. For the rest of the amplitude range ($0.9 < A < 1$), an exponentially large L_2 error demonstrates that all three approximations do not represent a good description for the FENO, and higher order perturbation solutions are needed instead. As a complement, very accurate asymptotic representations of the period are provided for the whole range of amplitudes of oscillation.

1. Introduction

Nonlinear phenomena is present in every branch of modern physics, being particularly important the emergence of nonlinear oscillators. Apparently, this notable interest lays in their wide range of applicability and their more efficient way to model reality. It is possible to find nonlinear oscillators in quantum mechanics [1,2], biology [3,4] and, of course, classical mechanics [5–7]. Although the most studied cases of nonlinearity are of polynomial-type, nonpolynomial nonlinearity (such as the type of the title problem) has emerged over the last decades as an active field of quantum mechanics, specially nonpolynomial oscillators [8]. They are important, for example in nonlinear Lagrangian field theory and nonlinear optics [9], as well as in elementary particle physics [10]. In the same sense, finite extensibility oscillators (FENOs) play an important role in the theory of macromolecules, particularly in the theory of polymer dynamics [11], DNA dynamics [12], and in the simulation of non-Newtonian fluids [13]. For chain models in polymer dynamics, the nonpolynomial nonlinearity arises from the physical situation where the bonds between molecules, modeled in a first simplified approximation by harmonic springs, cannot be extended to infinity. For small amplitudes of vibration, these oscillators can be well described by a linear spring

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connectivity between monomers; however, the description becomes unrealistic when the molecules are highly stretched and the forces are essentially nonlinear. From the large number of ways that may be possible to select to model a finite extensibility oscillator, a finite extensibility nonlinear elastic potential (FENE) [14] is chosen for the title problem. The reason for selecting this type of potential comes from the fact that it is widely used in computer simulations of polymers to prevent the overstretching of the chains, thus avoiding unphysical conformations. Despite its fundamental importance, we do not know of previous works on the problem, even in one dimension. However, the case of nonpolynomial (quantum) oscillators (NPO) with a saturable nonlinearity, that appears when modeling optical pulse propagation in doped fibers [15] or, for instance, in wave propagation in two dimensional quantum lattices [16] is closely related to it [17] but not the same. The main difference with FENO lies in that the saturable nonlinearity represents a kind softening (nonpolynomial) nonlinearity in NPO, whereas the FENO possesses a hardening one instead.

For the last decades, an enormous amount of work has been devoted to obtaining approximate analytical solutions to nonlinear oscillators, their amplitudes and periods of vibration. Among them, it is possible to mention: perturbation methods [18], where the solution is expressed as a power series in a small parameter; homotopy perturbation methods [19,20], which do not necessarily require a small parameter in the equation of motion to produce accurate results (a combination of the perturbation expansion method and homotopy techniques); harmonic balance [21,22], which proposes a periodic solution as a Fourier series and always equates higher order harmonics to zero; linearized harmonic balance [23,24] which is the linearized version of the harmonic balance method; the method of multiple scales [25], which introduces multiple time scales that are powers of a small parameter; Adomian decomposition method [26,27], based on transforming the original differential equation into an integral equation which is expanded into the so-called "Adomian's" polynomials; and iterative calculation techniques [28,29], which proposes a first order Taylor expansion of an artificial nonlinear term added to the original equation, provided that it is free from secular terms. From all of these techniques, we select a general decomposition method approach (GDM) and a linearized harmonic balance procedure (LHB), since they have proven to exhibit accurate results when applied to a large variety of velocity-dependent nonlinear problems [30,31,24,32], as it will be shown later.

The main objective of this work is, then, to obtain an approximate expression for the periodic solution of the FENO as well as for its period as a function of its amplitude of vibration. Then, this information could be used, for example, in a thorough study of the dynamics of a chain of these oscillators, which constitute the basis of a real polymer chain, a DNA molecule or a non-Newtonian fluid. Similarly, to know the period of the FENO (or frequency of oscillation) as a function of its amplitude could allow to analyze a forced FENO and to study the resonance conditions (frequencies at which the amplitude of the system is supposed to be unbounded only limited by the damping and the nonlinearity). Since the linearized frequency of the FENO (the frequency at small amplitude of oscillation) can differ greatly from the actual frequency for large amplitudes, it is not possible to address the problem of capturing these new resonances with the simplified linear model. Ultimately, these resonance conditions, which lead to super or sub-harmonic resonances, are crucial to determine the system's frequency response.

The paper is organized as follows. Section 2, presents the mathematical formulation of the problem, a stability analysis and the periodicity of solutions. Section 3 is devoted to obtaining approximate expressions for the amplitude of oscillation, frequency and period. To this end, the problem is presented within the generalized decomposition method in two different versions and the linearized harmonic balance procedure. Section 4 provides analytical asymptotic expansions for the period of the FENO for the whole range of amplitudes. The numerical results and comparisons between the three approximate analytical approaches and asymptotic expansions for the period are presented in Section 5. Finally, concluding remarks are presented and discussed in Section 6.

2. Mathematical model

The FENE potential is used in this work to model the finite extensibility of the oscillator. Mathematically, it is represented by the following expression:

$$V(x) = -\frac{1}{2}\ln(1-x^2) \tag{1}$$

where x represents the amplitude of the oscillator. Then, the equation of motion for the oscillator (being $x = 1$ its maximum possible amplitude) yields

$$\ddot{x} + \frac{x}{1-x^2} = 0 \quad x(0) = A;\ (A < 1);\ \dot{x}(0) = 0 \tag{2}$$

where the dot indicates differentiation with respect to t and A is a constant (initial amplitude).

2.1. Stability character and periodicity of solutions

As a first approximation to the solution of (2), a stability study of all possible solutions for different values of A is presented. First, it is possible to note that (2) can be written as a system of two coupled first-order differential equations (Hamilton's equations)

$$\dot{y} = -\frac{x}{1-x^2} \quad \dot{x} = y \tag{3}$$

Additionally, it is possible to obtain the trajectories through the 2-dimensional phase space dividing both expressions in (3):

$$\frac{dy}{dx} = -\frac{x}{1-x^2}\frac{1}{y}.$$

Integration of the above equation leads to an expression for $y = \dot{x}$ in terms of the total energy E and the potential energy $V(x)$ (FENE potential energy).

$$\dot{x} = \sqrt{2E - 2V(x)} = \sqrt{2E + \ln(1-x^2)} \tag{4}$$

The singular point or equilibrium point for this system is located at $x = 0$; $y = 0$ (velocity and acceleration are simultaneously zero) and the nullclines are: $dy/dx = 0$ (horizontal tangent) at $x = 0$ (y-axis) and $dy/dx = \infty$ (vertical tangent) at $y = 0$ (x-axis). Then, since $V(x) = -1/2\ln(1-x^2)$ is a minimum for $x = 0$, the equilibrium point is stable (in the sense of Liapunov) and a center [25]. As a consequence, the system performs a close trajectory surrounding the center if a small disturbance kicks it out. The motions corresponding to the closed curves are, then, periodic.

As an additional note, it is interesting to point out that (2) can be considered within the general case of a velocity-dependent frequency oscillator if the conservation of energy is used. From (4), it is possible to write $\frac{1}{1-x^2} = \frac{1}{1-A^2}e^{-\dot{x}^2}$. Finally, (2) results in

$$\ddot{x} + e^{-\dot{x}^2}\frac{x}{1-A^2} = 0$$

where $A = x(0)$ as usual.

3. Approximate analytical solutions

In this section, approximate analytical expressions for the amplitude of oscillation, frequency and period are obtained by means of the generalized decomposition method and, additionally, by a linearized harmonic balance procedure.

3.1. Generalized decomposition method

Two different versions of the generalized decomposition method as presented by [30,31] are employed. The first one consists basically in the introduction of an artificial ordering parameter in a nonlinear Volterra integral equation and the expansion of the solution as a power series of the ordering parameter. The second one is based on an artificial parameter-Lindstedt–Poincare technique and the expansion of the nonlinear term in a Taylor's series about the equilibrium point.

3.1.1. First version

In this subsection, the method as presented in [30] is followed. First, in order to adapt (2) to the required form, it must be rewritten to fit a linear harmonic oscillator in the RHS, plus a forcing term depending on displacement and acceleration in the LFS. This results in

$$\ddot{x} + \omega^2 x = +\omega^2 x - x + x^2\ddot{x} \quad x(0) = A;\ \dot{x}(0) = 0 \tag{5}$$

which was obtained from (2) by multiplication by $1 - x^2$ and the addition of a linear stiffness term. Upon making the change of variable $\theta = \omega t$, (5) yields

$$x'' + x = F(\theta, x, x'', \omega^2) = x - \frac{x}{\omega^2} + x^2 x'' \quad x(0) = A;\ \dot{x}(0) = 0 \tag{6}$$

where the prime represents differentiation with respect to θ, and ω is the unknown frequency of oscillation. Applying the method of variation of parameters, the solution to (6) is obtained:

$$x(\theta) = A\cos(\theta) + \int_0^\theta F(s, x, x'', \omega^2)\sin(\theta - s)ds. \tag{7}$$

An artificial parameter p is then introduced in (7) which yields

$$x(\theta) = A\cos(\theta) + p\int_0^\theta F(s, x, x'', \omega^2)\sin(\theta - s)ds; \tag{8}$$

clearly, (8) coincides with (7) upon setting $p = 1$. Now, (8) is appropriate to be solved by the generalized decomposition method. First, the solution $x(\theta)$ is expanded as

$$x(\theta) = \sum_{n=0}^{\infty} p^n x_n(\theta) \tag{9}$$

and also the square of the frequency of oscillation is expanded in the same form

$$\omega^2 = \sum_{n=0}^{\infty} p^n \omega_n^2 \tag{10}$$

Then, if (9) and (10) are substituted into (8), the final expressions at $\mathcal{O}(p^0)$, $\mathcal{O}(p^1)$, $\mathcal{O}(p^2)$ and $\mathcal{O}(p^3)$ are

$$x_0(\theta) = A\cos(\theta) \tag{11}$$

$$x_1(\theta) = \int_0^\theta F(s, x_0(s), x_0''(s), \omega_0^2)\sin(\theta - s)ds \tag{12}$$

$$x_2(\theta) = \int_0^\theta P(s, x_0(s), x_0''(s), \omega_0^2)\sin(\theta - s)ds$$

$$x_3(\theta) = \int_0^\theta H(s, x_0(s), x_0''(s), \omega_0^2)\sin(\theta - s)ds$$

The functions $P(s,x(s),x''(s),\omega^2)$ and $H(s,x(s),x''(s),\omega^2)$ result from a Taylor's series expansion of F at (x_0, x_0'', ω_0^2) and are given by

$$P = \frac{\partial F}{\partial x}x_1 + \frac{\partial F}{\partial x''}x_1'' + \frac{\partial F}{\partial(\omega^2)}\omega_1^2 \tag{13}$$

and

$$H = \frac{\partial F}{\partial x}x_2 + \frac{\partial F}{\partial x''}x_2'' + \frac{\partial F}{\partial \omega^2}\omega_2^2 + \frac{1}{2}\left\{\frac{\partial^2 F}{\partial x^2}x_1^2 + \frac{\partial^2 F}{\partial (x'')^2}(x_1'')^2 + \frac{\partial^2 F}{\partial (\omega^2)^2}\omega_1^4 + 2\frac{\partial^2 F}{\partial x \partial x''}x_1 x_1'' + 2\frac{\partial^2 F}{\partial x \partial \omega^2}x_1\omega_1^2 + 2\frac{\partial^2 F}{\partial x'' \partial \omega^2}x_1''\omega_1^2\right\}$$

Then, solving (12) one obtains

$$x_1(\theta) = \frac{1}{2}\left[A - \frac{A}{\omega_0^2} - \frac{3A^3}{4}\right]\theta\sin(\theta) - \frac{A^3}{32}(\cos(\theta) - \cos(3\theta)) \tag{14}$$

After eliminating the secular terms [33], the first correction to the frequency, ω_0^2 is obtained

$$\omega_0^2 = \frac{4}{4 - 3A^2} \tag{15}$$

from which it is possible to deduce that $A < 2/\sqrt{3} \approx 1.154$ in accordance with $A < 1$ from (2). For $\mathcal{O}(p^2)$, the solution reads

$$\begin{aligned} x_2(\theta) = {} & \frac{1}{2}\left[\frac{A\omega_1^2}{\omega_0^4} - \frac{A^3}{32}\left(1 - \frac{1}{\omega_0^2}\right) - \frac{A^5}{64}\right]\theta\sin(\theta) + \left[\frac{A^3}{256}\left(1 - \frac{1}{\omega_0^2}\right) - \frac{17A^5}{768}\right]\cos(\theta) \\ & + \left[\frac{A^3}{256}\left(1 - \frac{1}{\omega_0^2}\right) - \frac{19A^5}{1024}\right]\cos(3\theta) + \frac{11A^5}{3072}\cos(5\theta) \end{aligned} \tag{16}$$

Clearly, the secular term is eliminated if

$$\omega_1^2 = \frac{5A^4\omega_0^4}{128} = \frac{80}{128}\frac{A^4}{(4 - 3A^2)^2}.$$

Finally, for $\mathcal{O}(p^3)$ one obtains

$$\begin{aligned} x_3(\theta) = {} & \frac{1}{2}\left[\frac{18A^6}{3072} - \frac{\omega_2^2}{\omega_0^4} + \frac{59A^4}{3072}\left(1 - \frac{1}{\omega_0^2}\right) + \frac{96A^2}{3072}\frac{\omega_1^2}{\omega_0^4} + \frac{\omega_1^4}{\omega_0^6}\right]\theta\sin(\theta) \\ & + A^3\left[\frac{620A^2}{294912}\left(1 - \frac{1}{\omega_0^2}\right) - \frac{4449A^4}{294912} + \frac{1152}{294912}\frac{\omega_1^2}{\omega_0^4}\right]\cos(\theta) \\ & - A^3\left[\frac{64A^2}{32768}\left(1 - \frac{1}{\omega_0^2}\right) + \frac{128}{32768}\frac{\omega_1^2}{\omega_0^4} - \frac{351A^4}{32768}\right]\cos(3\theta) \\ & - A^5\left[\frac{11}{73728}\left(1 - \frac{1}{\omega_0^2}\right) - \frac{371A^2}{98304}\right]\cos(5\theta) + \frac{59A^7}{98304}\cos(7\theta); \end{aligned} \tag{17}$$

from which ω_2^2 can be easily derived, after the elimination of the secular term corresponding to the coefficient of $\theta\sin(\theta)$,

$$\omega_2^2 = A^6\omega_0^4\left(\frac{11}{512} + \frac{25}{16384}\omega_0^2A^2\right).$$

Summarizing, using (11), (14), (16) and (17) it is possible to write a four-term approximation to the solution of (5) as follows:

$$x(\theta) \approx \left[A - \frac{A^3}{32} - \frac{59A^5}{3072} - \frac{1313A^7}{98304}\right]\cos(\theta) + \left[\frac{A^3}{32} + \frac{A^5}{64} + \frac{298A^7}{32768}\right]\cos(3\theta) + \left[\frac{11A^5}{3072} + \frac{360A^7}{98304}\right]\cos(5\theta) + \frac{A^3}{32} + \frac{59A^5}{98304} \times \cos(7\theta).$$

For the frequency, the expression finally reads (upon setting $p = 1$)

$$\omega^2 \approx \frac{4}{4-3A^2}\left(1 + \frac{5}{128}A^4\frac{4}{4-3A^2}\right) + A^6\left(\frac{4}{4-3A^2}\right)^2\left(\frac{11}{512} + \frac{25}{16384}\left(\frac{4}{4-3A^2}\right)A^2\right)$$

giving for the period $T = 2\pi/\omega$ the following form:

$$T_{G1} = 2\pi\left(1 - \frac{3}{8}A^2 - \frac{23}{256}A^4 - \frac{91}{2048}A^6 - \frac{2713}{131072}A^8 + \cdots\right) \tag{18}$$

3.1.2. Second version

If a linear stiffness term, ω^2x, is introduced to (2), a second version of the generalized decomposition method [31] can be formulated. This gives

$$\ddot{x} + \omega^2x = \omega^2x - \frac{x}{1-x^2} \quad x(0) = A;\ (A < 1);\ \dot{x}(0) = 0 \tag{19}$$

Upon introducing $\theta = \omega t$ again, (19) reads

$$x'' + x \equiv F(\theta, x(\theta), x''(\theta), \omega^2) = x - \frac{1}{\omega^2}\frac{x}{1-x^2} \quad x(0) = A;\ (A < 1);\ \dot{x}(0) = 0. \tag{20}$$

where, as stated above, the primes represent differentiation with respect to θ. Following a similar analysis as in Section 3.1.1, $x(\theta)$ and ω are expanded as a power series in the same way. After introducing them in (8) and differentiating them with respect to θ twice, three ordinary differential equations for $x_0(\theta)$, $x_1(\theta)$ and $x_2(\theta)$ are obtained:

$$\begin{aligned} &x_0''(\theta) + x_0(\theta) = 0 \quad x_0(0) = A;\ \dot{x}_0(0) = 0 \\ &x_1''(\theta) + x_1(\theta) = F(x_0(\theta), x_0''(\theta), \omega_0^2) \quad x_1(0) = 0;\ \dot{x}_1(0) = 0 \\ &x_2''(\theta) + x_2(\theta) = P(x_0(\theta), x_0''(\theta), \omega_0^2) \quad x_2(0) = 0;\ \dot{x}_2(0) = 0 \end{aligned} \tag{21}$$

Note that F and P are given by (20) and (13) respectively.

The solution to (21) is simply

$$x_0(\theta) = A\cos(\theta). \tag{22}$$

After taking into account (20), the expression for $x_1(\theta)$ is

$$x_1''(\theta) + x_1(\theta) = x_0 - \frac{1}{\omega_0^2}\frac{x_0}{1-x_0^2}. \quad x_1(0) = 0;\ \dot{x}_1(0) = 0. \tag{23}$$

Finally, the differential equation for $x_2(\theta)$ is given by

$$x_2''(\theta) + x_2(\theta) = x_1\left(1 - \frac{1}{\omega_0^2}\frac{1+x_0^2}{(1-x_0^2)^2}\right) + \frac{\omega_1^2}{\omega_0^4}\frac{x_0}{1-x_0^2}. \quad x_2(0) = 0;\ \dot{x}_2(0) = 0. \tag{24}$$

The solution to (23) is obtained following Ramos [31]. Considering that $\frac{x}{1-x^2}$ is an analytic function of x, it can be expanded in a Taylor's series about $x = 0$ as

$$\frac{x}{1-x^2} = x\sum_{n=0}^{\infty}(x^2)^n$$

Then, introducing it into the RHS of (23) and using (22) one obtains

$$x_1''(\theta) + x_1(\theta) = A\left(1 - \frac{1}{\omega_0^2}\right)\cos(\theta) - \frac{1}{\omega_0^2}\sum_{n=1}^{\infty}A^{2n+1}\cos^{2n+1}(\theta) \quad x_1(0) = 0;\ \dot{x}_1(0) = 0 \tag{25}$$

With the help of Euler's formula, De Moivre's formula and Newton's binomial theorem [34], it is not difficult to write $\cos^{2n+1}(\theta)$ as

$$\cos^{2n+1}(\theta) = \sum_{k=0}^{n} a_{n,k}\cos(2n-2k-1)(\theta)$$

where $a_{n,k}$ is given by

$$a_{n,k} = \frac{1}{2^{2n}}\binom{2n+1}{k} \tag{26}$$

Substituting the above identity into (25) and eliminating the secular terms, the final expression for ω_0^2 reads

$$\omega_0^2 = \sum_{n=0}^{\infty} A^{2n+1}\frac{(2n+1)!}{2^{2n}(n+1)!(n!)} = 1 + \frac{3}{4}A^2 + \frac{10}{16}A^4 + \frac{35}{64}A^6 + \frac{126}{256}A^8 + \cdots \tag{27}$$

which represents the frequency at $\mathcal{O}(p^0)$. It is worth noting that the series for ω_0^2 (27) is a convergent series as it can be proven, for example, by the ratio test [35] (using $A < 1$). The expression for the period, $T = 2\pi/\omega$, then results:

$$T_{G2} = 2\pi\left(1 - \frac{3}{8}A^2 - \frac{13}{128}A^4 - \frac{55}{1024}A^6 - \frac{1149}{32768}A^8 + \cdots\right) \tag{28}$$

Finally, (25) can be written as

$$x_1''(\theta) + x_1(\theta) = -\sum_{n=1}^{\infty} H_n\cos((2n+1)\theta) \quad x_1(0) = 0;\ \dot{x}_1(0) = 0 \tag{29}$$

where the coefficient H_n is

$$H_n = \frac{1}{\omega_0^2}\sum_{k=n}^{\infty} A^{2k+1}\frac{(2k+1)!}{2^{2k}(k+1+n)!(k-n!)}$$

Solving (29), the solution to $\mathcal{O}(p^1)$ reads

$$x_1(\theta) = \sum_{n=1}^{\infty}\frac{H_n}{4n(n+1)}(\cos((2n+1)\theta) - \cos(\theta)) \tag{30}$$

Following a similar procedure, using (22) and (30) for $x_0(\theta)$ and $x_1(\theta)$ and inserting them into (24), one arrives at an equation for $x_2(\theta)$ with ω_1^2 as a function of A. However, the computations are too complex to be presented here and will be presented in a subsequent paper.

3.2. Linearized harmonic balance method

The linearized version of the harmonic balance procedure consists mainly in expressing the second approximation to the solution obtained within the harmonic balance method as the sum of the first approximation, which satisfies the initial conditions and an unknown function, $\delta x_1(t)$, which results after a linearization of the final expression in the equation of motion. Requiring that $\delta x_1(t)$ satisfies the homogeneous initial conditions, the harmonic balance method is applied again to obtain more accurate results. To this end, (2) is conveniently rewritten as a first step to apply the method:

$$(1 - x^2)\ddot{x} + x = 0 \quad x(0) = A;\ (A < 1);\ \dot{x}(0) = 0 \tag{31}$$

Next, following the lowest order harmonic balance method, a first initial approximation is proposed to be of the form $x_1(t) = A\cos(\omega t)$ where ω is the unknown frequency to be determined. The substitution of $x_1(t)$ into (31) leads to

$$\omega_{Hb1}(A)^2 = \frac{4}{4 - 3A^2} \tag{32}$$

which coincides with the frequency provided by (15). A more accurate frequency of oscillation is obtained if the following expression for the second-order approximation $x_2(t) = x_1(t) + \delta x_1(t)$ is replaced in (31) and second-order corrections in $(\delta x_1(t))^2$ are discarded. Finally, the frequency reads

$$\omega_{Lhb2}(A)^2 = \frac{80 - 46A^2 + \sqrt{4096 - 4352A^2 + 1236A^4}}{144 - 188A^2 + 55A^4}. \tag{33}$$

And the period results

$$T_{Lhb2} = 2\pi\left(1 - \frac{3}{8}A^2 - \frac{23}{256}A^4 - \frac{361}{8192}A^6 - \frac{3405}{131072}A^8 + \cdots\right) \tag{34}$$

Therefore, the second-order approximation to the solution is

$$x_2(t) = x_1(t) + \delta x_1(t) = A\cos(\omega_{Lhb2}t) + c_1(A)(\cos(\omega_{Lhb2}t) - \cos(3\omega_{Lhb2}t)) \tag{35}$$

where $c_1(A) = (\omega_{Lhb2}^2(A - \frac{3}{4}A^3) - A)/(1 - (1 + A^2/2)\omega_{Lhb2}^2)$.

4. Asymptotic representation of the period

Asymptotic representations of the period of nonlinear oscillators are very useful from a practical viewpoint since they provide, most of the times, very approximate analytical expressions of complex integrals which are very hard to obtain. This section is divided into two considering the magnitude of the amplitude of oscillation A. In the first subsection, amplitude values between $0 < A \leqslant 0.9$ are considered, and in the second subsection, A is taken for values very near to its maximum $A = 1$.

4.1. Small amplitudes of vibration

The exact value of the period for the FENO is, taking into account the initial conditions,

$$T(A) = 4\int_0^A \frac{dx}{\sqrt{2E + \ln(1 - x^2)}} \tag{36}$$

where E is the system's total energy. If a new constant G is defined as $G = \frac{1}{1-A^2}$ the period is now written as

$$T(A) = 4\int_0^A \frac{dx}{\sqrt{\ln(G(1 - x^2))}} \tag{37}$$

For small amplitudes $x \ll 1$, it is possible to express the term under the radical up to $\mathcal{O}(x^4)$ as $\ln(G(1 - x^2)) \approx \ln G - x^2 - x^4/2$. Then, applying the linear transformation $x = \sqrt{\ln G}\, u$, (37) transforms into

$$T(A) \approx T_s(A) = 4\int_0^{A/\sqrt{\ln G}} \frac{du}{\sqrt{1 - u^2 - au^4}} \tag{38}$$

where the constant $a = \ln G/2$ and the subscript s indicates small amplitude approximation. The last integral can be written in terms of an incomplete elliptic integral of the first kind $K(q;k)$ [34]. The result is given by

$$T_s(A) = 8\frac{1}{\sqrt{2 + 2\sqrt{1 + 4a}}} K\left(\frac{A}{2\sqrt{\ln G}}\sqrt{2 + 2\sqrt{1 + 4a}}; \sqrt{-1 - \left(\frac{1}{2a} - \frac{\sqrt{1 + 4a}}{2a}\right)}\right) \tag{39}$$

A second-order solution in A is possible to obtain expanding (39), so that

$$T_s(A) \approx T_{s1}(A) = 2\pi\left(1 - \frac{3}{8}A^2\right) \tag{40}$$

Unfortunately, it is difficult to get higher order approximations for the period starting from (39). Instead, higher order terms can be obtained considering (37) and rearranging the terms under the radical, after a Taylor's expansion:

$$\ln(G(1 - x^2)) \approx (A^2 - x^2) + \frac{1}{2}(A^4 - x^4) + \frac{1}{3}(A^6 - x^6) + \frac{1}{4}(A^8 - x^8) + \cdots \tag{41}$$

Substituting $x = Au$, the integral (39) for the period results

$$T(A) = 4\int_0^1 \frac{du}{\sqrt{(1 - u^2) + \frac{1}{2}A^2(1 - u^4) + \frac{1}{3}A^4(1 - u^6) + \frac{1}{4}A^6(1 - u^8) + \cdots}} \tag{42}$$

Again, the denominator of (42) can be approximated as follows:

$$\frac{1}{\sqrt{(1 - u^2)}} \times \frac{1}{\sqrt{1 + \frac{\frac{1}{2}A^2(1-u^4) + \frac{1}{3}A^4(1-u^6) + \frac{1}{4}A^6(1-u^8) + \cdots}{(1-u^2)}}} = \frac{1}{\sqrt{(1 - u^2)}} \times \left[1 + a_{s1}(u)A^2 + a_{s2}(u)A^4 + a_{s3}(4)A^6 + a_{s3}(4)A^8 + \cdots\right] \tag{43}$$

where

$$a_{s1}(u) = -\frac{(1 - u^2)}{4}; \quad a_{s2}(u) = -\left(\frac{7}{96} - \frac{7}{48}u^2 + \frac{7}{96}u^4\right); \quad a_{s3}(u) = -\left(\frac{5}{128} - \frac{1}{128}u^2 - \frac{1}{128}u^4 + \frac{5}{128}u^6\right);$$
$$a_{s4}(u) = -\left(\frac{787}{30720} - \frac{37}{7680}u^2 - \frac{13}{5120}u^4 - \frac{37}{7680}u^6 + \frac{787}{30720}u^8\right)$$

Finally, substituting (43) into the integral (37) and integrating for u, the approximate expression for the period up to eighth order in A is

$$T_{s2}(A) = 2\pi\left(1 - \frac{3}{8}A^2 - \frac{23}{256}A^4 - \frac{91}{2048}A^6 - \frac{21829}{786432}A^8\right) \tag{45}$$

4.2. Amplitudes near the maximum elongation of the oscillator, A = 1

Considering the expression for the exact period (36) and the conservation of energy (4), the following change of variable can be set

$$1 - x^2 = e^{-u^2}$$

then

$$dx = \frac{ue^{-u^2}}{\sqrt{1-e^{-u^2}}}du$$

With these relations, the period obtained is

$$T(A) = 4\int_0^{\sqrt{\ln G}} \frac{ue^{-u^2}du}{\sqrt{1-e^{-u^2}}\sqrt{2E-u}} \tag{46}$$

with G as defined above. Making use of the initial conditions, it can be easily shown that $\ln G = 2E$. After renaming $\ln G = b$, and a new change of variables $z = u^2$, the expression for the period reads

$$T(A) = 4\int_0^b \frac{e^{-\frac{z}{2}}dz}{2b^{1/2}\sqrt{e^z-1}\sqrt{1-\frac{z}{b}}} \tag{47}$$

To compute (47) it is convenient to make a power series expansion of the term $(1-\frac{z}{b})^{-1/2}$

$$\frac{1}{\sqrt{1-\frac{z}{b}}} = 1 + \frac{1}{2}\left(\frac{z}{b}\right) + \frac{3}{8}\left(\frac{z}{b}\right)^2 + \frac{5}{16}\left(\frac{z}{b}\right)^3 + \cdots \tag{48}$$

Then, inserting (48) into (47) results in

$$T(A) \approx T_l(A) = 4\int_0^b \frac{e^{-\frac{z}{2}}}{2b^{1/2}\sqrt{e^z-1}}\left(1 + \frac{1}{2}\left(\frac{z}{b}\right) + \frac{3}{8}\left(\frac{z}{b}\right)^2 + \frac{5}{16}\left(\frac{z}{b}\right)^3 + \cdots\right)dz \tag{49}$$

Finally, it is easy to see that it is possible to extend the upper limit of integration to infinity, since for $b \to \infty$ the integrand is $I \propto \frac{e^{-\frac{z}{2}}}{\sqrt{e^z-1}} \to e^{-z}$ and therefore only exponentially small errors occur [36]. Then, the expression for the period taking (49) into account and integrating up to $\mathcal{O}(\frac{z}{b})^3$ results in

$$T_{l\infty}(A) = \frac{2}{b^{1/2}}\left(2 + \frac{a_1}{b} + \frac{a_2}{b^2} + \frac{a_3}{b^3}\right) \tag{50}$$

where a_1, a_2, a_3 are constants given by

Table 1
Comparison between the exact period $T(A)$ (numerically integrated), the asymptotic representation of the period for small amplitudes $T_{s2}(A)$, the period obtained through a general decomposition method in its first version $T_{G1}(A)$ and in its second version T_{G2}, and the period calculated using the linearized harmonic balance $T_{Lhb2}(A)$. Relative error (re) is computed as differences (%) between $T(A)$ and all the other approximations.

A	$T(A)$	$T_{s2}(A)$	reT_{s2}	$T_{G1}(A)$	reT_{G1}	$T_{G2}(A)$	reT_{G2}	$T_{Lhb2}(A)$	reT_{Lhb2}
0.01	6.28295	6.28295	0	6.28295	0	6.28295	0	6.28295	0
0.1	6.25957	6.25957	0	6.25956	1.5910^{-4}	6.25956	1.5910^{-4}	6.25957	0
0.2	6.18802	6.18802	0	6.18802	0	6.18789	0.00210	6.18801	1.6110^{-4}
0.3	6.06634	6.06634	0	6.06634	0	6.06570	0.0105	6.06634	0
0.4	5.89047	5.89048	-1.6910^{-4}	5.89051	-6.79^4	5.88831	0.0366	5.89050	-5.0910^{-4}
0.5	5.65366	5.65381	−0.00265	5.65398	−0.00566	5.64792	0.101	5.65389	−0.00407
0.6	5.34482	5.34584	−0.0191	5.34659	−0.0331	5.33146	0.250	5.34614	−0.0247
0.7	4.94463	4.95021	−0.113	4.95277	−0.164	4.91565	0.586	4.95113	−0.131
0.8	4.41508	4.44155	−0.599	4.44899	−0.768	4.35312	1.40	4.44403	−0.655
0.825	4.25385	4.29254	−0.909	4.30206	−1.13	4.17845	1.77	4.29566	−0.983
0.85	4.07674	4.13334	−1.38	4.14543	−1.68	3.98446	2.26	4.13726	−1.484
0.875	3.87997	3.96310	−2.14	3.97834	−2.53	3.76598	2.94	3.96797	−2.27
0.9	3.65767	3.78085	−3.36	3.79994	−3.89	3.51491	3.90	3.78689	−3.53

$$a_1 = 2 - \ln 4; \quad a_2 = \frac{3}{8}\left[\frac{-2\pi^2}{3} + 2(8 - 4\ln 4 + (\ln 4)^2)\right];$$
$$a_3 = \frac{5}{16}[2(48 + \pi^2(-2 + \ln 4) - \ln 4(24 + \ln 4(-6 + \ln 4))) - 12\zeta(3)]$$

and $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$ is the Riemann Zeta function.

5. Comparison of numerical calculations

Table 1 and Fig. 1 show a comparison for $0 < A \leqslant 0.9$ between the exact period $T(A)$, obtained by means of an accurate numerical integration and four different ways to approximate this value: the asymptotic representation of the period for small amplitudes $T_{s2}(A)$, the period obtained with a general decomposition method in its first and second version, $T_{G1}(A)$ and $T_{G2}(A)$, and the calculations given by the linearized harmonic balance method $T_{Lhb2}(A)$. The relative error is computed as differences (%) between $T(A)$ and the other four approximate representations of the period, which are columns reT_{s2}, reT_G1, reT_{G2} and reT_{Lhb2} of Table 1. As it can be immediately seen from those values, the error never exceeds a maximum of 3.92% for all the different approaches, which represents a very good value bearing in mind that almost the whole allowed range of amplitudes of oscillation are considered. Moreover, for values of $A < 0.8$, the error is always less than 0.77 % for $T_{s2}(A)$, $T_{G1}(A)$ and $T_{Lhb2}(A)$. The lowest relative error in Table 1 is obtained with the asymptotic representation of the period, $T_{s2}(A)$. Then follow in increasing order, the linearized harmonic balance method ($T_{Lhb2}(A)$), the first version of the GDM ($T_{G1}(A)$), and finally the second version of the GDM ($T_{G2}(A)$).

Table 2 and Fig. 2 exhibit a comparison between the asymptotic representation of the period obtained in Section 4.2, $T_{l\infty}(A)$, and $T(A)$ for amplitudes $0.99 \leqslant A \leqslant 0.9999999$. Naturally, values for A between $0.999 \leqslant A \leqslant 0.9999999$ have only an academic interest and are going to be considered here as a demonstration of the accuracy of the asymptotic solution. Firstly, it must be noticed that the relative error never exceeds a maximum value of -3.19% for the considered range, and it improves its approximation for increasing values of A. This is in agreement with the analytic predictions since, for $T_{l\infty}(A)$ to give a good approximation for the period, $b = \ln(\frac{1}{1-A^2})$ must be high (>10). Moreover, for $0.999 \leqslant A \leqslant 0.9999999$ the agreement is excellent, always less than 0.5%.

In order to produce a global estimator of the accuracy of the solutions, the L_2 norm of a function $f(x)$ is used to compare the three different approximate solutions. The idea here is to know whether a good approximation to the period necessarily represents a satisfactory approximation to the solution. The L_2 norm of a function is defined as $\|f\|_{L_2} = \sqrt{\int_\Omega f^2(x)dx}$, and it represents the norm of a square-integrable function $f(x)$ in the domain Ω [37]. From this definition, it is possible to induce a distance between two functions ($f(x)$ and $g(x)$) or "L_2 error" as $\|f - g\|_{L_2}$. Fig. 3 shows the L_2 error between $f(A)$ and three approximate solutions of the FENO, f_{G1}, f_{G2} and f_{Lhb2} over one period, $T(A)$. As explained above, the "exact" solution $f(A)$ is obtained with an accurate numerical integration of (2). Two different situations can be observed in the figure. For $0 < A \leqslant 0.6$ the first version of the generalized decomposition method (f_{G1}) gives the best approximation to the numerical solution. Then follow, in decreasing order, f_{Lhb2} and f_{G2}. On the other hand, for $0.6 < A < 0.9$, f_{Lhb2} is the solution that shows the best agree-

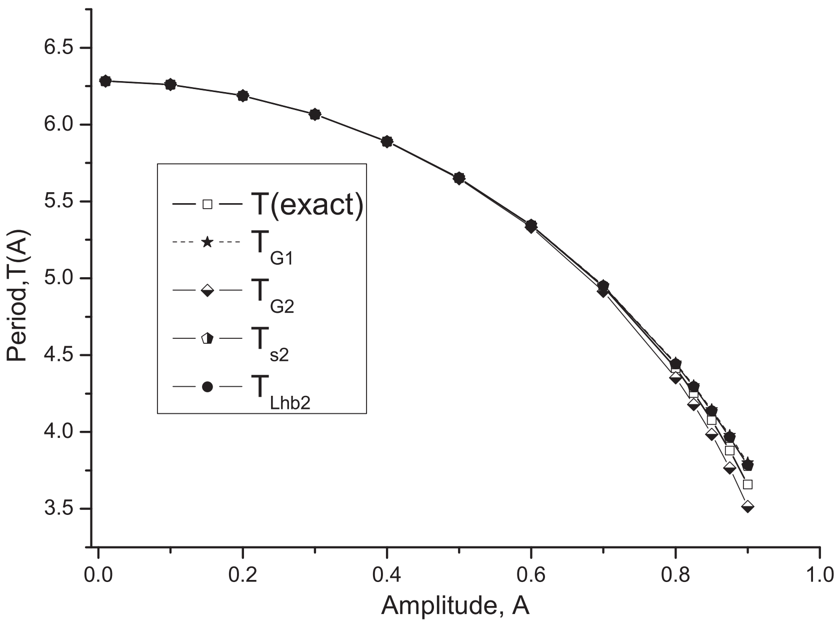

Fig. 1. Graphical representation of the exact period T(exact), an asymptotic representation of the period T_{s2}, the two versions of the GDM: first (T_{G1}) and second (T_{G2}), and the linearized version of the harmonic balance method T_{Lhb2} for $0 < A \leqslant 0.9$.

Table 2
Comparison between the exact period $T(A)$ and $T_{l\infty}(A)$ for $0.99 \leqslant A \leqslant 0.9999999$. Relative error (re) is computed as differences (%) between $T(A)$ and $T_{lb}(A)$.

A	$b = \ln\left(\frac{1}{1-A^2}\right)$	$T(A)$	$T_{l\infty}(A)$	re$T_{l\infty}(A)$
0.99	3.9170	2.26915	2.34159	−3.19
0.999	6.2151	1.71419	1.72219	−0.467
0.9999	8.5172	1.43181	1.43484	−0.212
0.99999	10.8197	1.25628	1.25787	−0.126
0.999999	13.1223	1.13333	1.13423	−0.0794
0.9999999	15.4249	1.04083	1.04137	−0.0519

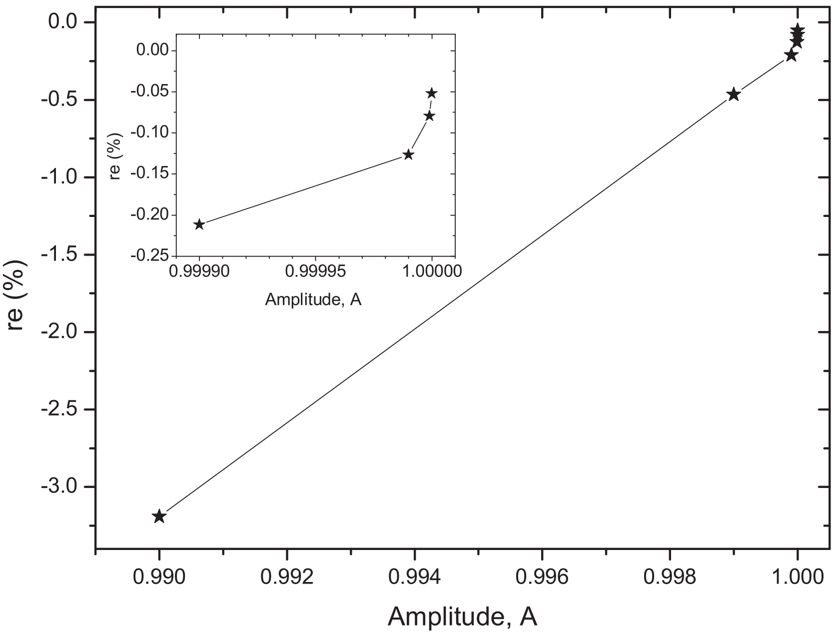

Fig. 2. Graphical representation of the exact period $T(A)$ and the asymptotic representation of the period, $T_{l\infty}(A)$ for $0.99 \leqslant A \leqslant 0.9999999$.

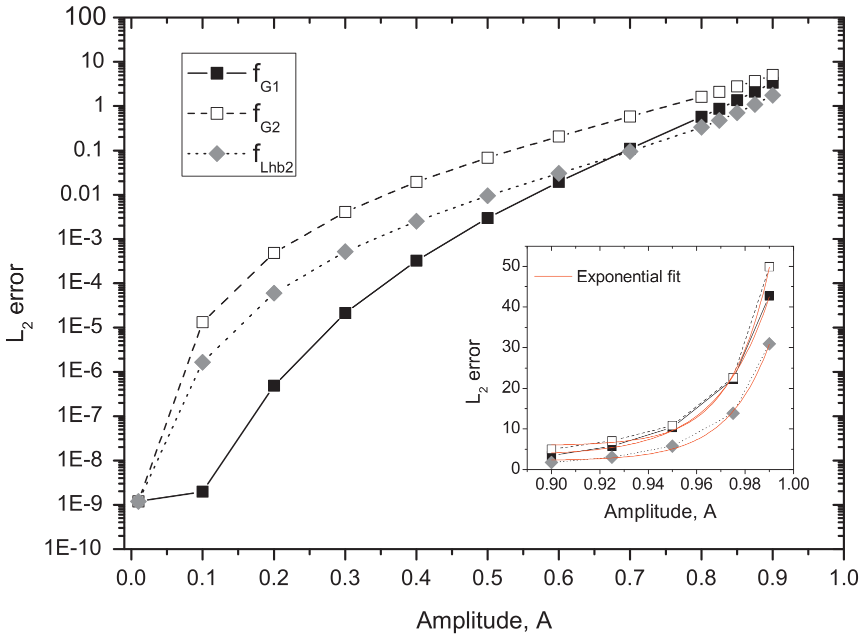

Fig. 3. L_2 error as a function of the amplitude A for the general decomposition method, first (f_{G1}) and second version (f_{G2}) and the linearized harmonic balance method f_{Lhb2} ($0 < A \leqslant 0.9$). The inset shows the L_2 error for $0.9 \leqslant A \leqslant 0.99$. It can be observed that the curves match exponential functions (solid traces) for the three cases (f_{G1},f_{G2},f_{Lhb2}).

ment with the numerical solution when compared to f_{G1} and alternatively to f_{G2}. It is possible then to conclude that all three perturbative schemes present also a good approximation to the solution itself for the considered range ($0 < A \leqslant 0.9$).

Finally, the inset of Fig. 3 shows the L_2 error for $0.9 < A \leqslant 0.99$. It can be observed that the error increases exponentially with A for all three cases. This clearly shows that, in this limit, the three approximations to the solution do not represent a good description for the FENO, and higher order perturbation solutions are needed to achieve a reasonable degree of accuracy.

6. Conclusions

In this work, approximate analytical expressions for the solution and the period of a finite extensibility nonlinear oscillator (FENO) are obtained by means of two perturbation methods: a general decomposition method (and a version of it) and a linearized harmonic balance procedure. Considering the perturbative characteristics of the proposed methods, the results show a very good agreement with the exact (numerically obtained) solution not only for the period, but also for the solution itself for amplitudes lower than 0.9 ($A \leqslant 0.9$). Within this range, it is possible to affirm the following. From the calculation of the period, the results for the three methods show the same tendency of increasing the relative error with increasing amplitude, but they never exceed a maximum error of 3.90% (for the second version of GDM at $A = 0.9$). It can also be observed that the linearized harmonic balance provides a slightly better approximation to the period than the other methods. From the point of view of the global quality of solutions (L_2 error), the results reveal the same tendency of increasing error with increasing amplitude in the three methods. However, the relative error is always bounded by a satisfactory value of 5% in the worst case (second version of the generalized decomposition method (f_{G2})) for $A = 0.9$. Detailed calculations show that (f_{G1}) gives the best approximation for $0 < A \leqslant 0.6$, and f_{Lhb2} is the solution that shows the best agreement for $0.6 < A \leqslant 0.9$. From this point of view, it is possible to arrive at the important conclusion that a properly linearized model of the FENO (linearized harmonic balance method) provides an excellent solution for a large range of amplitudes of oscillation ($0 < A \leqslant 0.9$).

For the rest of the amplitude the range ($0.9 < A < 1$), it is possible to conclude that the three methods cease to provide a good accuracy of the results for the orders of perturbation considered in the present work. This is evidenced by a calculated exponentially large L_2 error for all three approximations to the exact solution. Instead, an asymptotic representation of the period is presented for the same range. This solution yields a relative error lower than -3.19% for $A = 0.99$ and improves its accuracy as larger values of b (which increases with elongation) are considered. Moreover, the results show that the relative error never exceeds 0.5% for $0.999 \leqslant A \leqslant 0.9999999$. One must keep in mind that such large extensions of an oscillator might have only an academic interest since a rupture of any physical mechanism of the oscillator may take place if one approaches this limit.

A very interesting extension of this work would be to introduce damping into the model (of the viscous type or of the Coulomb type) to study how damping affects not only the natural frequency of the FENO, but also its periodic motion. This analysis, as stated in the introduction, would be the basis on which a profound study of the forced motion of the FENO may lay.

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References

[1] P. Maniadis, G. Kopidakis, S. Aubry, Classical and quantum targeted energy transfer between nonlinear oscillators, Physica D 188 (2004) 153–177.
[2] A. Pathak, S. Mandal, Classical and quantum oscillators of sextic and octic anharmonicities, Physics Letters A 298 (2002) 259–270.
[3] J.A. Acebrón, R. Spigler, Uncertainty in phase-frequency synchronization of large populations of globally coupled nonlinear oscillators, Physica D 141 (2000) 65–79.
[4] Sandra R.F.S.M. Gois, Marcelo A. Savi, An analysis of heart rhythm dynamics using a three-coupled oscillator model, Chaos, Solitons and Fractals 41 (2009) 2553–2565.
[5] J.I. Ramos, Linearized Galerkin and artificial parameter techniques for the determination of periodic solutions of nonlinear oscillators, Applied Mathematics and Computation 196 (2008) 483–493.
[6] Z. Guo, A.Y.T. Leung, The iterative homotopy harmonic balance method for conservative HelmholtzDuffing oscillators, Applied Mathematics and Computation 215 (2010) 3163–3169.
[7] R.E. Mickens, Quadratic non-linear oscillators, Journal of Sound and Vibration 270 (2004) 427–432.
[8] H. Scherrer, H. Risken, T. Leiber, Eigenvalues of the Schrödinger equation with rational potentials, Physical Review A 38 (1988) 3949–3959.
[9] H. Risken, H.D. Vollmer, The influence of higher order contributions to the correlation function of the intensity fluctuation in a laser near threshold, Zeitschrift fr Physik A 201 (1967) 323–330.
[10] S.N. Biswas, K. Datta, R.P. Saxena, P.K. Srivastava, V.S. Varma, Eigenvalues of λx^{2m} anharmonic oscillators, Journal of Mathematical Physics 14 (1973) 1190–1195.
[11] M. Febbo, A. Milchev, V. Rostiashvili, D. Dimitrov, T.A. Vilgis, Dynamics of a stretched nonlinear polymer chain, The Journal of Chemical Physics 129 (2008) 1549081–15490813.
[12] J.W. Hatfield, S.R. Quake, Dynamic properties of an extended polymer in solution, Physical Review Letters 82 (1999) 3548–3551.
[13] J. Koplik, J.R. Banavar, Extensional rupture of model non-Newtonian fluid filaments, Physical Review E 67 (2003) 115021–1150212.
[14] N. Kremer, Computational Soft Matter: From Synthetic Polymers to Proteins, NIC Series, vol. 23, John von Neumann Institute for Computing, 2004.

[15] S. Gatz, J. Herrmann, Soliton propagation and soliton collision in double-doped fibers with a non-Kerr-like nonlinear refractive-index change, Optics Letters 17 (1992) 84–486.
[16] T.R.O. Melvin, A.R. Champneys, P.G. Kevrekidis, J. Cuevas, Travelling solitary waves in the discrete Schrodinger equation with saturable nonlinearity: Existence, stability and dynamics, Physica D 237 (2008) 551–567.
[17] A. Khare, K. Rasmussen, M.R. Samuelsen, Avadh Saxena, Exact solutions of the saturable discrete nonlinear Schrödinger equation, Journal of Physics A 38 (2005) 807–814.
[18] R.E. Mickens, Construction of a perturbation solution for a non-linear, singular oscillator equation, Journal of Sound and Vibration 130 (1989) 513–515.
[19] J-H. He, Some asymptotic methods for strongly nonlinear equations, International Journal of Modern Physics 20 (2006) 1141–1199.
[20] J-H. He, Addendum: New interpretarion of homotoy perturbation methods, International Journal of Modern Physics 20 (2006) 2561–2568.
[21] H. Hu, J.H. Tang, Solution of a Duffing-harmonic oscillator by the method of harmonic balance, Journal of Sound and Vibration 294 (2006) 637–639.
[22] H.P.W. Gottlieb, Harmonic balance approach to periodic solutions of non-linear Jerk equations, Journal of Sound and Vibration 271 (2004) 671–683.
[23] B.S. Wu, C.W. Lim, Large amplitude nonlinear oscillations of a general conservative system, International Journal of Non-linear Mechanics 39 (2004) 859–870.
[24] A. Beléndez, D.I. Méndez, T. Beléndez, A. Hernández, M.L. Álvarez, Harmonic balance approaches to the nonlinear oscillators in which the restoring force is inversely proportional to the dependent variable, Journal of Sound and Vibration 314 (2008) 775–782.
[25] A.H. Nayfeh, D.T. Mook, Nonlinear Oscillations, John Wiley and Sons, 1979.
[26] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer, Boston, MA, 1994.
[27] M.M. Hosseini, Adomian decomposition method with Chebyshev polynomials, Applied Mathematics and Computation 175 (2006) 1685–1693.
[28] R.E. Mickens, Harmonic balance and iteration calculations of periodic solutions to $\ddot{y} + y^{-1} = 0$, Journal of Sound and Vibration 306 (2007) 968–972.
[29] R.E. Mickens, A generalized iteration procedure for calculating approximations to periodic solutions of truly nonlinear oscillators, Journal of Sound and Vibration 287 (2005) 1045–1051.
[30] J.I. Ramos, Generalized decomposition method for singular oscillators, Chaos, Solitons and Fractals 42 (2009) 1149–1155.
[31] J.I. Ramos, Generalized decomposition method for nonlinear oscillators, Chaos, Solitons and Fractals 41 (2009) 1078–1084.
[32] A. Beléndez, C. Pascual, D.I. Mendez, C. Neipp, Solution of the relativistic (an)harmonic oscillator using the harmonic balance method, Journal of Sound and Vibration 311 (2008) 1447–1456.
[33] L. Meirovitch, Methods of Analytical Dynamics, Dover, Mineola, NY, 2003.
[34] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, Inc., 1996.
[35] T.M. Apostol, Mathematical analysis, second ed., Addison-Wesley, 1974.
[36] A. Beléndez, A. Hernández, T. Beléndez, C. Neipp, A. Márquez, Asymptotic representations of the period for the nonlinear oscillator, Journal of Sound and Vibration 299 (2007) 403–408.
[37] K. Rektorys, Variational methods in mathematics, science and engineering, D. Reidel, Dordrecht, Holland, 1980.