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Research Article

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On viscosity and weak solutions for non-homogeneous p-Laplace equations

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Abstract: In this manuscript, we study the relation between viscosity and weak solutions for non-homogeneous p-Laplace equations with lower-order term depending on x, u and ∇u. More precisely, we prove that any locally bounded viscosity solution constitutes a weak solution, extending results presented in Juutinen, Lindqvist and Manfredi [9], and Julin and Juutinen [6]. Moreover, we provide a converse statement in the full case under extra assumptions on the data.

Keywords: Quasilinear equations with p-Laplacian, weak solutions, viscosity solutions, non-homogeneous equation

MSC 2010: 35J92, 35J70, 35D40, 35D30

1 Introduction and main results

In this work, we consider the following degenerate (or singular) elliptic equations of p-Laplacian type:

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(x, u, \nabla u), \tag{1.1}$$

defined in an open and bounded set $\Omega \subset \mathbb{R}^n$ and for $1 < p < \infty$. The modulus of ellipticity of the p-Laplace operator is $|\nabla u|^{p-2}$. When $p > 2$, the modulus vanishes whenever $\nabla u = 0$, and the equation is called degenerate at those points where that occurs. On the other hand, for $p < 2$, the modulus becomes infinite when $\nabla u = 0$, and the equation is called singular at those points. Observe that the case $p = 2$ is just the linear case and corresponds to the Laplace operator.

Different notions of solutions have been formulated for equation (1.1). We are interested in the relation between Sobolev weak solutions and viscosity solutions. For the homogeneous p-Laplace equation, this relation has already been studied by Juutinen, Lindqvist and Manfredi in [9], via the notion of p-harmonic, p-subharmonic and p-superharmonic functions. Roughly speaking, a p-harmonic function is a continuous function which solves, weakly, the homogeneous p-Laplace equation, and a p-superharmonic (p-subharmonic) function is a lower (upper) semicontinuous function that admits comparison with p-harmonic functions from below (above).

In [9], Juutinen, Lindqvist and Manfredi showed that the notion of p-harmonic solution is equivalent to the notion of viscosity solution. Moreover, it was shown in [13] that locally bounded p-harmonic functions are weak solutions. Conversely, every weak solution to the homogeneous p-Laplace equation has a representative which is lower semicontinuous and it is p-harmonic. We refer the interested reader to [5] for further details. In this way, there is an equivalence between the notion of weak and viscosity solutions for the homogeneous

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framework. It is worth to mention that a different and simpler proof of this equivalence was stated by Julin and Juutinen in [6] by using inf and sup convolutions. In turn, this reasoning was extended in [10] to more general second-order differential equations.

For the non-homogeneous case, the notion of p-harmonic functions is lost and we need to study directly the link between viscosity and Sobolev weak solutions. In [6], the authors showed that viscosity solutions of (1.1) are weak solutions in the case where f is continuous and depends only on x.

Our main goal in the present manuscript is to prove the equivalence of these two notions of solutions for the general structure (1.1). The implication that viscosity solutions are weak solutions is partially based on the work [6], but the non-homogeneous nature of the equation under consideration requires some extra effort to deal with the lower-order term.

On the other hand, the converse statement relies on comparison principles for weak solutions. To the best of our knowledge, the available comparison results for the full case $f = f(x, s, \eta)$ require additional limitations in the degenerate case which do not appear in the singular context (compare Theorem A.1 and Theorem A.2). Moreover, we believe that the assumption that weak subsolutions and weak supersolutions belong to $\mathcal{C}^1$ or to the Sobolev space $W^{1,\infty}_{\text{loc}}$ in order to have comparison is not a strong limitation since we are interested in the equivalence of weak and viscosity solutions, and for weak solutions the $\mathcal{C}^{1,\alpha}$-regularity holds (see [4, 16]). Finally, in the quasi-linear case $f = f(x, u)$ there is no need to impose higher regularity than $W^{1,p}_{\text{loc}} \cap \mathcal{C}$ on the solutions. We refer the reader to [15] for a survey of maximum principles and comparison results for general structures in divergence form.

Finally, we stress that the equivalence between weak and viscosity solutions may be used to prove relevant properties on the solutions. As an example, in [7], Juutinen and Lindqvist prove a Radó's-type theorem for p-harmonic functions. Roughly speaking, they state that if a function u solves, weakly, the homogeneous p-Laplace equation in the complement of the set where u vanishes, then it is a solution in the whole set. It is an open problem to obtain a similar result for equations like (1.1). We shall return to this issue in a subsequent paper.

We recall that the p-Laplace operator is defined as

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

Let us state the different type of solutions to (1.1) we will manage.

Definition 1.1 (Sobolev weak solution). *A function $u \in W^{1,p}_{\text{loc}}(\Omega)$ is a weak supersolution to* (1.1) *if*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\psi \geq \int_\Omega f(x, u, \nabla u)\psi$$

for all non-negative $\psi \in \mathcal{C}^\infty_0(\Omega)$. On the other hand, u is a weak subsolution if $-u$ is a weak supersolution of the equation $-\Delta_p u = -f(x, -u, -\nabla u)$. We call u a weak solution if it is both a weak subsolution and a weak supersolution to (1.1).

Due to the non-homogeneous nature of (1.1), viscosity solutions are stated as in [6], considering semicontinuous envelopes of the p-Laplace operator. More precisely, we have the following definition.

Definition 1.2. *A lower semicontinuous function $u : \Omega \to (-\infty, +\infty]$ is a viscosity supersolution to* (1.1) *if $u \not\equiv +\infty$ and for every $\phi \in \mathcal{C}^2(\Omega)$ such that $\phi(x_0) = u(x_0)$, $u(x) \geq \phi(x)$ and $\nabla\phi(x) \neq 0$ for all $x \neq x_0$, there holds*

$$\lim_{r\to 0} \sup_{x\in B_r(x_0)\setminus\{x_0\}} (-\Delta_p\phi(x)) \geq f(x_0, u(x_0), \nabla\phi(x_0)). \tag{1.2}$$

A function u is a viscosity subsolution if $-u$ is a viscosity supersolution to the equation $-\Delta_p u = -f(x, -u, -\nabla u)$, and it is a viscosity solution if it is both a viscosity sub- and supersolution.

Remark 1.3. Notice that condition (1.2) is established this way to avoid the problems derived from having $\nabla\phi(x_0) = 0$ in the case $1 < p < 2$. If $p \geq 2$, this condition can be simply replaced by

$$-\Delta_p\phi(x_0) \geq f(x_0, u(x_0), \nabla\phi(x_0)).$$

We now list the main contributions of our work. The results are stated for supersolutions, but they hold for subsolutions as well.

Theorem 1.4. *Let $1 < p < \infty$. Assume that $f = f(x, s, \eta)$ is uniformly continuous in $\Omega \times \mathbb{R} \times \mathbb{R}^n$, non-increasing in s, and satisfies the growth condition*

$$|f(x, s, \eta)| \le \gamma(|s|)|\eta|^{p-1} + \phi(x), \tag{1.3}$$

where $\gamma \ge 0$ is continuous, and $\phi \in L^\infty_{\mathrm{loc}}(\Omega)$. Hence, if $u \in L^\infty_{\mathrm{loc}}(\Omega)$ is a viscosity supersolution to (1.1)*, then it is a weak supersolution to* (1.1).

A converse of Theorem 1.4 is given below.

Theorem 1.5. *Assume that $f = f(x, s, \eta)$ is continuous in $\Omega \times \mathbb{R} \times \mathbb{R}^n$, non-increasing in s, and locally Lipschitz continuous with respect to η. Hence we have the following:*
(i) *If $1 < p \le 2$ and if $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ is a weak supersolution to* (1.1)*, then it is a viscosity supersolution to* (1.1) *in Ω.*
(ii) *If $p > 2$, $f(x, s, 0) = 0$ for $x \in \Omega$ and $s \in \mathbb{R}$, and if $u \in \mathcal{C}^1(\Omega)$ is a weak supersolution to* (1.1)*, then it is a viscosity supersolution to* (1.1) *in Ω.*
(iii) *Finally, if $p > 2$ and if $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ is a weak supersolution to* (1.1)*, with $\nabla u \neq 0$ in Ω, then it is a viscosity supersolution to* (1.1).

Remark 1.6. According to recent results (see [11]), it is possible to weak the locally Lipschitz assumption in Theorem 1.5 when f takes some particular forms or it satisfies extra convexity and coercivity assumptions. For instance, as a consequence of the results in [11], if

$$f(x, \eta) = c|\eta|^q + \phi(x), \quad \phi > 0,\ c \in \mathbb{R},\ \eta \in \mathbb{R}^n, \text{ and } q \in [1, p],$$

then Theorem 1.5 holds for bounded supersolutions u in $W^{1,p}(\Omega) \cap \mathcal{C}(\Omega)$. It is also a consequence of [11, Theorem 1.3] that the same conclusion is obtained when

$$f(x, s, \eta) = h(s)|\eta|^{p-1} + \phi(x),$$

where $h \ge 0$ is decreasing and $\phi \ge 0$.

In view of the available regularity theory for weak solutions of (1.1), we have the following equivalence.

Corollary 1.7. *Let $1 < p < \infty$. Assume that $f = f(x, s, \eta)$ is uniformly continuous, locally Lipschitz in η, non-increasing in s, and satisfies the growth condition* (1.3)*. Additionally, assume that $f(x, s, 0) = 0$ for $x \in \Omega$ and $s \in \mathbb{R}$ when $p > 2$. Then u is a weak solution to* (1.1) *if and only if it is a viscosity solution to* (1.1).

We point out that, in the degenerate case, it is possible to remove the assumption $f(x, s, 0) = 0$ by imposing the non-vanishing of the gradient of the weak solution in the whole Ω. This is a straightforward consequence of Theorem 1.5 (iii).

In the particular case where f does not depend on η, we have the following converse to Theorem 1.4 which does not require the locally Lipschitz regularity of the solutions.

Theorem 1.8. *Let $1 < p < \infty$. Suppose that $f = f(x, s)$ is continuous in $\Omega \times \mathbb{R}$ and non-increasing in s. If $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap \mathcal{C}(\Omega)$ is a weak supersolution to* (1.1)*, then it is a viscosity supersolution to* (1.1).

Let us briefly discuss the above hypotheses on f. Firstly, assuming that f is non-increasing and introducing the operator

$$F(x, s, \eta, \mathcal{X}) := -|\eta|^{p-2}\Big(\mathrm{tr}(\mathcal{X}) + \frac{p-2}{|\eta|^2}\mathcal{X}\eta \cdot \eta\Big) - f(x, s, \eta), \quad p \ge 2,$$

we derive that F is proper, that is, F is non-increasing in $\mathcal{X}$ and non-decreasing in s, which is a standard and useful assumption in the theory of viscosity solutions [1]. For instance, it allows to get the equivalence between classical solutions ($\mathcal{C}^2$ functions which satisfy the equations pointwise) and $\mathcal{C}^2$ viscosity solutions.

On the other hand, the growth property (1.3) implies the $\mathcal{C}^{1,\alpha}$-regularity of weak solutions to (1.1) (see [4, 16, 17]). Moreover, under a regular Dirichlet boundary condition $\varphi \in \mathcal{C}^{1,\alpha}$, the $\mathcal{C}^{1,\alpha}$-regularity up to the boundary of weak solutions follows. For further details, see the reference [12]. Finally, the extra assumption $f(x, s, 0) = 0$ appearing in Theorem 1.5 in the degenerate case is used to remove critical sets of points of the weak solution (see reference [8]). Hence, it allows the application of comparison results without assuming the non-vanishing of the gradients. We point out that other properties of $f = f(x, s, \eta)$, as more regularity on s and η and convexity-like conditions, may be employed to ensure comparison for weak solutions. We refer the reader to [11] and the references therein for more details.

It is worth mentioning that many equations appearing in the literature have the structure of (1.1) with the lower-order term satisfying the above assumptions on f. We refer the reader to [2, 3, 14, 15] and the references therein for examples of such f.

The paper is organized as follows: In Section 2, we provide some preliminary results concerning properties of infimal convolutions (which will be the main tool in the proof of Theorem 1.4) and a convergence result. In addition, we prove a Caccioppoli-type estimate that will provide important uniform bounds, fundamental when using approximation arguments. This result is interesting in itself.

Section 3 contains the proof of the main result of the paper, Theorem 1.4, that states under which conditions on the non-homogeneous function f in (1.1) viscosity solutions are actually weak solutions. This proof is divided into two major cases: the singular and the degenerate scenario, thus, although both cases rely on the same idea, different approximations and estimates are needed depending on the range of p.

In Section 4, we prove the reverse statement, that is, weak solutions of (1.1) are viscosity solutions. This result is based on comparison arguments, and this will determine the conditions we will need to impose on f. Finally, in Appendix A we give, for the sake of completeness, precise references and state the comparison results that we use in Section 4.

2 Preliminary results

2.1 Infimal convolution

Let us define the infimal convolution of a function u as

$$u_\varepsilon(x) := \inf_{y\in\Omega}\Big(u(y) + \frac{|x-y|^q}{q\varepsilon^{q-1}}\Big), \tag{2.1}$$

where $q \geq 2$ and $\varepsilon > 0$.

We recall some useful properties of u_ε. Let $u : \Omega \to \mathbb{R}$ be bounded and lower semicontinuous in Ω. It is well known that u_ε is an increasing sequence of semiconcave functions in Ω, which converges pointwise to u. Hence, u_ε is locally Lipschitz and twice differentiable a.e. in Ω. Moreover, it is possible to write

$$u_\varepsilon(x) = \inf_{y\in B_{r(\varepsilon)}(x)\cap\Omega}\Big(u(y) + \frac{|x-y|^q}{q\varepsilon^{q-1}}\Big)$$

for $r(\varepsilon) \to 0$ as $\varepsilon \to 0$. For these and further properties, see [6, Lemma A.1.] and [1].

The next lemma is the counterpart of [6, Lemma A.1 (iii)] for our setting.

Lemma 2.1. *Suppose that $u : \Omega \to \mathbb{R}$ is bounded and lower semicontinuous in Ω. Let $f = f(x, s, \eta)$ be continuous in $\Omega \times \mathbb{R} \times \mathbb{R}^n$ and non-increasing in s. If u is a viscosity supersolution to*

$$-\Delta_p u = f(x, u, \nabla u) \tag{2.2}$$

in Ω for $1 < p < \infty$, then u_ε is a viscosity supersolution to

$$-\Delta_p u_\varepsilon = f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)$$

in $\Omega_\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > r(\varepsilon)\}$, where

$$f_\varepsilon(x, s, \eta) := \inf_{y\in B_{r(\varepsilon)}(x)} f(y, s, \eta).$$

Proof. We start by noticing that

$$u_\varepsilon(x) = \inf_{z\in B_{r(\varepsilon)}(0)}\Big(u(z+x) + \frac{|z|^q}{q\varepsilon^{q-1}}\Big), \quad x \in \Omega_\varepsilon.$$

Let us see first that for every $z \in B_{r(\varepsilon)}(0)$, the function

$$\phi_z(x) := u(z+x) + \frac{|z|^q}{q\varepsilon^{q-1}}$$

is a viscosity supersolution to $-\Delta_p\phi_z = f_\varepsilon$ in Ω_ε. Indeed, let $x_0 \in \Omega_\varepsilon$ and $\varphi \in \mathcal{C}^2(\Omega_\varepsilon)$ so that

$$\min_{\Omega_\varepsilon}(\phi_z - \varphi) = (\phi_z - \varphi)(x_0) = 0.$$

We assume that $\nabla\varphi(x) \neq 0$ for all $x \neq x_0$ if $1 < p < 2$. Making $y := z + x$, $y_0 := z + x_0$ and

$$\tilde\varphi(y) := \varphi(y - z) - \frac{|z|^q}{q\varepsilon^{q-1}},$$

we derive that $u - \tilde\varphi$ has a local minimum at y_0, and indeed $(u - \tilde\varphi)(y_0) = 0$. Since u is a viscosity supersolution to (2.2), there follows

$$\lim_{\rho\to 0}\sup_{x\in B_\rho(y_0)\setminus\{y_0\}}(-\Delta_p\tilde\varphi(x)) \geq f(y_0, \tilde\varphi(y_0), \nabla\tilde\varphi(y_0)).$$

Therefore,

$$\begin{aligned}
\lim_{\rho\to 0}\sup_{x\in B_\rho(x_0)\setminus\{x_0\}}(-\Delta_p\varphi(x)) &= \lim_{\rho\to 0}\sup_{x\in B_\rho(y_0)\setminus\{y_0\}}(-\Delta_p\tilde\varphi(x)) \\
&\geq f(y_0, \tilde\varphi(y_0), \nabla\tilde\varphi(y_0)) \\
&= f(z + x_0, \tilde\varphi(z + x_0), \nabla\varphi(x_0)) \\
&= f\Big(z + x_0, \varphi(x_0) - \frac{|z|^q}{q\varepsilon^{q-1}}, \nabla\varphi(x_0)\Big) \\
&\geq f(z + x_0, \varphi(x_0), \nabla\varphi(x_0)) \\
&\geq f_\varepsilon(x_0, \varphi(x_0), \nabla\varphi(x_0)),
\end{aligned} \tag{2.3}$$

where we have used that f is non-increasing in the second variable. Let us see now that, since u_ε is an infimum of supersolutions, it is itself a supersolution (observe that u_ε is continuous, since it is locally Lipschitz). Let $x_0 \in \Omega_\varepsilon$ and $\phi \in \mathcal{C}^2(\Omega_\varepsilon)$ so that

$$\min_{\Omega_\varepsilon}(u_\varepsilon - \phi) = (u_\varepsilon - \phi)(x_0) = 0. \tag{2.4}$$

Again, $\nabla\phi(x) \neq 0$ for all $x \neq x_0$ in the singular scenario. Moreover, we may assume that the minimum is strict. For each n, there exists $z_n \in B_{r(\varepsilon)}(0)$ such that

$$u(z_n + x_0) + \frac{|z_n|^q}{q\varepsilon^{q-1}} < u_\varepsilon(x_0) + \frac{1}{n}. \tag{2.5}$$

Let x_n be a sequence of points in $\overline{B}_r(x_0) \subset \Omega_\varepsilon$ so that

$$u(z_n + x_n) + \frac{|z_n|^q}{q\varepsilon^{q-1}} - \phi(x_n) \leq u(z_n + x) + \frac{|z_n|^q}{q\varepsilon^{q-1}} - \phi(x)$$

for all $x \in \overline{B}_r(x_0)$, i.e., $(\phi_{z_n} - \phi)$ has a minimum in $\overline{B}_r(x_0)$ at x_n. Up to a subsequence, $x_n \to y_0$ as $n \to \infty$. Furthermore, by (2.5),

$$\begin{aligned}
u_\varepsilon(x_n) - \phi(x_n) &\leq u(z_n + x_n) + \frac{|z_n|^q}{q\varepsilon^{q-1}} - \phi(x_n) \\
&\leq u(z_n + x_0) + \frac{|z_n|^q}{q\varepsilon^{q-1}} - \phi(x_0) \\
&\leq u_\varepsilon(x_0) + \frac{1}{n} - \phi(x_0).
\end{aligned} \tag{2.6}$$

Taking liminf and using the lower semicontinuity of u_ε, we derive

$$u_\varepsilon(y_0) - \phi(y_0) \le u_\varepsilon(x_0) - \phi(x_0).$$

Since the minimum in (2.4) is strict, we must have $y_0 = x_0$. Moreover, taking

$$\varphi(x) := \phi(x) + (\phi_{z_n} - \phi)(x_n)$$

in (2.3), we have

$$\lim_{\rho\to 0} \sup_{x\in B_\rho(x_n)\setminus\{x_n\}} (-\Delta_p \phi(x)) \ge f\Big(z_n + x_n, u(z_n + x_n) + \frac{|z_n|^q}{q\varepsilon^{q-1}}, \nabla\phi(x_n)\Big).$$

Since f is non-increasing with respect to the second variable, by (2.6) we obtain

$$\lim_{\rho\to 0} \sup_{x\in B_\rho(x_n)\setminus\{x_n\}} (-\Delta_p \phi(x)) \ge f\Big(z_n + x_n, u_\varepsilon(x_0) + \frac{1}{n} - \phi(x_0) + \phi(x_n), \nabla\phi(x_n)\Big).$$

As $n \to \infty$, there holds

$$\lim_{\rho\to 0} \sup_{x\in B_\rho(x_0)\setminus\{x_0\}} (-\Delta_p \phi(x)) \ge f(z' + x_0, u_\varepsilon(x_0), \nabla\phi(x_0))$$

for some $z' \in \overline{B_r(0)}$. Therefore,

$$\lim_{\rho\to 0} \sup_{x\in B_\rho(x_0)\setminus\{x_0\}} (-\Delta_p \phi(x)) \ge f_\varepsilon(x_0, \phi(x_0), \nabla\phi(x_0)),$$

and we conclude that u_ε is a viscosity supersolution of

$$-\Delta_p u_\varepsilon = f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \quad \text{in } \Omega_\varepsilon,$$

as desired. □

The next lemma states the weak convergence of the lower-order terms in the particular situation of infimal convolutions.

Lemma 2.2. *Let $f = f(x, s, \eta)$ be a uniformly continuous function, which satisfies the growth condition* (1.3). *Assume that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is locally bounded and lower semicontinuous in Ω. For each $\varepsilon > 0$ define u_ε as in* (2.1) *and f_ε as in Lemma 2.1. Then, if ∇u_ε converges to ∇u in $L^p_{\mathrm{loc}}(\Omega)$, the following holds:*

$$\lim_{\varepsilon\to 0} \int_\Omega f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\psi\, dx = \int_\Omega f(x, u, \nabla u)\psi\, dx$$

for every non-negative $\psi \in \mathcal{C}^\infty_0(\Omega)$.

Proof. Let $\psi \in \mathcal{C}^\infty_0(\Omega)$ and denote $K :=\mathrm{spt}(\psi)$. Consider $\varepsilon > 0$ small enough so that

$$K \subset K' \subset \Omega,$$

where $K' := \overline{\bigcup_{x\in K} B_{r(\varepsilon)}(x)}$. Since f is uniformly continuous in $K' \times \mathbb{R} \times \mathbb{R}^n$, for every $\rho > 0$ there exists $\delta > 0$ such that

$$|f(x, u_\varepsilon(x), \nabla u_\varepsilon(x)) - f(y, u_\varepsilon(x), \nabla u_\varepsilon(x))| < \rho \quad \text{if } |x - y| < \delta,\ x, y \in K'.$$

Choose $\varepsilon_0 > 0$ so that $r(\varepsilon) < \delta$ for every $\varepsilon < \varepsilon_0$. Thus, from the previous inequality we get

$$f(x, u_\varepsilon(x), \nabla u_\varepsilon(x)) < \rho + f(y, u_\varepsilon(x), \nabla u_\varepsilon(x))$$

for every $x \in K$ and $y \in B_{r(\varepsilon)}(x)$. In particular,

$$f(x, u_\varepsilon(x), \nabla u_\varepsilon(x)) < \rho + f_\varepsilon(x, u_\varepsilon(x), \nabla u_\varepsilon(x)),$$

and therefore

$$0 \le |f(x, u_\varepsilon(x), \nabla u_\varepsilon(x)) - f_\varepsilon(x, u_\varepsilon(x), \nabla u_\varepsilon(x))| < \rho.$$

Hence we arrive at the estimate

$$\int_\Omega |f(x, u_\varepsilon, \nabla u_\varepsilon) - f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)|\psi \, dx \le \rho \|\psi\|_{L^\infty(K)} |K|. \tag{2.7}$$

On the other hand, due to the continuity of f and the convergences of u_ε and ∇u_ε,

$$f(x, u_\varepsilon(x), \nabla u_\varepsilon(x)) \to f(x, u(x), \nabla u(x)) \quad \text{a.e. in } \Omega.$$

Observe that

$$u_{\varepsilon_0} \le u_\varepsilon \le u \quad \text{for all } \varepsilon \le \varepsilon_0.$$

Since u_{ε_0}, u belong to $L^\infty_{\mathrm{loc}}(\Omega)$, there exists a uniform constant $C > 0$ so that

$$\|u_\varepsilon\|_{L^\infty(K)} \le C, \quad \varepsilon \le \varepsilon_0.$$

Thus, in view of the growth estimate on f and the continuity of γ, we have, for an appropriate positive constant C,

$$|f(x, u_\varepsilon(x), \nabla u_\varepsilon(x))| \le C|\nabla u_\varepsilon(x)|^{p-1} + \phi(x). \tag{2.8}$$

Since $|\nabla u_\varepsilon|^{p-1} \in L^{p/(p-1)}_{\mathrm{loc}}(\Omega)$, Hölder's inequality and the strong convergence of ∇u_ε imply

$$\int_K |\nabla u_\varepsilon|^{p-1} \le C\|\nabla u_\varepsilon\|^{p-1}_{L^p(K)} \le C \quad \text{for all } \varepsilon.$$

By (2.7), (2.8) and the Lebesgue dominated convergence theorem, we conclude

$$\lim_{\varepsilon \to 0} \int_K f(x, u_\varepsilon, \nabla u_\varepsilon)\psi \, dx = \int_K f(x, u, \nabla u)\psi \, dx. \qquad \square$$

2.2 A Caccioppoli's estimate

In the next lemma we provide a Caccioppoli's estimate for the L^p_{loc}-norm of the gradients of weak solutions.

Lemma 2.3. *Let $u \in W^{1,p}(\Omega)$ be a locally bounded weak supersolution to* (1.1)*. Assume that f is continuous in $\Omega \times \mathbb{R} \times \mathbb{R}^n$ and satisfies the growth bound* (1.3)*. Then there exists a constant $C = C(p, \Omega, \phi, \gamma) > 0$ such that for all test function $\xi \in \mathcal{C}^\infty_0(\Omega)$, $0 \le \xi \le 1$, we have*

$$\int_\Omega |\nabla u|^p \xi^p \, dx \le C\Big[(\operatorname{osc}_K u)^p \int_\Omega (|\nabla \xi|^p + 1) \, dx + \operatorname{osc}_K u\Big],$$

where $\operatorname{osc}_K u := \sup_K u - \inf_K u$, *and* $K := \operatorname{spt}(\xi)$.

Proof. Let $\xi \in \mathcal{C}^\infty_0(\Omega)$ and $K \subset \Omega$ be as in the lemma. Consider the test function

$$\psi(x) := (\sup_K u - u(x))\xi^p(x), \quad x \in \Omega.$$

Then

$$\begin{aligned}\int_\Omega f(x, u, \nabla u)\psi \, dx &\le \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla \psi \, dx \\ &= -\int_\Omega |\nabla u|^{p-2}\nabla u \cdot [\xi^p \nabla u - p\xi^{p-1}\nabla \xi(\sup_K u - u)] \, dx.\end{aligned}$$

Therefore,

$$\int_\Omega |\nabla u|^p \xi^p \, dx \le p \int_\Omega \xi^{p-1} |\nabla u|^{p-2} \nabla u \cdot \nabla \xi (\sup_K u - u) \, dx - \int_\Omega f(x, u, \nabla u) \psi \, dx. \tag{2.9}$$

Observe that

$$\int_\Omega ||\nabla u|^{p-2} \nabla u| \, dx \le \int_\Omega |\nabla u|^{p-1} \, dx,$$

which shows that $|\nabla u|^{p-2}\nabla u \in L^{p/(p-1)}(\Omega)$. Hence, Young's inequality

$$ab \le \delta a^q + \delta^{-1/(q-1)} b^{q'},$$

where q and q' are conjugate exponents, implies that the first integral on the right-hand side of (2.9) may be bounded by

$$\delta \int_\Omega |\nabla u|^p \xi^p + \delta^{1-p} \int_\Omega p^p |\nabla \xi|^p (\operatorname{osc}_K u)^p \, dx.$$

Moreover, by (1.3) we have

$$f(x, u(x), \nabla u(x)) \ge -\gamma_\infty |\nabla u(x)|^{p-1} - \|\phi\|_{L^\infty(K)} \tag{2.10}$$

for all x in the support of ξ, where $\gamma_\infty := \sup_{x\in K} |\gamma(u(x))|$. Therefore, the second integral in (2.9) is estimated from above by

$$\gamma_\infty \int_\Omega |\nabla u|^{p-1} (\sup_K u - u) \xi^p \, dx + C(\Omega, \phi) \operatorname{osc}_K u,$$

where $C(\Omega, \phi)$ is a positive constant. The assumption $\xi \le 1$ and Young's inequality yield

$$\begin{aligned} \gamma_\infty \int_\Omega |\nabla u|^{p-1} (\sup_K u - u) \xi^p \, dx &\le \gamma_\infty \int_\Omega |\nabla u|^{p-1} (\sup_K u - u) \xi^{p-1} \, dx \\ &\le \delta \int_\Omega |\nabla u|^p \xi^p \, dx + \delta^{1-p} C(p, \Omega, \gamma) (\operatorname{osc}_K u)^p. \end{aligned}$$

Therefore,

$$\int_\Omega |\nabla u|^p \xi^p \, dx \le 2\delta \int_\Omega |\nabla u|^p \xi^p \, dx + \delta^{1-p} C(p, \Omega, \gamma, \phi) \Big[(\operatorname{osc}_K u)^p \int_\Omega (|\nabla \xi|^p + 1) \, dx + \operatorname{osc}_K u \Big].$$

Taking $\delta < 1/2$, we derive Caccioppoli's estimate. □

3 Proof of Theorem 1.4

3.1 Degenerate case

We begin with the range $p \ge 2$.

Proof of Theorem 1.4. Let u_ε be the infimal convolution defined in (2.1) with $q = 2$. Then

$$\phi(x) := u_\varepsilon(x) - C|x|^2$$

is concave in $\Omega_{r(\varepsilon)}$ (see [6, Lemma A.2.]). By Aleksandrov's Theorem, ϕ is twice differentiable almost everywhere in $\Omega_{r(\varepsilon)}$, and so is u_ε. Therefore by Lemma 2.1,

$$-\Delta_p u_\varepsilon(x) \ge f_\varepsilon(x, u_\varepsilon(x), \nabla u_\varepsilon(x))$$

a.e. in $\Omega_{r(\varepsilon)}$. Furthermore,

$$\int_\Omega |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \cdot \nabla\psi \geq \int_\Omega (-\Delta_p u_\varepsilon)\psi$$

for all non-negative test functions ψ (see the proof of [6, Theorem 3.1]). Hence, we derive

$$\int_\Omega f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\psi\, dx \leq \int_\Omega |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \cdot \nabla\psi\, dx$$

for all non-negative test functions ψ and all $\varepsilon > 0$. We claim that, as $\varepsilon \to 0$, there holds

$$\int_\Omega f(x, u, \nabla u)\psi\, dx \leq \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\psi\, dx.$$

To prove the claim, observe first that Caccioppoli's estimate allows us to conclude that

$$|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon$$

converges weakly in $L^{p/(p-1)}_{\mathrm{loc}}(\Omega)$. Indeed, for any compact set $K \subset \Omega$, choose an open set $U \subset \Omega$ containing K and a non-negative test function $0 \leq \xi \leq 1$ so that

$$K \subset K' := \operatorname{spt}\xi \subset U$$

and $\xi = 1$ in K. Then

$$\int_K ||\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon|^{p/(p-1)}\, dx \leq \int_K |\nabla u_\varepsilon|^p\, dx \leq \int_\Omega |\nabla u_\varepsilon|^p \xi^p\, dx. \tag{3.1}$$

Observe that since f satisfies (1.3), the lower term f_ε verifies the bound (2.10). Therefore, Lemma 2.3 applies and the right-hand side of (3.1) is bounded from above by

$$C\Big[(\operatorname{osc}_{K'} u_\varepsilon)^p \int_\Omega (|\nabla\xi|^p + 1)\, dx + \operatorname{osc}_{K'} u_\varepsilon\Big]. \tag{3.2}$$

Moreover, since u_ε is an increasing sequence and converges pointwise to u in Ω, we have

$$\operatorname{osc}_{K'} u_\varepsilon \leq \sup_{K'} u - \inf_{K'} u_{\varepsilon_0}$$

for all $\varepsilon < \varepsilon_0$. Then in view of (3.1), (3.2) and the above comments, we can find a uniform bound for the integrals

$$\int_K ||\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon|^{p/(p-1)}\, dx, \qquad \int_\Omega |\nabla u_\varepsilon|^p \xi^p\, dx.$$

Hence $|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon$ converges weakly in $L^{p/(p-1)}_{\mathrm{loc}}(\Omega)$, and ∇u_ε converges weakly in $L^p_{\mathrm{loc}}(\Omega)$. Since u_ε converges pointwise to u, we derive that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and u_ε converges weakly in $W^{1,p}_{\mathrm{loc}}(\Omega)$ to u.

More can be said: ∇u_ε converges strongly in $L^p_{\mathrm{loc}}(\Omega)$ to ∇u. Indeed, take

$$\phi(x) := (u(x) - u_\varepsilon(x))\theta(x), \quad x \in \Omega,$$

where θ is a non-negative smooth test function compactly supported in Ω. From

$$\int_\Omega |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \cdot \nabla\phi\, dx \geq \int_\Omega f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\phi\, dx$$

we get

$$\int_\Omega \big[|\nabla u|^{p-2}\nabla u - |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\big] \cdot \nabla(u - u_\varepsilon)\theta\, dx \leq -\int_\Omega f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\phi\, dx + \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla(u - u_\varepsilon)\theta\, dx. \tag{3.3}$$

By the weak convergence of u_ε to u in $W^{1,p}_{\mathrm{loc}}(\Omega)$, the last integral in (3.3) tends to 0 as $\varepsilon \to 0$. The left-hand side is given by

$$\int_\Omega \theta[|\nabla u|^{p-2}\nabla u - |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon] \cdot \nabla(u - u_\varepsilon)\,dx + \int_\Omega (u - u_\varepsilon)[|\nabla u|^{p-2}\nabla u - |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon] \cdot \nabla\theta\,dx. \tag{3.4}$$

The second integral in (3.4) is estimated in absolute value by

$$\|\nabla\theta\|_{L^\infty(\Omega)}\Big(\int_{\operatorname{spt}\theta} |u - u_\varepsilon|^p\,dx\Big)^{1/p}\Big[\Big(\int_{\operatorname{spt}\theta} |\nabla u|^p\,dx\Big)^{(p-1)/p} + \Big(\int_{\operatorname{spt}\theta} |\nabla u_\varepsilon|^p\,dx\Big)^{(p-1)/p}\Big],$$

which tends to 0 as $\varepsilon \to 0$. Moreover, since

$$-\int_{\operatorname{spt}\theta} f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\phi\,dx \le \gamma_\infty \int_\Omega |\nabla u_\varepsilon|^{p-1}(u - u_\varepsilon)\theta\,dx + \|\phi\|_{L^\infty(\operatorname{spt}(\theta))}\int_\Omega (u - u_\varepsilon)\theta\,dx,$$

with $\gamma_\infty := \sup_{x\in\operatorname{spt}(\theta)}|\gamma(u_\varepsilon(x))|$, does not depend on ε, it also holds that

$$\limsup_{\varepsilon\to 0}\Big[-\int_{\operatorname{spt}\theta} f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\phi\,dx\Big] = 0.$$

Hence

$$\lim_{\varepsilon\to 0}\int_\Omega \theta[|\nabla u|^{p-2}\nabla u - |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon] \cdot \nabla(u - u_\varepsilon)\,dx = 0, \tag{3.5}$$

where we have used the fact that the integrand is always non-negative. Therefore, using the inequality

$$2^{p-2}|\nabla u(x) - \nabla u_\varepsilon(x)|^p \le [|\nabla u(x)|^{p-2}\nabla u(x) - |\nabla u_\varepsilon(x)|^{p-2}\nabla u_\varepsilon(x)] \cdot \nabla(u(x) - u_\varepsilon(x))$$

valid for all $p \ge 2$, and (3.5), we conclude the strong convergence of ∇u_ε in $L^p_{\mathrm{loc}}(\Omega)$. Finally, (3.5) together with [5, Lemma 3.73]), implies

$$|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \rightharpoonup |\nabla u|^{p-2}\nabla u \quad \text{in } L^{p/(p-1)}_{\mathrm{loc}}(\Omega),$$

and, in turn, the strong convergence of the gradients ∇u_ε and Lemma 2.2 gives

$$\lim_{\varepsilon\to 0}\int_\Omega f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\psi\,dx = \int_\Omega f(x, u, \nabla u)\psi\,dx.$$

This ends the proof of the claim and we deduce that u is a weak supersolution. □

3.2 The singular case: $1 < p < 2$

Consider now the infimal convolution given in (2.1) choosing $q > p/(p-1)$, i.e.,

$$u_\varepsilon(x) := \inf_{y\in\Omega}\Big(u(y) + \frac{|x - y|^q}{q\varepsilon^{q-1}}\Big). \tag{3.6}$$

Notice that $q > 2$ for $1 < p < 2$.

We need the following auxiliary result, which is an adaptation of [6, Lemma 4.3].

Lemma 3.1. *Suppose that u is a bounded viscosity supersolution to* (1.1). *If there is $\hat{x} \in \Omega_{r(\varepsilon)}$ such that u_ε is differentiable at $\hat{x}$ and $\nabla u_\varepsilon(\hat{x}) = 0$, then $f_\varepsilon(\hat{x}, u_\varepsilon(\hat{x}), \nabla u_\varepsilon(\hat{x})) \le 0$.*

Proof. From [6, Lemma 4.3] we know that $u_\varepsilon(\hat{x}) = u(\hat{x})$, and hence

$$u(y) + \frac{|\hat{x} - y|^q}{q\varepsilon^{q-1}} \ge u(\hat{x}) \quad \text{for every } y \in \Omega.$$

Define

$$\psi(y) := u(\hat{x}) - \frac{|\hat{x} - y|^q}{q\varepsilon^{q-1}}, \quad y \in \Omega,$$

which satisfies $\psi \in \mathcal{C}^2(\Omega)$, $\nabla\psi(\hat{x}) = 0$ and

$$\lim_{r\to 0} \sup_{y\in B_r(\hat{x})\setminus\{\hat{x}\}} (-\Delta_p\psi(y)) = 0 \tag{3.7}$$

in view of $q > p/(p-1)$. Since $\psi(\hat{x}) = u(\hat{x})$, $\psi(y) \le u(y)$ for $y \in \Omega$, $\nabla\psi(x) \ne 0$ for all $x \ne \hat{x}$, and u is a viscosity supersolution to (1.1),

$$\lim_{r\to 0} \sup_{y\in B_r(\hat{x})\setminus\{\hat{x}\}} (-\Delta_p\psi(y)) \ge f(\hat{x}, \psi(\hat{x}), \nabla\psi(\hat{x})).$$

Noticing that $\psi(\hat{x}) = u_\varepsilon(\hat{x})$ and $\nabla\psi(\hat{x}) = \nabla u_\varepsilon(\hat{x}) = 0$, by (3.7) we conclude

$$0 \ge f(\hat{x}, \psi(\hat{x}), \nabla\psi(\hat{x})) = f(\hat{x}, u_\varepsilon(\hat{x}), \nabla u_\varepsilon(\hat{x})) \ge f_\varepsilon(\hat{x}, u_\varepsilon(\hat{x}), \nabla u_\varepsilon(\hat{x})),$$

as desired. □

We can prove now Theorem 1.4 in the case $1 < p < 2$.

Proof of Theorem 1.4. Let u_ε be defined in (3.6). Proceeding as in the degenerate case, by Aleksandrov's theorem and Lemma 2.1 we obtain

$$-\Delta_p u_\varepsilon \ge f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)$$

a.e. in $\Omega_{r(\varepsilon)} \setminus \{\nabla u_\varepsilon = 0\}$. Performing the same approximation argument as in the proof of [6, Theorem 4.1], we reach that

$$\int_\Omega |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \cdot \nabla\psi \, dx \ge \int_{\Omega\setminus\{\nabla u_\varepsilon = 0\}} f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\psi \, dx$$

for every $\psi \in \mathcal{C}_0^\infty(\Omega)$, $\psi \ge 0$. Therefore and since by Lemma 3.1 we know $f_\varepsilon \le 0$ in the set $\{x \in \Omega : \nabla u_\varepsilon(x) = 0\}$, we get

$$\int_\Omega |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \cdot \nabla\psi \, dx \ge \int_\Omega f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\psi \, dx. \tag{3.8}$$

Repeating the proof for the case $p \ge 2$ (and noticing that Lemma 2.3 works for every $1 < p < \infty$), we obtain the uniform boundedness of ∇u_ε in $L^p_{\mathrm{loc}}(\Omega)$ and

$$\lim_{\varepsilon\to 0} \int_K [|\nabla u|^{p-2}\nabla u - |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon] \cdot \nabla(u - u_\varepsilon) \, dx = 0 \tag{3.9}$$

for any compact set $K \subset \Omega$, and from here the convergence

$$|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \rightharpoonup |\nabla u|^{p-2}\nabla u \quad \text{in } L^{p/(p-1)}_{\mathrm{loc}}(\Omega). \tag{3.10}$$

Using Hölder's inequality and the vector inequality (see [4, Chapter I])

$$\frac{|a-b|^2}{(|a|+|b|)^{2-p}} \le C(|a|^{p-2}a - |b|^{p-2}b) \cdot (a-b), \quad 1 < p < 2,$$

with $C = C(n, p)$ and $a, b \in \mathbb{R}^n$, we obtain

$$\begin{aligned}\int_K |\nabla u - \nabla u_\varepsilon|^p \, dx &\le \Big(\int_K \frac{|\nabla u - \nabla u_\varepsilon|^2}{(|\nabla u| + |\nabla u_\varepsilon|)^{2-p}} \, dx\Big)^{p/2} \Big(\int_K (|\nabla u| + |\nabla u_\varepsilon|)^p \, dx\Big)^{(2-p)/2} \\ &\le C\Big(\int_K \frac{|\nabla u - \nabla u_\varepsilon|^2}{(|\nabla u| + |\nabla u_\varepsilon|)^{2-p}} \, dx\Big)^{p/2} \\ &\le C\Big(\int_K [|\nabla u|^{p-2}\nabla u - |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon] \cdot \nabla(u - u_\varepsilon) \, dx\Big)^{p/2}.\end{aligned}$$

Thus, from (3.9) we deduce that ∇u_ε converges to ∇u in $L^p_{\mathrm{loc}}(\Omega)$. By (3.10) and Lemma 2.2 we can pass to the limit in (3.8) to conclude

$$\int_\Omega |\nabla u|^{p-1}\nabla u \cdot \nabla\psi \, dx \ge \int_\Omega f(x, u, \nabla u)\psi \, dx.$$

□

4 Proofs of Theorem 1.5 and Theorem 1.8

Proof of Theorem 1.5 (i). Let $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ be a weak supersolution to (1.1). To reach a contradiction, assume that u is not a viscosity supersolution. By assumption, there exist $x_0 \in \Omega$ and $\varphi \in \mathcal{C}^2(\Omega)$ so that $\nabla\varphi(x) \neq 0$ for all $x \neq x_0$,

$$u(x_0) = \varphi(x_0), \qquad u(x) > \varphi(x) \quad \text{for all } x \neq x_0, \tag{4.1}$$

and

$$\lim_{r\to 0} \sup_{x\in B_r(x_0)\setminus\{x_0\}} (-\Delta_p\varphi(x)) < f(x_0, u(x_0), \nabla\varphi(x_0)).$$

Moreover, by the $W^{1,\infty}_{\mathrm{loc}}$-regularity of u, we may assume that u is continuous in Ω. Thus, the map

$$x \to f(x, u(x), \nabla\varphi(x))$$

is continuous in Ω, and (4.1) yields

$$\lim_{r\to 0} \sup_{x\in B_r(x_0)\setminus\{x_0\}} \big[-\Delta_p\varphi(x) - f(x, u(x), \nabla\varphi(x))\big] < 0.$$

Hence, there exists some $r_0 > 0$ so that

$$-\Delta_p\varphi \le f(x, u(x), \nabla\varphi(x)), \quad x \in B_{r_0}(x_0) \setminus \{x_0\}. \tag{4.2}$$

Let

$$m := \inf_{\partial B_{r_0}(x_0)} (u - \varphi).$$

Then by (4.1) we have $m > 0$. Consider

$$\tilde{\varphi}(x) := \varphi(x) + m, \quad x \in \Omega.$$

By (4.2), $\tilde{\varphi}$ is a weak subsolution to

$$-\Delta_p v = \tilde{f}(x, \nabla v) \tag{4.3}$$

in $B_{r_0}(x_0)$, where $\tilde{f}(x, \eta) := f(x, u(x), \eta)$. Observe that $\tilde{f}$ is continuous in $\Omega \times \mathbb{R}^n$ and locally Lipschitz in η. Moreover, in the weak sense, we have

$$-\Delta_p u \ge f(x, u, \nabla u) = \tilde{f}(x, \nabla u),$$

which shows that u is a weak supersolution to (4.3). In addition, $u \ge \tilde{\varphi}$ on $\partial B_{r_0}(x_0)$. By the comparison Theorem A.2, we conclude that $u \ge \tilde{\varphi}$ in $B_{r_0}(x_0)$. This contradicts (4.1). □

Proof of Theorem 1.5 (ii). For a given weak supersolution $u \in \mathcal{C}^1(\Omega)$, by following the lines above and appealing to the comparison Theorem A.1, we can show that u is a viscosity supersolution in the non-critical set

$$\{x \in \Omega : \nabla u(x) \neq 0\}.$$

By [8, Corollary 4.4], which holds true for sub- and supersolutions, u is a viscosity supersolution in the whole set Ω. □

Proof of Theorem 1.5 (iii). The proof follows similarly just by using the assumption $\nabla u \neq 0$ in Ω together with the comparison Theorem A.1. □

Remark 4.1. Observe that the $W^{1,\infty}_{\mathrm{loc}}$-regularity of u is only needed to apply the comparison principles. In the rest of the proof, the continuity of u would be enough.

Proof of Theorem 1.8. The result follows by reproducing the proof of Theorem 1.5 using the comparison Theorem A.3 instead of Theorems A.2 and A.1. □

A Appendix

A.1 Comparison principles for weak solutions

In this section, we provide the comparison principles for weak solutions of (1.1) that we use in the proof of Theorem 1.5. As we pointed out in Section 1, other comparison results may be employed (see [11]).

The first one is contained in [15, Corollary 3.6.3].

Theorem A.1. *Assume that $f = f(x, s, \eta)$ is continuous in $\Omega \times \mathbb{R} \times \mathbb{R}^n$, non-increasing in s, and locally Lipschitz continuous with respect to η in $\Omega \times \mathbb{R} \times \mathbb{R}^n$. Let $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ be a weak supersolution and let $v \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ be a weak subsolution to* (1.1) *in Ω for $1 < p < \infty$. Assume that $|\nabla u| + |\nabla v| > 0$ in Ω. If $u \geq v$ on $\partial\Omega$, then $u \geq v$ in Ω.*

In the singular case, the assumptions on the gradients may be removed. See [15, Corollary 3.5.2].

Theorem A.2. *Let $1 < p \leq 2$. Assume that $f = f(x, s, \eta)$ is continuous in $\Omega \times \mathbb{R} \times \mathbb{R}^n$, non-increasing in s, and that it is locally Lipschitz continuous in η on compact subsets of its variables. Then if $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ is a weak supersolution and if $v \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ is a weak subsolution to* (1.1) *in Ω so that $u \geq v$ on $\partial\Omega$, then $u \geq v$ in Ω.*

Finally, in the case where f does not depend on η, we have the following result (see [15, Corollary 3.4.2]).

Theorem A.3. *Suppose that $f = f(x, s)$ is continuous in $\Omega \times \mathbb{R}$ and non-increasing in s. Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap \mathcal{C}(\Omega)$ be a supersolution and $v \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap \mathcal{C}(\Omega)$ a subsolution so that $u \geq v$ on $\partial\Omega$. Then $u \geq v$ in Ω.*

Remark A.4. In [15], Theorem A.1 is stated in a more general framework of equations in divergence form as

$$\operatorname{div}(A(x, u, \nabla u)) = f(x, u, \nabla u),$$

where the operator $A = A(x, s, \eta)$ is assumed to be continuous in $\Omega \times \mathbb{R} \times \mathbb{R}^n$, continuously differentiable with respect to s and η for all s and all $\eta \neq 0$, and elliptic in the sense that $\nabla A(x, s, \eta)$ is positive definite in $\Omega \times \mathbb{R} \times (\mathbb{R}^n \setminus \{0\})$. In the particular case of the p-Laplace operators

$$A_p(\eta) = |\eta|^{p-2}\eta, \qquad p \geq 2,$$
$$A_p(\eta) = \begin{cases} |\eta|^{p-2}\eta & \text{if } \eta \neq 0, \\ 0, & \text{for } \eta = 0, \end{cases} \qquad 1 < p \leq 2,$$

all of the assumptions above are satisfied. The positive definiteness of ∇A_p is a consequence of

$$\sum_{i,j=1}^{n} \frac{\partial A^i}{\partial \eta_j}(\eta)\xi_i\xi_j \geq c|\eta|^{p-2}|\xi|^2$$

for a positive constant c. Finally, observe that A is uniformly elliptic for $0 < |\eta| \leq C$ if $p \leq 2$. This allows the improved comparison result in Theorem A.2.

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