Physica A 462 (2016) 930–939

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Unified entropic measures of quantum correlations induced by local measurements

G.M. Bosyk [a,*], G. Bellomo [a], S. Zozor [b], M. Portesi [a], P.W. Lamberti [c]

[a] Instituto de Física La Plata, UNLP, CONICET, Facultad de Ciencias Exactas, Casilla de Correo 67, 1900 La Plata, Argentina

[b] Laboratoire Grenoblois d'Image, Parole, Signal et Automatique (GIPSA-Lab, CNRS), 11 rue des Mathématiques, 38402 Saint Martin d'Hères, France

[c] Facultad de Matemática, Astronomía y Física, Universidad Nacional de Córdoba, and CONICET, Avenida Medina Allende S/N, Ciudad Universitaria, X5000HUA, Córdoba, Argentina

HIGHLIGHTS

- We propose a family of quantum correlation measures for bipartite quantum systems.
- We extend previous approaches by using quantum unified entropies.
- The measures are invariant under addition of an uncorrelated ancilla.
- We give some relationships between total and semiquantum correlations.
- We obtain analytical results for pseudopure, Werner and isotropic states.

ARTICLE INFO

Article history:
Received 27 April 2016
Available online 29 June 2016

Keywords:
Quantum correlations
Quantum unified entropies
Local projective measurements

ABSTRACT

We introduce quantum correlation measures based on the minimal change in unified entropies induced by local rank-one projective measurements, divided by a factor that depends on the generalized purity of the system in the case of nonadditive entropies. In this way, we overcome the issue of the artificial increasing of the value of quantum correlation measures based on nonadditive entropies when an uncorrelated ancilla is appended to the system, without changing the computability of our entropic correlation measures with respect to the previous ones. Moreover, we recover as limiting cases the quantum correlation measures based on von Neumann and Rényi entropies (i.e., additive entropies), for which the adjustment factor becomes trivial. In addition, we distinguish between total and semiquantum correlations and obtain some inequalities between them. Finally, we obtain analytical expressions of the entropic correlation measures for typical quantum bipartite systems.

1. Introduction

Quantum correlations lie at the heart of the difference between classical and quantum worlds. Two paradigms, at least, address this issue beyond the usual entangled–separable distinction [1]. For instance steering correlations, the origins of which can be found in the seminal works by Einstein, Podolsky and Rosen [2] and by Schrödinger [3], have recently been formulated in an operational way [4]. These correlations intermediate between entanglement and nonlocality (i.e., a

* Corresponding author.
E-mail address: gbosyk@fisica.unlp.edu.ar (G.M. Bosyk).

http://dx.doi.org/10.1016/j.physa.2016.06.131
0378-4371/

violation of Bell inequalities [5]). Moreover it is possible to identify quantum correlations even in separable states. This has been firstly observed by Ollivier and Zurek [6] and by Henderson and Vedral [7], who derived the quantum discord as a signature of quantum correlations in bipartite systems. The original definition of discord relies on the difference between two extensions of the classical mutual information to the quantum case. A generalization of discord using entropic forms other than von Neumann entropy [8], naively replacing it by general entropies like Rényi [9] or Tsallis [10] ones, as proposed in Ref. [11], fails as it has been shown in Refs. [12,13].

Here, we aim to obtain quantum correlation measures by using general entropic forms, namely (q, s)-entropies (or unified entropies) [14,15]. To avoid the difficulty discussed in Refs. [12,13], we follow an alternative approach inspired by the work of Luo [16]. We propose as quantum correlation measures the minimal change in unified entropies induced by a local rank-one measurement, divided by a factor that depends on the generalized purity only in the case of nonadditive entropies (this adjusting factor becomes trivial for additive entropies). Several quantum correlation measures discussed in the literature, like [17–29] among others, are particular cases of (or close to) our proposal (see Refs. [30] for recent reviews of quantum correlations). Indeed, the case of trace form entropies [31], which are nonadditive (except for the von Neumann case), has been dealt with in Refs. [22–24] and deserves a particular mention. These entropic quantum correlation measures artificially increase when an uncorrelated ancilla is appended to the system (the geometric discord [20] has the same issue, as it has been pointed out in Ref. [32]). The nonadditivity of trace form entropies is the cause of this problem. We solve this drawback in the case of (q, s)-entropies by introducing a generalized purity factor, similarly to what has been done with the geometric discord, that is, dividing it by the purity [25]. In this way, we obtain a family of (q, s)-entropic measures of quantum correlations that are invariant under the addition of an uncorrelated ancilla, both in the cases of additive and nonadditive entropies. We notice that the computability of our entropic quantum correlation measures remains equal to the previous ones [22–24], since the adjustment factor is simply the trace of a power of the density operator.

The outline of the work is as follows. Our proposal and main results are given in Section 2. In 2.1, we present a review of the notion and some properties of the unified (q, s)-entropies and majorization, and we introduce a family of entropic measures of disturbance due to a projective measurement. In 2.2, we present the general entropic quantum correlation measures by quantifying disturbances due to local projective measurements, distinguishing between total and semiquantum correlations. Besides, we provide basic properties that justify our proposal. In 2.3, we find a lower bound for the entropic quantum correlations in terms of generalized entanglement entropies. In 2.4, we give some interesting inequalities between total and semiquantum measures. Typical examples where we apply our correlation measures are given in Section 3, and finally some conclusions are drawn in Section 4.

2. Entropic measures of quantum correlations

2.1. Unified entropies, majorization and (q, s)-disturbances

Let a quantum system be described by a density operator ρ, that is, a trace-one positive semidefinite operator acting on an N-dimensional Hilbert space, $\mathcal{H}_N$. The quantum unified (q, s)-entropies for the state ρ are defined as [14,15]

$$S_{(q,s)}(\rho) = \frac{(\operatorname{Tr}\rho^q)^s - 1}{(1-q)s}, \tag{1}$$

for entropic indices $q > 0$, $q \neq 1$, and $s \neq 0$. The limiting case $q \to 1$, for any s, corresponds to von Neumann entropy [8],

$$\lim_{q\to 1} S_{(q,s)}(\rho) \equiv S(\rho) = -\operatorname{Tr}\rho \ln \rho, \tag{2}$$

while for vanishingly small s, the quantum versions of Rényi entropies [9] are recovered,

$$\lim_{s\to 0} S_{(q,s)}(\rho) \equiv S^{\mathrm{R}}_q(\rho) = \frac{\ln \operatorname{Tr}\rho^q}{1-q}. \tag{3}$$

Besides, setting $s = 1$ gives rise to the quantum versions of Tsallis entropies [10],

$$S_{(q,1)}(\rho) \equiv S^{\mathrm{T}}_q(\rho) = \frac{\operatorname{Tr}\rho^q - 1}{1-q}. \tag{4}$$

Another interesting subfamily is that obtained with $q = 2$, as the entropies $S_{(2,s)}(\rho)$ are directly related to the purity of the state, $\operatorname{Tr}\rho^2$; in particular $S^R_2(\rho) = -\ln \operatorname{Tr}\rho^2$ and $S^T_2(\rho) = 1 - \operatorname{Tr}\rho^2$.

A feature of (q, s)-entropies is their nonadditive character [14], which is reflected in the sum rule for product states $\rho^A \otimes \rho^B$ acting on a Hilbert space $\mathcal{H}_{N^A} \otimes \mathcal{H}_{N^B}$,

$$S_{(q,s)}(\rho^A \otimes \rho^B) = S_{(q,s)}(\rho^A) + S_{(q,s)}(\rho^B) + (1-q)s\, S_{(q,s)}(\rho^A) S_{(q,s)}(\rho^B). \tag{5}$$

Notice that in the cases $q = 1$ or $s = 0$, one recovers the additivity of von Neumann and Rényi entropies.

A closely related concept to entropy is that of majorization (see e.g. Ref. [33]). Let us consider two density operators ρ and σ, and let the probability vectors p and q be formed with the eigenvalues of ρ and σ, respectively, sorted in decreasing order. Then ρ is majorized by σ, denoted as $\rho \prec \sigma$, when

$$\sum_{i=1}^{n} p_i \leq \sum_{i=1}^{n} q_i \quad \forall\, n = 1, \ldots, N-1, \quad \text{and} \quad \sum_{i=1}^{N} p_i = \sum_{i=1}^{N} q_i = 1, \tag{6}$$

where $N = \max\{\operatorname{rank}\rho, \operatorname{rank}\sigma\}$ and rank denotes the rank of a density operator. Notice that if $\operatorname{rank}\rho \leq \operatorname{rank}\sigma$ we complete the vector p with 0 entries to have the same length as q, and vice versa. This has no impact in the value of the unified entropies due to the expansibility property.

It can be shown that (q, s)-entropies preserve the majorization relation (see e.g. Refs. [15,34]), that is,

$$\text{if } \rho \prec \sigma \quad \text{then } S_{(q,s)}(\rho) \geq S_{(q,s)}(\sigma), \tag{7}$$

with equality if and only if ρ and σ have the same eigenvalues. We observe that the reciprocal does not hold in general, which means that majorization is stronger (as an order relation) than entropic behaviour for a single choice of the entropic indices.

Now, using the Schur-concavity (7) it is straightforward to show that (q, s)-entropies are lower and upper bounded:

$$0 \leq S_{(q,s)}(\rho) \leq \frac{N^{(1-q)s} - 1}{(1-q)s}, \tag{8}$$

where the first inequality is obtained for pure states, whereas the second one for the maximally mixed state $\rho^* = \frac{I}{N}$.

It can be shown that the eigenvalues of a density operator ρ are invariant under arbitrary unitary transformations U, in other words ρ and $U\rho U^\dagger$ have the same eigenvalues. Hence the (q, s)-entropies are invariant under unitary transformations,

$$S_{(q,s)}(\rho) = S_{(q,s)}(U\rho U^\dagger). \tag{9}$$

Moreover, we show below that the change in entropy due to local measurements plays a key role in order to quantify quantum correlations. Let us recall the action of any bistochastic map over an arbitrary state. A bistochastic (or completely positive, trace-preserving unital) map $\mathcal{E}$ can be written in the Kraus form as $\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$ with both sets of positive operators $\left\{E_k^\dagger E_k\right\}$ and $\left\{E_k E_k^\dagger\right\}$ summing up to the identity (see e.g. Ref. [35]). Notice that this map leaves the maximally mixed state, ρ^*, invariant, i.e., $\mathcal{E}(\rho^*) = \rho^*$. It can be shown that $\mathcal{E}(\rho) \prec \rho$ if and only if $\mathcal{E}$ is a bistochastic map [36]. In other words, for bistochastic maps the final state $\mathcal{E}(\rho)$ is more disordered (in terms of majorization) than the initial state ρ. As a consequence of this ordering and the Schur-concavity of the (q, s)-entropies, we have

$$S_{(q,s)}(\mathcal{E}(\rho)) \geq S_{(q,s)}(\rho), \tag{10}$$

where the equality is attained if and only if $\mathcal{E}(\rho) = U\rho U^\dagger$.

Hereafter, we are only interested in rank-one projective measurements without postselection, that is, a set of orthogonal rank-one projectors $\Pi = \{P_i = |i\rangle\langle i|\}$, where $P_i P_{i'} = \delta_{ii'} P_i$ and $\sum_{i=1}^N P_i = I$, being $\{|i\rangle\}$ an orthonormal basis of $\mathcal{H}_N$. The state after a rank-one projective measurement Π is equal to $\Pi(\rho) = \sum_{i=1}^N P_i \rho P_i = \sum_{i=1}^N p_i |i\rangle\langle i|$ with $p_i = \langle i|\rho|i\rangle$. As projective measurements are particular cases of bistochastic maps, we have also an inequality similar to (10) for Π. Thus, we propose to use the difference of quantum (q, s)-entropies between the final and initial states (adequately rescaled by a factor depending on the generalized purity) as a signature of the disturbance of the state of a system due to the measurement, that is

$$D^{\Pi}_{(q,s)}(\rho) = \frac{S_{(q,s)}(\Pi(\rho)) - S_{(q,s)}(\rho)}{(\operatorname{Tr}\rho^q)^s}. \tag{11}$$

For any choice of the entropic indices this quantity is nonnegative, and vanishes if and only if the measurement does not disturb the state, i.e., $\Pi(\rho) = \rho$, which happens when measuring in the basis that diagonalizes ρ. Notice that the rescaling factor plays no role for von Neumann and Rényi entropies, which correspond to the cases of additive entropies. Below we clarify the importance of rescaling with $(\operatorname{Tr}\rho^q)^s$ when dealing with quantum correlation measures based on nonadditive entropies.

Finally, notice that two interesting cases arise from definition (11). The first one consists in considering the von Neumann entropy, in this case the disturbance can be recast as the quantum relative entropy (or quantum Kullback–Leibler divergence) between ρ and $\Pi(\rho)$, that is

$$D^{\Pi}_{(1,s)}(\rho) \equiv D^{\Pi}(\rho) = S\left(\rho \parallel \Pi(\rho)\right), \tag{12}$$

where $S(\rho \parallel \sigma) = \operatorname{Tr}(\rho(\ln\rho - \ln\sigma))$ is the quantum relative entropy. The second one comes from evaluating (11) for Tsallis entropy with entropic index equal to 2, for which the disturbance expresses in terms of the Hilbert–Schmidt distance between ρ and $\Pi(\rho)$ divided by the purity of ρ,

$$D^{\Pi}_{(2,1)}(\rho) \equiv D^{\Pi}_2(\rho) = \frac{\|\rho - \Pi(\rho)\|^2}{\operatorname{Tr}\rho^2}, \tag{13}$$

where $\|A\| = \sqrt{\operatorname{Tr} A^\dagger A}$ is the Hilbert–Schmidt norm of operator A.

2.2. Quantum correlations from disturbance due to a local projective measurement

Let us consider a bipartite quantum system AB with density operator ρ^{AB} acting on a product finite dimensional Hilbert space, $\mathcal{H}_{N^{AB}} = \mathcal{H}_{N^A} \otimes \mathcal{H}_{N^B}$, where $N^{AB} = N^A N^B$. Following Ref. [16], we consider the local rank-one projective measurements (without postselection), $\Pi^A = \{P_i^A \otimes I^B\}$, $\Pi^B = \{I^A \otimes P_j^B\}$ and $\Pi^{AB} = \{P_i^A \otimes P_j^B\}$, where $\{P_i^A\}$ and $\{P_j^B\}$ are two sets of orthogonal rank-one projectors that sum to the identity, I^A and I^B, respectively. Then, the resulting states after these measurements are

$$\Pi^A(\rho^{AB}) = \sum_i P_i^A \otimes I^B \rho^{AB} P_i^A \otimes I^B = \sum_i p_i^A P_i^A \otimes \rho^{B|i}, \tag{14}$$

$$\Pi^B(\rho^{AB}) = \sum_j I^A \otimes P_j^B \rho^{AB} I^A \otimes P_j^B = \sum_j p_j^B \rho^{A|j} \otimes P_j^B, \tag{15}$$

$$\begin{aligned}\Pi^{AB}(\rho^{AB}) &= \Pi^A \circ \Pi^B(\rho^{AB}) = \Pi^B \circ \Pi^A(\rho^{AB}) \\ &= \sum_{ij} P_i^A \otimes P_j^B \rho^{AB} P_i^A \otimes P_j^B = \sum_{ij} p_{ij}^{AB} P_i^A \otimes P_j^B,\end{aligned} \tag{16}$$

where $\rho^{B|i} = \frac{\mathrm{Tr}_A\left(P_i^A \otimes I^B \rho^{AB}\right)}{p_i^A}$ with $p_i^A = \mathrm{Tr}\left(P_i^A \otimes I^B \rho^{AB}\right)$, $\rho^{A|j} = \frac{\mathrm{Tr}_B\left(I^A \otimes P_j^B \rho^{AB}\right)}{p_j^B}$ with $p_j^B = \mathrm{Tr}\left(I^A \otimes P_j^B \rho^{AB}\right)$ and $p_{ij}^{AB} = \mathrm{Tr}\left(P_i^A \otimes P_j^B \rho^{AB}\right)$. According to Luo [16], these states are called classical–quantum (CQ), quantum–classical (QC) and classical–classical (CC) correlated states with respect to the local measurements Π^A, Π^B and Π^{AB}, respectively. A state is said CQ correlated if there is a local projective measurement over A that does not disturb it, i.e. $\Pi^A(\rho^{AB}) = \rho^{AB}$; analogously, a QC correlated state satisfies $\Pi^B(\rho^{AB}) = \rho^{AB}$, while CC correlated states are those that obey $\Pi^{AB}(\rho^{AB}) = \rho^{AB}$. All these states are separable (i.e., nonentangled), as they are convex combinations of product states [1]; however we recall that not all separable states are of the forms (14)–(16). Moreover, the sets formed by all CQ, QC and CC correlated states, denoted as Ω^A, Ω^B and Ω^{AB}, respectively, are not convex in contrast to the set of separable states. Notice that Ω^A and Ω^B are the sets of zero quantum discord states with respect to $\mathcal{H}_{N^A}$ and $\mathcal{H}_{N^B}$ respectively [20,37], and that $\Omega^{AB} = \Omega^A \cap \Omega^B$ [26]. In the sequel, for the sake of brevity, we will use L to denote either A or B (but not AB), and K when we consider A, B, or AB.

Now, we can use (11) to quantify the disturbance due to the local projective measurement Π^K,

$$D_{(q,s)}^{\Pi^K}(\rho^{AB}) = \frac{S_{(q,s)}\left(\Pi^K(\rho^{AB})\right) - S_{(q,s)}(\rho^{AB})}{\left(\mathrm{Tr}(\rho^{AB})^q\right)^s}. \tag{17}$$

We refer to $D_{(q,s)}^{\Pi^L}$ as unilocal disturbances, whereas $D_{(q,s)}^{\Pi^{AB}}$ are called bilocal disturbances.

In order to obtain a measurement-independent signature of quantum correlations, one takes the minimum of the disturbances (17) over the set of local measurements, that is

$$D_{(q,s)}^{K}(\rho^{AB}) = \min_{\Pi^K} D_{(q,s)}^{\Pi^K}(\rho^{AB}). \tag{18}$$

The following properties justify our proposal (18) as measures of quantum correlations:

(i) nonnegativity: $D_{(q,s)}^{K}(\rho^{AB}) \geq 0$ with equality if and only if $\rho^{AB} \in \Omega^K$. Accordingly, $D_{(q,s)}^{L}$ are semiquantum correlation measures (with respect to $\mathcal{H}_{N^L}$), whereas $D_{(q,s)}^{AB}$ are total quantum correlation measures;

(ii) invariance under local unitary operators: $D_{(q,s)}^{K}(U \otimes V \rho^{AB} U^\dagger \otimes V^\dagger) = D_{(q,s)}^{K}(\rho^{AB})$, where U and V are unitary operations over A and B respectively; and

(iii) invariance when an uncorrelated ancilla is appended to the system: $D_{(q,s)}^{K}(\rho^{AB} \otimes \rho^C) = D_{(q,s)}^{K}(\rho^{AB})$ for bipartitions $A|BC$ or $B|AC$ (for the bipartition $AB|C$ the quantum correlation measures naturally vanish).

The first property is a direct consequence of majorization relation between the states after and before local projective measurements. The second one can be proved from the definition of our measure, Eq. (18), noting that $\Pi^K(U \otimes V \rho^{AB} U^\dagger \otimes V^\dagger) = U \otimes V \tilde{\Pi}^K(\rho^{AB}) U^\dagger \otimes V^\dagger$, with $\tilde{\Pi}^K = U^\dagger \otimes V^\dagger \Pi^K U \otimes V$, and recalling the invariance of (q,s)-entropies under unitary transformations. The third property is more subtle and it is related to the sum rule (5) of the (q,s)-entropies. Indeed, the generalized purity factor $\left(\mathrm{Tr}(\rho^{AB})^q\right)^s$ plays a crucial role to fulfill this property in the case of nonadditive entropies, without affecting the complexity of computability of the measures. In general, this property has not been taken into account in the literature of nonadditive entropic measures of quantum correlations. For instance, entropic quantum correlation measures based on the difference of trace form entropies,[1] i.e., $S_\phi(\rho) = \mathrm{Tr}\,\phi(\rho)$ with ϕ concave and $\phi(0) = 0$ [31], have been dealt with in Refs. [22,24]. However, these measures are not invariant when an uncorrelated ancilla is appended to the system, except for the von Neumann case. This is direct consequence of nonadditivity of trace form entropies. For a more

[1] Notice that (q,s)-entropies reduce to a trace form only if $s = 1$ (Tsallis entropies).

general discussion about necessary and reasonable conditions of quantum correlation measures, see Ref. [38]. Moreover, our semiquantum correlation measures can be also interpreted as a quantum deviation from the Bayes rule in a way similar to that discussed in Ref. [24].

We remark that our quantum correlation measures include some important cases already discussed in the literature. The first one consists in evaluating (18) for the von Neumann entropy. In this case we reobtain the so-called information deficit [18], which can be rewritten in terms of the minimal relative entropy over the sets Ω^K [19],

$$D^K(\rho^{AB}) = \min_{\Pi^K} S\left(\rho^{AB} \| \Pi^K(\rho^{AB})\right) = \min_{\chi^{AB}\in\Omega^K} S\left(\rho^{AB} \| \chi^{AB}\right). \tag{19}$$

The second one arises when evaluating (18) for the Tsallis entropy with entropic index equal to 2. This case is close to the geometric discord [20],

$$D_G^K(\rho^{AB}) = \min_{\chi^{AB}\in\Omega^K} \|\rho^{AB} - \chi^{AB}\|^2. \tag{20}$$

Indeed, using the expression of D_G^K in terms of local projective measurements given in Ref. [21], we obtain

$$D_2^K(\rho^{AB}) = \frac{\min\limits_{\Pi^K} \|\rho^{AB} - \Pi^K(\rho^{AB})\|^2}{\mathrm{Tr}(\rho^{AB})^2} = \frac{D_G^K(\rho^{AB})}{\mathrm{Tr}(\rho^{AB})^2}. \tag{21}$$

Notice that D_G^K is not invariant when an uncorrelated ancilla is appended to the system [32]. The purity rescaled factor solves this issue [25], although there is not the unique way to do it (see e.g. Refs. [25,39]). Finally, notice that in the case of Rényi entropies, which have recently been introduced in Ref. [27], our measure fulfills the desired invariance property when appending an uncorrelated ancilla to the system.

2.3. Lower bound of (18) and its relation with entanglement

First, let us note that since QC, CQ and CC correlated states (14)–(16) are separable, they fulfill some general entropic separability inequalities (see e.g. Ref. [34]),

$$S_{(q,s)}(\Pi^K(\rho^{AB})) \geq \max\left\{S_{(q,s)}(\mathrm{Tr}_A\, \Pi^K(\rho^{AB})), S_{(q,s)}(\mathrm{Tr}_B\, \Pi^K(\rho^{AB}))\right\}. \tag{22}$$

On the other hand, the corresponding final reduced states are

$$\mathrm{Tr}_A\, \Pi^A(\rho^{AB}) = \rho^B \quad \text{and} \quad \mathrm{Tr}_B\, \Pi^A(\rho^{AB}) = \mathrm{Tr}_B\, \Pi^{AB}(\rho^{AB}) = \sum_i p_i^A P_i^A = \rho^A_{\mathrm{diag}}, \tag{23}$$

$$\mathrm{Tr}_B\, \Pi^B(\rho^{AB}) = \rho^A \quad \text{and} \quad \mathrm{Tr}_A\, \Pi^B(\rho^{AB}) = \mathrm{Tr}_A\, \Pi^{AB}(\rho^{AB}) = \sum_j p_j^B P_j^B = \rho^B_{\mathrm{diag}} \tag{24}$$

where ρ^L_{diag} denotes the diagonal of ρ^L in the basis underlying by $\{P_i^L\}$. Since $\rho^L_{\mathrm{diag}} \prec \rho^L$ (see e.g. Ref. [35]) and due to the Schur-concavity of the (q, s)-entropies, inequality (22) reduces to

$$S_{(q,s)}(\Pi^K(\rho^{AB})) \geq \max\left\{S_{(q,s)}(\rho^A), S_{(q,s)}(\rho^B)\right\}. \tag{25}$$

Thus, plugging (25) into (17) to lowerbound $D^{\Pi^K}_{(q,s)}(\rho^{AB})$ and taking the minimum, we obtain that the quantum correlation measures are lower bounded, as follows

$$D^K_{(q,s)}(\rho^{AB}) \geq \max\left\{\frac{S_{(q,s)}(\rho^A) - S_{(q,s)}(\rho^{AB})}{\left(\mathrm{Tr}(\rho^{AB})^q\right)^s}, \frac{S_{(q,s)}(\rho^B) - S_{(q,s)}(\rho^{AB})}{\left(\mathrm{Tr}(\rho^{AB})^q\right)^s}\right\}. \tag{26}$$

Notice that this lower bound could be nontrivial only for entangled states; indeed, the right hand side of (26) is negative for separable states. A similar result has already been obtained in the case of trace form entropies [22].

Now, let us consider a pure state $\rho^{AB} = |\Psi^{AB}\rangle\langle\Psi^{AB}|$. Let us suppose that

$$|\psi^{AB}\rangle = \sum_{k=1}^{n} \sqrt{\lambda_k} |k^A\rangle \otimes |k^B\rangle \tag{27}$$

is the Schmidt decomposition of $|\psi^{AB}\rangle$ ($n \leq \min\{N^A, N^B\}$ and $\{|k^L\rangle\}$ are an orthonormal set). Thus, it can be shown that the reduced states $\rho^A = \mathrm{Tr}_B\, |\Psi^{AB}\rangle\langle\Psi^{AB}|$ and $\rho^B = \mathrm{Tr}_A\, |\Psi^{AB}\rangle\langle\Psi^{AB}|$ have the same unified entropy and, as a consequence, the lower bound (26) reduces to $S_{(q,s)}(\rho^A) = S_{(q,s)}(\rho^B)$ for pure states ρ^{AB}. Moreover, this bound is saturated when the local measurements are taken on the Schmidt basis. After these measurements, i.e., choosing the local projectors as $P_k^L = |k^L\rangle\langle k^L|$ (completed to obtain N^L projector), the state is given by $\Pi^K(\rho^{AB}) = \sum_k \lambda_k P_k^A \otimes P_k^B$, with unified entropies $S_{(q,s)}(\Pi^K(\rho^{AB})) = S_{(q,s)}(\rho^A) = S_{(q,s)}(\rho^B)$. Therefore, we obtain that for pure states the entropic quantum correlation measures become a generalization of the entanglement entropy,

$$D^K_{(q,s)}(\rho^{AB}) = S_{(q,s)}(\rho^A) = S_{(q,s)}(\rho^B), \tag{28}$$

which for the von Neumann entropy reduces to the standard one [40].

2.4. Inequalities between total and semiquantum correlations

It is possible to find some interesting inequalities between total and semiquantum correlations when bilocal disturbances, $D^{\Pi^{AB}}_{(q,s)}(\rho^{AB})$, are rewritten in terms of unilocal disturbances,

$$D^{\Pi^{AB}}_{(q,s)}(\rho^{AB}) = D^{\Pi^{A}}_{(q,s)}(\rho^{AB}) + \pi^{\Pi^A}_{(q,s)}\, D^{\Pi^{B}}_{(q,s)}(\Pi^A(\rho^{AB})), \tag{29}$$

$$D^{\Pi^{AB}}_{(q,s)}(\rho^{AB}) = D^{\Pi^{B}}_{(q,s)}(\rho^{AB}) + \pi^{\Pi^B}_{(q,s)}\, D^{\Pi^{A}}_{(q,s)}(\Pi^B(\rho^{AB})), \tag{30}$$

where $\pi^{\Pi}_{(q,s)} = \left(\frac{\mathrm{Tr}(\Pi(\rho^{AB}))^q}{\mathrm{Tr}(\rho^{AB})^q}\right)^s$ (for sake of brevity, we omit the dependence of this factor on the state). This quantity, $\pi^{\Pi}_{(q,s)}$, is nonnegative but it can take values below or above 1, depending on the value of the entropic index q. As $\Pi(\rho) \prec \rho$, we have that $\Pi(\rho)^q \prec \rho^q$ if $q \geq 1$, whereas, $\rho \prec \Pi(\rho)$ holds if $0 \leq q < 1$. Thus, $\pi^{\Pi}_{(q,s)} \in (0, 1]$ if $q \geq 1$, else $\pi^{\Pi}_{(q,s)} \geq 1$. In particular, for Rényi entropies the factor is always equal to 1.

Now, let us consider two possible measurement scenarios:

- $\Pi^{AB}_0 = \Pi^A_0 \circ \Pi^B_0$ is a bilocal measurement that minimizes the total quantum correlation measure, i.e., $D^{\Pi^{AB}_0}_{(q,s)}(\rho^{AB}) = D^{AB}_{(q,s)}(\rho^{AB})$,
- $\Pi^{AB}_1 = \Pi^A_1 \circ \Pi^B_1$, where Π^L_1 optimizes the unilocal disturbances, i.e., $D^{\Pi^L_1}_{(q,s)}(\rho^{AB}) = D^{L}_{(q,s)}(\rho^{AB})$.

Applying Eqs. (29)–(30) to both scenarios, we obtain

$$D^{AB}_{(q,s)}(\rho^{AB}) = D^{\Pi^A_0}_{(q,s)}(\rho^{AB}) + \pi^{\Pi^A_0}_{(q,s)}\, D^{\Pi^B_0}_{(q,s)}(\Pi^A_0(\rho^{AB})) = D^{\Pi^B_0}_{(q,s)}(\rho^{AB}) + \pi^{\Pi^B_0}_{(q,s)}\, D^{\Pi^A_0}_{(q,s)}(\Pi^B_0(\rho^{AB})), \tag{31}$$

and

$$D^{\Pi^{AB}_1}_{(q,s)}(\rho^{AB}) = D^{A}_{(q,s)}(\rho^{AB}) + \pi^{\Pi^A_1}_{(q,s)}\, D^{\Pi^B_1}_{(q,s)}(\Pi^A_1(\rho^{AB})) = D^{B}_{(q,s)}(\rho^{AB}) + \pi^{\Pi^B_1}_{(q,s)}\, D^{\Pi^A_1}_{(q,s)}(\Pi^B_1(\rho^{AB})). \tag{32}$$

Using that $D^{AB}_{(q,s)}(\rho^{AB}) \leq D^{\Pi^{AB}_1}_{(q,s)}(\rho^{AB})$ (and the analogous relations for the unilocal disturbances) on Eqs. (31)–(32) respectively, it can be shown that $D^{AB}_{(q,s)}(\rho^{AB})$ is lower and upper bounded as follows,

$$D^{AB}_{(q,s)}(\rho^{AB}) \geq \max\{D^{A}_{(q,s)}(\rho^{AB}) + \pi^{\Pi^A_0}_{(q,s)}\, D^{\Pi^B_0}_{(q,s)}(\Pi^A_0(\rho^{AB})), D^{B}_{(q,s)}(\rho^{AB}) + \pi^{\Pi^B_0}_{(q,s)}\, D^{\Pi^A_0}_{(q,s)}(\Pi^B_0(\rho^{AB}))\}, \tag{33}$$

$$D^{AB}_{(q,s)}(\rho^{AB}) \leq \min\{D^{A}_{(q,s)}(\rho^{AB}) + \pi^{\Pi^A_1}_{(q,s)}\, D^{\Pi^B_1}_{(q,s)}(\Pi^A_1(\rho^{AB})), D^{B}_{(q,s)}(\rho^{AB}) + \pi^{\Pi^B_1}_{(q,s)}\, D^{\Pi^A_1}_{(q,s)}(\Pi^B_1(\rho^{AB}))\}. \tag{34}$$

In particular, given that the nonoptimal unilocal disturbances in (33) are nonnegative, we naturally obtain that total quantum correlation is greater than or equal to the semiquantum ones,

$$D^{AB}_{(q,s)}(\rho^{AB}) \geq \max\{D^{A}_{(q,s)}(\rho^{AB}), D^{B}_{(q,s)}(\rho^{AB})\}. \tag{35}$$

This result can be also obtained more directly from the fact that $S_{(q,s)}(\Pi^{AB}_0(\rho^{AB})) \geq S_{(q,s)}(\Pi^L_1(\rho^{AB}))$. Notice that (35) is in accordance with the inclusion relations among the sets of CQ, QC and CC correlated states, i.e., $\Omega^{AB} = \Omega^A \cap \Omega^B \subset \Omega^L$.

Moreover, noting that $2\, D^{\Pi^{AB}_1}_{(q,s)}(\rho^{AB}) \geq D^{AB}_{(q,s)}(\rho^{AB}) + D^{\Pi^{AB}_1}_{(q,s)}(\rho^{AB}) \geq 2\, D^{AB}_{(q,s)}(\rho^{AB})$ we can deduce from Eqs. (31)–(32) the following inequality for the sum of semiquantum correlations:

$$D^{AB}_{(q,s)}(\rho^{AB}) + \Delta_0 \geq D^{A}_{(q,s)}(\rho^{AB}) + D^{B}_{(q,s)}(\rho^{AB}) \geq D^{AB}_{(q,s)}(\rho^{AB}) + \Delta_1, \tag{36}$$

where we defined the quantities $\Delta_i := D^{\Pi^{AB}_i}_{(q,s)}(\rho^{AB}) - \pi^{\Pi^B_i}_{(q,s)}\, D^{\Pi^A_i}_{(q,s)}(\Pi^B_i(\rho^{AB})) - \pi^{\Pi^A_i}_{(q,s)}\, D^{\Pi^B_i}_{(q,s)}(\Pi^A_i(\rho^{AB}))$, with $i = 0, 1$.

Notice that for CQ and QC correlated states, one has $\Delta_1 = 0$, Π^L_1 being defined by the set $\{P^L_i\}$ so that it does not disturb the joint state. Finally, notice that for CC correlated states, all quantities in (36) vanish. Therefore, from these observations together with (35), we obtain

- if $D^{A}_{(q,s)}(\rho^{AB}) = 0$, then $D^{AB}_{(q,s)}(\rho^{AB}) = D^{B}_{(q,s)}(\rho^{AB})$,
- if $D^{B}_{(q,s)}(\rho^{AB}) = 0$, then $D^{AB}_{(q,s)}(\rho^{AB}) = D^{A}_{(q,s)}(\rho^{AB})$,
- if $D^{AB}_{(q,s)}(\rho^{AB}) = 0$, then $D^{A}_{(q,s)}(\rho^{AB}) = D^{B}_{(q,s)}(\rho^{AB}) = 0$.

Furthermore, a triangle-like inequality between total and semiquantum correlations,

$$D^{A}_{(q,s)}(\rho^{AB}) + D^{B}_{(q,s)}(\rho^{AB}) \geq D^{AB}_{(q,s)}(\rho^{AB}), \tag{37}$$

is trivially satisfied for CQ, QC and CC correlated states. The validity of the triangle-like inequality (37) in the general case relies on the sign of Δ_1. If $\Delta_1 \geq 0\, \forall \rho^{AB}$, the inequality is generally true. On the contrary, if $\Delta_1 < 0$ for some ρ^{AB} then it could be the case that the inequality does not hold for those states.

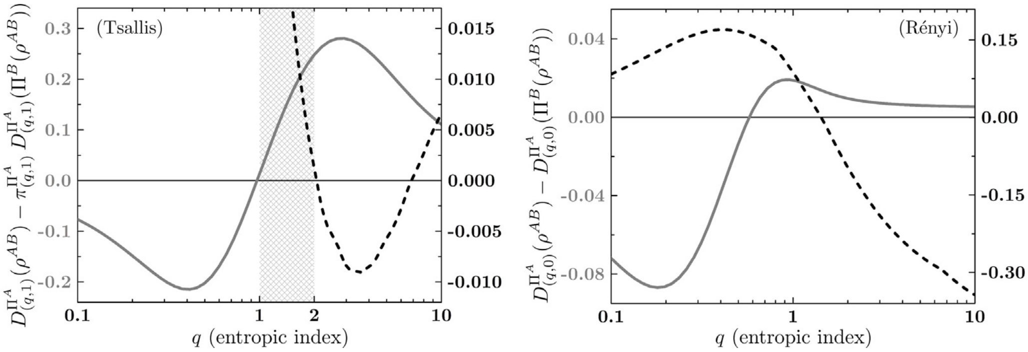

Fig. 1. Minimal differences between $D^{\Pi^A}_{(q,s)}(\rho^{AB})$ and $\pi^{\Pi^A}_{(q,s)} D^{\Pi^A}_{(q,s)}(\Pi^B(\rho^{AB}))$ computed for 10^3 random local projective measurements $\Pi^{A(B)}$, using Tsallis entropies (left inset) and Rényi entropies (right inset). Each line corresponds to a random two-qubit state. Notice that a wide range of values of the entropic index, q, yields negative values for these differences, implying a violation of the contractivity property under local projective measurements (see relations (38)–(39) and text for details). For $q = 1$ both measures converge to the von Neumann-based one, which fulfills the contractivity property. The same happens for Tsallis with $q = 2$, corresponding to the Hilbert–Schmidt distance. Interestingly, in the Tsallis case we have been unable to find a counterexample to the mentioned contractivity for $q \in (1, 2)$ (shaded region of the left inset).

Although the most general conditions for the validity of the triangle-like inequality (37) are hard to analyze, we can link the validity of (37) with a kind of local contractivity property of the unilocal disturbances. Specifically, let us assume as valid the following inequalities:

$$\pi^{\Pi^B_j}_{(q,s)} D^{\Pi^A_i}_{(q,s)}(\Pi^B_j(\rho^{AB})) \leq D^{\Pi^A_i}_{(q,s)}(\rho^{AB}), \tag{38}$$

$$\pi^{\Pi^A_j}_{(q,s)} D^{\Pi^B_i}_{(q,s)}(\Pi^A_j(\rho^{AB})) \leq D^{\Pi^B_i}_{(q,s)}(\rho^{AB}). \tag{39}$$

Then, we have

$$\pi^{\Pi^B_1}_{(q,s)} D^{\Pi^A_1}_{(q,s)}(\Pi^B_1(\rho^{AB})) \leq D^{\Pi^A_1}_{(q,s)}(\rho^{AB}) = D^A_{(q,s)}(\rho^{AB}), \tag{40}$$

$$\pi^{\Pi^A_1}_{(q,s)} D^{\Pi^B_1}_{(q,s)}(\Pi^A_1(\rho^{AB})) \leq D^{\Pi^B_1}_{(q,s)}(\rho^{AB}) = D^B_{(q,s)}(\rho^{AB}), \tag{41}$$

and, replacing any of these relations in (32), we obtain

$$D^{\Pi^{AB}_1}_{(q,s)}(\rho^{AB}) \leq D^A_{(q,s)}(\rho^{AB}) + D^B_{(q,s)}(\rho^{AB}). \tag{42}$$

Finally, recalling that $D^{AB}_{(q,s)}(\rho^{AB}) \leq D^{\Pi^{AB}_1}_{(q,s)}(\rho^{AB})$, it follows the triangle-like inequality (37).

Thus, we are able to link the validity of the triangle-like inequality, for all states and any entropic indices, with a kind of contractivity under local projective measurements of unilocal disturbances (38)–(39). Indeed, for von Neumann entropy, inequalities (38)–(39) are particular cases of contractivity under trace-preserving completely positive maps of the quantum relative entropy [41]. Otherwise, for Tsallis entropy of entropic index 2, inequalities (38)–(39) are particular cases of contractivity under projective measurements of the Hilbert–Schmidt distance [42]. Therefore, in both cases the triangle-like inequality is satisfied (notice that for the latter, this result has been proved in an alternative way [26]). Unfortunately, the local contractivity is not valid for general entropic functionals. Indeed, we show that is the case for a wide range of the entropic index of the Rényi and Tsallis entropies in Fig. 1.

3. Examples

3.1. Mixtures of a pure state and the maximally mixed one

An interesting example where the computations can be carried out analytically involves the family of *pseudopure* states, given by mixtures of an arbitrary pure state, $|\psi^{AB}\rangle \in \mathcal{H}_{N^A} \otimes \mathcal{H}_{N^B}$, with the maximally mixed state, yielding

$$\rho^{AB}_p = (1-p)\frac{I^{AB}}{N^{AB}} + p|\psi^{AB}\rangle\langle\psi^{AB}|, \tag{43}$$

with $0 \leq p \leq 1$ (remind that $N^{AB} = N^A N^B$). The spectrum of ρ^{AB} is given by the eigenvalue $(1-p)/N^{AB} + p$, with multiplicity 1, and the eigenvalues $(1-p)/N^{AB}$, with multiplicity $N^{AB} - 1$. The measurements that optimize both the unilocal and the bilocal quantifiers are unique (do not depend on the entropic form) and are given by the local Schmidt basis [22]. This entropic-independent measurement fact is not a universal property, but depends on the particular states. In this case, measuring in the Schmidt basis yields a final spectrum that is majorized by any other spectrum corresponding to any other measurement, implying the entropic-independent optimization. After the measurement, the spectrum is given by the eigenvalue $(1-p)/N^{AB}$, with multiplicity $N^{AB}-n$, and the eigenvalues $(1-p)/N^{AB}+p\lambda_k$ with $1\leq k\leq n$, where n is the Schmidt number and λ_k the square of Schmidt coefficients (27). Using Eq. (18), we obtain

$$D^K_{(q,s)}(\rho^{AB}_p)=\frac{1}{(1-q)s}\left[\left(\frac{(N^{AB}-n)(1-p)^q+\sum_{k=1}^{n}[1+(N^{AB}\lambda_k-1)p]^q}{(N^{AB}-1)(1-p)^q+[1+(N^{AB}-1)p]^q}\right)^s-1\right] \tag{44}$$

for the generalized quantum correlations of pseudopure states. It is remarkable that, in this particular case and given the collapse of the semiquantum and total quantifiers, the triangle-like inequality (37) holds for the most general (q,s)-entropic forms.

In particular, when $|\psi^{AB}\rangle$ is a maximally entangled state, with $N^A=N^B=N$, states ρ^{AB}_p constitute a family of isotropic states, ρ^I_p. In that case, $\forall k,\ \lambda_k=N^{-1}$, $n=N$, and the generalized quantum correlations are

$$D^K_{(q,s)}(\rho^I_p)=\frac{1}{(1-q)s}\left[\left(\frac{(N^2-N)(1-p)^q+N[1+(N-1)p]^q}{(N^2-1)(1-p)^q+[1+(N^2-1)p]^q}\right)^s-1\right]. \tag{45}$$

Specializing this for Tsallis and Rényi entropies one obtains, respectively,

$$D^K_{(q,1)}(\rho^I_p)=\frac{1}{1-q}\frac{1}{N^{2q}}\left[N(1-p+Np)^q-(1-p+N^2p)^q(N-1)(1-p)^q\right], \tag{46}$$

$$D^K_{(q,0)}(\rho^I_p)=\frac{1}{1-q}\ln\left[\frac{N(1-p+Np)^q+(N^2-N)(1-p)^q}{(1-p+N^2p)^q+(N^2-1)(1-p)^q}\right]. \tag{47}$$

3.2. Werner and isotropic states

Although isotropic states are particular cases of Eq. (43), i.e., mixtures of a pure state and the maximally mixed one, we aim to show that both isotropic [43] and Werner states [1], due to their symmetries, are independent of the local measurements performed. A Werner state is an $N\times N$ dimensional bipartite quantum state that is invariant under local unitary transformations of the form $U\otimes U$, with U an arbitrary unitary acting on N dimensional systems, that is, $\rho^W=U\otimes U\rho U^\dagger\otimes U^\dagger$. On the other hand, an $N\times N$-dimensional isotropic state is invariant under arbitrary local unitaries of the form $U\otimes U^*$, that is, $\rho^I=U\otimes U^*\rho U^\dagger\otimes(U^*)^\dagger$. They can be parametrized, respectively, as

$$\rho^W_x=\frac{N-x}{N^3-N}I+\frac{Nx-1}{N^3-N}F, \tag{48}$$

with $F=\sum_{ij}|ij\rangle\langle ji|$, $1\leq i,j\leq N$, $x\in[-1,1]$, and

$$\rho^I_y=\frac{1-y}{N^2-1}I+\frac{N^2y-1}{N^2-1}|\psi^+\rangle\langle\psi^+|, \tag{49}$$

with $|\psi^+\rangle=\frac{1}{\sqrt{N}}\sum_{i=1}^N|ii\rangle$ and $y\in[\frac{1}{N^2},1]$. Notice that both definitions of isotropic states – the one derived from Eq. (43) and the one given by Eq. (49) – coincide under the identification $p=\frac{N^2y-1}{N^2-1}$ and $|\psi^{AB}\rangle=|\psi^+\rangle$.

To see that any local measurement yields the same disturbance over these families of states, let us consider Π^A_1 as the optimal unilocal measurement over A. Any other local measurement is achieved by a unitary transformation over Π^A_1 as $\Pi^A_V=V\otimes I^B\,\Pi^A_1\,V^\dagger\otimes I^B$, with V an arbitrary unitary over A. Then, using the invariance properties of Werner states, the action of Π^A_V is $\Pi^A_V(\rho^W)=V\otimes V\,\Pi^A_1\,V^\dagger\otimes V^\dagger$. Analogous results hold for isotropic states and measurements over B. Invoking the unitary invariance of (q,s)-entropies one has that the minimum in (18) is attained for any local projective measurement. To prove that nothing changes when considering bilocal measurements, it is sufficient to observe that after any local measurement the state becomes a CC correlated state. Thus, given that the total disturbance can be computed via the partial disturbances (see Eqs. (29)–(30)), the total quantum correlations are equal to the semiquantum ones.

In order to find an explicit formula of the generalized correlations, it is easier to measure on the standard basis (the ones used to define F in Werner states and $|\psi^+\rangle$ in isotropic states), readily obtaining

$$D^K_{(q,s)}(\rho^W_x)=\frac{1}{(1-q)s}\times\left[\left(\frac{2[(N-1)^q(x+1)^q+(N-1)(N-x)^q]}{2(N-1)^q(x+1)^q+(N-1)\left[\left(N-x+\frac{1}{2}Nx-\frac{1}{2}\right)^q+\left(N-x-\frac{1}{2}Nx+\frac{1}{2}\right)^q\right]}\right)^s-1\right] \tag{50}$$

and

$$D^K_{(q,s)}(\rho^I_y) = \frac{1}{(1-q)s}\left[\left(\frac{N\left[(N-1)(1-y)^q + \left(1-y+Ny-\frac{1}{N}\right)^q\right]}{(N^2-1)^q y^q + (N^2-1)(1-y)^q}\right)^s - 1\right]. \tag{51}$$

Again, it is interesting to observe that these families of states are among the ones that satisfy the triangle-like inequality (Eq. (37)) for any (q, s)-entropy.

4. Concluding remarks

In this work we address the problem of quantifying quantum correlations beyond discord. Specifically, following Ref. [16], we obtain entropic measures of bipartite quantum correlations by quantifying the system's states disturbance under local measurements. Our measures are based on very general entropic forms given by the quantum (q, s)-entropies [14,15]. As a consequence, we obtain quantum correlation measures, which include as particular cases or are close to several other measures previously discussed in the literature [17–27]. Our main contribution is to propose such quantum correlation measures based on quantum unified (q, s)-entropies that are: (i) nonnegative and vanish only for QC, CQ and CC correlated states, (ii) invariant under local unitary operators, and (iii) invariant under the addition of an uncorrelated ancilla. Regarding the last property, we show that for $q \not\to 1$ or $s \not\to 0$, that is when the (q, s)-entropies are nonadditive, it is necessary to rescale the disturbances by a generalized purity factor in order to avoid undesirable effects of previous entropic based correlation measures [22–24].

Moreover, we distinguish between total and semiquantum correlations, and we naturally obtain that the former are greater than the latter. In addition, we show that a triangle-like inequality is fulfilled for certain families of states, namely QC, CQ and CC correlated states, as well as, Werner and Isotropic states, for any entropic measures. In the general case, we only prove this for the von Neumann and Tsallis with entropic index of order 2, which follows from the contractivity property under a projective measurement of quantum relative entropy and Hilbert–Schmidt distance, respectively. We provide numerical counterexamples where the local contractivity property of unilocal disturbances fails in a wide range of the entropic index of Rényi and Tsallis entropies, but it remains open if the triangle-like inequality is fulfilled for other entropic measures.

Finally, we provide analytical expressions of the entropic correlation measures for pseudopure, Werner and isotropic states. For these families of states, the optimal measurement of unilocal and bilocal disturbances is independent of the entropic form.

Acknowledgments

GMB, GB, MP and PWL acknowledge CONICET and UNLP (Argentina), and MP and PWL also acknowledge SECyT-UNC (Argentina) for financial support. SZ is grateful to the University of Grenoble-Alpes (France) for the AGIR financial support.

References

[1] R.F. Werner, Phys. Rev. A 40 (1989) 4277.
[2] A. Einstein, B. Podolsky, N. Rosen, Phys. Rev. 47 (1935) 777.
[3] E. Schrödinger, Naturwissenschaften 23 (1935) 844.
[4] H.M. Wiseman, S.J. Jones, A.C. Doherty, Phys. Rev. Lett. 98 (2007) 140402.
[5] J.S. Bell, Physics 1 (1964) 195.
[6] H. Ollivier, W.H. Zurek, Phys. Rev. Lett. 88 (2001) 017901.
[7] L. Henderson, V. Vedral, J. Phys. A 34 (2001) 6899.
[8] J. von Neumann, Gótt. Nachr. 1 (1927) 273.
[9] A. Rényi, Fourth Berkeley Symp. Math. Stat. Probability 1 (1961) 547.
[10] C. Tsallis, J. Stat. Phys. 52 (1988) 479.
[11] X.-W. Hou, Z.-P. Huang, S. Chen, Eur. Phys. J. D 68 (2014) 1.
[12] J. Jurkowski, Int. J. Quantum Inf. 11 (2013) 1350013.
[13] G. Bellomo, A. Plastino, A.P. Majtey, A.R. Plastino, Eur. Phys. J. D 68 (2014) 337.
[14] X. Hu, Z. Ye, J. Math. Phys. 47 (2006) 023502.
[15] A.E. Rastegin, J. Stat. Phys. 143 (2011) 1120.
[16] S. Luo, Phys. Rev. A 77 (2008) 022301.
[17] A.K. Rajagopal, R.W. Rendell, Phys. Rev. A 66 (2002) 022104.
[18] M. Horodecki, P. Horodecki, R. Horodecki, J. Oppenheim, A. Sen(De), U. Sen, B. Synak-Radtke, Phys. Rev. A 71 (2005) 062307.
[19] K. Modi, T. Paterek, W. Son, V. Vedral, M. Williamson, Phys. Rev. Lett. 104 (2010) 080501.
[20] B. Dakić, V. Vedral, Č. Brukner, Phys. Rev. Lett. 105 (2010) 190502.
[21] S. Luo, S. Fu, Phys. Rev. A 82 (2010) 034302.
[22] R. Rossignoli, N. Canosa, L. Ciliberti, Phys. Rev. A 82 (2010) 052342.
[23] R. Rossignoli, N. Canosa, L. Ciliberti, Phys. Rev. A 84 (2011) 052329.
[24] A.C.S. Costa, R.M. Angelo, Phys. Rev. A 87 (2013) 032109.
[25] T. Tufarelli, T. MacLean, D. Girolami, R. Vasile, G. Adesso, J. Phys. A 46 (2013) 275308.
[26] M.-J. Zhao, T.-G. Zhang, Z.-G. Li, X. Li-Jost, S.-M. Fei, D.-S. Zhong, Internat. J. Theoret. Phys. 52 (2013) 2379.
[27] N. Canosa, L. Ciliberti, R. Rossignoli, Entropy 17 (2015) 1634.

[28] A. Misra, A. Biswas, A.K. Pati, A. Sen(De), U. Sen, Phys. Rev. E 91 (2015) 052125.
[29] G. Adesso, T.R. Bromley, M. Cianciaruso, arXiv:1605.00806.
[30] K. Modi, A. Brodutch, H. Cable, T. Paterek, V. Vedral, Rev. Modern Phys. 104 (2012) 1655.
[31] A. Wehrl, Rev. Modern Phys. 50 (1978) 221.
[32] M. Piani, Phys. Rev. A 86 (2012) 034101.
[33] A.W. Marshall, I. Olkin, B.C. Arnold, Inequalities: Theory of Majorization and its Applications, second ed., Springer Verlag, New York, 2011.
[34] G.M. Bosyk, S. Zozor, F. Holik, M. Portesi, P.W. Lamberti, Quantum Inf. Process. (2016) 1–28. in press, http://dx.doi.org/10.1007/s11128-016-1329-5.
[35] I. Bengtsson, K. Życzkowski, Geometry of Quantum States: An Introduction to Quantum Entanglement, Cambridge University Press, Cambridge, 2006.
[36] A. Chefles, Phys. Rev. A 65 (2002) 052314.
[37] A. Datta, arXiv:1003.5256v2.
[38] A. Brodutch, K. Modi, Quantum Inf. Comput. 12 (2012) 721.
[39] L. Chang, S. Luo, Phys. Rev. A 87 (2013) 062303.
[40] C.H. Bennett, H. Bernstein, S. Popescu, B. Schumacher, Phys. Rev. A 53 (1996) 2046.
[41] G. Lindblad, Comm. Math. Phys. 40 (1975) 147.
[42] B. Tamir, E. Cohen, J. Quantum Inf. Sci. 5 (2015) 127.
[43] M. Horodecki, P. Horodecki, Phys. Rev. A 59 (1999) 4206.