

Prioritized and Non-prioritized Multiple Change on Belief Bases

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Abstract In this article we explore multiple change operators, i.e., operators in which the epistemic input is a set of sentences instead of a single sentence. We propose two types of change: prioritized change, in which the input set is fully accepted, and symmetric change, where both the epistemic state and the epistemic input are equally treated. In both kinds of operators we propose a set of postulates and we present different constructions: kernel changes and partial meet changes.

Keywords Belief revision · Merging · Multiple change · Belief bases

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1 Introduction

1.1 Motivation

The logic of theory change deals with the problem of modeling the dynamics of knowledge, that is, how the beliefs of a cognitive agent are modified in some dynamic scenario. Usually, the new information to be adopted is assumed to be one single sentence, as in classical AGM theory [1, 2, 22, 23]. Our concern in this paper is to study multiple change operations, i.e., belief change processes that are induced simultaneously by sets of sentences [19]. In particular, we will not consider the conjunction of all new sentences as the new piece of information, since this would reduce the problem to change by singletons.

A good example could be presented in a multi-agent system: suppose a system in which an agent C is the coordinator of a set of agents. This agent C decides the destiny of some actions using information from the other agents. Periodically, agent C needs to query them in order to decide which action will be performed. Every time C receives information of the agents, it could be seen as a case of multiple revision. Moreover, this example is a good justification for the use of multiple change instead of the conjunction of the new sentences. For instance, if every agent has a credibility degree, we could preserve this information if we represent the new information by a set of tuples (α, I) where α represents the belief and I the informant agent. At the knowledge level, the agent believes in α ; at the symbolic level, we add an agent identifier to specify the reliability of the source.¹ Changing the credibility of agents we could change the entrenchment or plausibility of their associated beliefs. So, most importantly, multiple change operators offer the possibility to consider incoming information as independent pieces of information which might be treated differently during the change process. However, using this representation (a tuple (α, I) instead of a single sentence α) should lead to a non-prioritized revision although this problem is not addressed in the paper.

In this paper, we will consider two different types of multiple change. In prioritized change operators, all new beliefs are supposed to be fully accepted. We do not consider partial acceptance (that is, to accept a proper subset of the new beliefs) neither disjunctive acceptance (that is, to accept $\{\alpha \vee \beta\}$ instead of $\{\alpha, \beta\}$). The other change operators are merging operators, allowing prior and new beliefs to play symmetric roles under the change process. For each type of change, we propose two operators, one being based on kernels and the other making use of remainder sets. As the main contribution of this paper, we characterize all revision and merging operators by sets of postulates. Following the suggestion of [16], we assume that the paradigm or scenario underlying the

¹The distinction about *knowledge* and *symbolic level* was proposed by [37]. According to Newell, the knowledge level lies above the symbolic level where all of the knowledge in the system is represented. An implicit assumption in Artificial Intelligence is that a knowledge-level system must contain a symbol system. Following Newell's unified theory of cognition, the symbol level corresponds to the cognitive part and the knowledge level to the rational part.

belief change process could be a computational agent in a multi-agent system who tries to incorporate information from multiple sources (in the case of prioritized multiple revision), or an agent who tries to merge its information with the information provided by another reliable agent (in the case of merge).

1.2 Preliminaries and Notation

We will adopt a propositional language \mathcal{L} with a complete set of boolean connectives: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$. Formulae in \mathcal{L} will be denoted by lowercase Greek characters: $\alpha, \beta, \delta, \dots, \omega$. Sets of sentences in \mathcal{L} will be denoted by uppercase Latin characters: A, B, C, \dots, Z . The symbol \top represents a tautology or *truth*. The symbol \perp represents a contradiction or *falsum*. The characters γ and σ will be reserved to represent selection and incision functions for change operators respectively. We also use a consequence operator Cn . Cn takes sets of sentences in \mathcal{L} and produces new sets of sentences. The operator Cn satisfies *inclusion* ($A \subseteq Cn(A)$), *idempotence* ($Cn(A) = Cn(Cn(A))$), and *monotony* (if $A \subseteq B$ then $Cn(A) \subseteq Cn(B)$). We will assume that the consequence operator includes classical consequences and verifies the standard properties of *supraclassicality* (if α can be derived from A by deduction in classical logic, then $\alpha \in Cn(A)$), *deduction* ($\beta \in Cn(A \cup \{\alpha\})$ if and only if $(\alpha \rightarrow \beta) \in Cn(A)$) and *compactness* (if $\alpha \in Cn(A)$ then $\alpha \in Cn(A')$ for some finite subset A' of A). In general, we will write $\alpha \in Cn(A)$ as $A \vdash \alpha$.

1.3 Belief Bases vs. Belief Sets

When we talk about change we also have to talk about the object of change. Generally, this object of change is the epistemic state of an agent [30]. Different constructions can be used as models of epistemic states, two of the most commonly used models being belief sets (sets of sentences closed under logical consequence), and belief bases (sets of sentences not necessarily closed).

Clearly, the representation of epistemic states by belief sets has many disadvantages from a computational point of view. A belief set is a very large entity. For any two sentences α, β in a belief set \mathbf{K} then so are $\alpha \vee \beta, \alpha \vee \neg\beta, \alpha \vee \beta \vee \delta, \dots$ because they are logical consequences of \mathbf{K} . If the language is sufficiently rich, the belief set will contain innumerable many sentences that the believer has never thought of [30].

It is more natural to think of the epistemic state as being represented by a limited number of sentences that may (roughly speaking) correspond to the explicit beliefs. Changes can operate on this smaller set, rather than directly on the belief set. Such a model is much closer to computational applications such as knowledge based systems, deductive databases, multi-agent systems, etc. Moreover, some implementations of changes can be improved using the method of *prime-implicants/implicates* of [5]. This method avoids multiple

satisfiability checking (which is an **NP-Complete** problem)² by expressing beliefs in a specific syntax. The prime canonical forms are important in knowledge representation, because theories compiled into them can be queried in polynomial time for consistency, validity, clause entailment, implicants, equivalence, sentential entailment and model enumeration [6]. Bienvenu et al. [4] propose a way to define revision operators which are formula-based yet syntax-insensitive. They define the revision operators in two steps: first replacing the belief base by its set of prime implicates and then applying two specific revision operators. Recently, in [3], sound and complete algorithms for generating and recognizing prime implicates were presented, showing that prime implicate recognition task is a **PSpace-Complete** problem.³

The distinction between belief sets and belief bases is similar to the distinction between the coherence approach and the foundational approach to belief revision. The *coherence approach* focuses on logical relations among beliefs rather than on inferential relations, that is, no belief is more fundamental than others [9]. In the coherentist approach, beliefs provide each other with mutual support; therefore, a belief set represents the limit case of this approach. On the other hand, the *foundational approach* divides beliefs into two classes: explicit beliefs and implicit beliefs. The explicit beliefs are considered as “self-justified beliefs” whereas the implicit beliefs are seen as beliefs derived, justified or supported by the explicit beliefs. The foundational approach provides explanation of beliefs by requiring that each belief is supportable by means of non-circular arguments from explicit or basic beliefs [9]. Another feature of the foundational approach is that a belief α may be justified or derived by several independent beliefs, so that even if some of the justifications for α are removed, the belief α may be retained because it is supported by other beliefs.⁴ A belief base is considered to be a good candidate for change according to the foundational approach.

We may find several advantages and disadvantages for these approaches. While a definitive conclusion about whether either of these approaches is better than the other awaits answers, we believe that the foundational approach by means of belief bases is better in computational environments such as argumentative systems [20, 42], truth maintenance systems [8], and multi-agent

²In terms of language recognition, **NP** is the class of languages decided by nondeterministic Turing machines in polynomial time [39]. From the perspective of decision problems (a question in some formal system with a yes-or-no answer), **NP** is the set of all decision problems where the “yes”-instances can be recognized in polynomial time by a non-deterministic Turing machine. A problem P is **NP-Complete** if P is a problem in **NP** and all problems in **NP** can be reduced/transformed to P . One of the most important characteristics of **NP-complete** problems is that no efficient solution to them is known.

³In terms of decision problems, **PSpace** is the set of all decision problems which can be solved by a Turing Machine using a polynomial amount of space. The hardest problems in **PSpace** are the **PSpace-Complete** problems. It is suspected that some problems of **PSpace-Complete** are problems in **PSpace** but not in **NP**.

⁴In the coherentist approach, if some justification for α is preserved then α is also preserved.

systems [11, 44]. Although the foundational approach entails syntax dependence of the belief base, this disadvantage can be overcome by *normalizing* the belief base using the method of *prime-implicants/implicates* [5].

1.4 Overview

This article is organized as follows. Section 2 presents different possible operators for multiple change, giving characteristic postulates. In the following two sections, we define two prioritized, respectively symmetric multiple revision operators: multiple kernel revision operator and multiple partial meet revision operator in Section 3, and kernel merge and partial meet merge in Section 4. The two latter mentioned sections also contain some connections with specific related works. Section 5 addresses some related work about multiple change, and in the last section, we conclude and mention plans for future work. All proofs can be found in the [Appendix](#).

2 Multiple Changes

In this section we will briefly introduce and discuss multiple change operators. Moreover, we will propose several postulates to describe multiple change that adequately generalize postulates for changes by single sentences.

2.1 Prioritized Revision vs. Merging

Suppose that we want to change⁵ a set K with respect to a set of sentences A . We may distinguish between different kinds of multiple change.

1. *Prioritized change (revision)*: all sentences of A are to be included in the new belief base.
2. *Symmetric change (merge)*: K and A are symmetrically treated (i.e., sentences of K and A could be accepted or rejected).
3. *Selective change*: some sentences of A could be accepted, some others could be rejected.⁶

In this work, we will deal with prioritized multiple revision and symmetric revision, leaving the selective multiple revision as a subject of future work. The term “multiple revision” is used to refer to operations of revision allowing simultaneous revision by more than one sentence. It should be distinguished from “repeated” or “iterated” revision, that is, the application of two or more revisions in a sequence.

⁵When we say “change” we may refer to a revision or merge.

⁶Selective revision was proposed by [14]. However, this model makes use of a single sentence (instead of a set) as epistemic input.

For prioritized multiple revision, we might think of making use of classical AGM revision (i.e., revision by a single sentence) via applying sentential negation:

Definition 1 (Hansson [25]) The function \neg is a mapping from set of sentences to sentences such that, for every finite set A holds:

1. $\neg\emptyset = \perp$.
2. If $A = \{\alpha\}$ is a singleton then $\neg A = \neg\alpha$.
3. If $A = \{\alpha_1, \dots, \alpha_m\}$ for some $m > 1$, then $\neg A = \neg\alpha_1 \vee \dots \vee \neg\alpha_m$.

$\neg A$ is the *sentential negation* of A .

Let \div be an AGM contraction operator for K . A way to define a multiple revision is by using a Generalized Levi Identity:⁷

$$K * A = (K \div \neg A) \cup \{A\}$$

Then, the revision can be trivially achieved by expansion, and the axiomatic characterization could be straightforwardly obtained from the corresponding characterizations of the traditional models.

The aim of our work is not to define revisions from contractions, but rather to construct and axiomatically characterize multiple revision operators in a direct way. For slight, but substantial differences between contraction by single sentences and contraction by sets of sentences, cf. [19].

2.2 Postulates for Multiple Revision

Let K , A and B be sets of sentences, and $*$ be a binary multiple revision operator that takes a belief base and a set of sentences as inputs. We propose the following postulates as reasonable postulates for multiple revision operations:

- Inclusion $K * A \subseteq K \cup A$.
- Success $A \subseteq K * A$.
- Weak Success If A is consistent then $A \subseteq K * A$.
- Relative Success $A \subseteq K * A$ or $K * A = K$.
- Consistency If A is consistent then $K * A$ is consistent.
- Vacuity 1 If A is inconsistent then $K * A = K$.
- Vacuity 2 If $K \cup A \not\vdash \perp$ then $K * A = K \cup A$.
- Uniformity 1 Given A and B two consistent sets, for all subset X of K , if $(X \cup A) \vdash \perp$ if and only if $(X \cup B) \vdash \perp$ then $K \setminus (K * A) = K \setminus (K * B)$.

⁷The identity used in this paper is in fact a generalization of the (actual) Levi Identity to the case of multiple operators on belief bases. In Hansson's PhD thesis the (generalization) of the Levi Identity to the case of multiple operator on belief sets has been called *Levi identity for sets* [24].

- Uniformity 2 Given A and B two consistent sets, for all subset X of K , if $(X \cup A) \vdash \perp$ if and only if $(X \cup B) \vdash \perp$ then $K \cap (K * A) = K \cap (K * B)$.
- Relevance If $\alpha \in K \setminus (K * A)$ then there is a set C such that $K * A \subseteq C \subseteq (K \cup A)$, C is consistent with A but $C \cup \{\alpha\}$ is inconsistent with A .
- Core-Retainment If $\alpha \in K \setminus (K * A)$ then there is a set C such that $C \subseteq (K \cup A)$, C is consistent with A but $C \cup \{\alpha\}$ is inconsistent with A .

Inclusion says that any change operator between two arbitrary sets is included in the (unrestricted) union of them. Success gives priority to the new information. Weak Success gives priority to the new information only when it is consistent. Relative Success establishes that all sentences of the input set are added to the changed set or nothing is changed (except in the limit case in which some sentences of the input set are in the belief base). Consistency determines that the changed set is consistent whenever the input set is consistent. Vacuity 1 says that nothing is changed if the input set is inconsistent. Vacuity 2 establishes that if the input set is consistent with the original belief base, then the changed set is equal to the union of them. Uniformity 1 determines that if two consistent sets are consistent with the same subsets of the original belief base K then the respective erased sentences of K should be identical. Uniformity 2 determines that if two consistent sets are consistent with the same subsets of the original belief base K then the respective preserved sentences of K should be identical. Uniformity 2 is an adaptation of Uniformity [26] which is used in revisions where the epistemic input is just a single sentence. Relevance and Core-Retainment were adapted from [30] and they express the intuition that nothing is removed from the original belief base unless its removal in some way contributes to making the new belief base consistent. Except weak success and uniformity 1, these postulates are adaptations of similar postulates from singleton internal revision [29, 30].

2.3 Postulates for Merge

Let K, H, A and B be sets of sentences and \circ be a binary merge operator that takes sets of sentences as inputs. We propose the following postulates to classify reasonable merge operations:

- Inclusion $K \circ A \subseteq K \cup A$.
- Symmetry $K \circ A = A \circ K$.
- Strong Consistency $K \circ A$ is consistent.
- Congruence If $K \cup A = H \cup B$ then $K \circ A = H \circ B$.
- Vacuity If $K \cup A \not\vdash \perp$ then $K \circ A = K \cup A$.

Reversion	If $K \cup A$ and $H \cup B$ have the same set of minimal inconsistent subsets then $(K \cup A) \setminus (K \circ A) = (H \cup B) \setminus (H \circ B)$. ⁸
Global Relevance	If $\alpha \in (K \cup A) \setminus (K \circ A)$ then there is a set C such that $K \circ A \subseteq C \subseteq (K \cup A)$, C is consistent but $C \cup \{\alpha\}$ is inconsistent.
Global Core-Retainment	If $\alpha \in (K \cup A) \setminus (K \circ A)$ then there is a set C such that $C \subseteq (K \cup A)$, C is consistent but $C \cup \{\alpha\}$ is inconsistent.

Again, Inclusion makes the union of two sets to be the upper bound of any merge operation between them. Symmetry establishes that both belief bases are considered of equal importance. Strong Consistency requires that the changed belief base must be consistent. Congruence requires that the result of merge should not depend on syntactic properties of the sentences to be added or erased: only their logical content should count. Vacuity says that, if two sets are consistent then the merged set is equal to the union between them. Reversion says that, if two pairs of sets contain the same minimally inconsistent subsets then the sentences erased in the respective changes are the same. Global Relevance and Global Core-Retainment express the intuition that nothing is removed from the union of the original belief base and the input set unless its removal in some way contributes to making the new belief base consistent. Inclusion, Symmetry, Congruence and Global Relevance have been proposed in [18]; Reversion and Global Core-Retainment have been proposed in [12].

The following lemma shows some relations among postulates.

Lemma 1

- If an operator satisfies relevance then it satisfies core-retainment.
- If an operator satisfies global relevance then it satisfies global core-retainment.
- If an operator satisfies vacuity 1 and weak success then it satisfies relative success.
- An operator satisfies uniformity 1 if and only if it satisfies uniformity 2.
- If an operator satisfies inclusion, weak success and core-retainment then it satisfies vacuity 2.
- If an operator satisfies reversion and inclusion then it satisfies congruence.
- If an operator satisfies congruence then it satisfies symmetry.

⁸Given a set of sentences S , a *minimal inconsistent subset* of S is a set S' such that $S' \subseteq S$, $S' \vdash \perp$, and if $S'' \subset S'$ then $S'' \not\vdash \perp$. Making use of the notation introduced in Definition 11 this postulate can be formulated in the following alternative way: If $(K \cup A)\perp\perp = (H \cup B)\perp\perp$ then $(K \cup A) \setminus (K \circ A) = (H \cup B) \setminus (H \circ B)$.

3 Prioritized Multiple Revision

Suppose that we want to revise an arbitrary set K with respect to another arbitrary set A without using sentential negation. We may construct two kinds of multiple revision by generalizing techniques from classical belief (base) revision [10, 30]:

1. Multiple Kernel Revision.⁹
2. Multiple Partial Meet Revision.

These two revision operators are syntax dependent and they are based in the foundational approach. In order to define these constructions, we need to define two concepts: A -consistent-remainders and A -inconsistent-kernels.

3.1 Kernels and Remainder Sets

Definition 2 Let K, A be sets of sentences, where A is consistent. The set of A -inconsistent-kernels of K , noted by $K \perp\!\!\!\perp A$, is the set of sets X such that:

1. $X \subseteq K$.
2. $X \cup A$ is inconsistent.
3. For any X' such that $X' \subset X \subseteq K$ then $X' \cup A$ is consistent.

That is, given a consistent set A , $K \perp\!\!\!\perp A$ is the set of minimal K -subsets inconsistent with A .

Definition 3 Let K, A be sets of sentences, where A is consistent. The set of A -consistent-remainders of K , noted by $K \perp\!\!\!\top A$, is the set of sets X such that:

1. $X \subseteq K$.
2. $X \cup A$ is consistent.
3. For any X' such that $X \subset X' \subseteq K$ then $X' \cup A$ is inconsistent.

That is, $K \perp\!\!\!\top A$ is the set of maximal K -subsets consistent with A .

Example 1 Suppose that $K = \{p, p \rightarrow q, q, \neg r\}$ and $A = \{\neg q, r\}$. Then we have that:

- $K \perp\!\!\!\perp A = \{\{p, p \rightarrow q\}, \{q\}, \{\neg r\}\}$.
- $K \perp\!\!\!\top A = \{\{p\}, \{p \rightarrow q\}\}$.

Observation 1 Let K, A be sets of sentences, A be consistent. Suppose that $\alpha \in K$ and $\alpha \in A$. Then $\alpha \notin \bigcup(K \perp\!\!\!\perp A)$ and, therefore, $A \cap \bigcup(K \perp\!\!\!\perp A) = \emptyset$.

Observation 2 Let K, A be sets of sentences, A be consistent. Suppose that $\alpha \in K$ and $\alpha \in A$. Then $\alpha \in X$ for all $X \in K \perp\!\!\!\top A$ and, therefore, $\alpha \in \bigcap(K \perp\!\!\!\top A)$.

⁹The concept of kernel contraction was introduced in [27].

Now, we will define two kinds of prioritized multiple revision operators. However, both operators are not fully prioritized because they do not produce changes when the new set of beliefs is inconsistent. This intuition is captured by the postulate of vacuity 1.

3.2 Prioritized Multiple Kernel Revision

The first construction of multiple revision by a set of sentences is based on the concept of a A -inconsistent-kernels. In order to complete the construction, we must define an incision function that cuts in every inconsistent-kernel.

Definition 4 Let K be a set of sentences. σ is a *consolidated incision function* for K ($\sigma : 2^{2^{\mathcal{L}}} \Rightarrow 2^{\mathcal{L}}$) if and only if, for all consistent set A :

1. $\sigma(K \perp\!\!\!\perp A) \subseteq \bigcup K \perp\!\!\!\perp A$.
2. If $X \in K \perp\!\!\!\perp A$ then $X \cap (\sigma(K \perp\!\!\!\perp A)) \neq \emptyset$.

From Observation 1 and Definition 4, it follows that all the sentences of A are ‘protected’, in the sense that they can not be considered for removing by the consolidated incision function. That is, a consolidated incision function selects among the sentences of $K \setminus A$ that make $K \cup A$ inconsistent.

Example 2 Let $K = \{p, p \rightarrow q, r, s, s \rightarrow t\}$ and $A = \{p, \neg q, \neg r\}$. It follows that $K \perp\!\!\!\perp A = \{\{p \rightarrow q\}, \{r\}\}$. The one possible consolidated incision function selects $\{p \rightarrow q, r\}$; p could not be selected by σ since p does not belong to any $X \in K \perp\!\!\!\perp A$.

Definition 5 Let K be a set of sentences and σ a consolidated incision function for K . The *prioritized multiple kernel revision* on K that is generated by σ is the operator $*_{\sigma}$ such that, for all set A ($*_{\sigma} : 2^{\mathcal{L}} \times 2^{\mathcal{L}} \Rightarrow 2^{\mathcal{L}}$):

$$K *_{\sigma} A = \begin{cases} (K \setminus \sigma(K \perp\!\!\!\perp A)) \cup A & \text{if } A \text{ is consistent} \\ K & \text{otherwise} \end{cases}$$

An operator $*$ is a *prioritized multiple kernel revision* for K if and only if there is a consolidated incision function σ for K such that for all sets A , $K * A = K *_{\sigma} A$.

Example 3 Given $K = \{p, p \rightarrow q, q, \neg r\}$, $A = \{\neg q, r\}$. and $K \perp\!\!\!\perp A = \{\{p, p \rightarrow q\}, \{q\}, \{\neg r\}\}$. We have two possible results for the consolidated incision function and its associated multiple kernel revision operator:

- $\sigma_1(K \perp\!\!\!\perp A) = \{p, q, \neg r\}$ and $K *_{\sigma_1} A = \{p \rightarrow q, \neg q, r\}$.
- $\sigma_2(K \perp\!\!\!\perp A) = \{p \rightarrow q, q, \neg r\}$ and $K *_{\sigma_2} A = \{p, \neg q, r\}$.

Theorem 1 *An operator $*$ is a prioritized multiple kernel revision for K if and only if it satisfies inclusion, consistency, weak success, vacuity 1, uniformity 1 (and uniformity 2), and core-retainment.*

From the definition and the axiomatization, we want to remark some important differences with AGM revisions: a) the epistemic state is a belief base, b) the epistemic input is a set of sentences, c) the operator is more intuitive since it satisfies Weak Success and Vacuity 1. Weak Success ensures the input set is accepted when it is consistent. Vacuity 1 ensures that the belief base remains unchanged when the input set is inconsistent. These two postulates resolve a controversial point of AGM revisions such as success which forces α to be accepted in the revision of K by α even though α might be inconsistent. Some models of change in which a weaker version of the success postulate is considered can be found, for example, in [12, 14, 31, 36].

3.3 Prioritized Multiple Partial Meet Revision

The second construction of multiple revision by a set of sentences is based on the concept of a A -consistent-remainders. In order to complete the construction, we must define a selection function that selects the ‘best’ consistent remainders.

Definition 6 Let K be a set of sentences. γ is a *consolidated selection function* for K ($\gamma : 2^{2^{\mathcal{C}}} \Rightarrow 2^{2^{\mathcal{C}}}$) if and only if, for all set A :

1. If $K \perp_{\top} A \neq \emptyset$ then $\emptyset \neq \gamma(K \perp_{\top} A) \subseteq K \perp_{\top} A$.
2. If $K \perp_{\top} A = \emptyset$ then $\gamma(K \perp_{\top} A) = \{K\}$.

From Observation 2 and Definition 6, it follows that all the sentences of $K \cap A$ are ‘protected’, in the sense that they are included in the intersection of any collection of remainders. That is, a consolidated selection function selects a subset of the set $K \perp_{\top} A$ whose elements all contain the set $K \cap A$.

Example 4 Let $K = \{p, p \rightarrow q, q, s, s \rightarrow \neg r, u, u \rightarrow \neg r, t\}$ and $A = \{p, \neg q, s, r\}$. It follows that $K \perp_{\top} A = \{\{p, s, u, t\}, \{p, s, u \rightarrow \neg r, t\}\}$. That is, p and s are in every A -consistent-remainder, so p and q will be in $\bigcap \gamma(K \perp_{\top} A)$ for any consolidated selection function γ for K .

Definition 7 Let K be a set of sentences and γ a consolidated selection function for K . The *prioritized multiple partial meet revision* on K that is generated by γ is the operator $*_{\gamma}$ such that, for all set A ($*_{\gamma} : 2^{\mathcal{C}} \times 2^{\mathcal{C}} \Rightarrow 2^{\mathcal{C}}$):

$$K *_{\gamma} A = \begin{cases} \bigcap \gamma(K \perp_{\top} A) \cup A & \text{if } A \text{ is consistent} \\ K & \text{otherwise} \end{cases}$$

An operator $*$ is a *prioritized multiple partial meet revision* on K if and only if there is a consolidated selection function γ for K such that for all sets A , $K * A = K *_{\gamma} A$.

Example 5 Given $K = \{p, p \rightarrow q, q, \neg r\}$, $A = \{\neg q, r\}$. and $K \perp_{\top} A = \{\{p\}, \{p \rightarrow q\}\}$. We have three possible results for the consolidated selection function and its associated multiple partial meet revision operator:

- $\gamma_1(K \perp_{\top} A) = \{\{p\}\}$ and $K *_{\gamma_1} A = \{p, \neg q, r\}$.
- $\gamma_2(K \perp_{\top} A) = \{\{p \rightarrow q\}\}$ and $K *_{\gamma_2} A = \{p \rightarrow q, \neg q, r\}$.
- $\gamma_3(K \perp_{\top} A) = \{\{p\}, \{p \rightarrow q\}\}$ and $K *_{\gamma_3} A = \{\neg q, r\}$.

Theorem 2 *An operator $*$ is a prioritized multiple partial meet revision for K if and only if it satisfies inclusion, consistency, weak success, vacuity 1, uniformity 2 (and uniformity 1), and relevance.*

It follows immediately from the the representation Theorems 1 and 2 and from Lemma 1, that the multiple partial meet revision operators satisfy all the postulates that axiomatically characterize the multiple kernel revision operators. Thus, the following corollary holds.

Corollary 1 *Let K be a belief base. If $*$ is a multiple partial meet revision for K then $*$ is a multiple kernel revision for K .*

The following example (which has been adapted from a similar one presented in [30, page 91]) clarifies that the converse of the above corollary does not hold, i.e., that some multiple kernel revision operators are not multiple partial meet revision operators.

Example 6 Let p, q and r be logically independent sentences and $K = \{p, q, r\}$. Suppose that $A = \{\neg p \vee \neg q, \neg p \vee \neg r\}$. Suppose that we are applying a prioritized multiple kernel revision. Then we have that:

$$K \perp_{\perp} A = \{\{p, q\}, \{p, r\}\}$$

Suppose that $\sigma(K \perp_{\perp} A) = \{p, r\}$. Then, prioritized multiple kernel revision of K with respect to A is:

$$K *_{\sigma} A = (K \setminus \sigma(K \perp_{\perp} A)) \cup A = \{q, \neg p \vee \neg q, \neg p \vee \neg r\}$$

Now, suppose that we are applying a prioritized multiple partial meet revision. Then we have that:

$$K \perp_{\top} A = \{\{p\}, \{q, r\}\}$$

If γ is a selection function for K , then $K *_{\gamma} A = \bigcap (K \perp_{\top} A) \cup A$. We have three cases:

$$\gamma_1(K \perp_{\top} A) = \{\{p\}\} \text{ and } K *_{\gamma_1} A = \{p, \neg p \vee \neg q, \neg p \vee \neg r\}.$$

$$\gamma_2(K \perp_{\top} A) = \{\{q, r\}\} \text{ and } K *_{\gamma_2} A = \{q, r, \neg p \vee \neg q, \neg p \vee \neg r\}.$$

$$\gamma_3(K \perp_{\top} A) = \{\{p\}, \{q, r\}\} \text{ and } K *_{\gamma_3} A = \{\neg p \vee \neg q, \neg p \vee \neg r\}.$$

Thus, we can immediately conclude that $*_{\sigma}$ is not a multiple partial meet revision.

From Corollary 1 and Example 6 it follows that kernel operators are more general than partial meet operators. That is, every partial meet operator can be modeled by a kernel operator but some kernel operators can not be modeled by partial meet operators.

3.4 Connection with AGM revision

In order to show the interrelation of these new operators with AGM revision we will define a revision operator on belief sets from a multiple revision on belief bases. This idea is similar to the one present in Fuhrmann’s work [17], namely where contractions for the logical closures of belief bases are defined.

Definition 8 Let K be a belief base. Then $\mathbf{K} = Cn(K)$ is the *associated belief set generated from the base K* by closing K under a logical consequence operator Cn .

For a (finite) set A of sentences, let $\bigwedge A$ denote the conjunction over all elements of A .

Definition 9 Let K be a belief base and let $*$ be a prioritized multiple kernel (or partial meet) revision for K . Let A be a set of sentences, $\mathbf{K} = Cn(K)$ and $\alpha = \bigwedge A$. The *associated revision operator \odot* for \mathbf{K} is defined as:

$$\mathbf{K} \odot \alpha = \begin{cases} Cn(K * A) & \text{if } A \text{ is consistent} \\ Cn(K \cup A) & \text{otherwise} \end{cases}$$

Note that we take the belief base K and a multiple revision operator for K as the reference point, and the revision of the belief set $\mathbf{K} = Cn(K)$ as the derived technique. This is necessary to avoid the problem of well-definedness of the associated revision operator. If we take the belief set \mathbf{K} as the reference point then two different bases K, K' with $\mathbf{K} = Cn(K) = Cn(K')$ might give rise to different revision operators. That is, we define a revision for $\mathbf{K} = Cn(K)$ from K and a multiple revision operator $*$ for K . The set K is fixed and the revision of the \mathbf{K} depends on K and $*$.

Alchourrón et al. [1] have proposed a basic set of postulates for revision on belief sets. Let \mathbf{K} be a belief set, \otimes a revision operator on \mathbf{K} and α, β sentences of \mathcal{L} . The basic postulates for revision are:

- AGM-Closure $\mathbf{K} \otimes \alpha = Cn(\mathbf{K} \otimes \alpha)$.
- AGM-Success $\alpha \in \mathbf{K} \otimes \alpha$.
- AGM-Inclusion $\mathbf{K} \otimes \alpha \subseteq Cn(\mathbf{K} \cup \{\alpha\})$.
- AGM-Vacuity If $\mathbf{K} \not\vdash \neg \alpha$ then $\mathbf{K} \otimes \alpha = Cn(\mathbf{K} \cup \{\alpha\})$.

AGM-Consistency If $\not\vdash \neg\alpha$ then $\mathbf{K} \circledast \alpha \not\vdash \perp$.
 AGM-Extensionality If $\vdash \alpha \leftrightarrow \beta$ then $\mathbf{K} \circledast \alpha = \mathbf{K} \circledast \beta$.¹⁰

It is easy to show that, if \circledast is a revision operator defined as in Definition 9, then it satisfies the basic postulates for AGM revision.

Proposition 1 *Let \ast be a prioritized multiple kernel (or partial meet) revision for a belief base K . Let \circledast be a revision operator for the belief set $\mathbf{K} = Cn(K)$ defined according to Definition 9. Then \circledast satisfies AGM-closure, AGM-success, AGM-inclusion, AGM-vacuity, AGM-consistency and AGM-extensionality.*

Despite this result, we believe it is important to consider multiple revisions rather than revisions by a single sentence. Suppose that we want to revise a set K with respect to a set $A = \{p, q, r\}$. Suppose also that beliefs are informed by agents: p and q are informed by the agent Black, and r is informed by the agent White. With the passage of time, the agent White loses credibility and, therefore, the credibility of r could be decreased. If we make a revision of K with respect to $\bigwedge A = p \wedge q \wedge r$ then we would not have the possibility to distinguish the credibility of r from the credibility of p and q .

As it was pointed out in [13], in a revision general process, the new information I may come in very different shapes and forms. In the simplest scenario, I is a propositional fact. This is the scope of the basic AGM theory. But I might be much more complex: it can be equipped with a degree of plausibility or with an informant agent, or have the form of a rule, or consist of a set of such entities. For the further processing of I , it is crucial for the agent to know its origin, as this knowledge will influence decisively her willingness to adopt I . For instance, if I is based on an observation made by the agent herself, she will usually be convinced of it being true. However, if I is conveyed to her by another agent, then the agent could require some justification for I . That is one reason why we believe it is important to define multiple revisions instead of single revisions.

3.5 Connection with Konieczny & Pino Pérez Merging with Integrity Constraints

Konieczny and Pino Pérez have been intensely working on merging operators, in particular with respect to axiomatic characterizations. In [33], they propose merging operators as a generalization of AGM revision operators and set up a logical framework for belief base merging in the presence of integrity

¹⁰The name *extensionality* was proposed in [21]. A logic is extensional if it allows logically equivalent sentences to be freely substituted for each other. This postulate is also called *preservation* because logical equivalence is preserved [30, page 69].

constraints (IC) where there is no preference over the belief bases. They define IC merging operators and a set of postulates to classify three different merging operators: i) Σ operators, which resolve conflicts using *majority*, ii) *Max* operators, which are very close to the minimax rule used in decision theory, and iii) *GMax* operators, which are operators for *arbitration* with a more consensual behavior, trying to satisfy each agent’s goals as far as possible. In particular, they propose a way to build an IC merging operator from a classical revision operator. We look on this connection from the other side and explain in the following how our prioritized multiple revision operators can be seen as a particular class of IC merging operators.

In [33] the authors suggested that IC are a finite set of formulae (a belief base) and they usually denote IC as μ . Let ψ_1, \dots, ψ_n be n belief bases (not necessarily different). According to [33], a *belief set* Ψ is a multi-set consisting of n belief bases: $\Psi = \{\psi_1, \dots, \psi_n\}$. $\bigwedge \Psi$ denotes the conjunction of all formulas of all belief bases of Ψ , and the union of the belief bases of Ψ is denoted by $\sqcup \Psi$. Usually, Δ represents a merging operator.

Definition 10 Δ_μ is said to be an IC merging operator if and only if it satisfies the following basic postulates (IC7 and IC8 are not presented since they are a generalization of supplementary postulates for revision):

- (IC0) $\Delta_\mu(\Psi) \vdash \mu$.
- (IC1) If μ is consistent then $\Delta_\mu(\Psi)$ is consistent.
- (IC2) If $\bigwedge \Psi$ is consistent with μ then $\Delta_\mu(\Psi) = \bigwedge \Psi \wedge \mu$.
- (IC3) If $\Psi_1 \leftrightarrow \Psi_2$ and $\mu_1 \leftrightarrow \mu_2$ then $\Delta_{\mu_1}(\Psi_1) = \Delta_{\mu_2}(\Psi_2)$.
- (IC4) If $\varphi \vdash \mu$ and $\varphi' \vdash \mu$ then $\Delta_\mu(\varphi \sqcup \varphi') \wedge \varphi$ is consistent if and only if $\Delta_\mu(\varphi \sqcup \varphi') \wedge \varphi'$ is consistent.
- (IC5) $\Delta_\mu(\Psi_1) \wedge \Delta_\mu(\Psi_2) \vdash \Delta_\mu(\Psi_1 \sqcup \Psi_2)$.
- (IC6) If $\Delta_\mu(\Psi_1) \wedge \Delta_\mu(\Psi_2)$ is consistent then it holds that $\Delta_\mu(\Psi_1 \sqcup \Psi_2) \vdash \Delta_\mu(\Psi_1) \wedge \Delta_\mu(\Psi_2)$.

(IC0) assures that the result of the merging satisfies the integrity constraints. (IC1) states that if the integrity constraints are consistent, then the result of merging will be consistent. (IC2) establishes that if possible, the result of merging is the conjunction of the belief bases with the integrity constraints. (IC3) is the principle of irrelevance of syntax. (IC4) is the fairness postulate, stating that merging operators must not give undue preference to any of the belief bases. (IC5) and (IC6) together state that if one could find two subgroups which agree in at least one alternative, then the result of global merging will be exactly those alternatives the two groups agree on.

In our approach, if we consider A as a set of integrity constraints and a belief set consisting of just one belief base K , then (IC0) corresponds to our *success* postulate, (IC1) corresponds to *weak success*, (IC2) corresponds to *vacuity 2*. The other postulates can not be compared because they are formulated for the case in which the belief set contains two or more belief bases.

Finally, we do not compare our multiple revision operators with *iterated belief revision*, that is, the repeated application of belief revision operators. It is easy to check that, if $A = \{\alpha_1, \dots, \alpha_n\}$ is a set of logically dependent sentences, then $K * A$ is not identical to $(\dots((K * \{\alpha_1\}) * \{\alpha_2\}) * \dots) * \{\alpha_n\}$.

4 Merging Operators

Contrary to the absolute preference for new information implicit in the idea of AGM revision, we may change our beliefs in a non prioritized way. Some of the most relevant works about non-prioritized change on belief sets are: *credibility limited revision* [31], *selective revision* [14] and *screened revision* [36]; a work about non-prioritized change on belief bases is [12]. In this section, we will focus on a special kind of non prioritized change on belief bases called *merging*.

Merging is a problem that occurs in several situations, for instance, in computer science fields where it is necessary to coherently synthesize several sources of information. Merging multiple sources of information can be applied in distributed databases, multi-agent systems, expert systems, data warehousing, etc. In these situations, it is important to integrate multiple databases into a single and preferably consistent database. One of the most important applications of merging is in the integration of expert systems, where each one has a belief base representing its belief state. In some situations, belief states can be represented with a propositional language; in other cases, a subset of a first order logic (such as in logic programs) can be required.

Merging opens the possibility that new evidence is partially or even completely ignored if old information is suitably well-entrenched. The merge operation joins old and new information to a consistent whole without giving possibly undue precedence to the one or the other. The result of a merge operation is only dictated by the informational value rather than the novelty [18]. Konieczny and Pino Pérez [33] focused on belief merging from a semantical point of view. Here, we present a syntactic approach defining kernel merge and partial meet merge, in analogy to the respective notions for revision in the previous section.

4.1 Motivation and Technical Preliminaries

We will present some preliminary concepts in order to define two kinds of merging operators.

Definition 11 (Hansson [27]) Let K be a belief base and α a sentence. Then $K \perp\!\!\!\perp \alpha$ is the set such that $K' \in K \perp\!\!\!\perp \alpha$ if and only if $K' \subseteq K$, $K' \vdash \alpha$, and if

$K'' \subset K'$ then $K'' \not\vdash \alpha$. The set $K \perp\!\!\!\perp \alpha$ is called the *kernel set of K with respect to α* , and its elements are called the α -kernels of K .

So, the α -kernels of K are the minimal subsets of K that entail α . This definition is similar to the definition of prime implicants [41] with syntax dependency. The concept of α -kernel was introduced by Hansson in the definition of kernel contraction [27].

Definition 12 (Alchourr3n et al. [1]) Let K be a set of sentences and α a sentence. Then $K \perp\!\!\!\perp \alpha$ is the set of all X such that $X \in K \perp\!\!\!\perp \alpha$ if and only if $X \subseteq K$, $X \not\vdash \alpha$ and if $X \subset X' \subseteq K$ then $X' \vdash \alpha$. The set $K \perp\!\!\!\perp \alpha$ is called the *remainder set of K with respect to α* , and its elements are called the α -remainders of K .

So, the α -remainders of K are the maximal subsets of K that fail to entail α .

4.2 Kernel Merge

In order to define the operator of kernel merge we need to use an incision function. This function selects sentences to be removed from $K \cup A$ and it is called incision function because it makes an incision in every \perp -kernel.

Definition 13 Let K and A be two belief bases. A *general incision function* for K and A is a function σ ($\sigma : 2^{2^{\mathcal{L}}} \Rightarrow 2^{\mathcal{L}}$) such that for any sets $K, A \subseteq \mathcal{L}$, the following hold:

- 1) $\sigma((K \cup A) \perp\!\!\!\perp) \subseteq \bigcup((K \cup A) \perp\!\!\!\perp)$.
- 2) If $X \in (K \cup A) \perp\!\!\!\perp$ and $X \neq \emptyset$ then $(X \cap \sigma((K \cup A) \perp\!\!\!\perp)) \neq \emptyset$.

In the limit case in which $(K \cup A) \perp\!\!\!\perp = \emptyset$, we have $\sigma((K \cup A) \perp\!\!\!\perp) = \emptyset$.

Definition 14 Let K and A be belief bases and σ a general incision function. The operator \circ_σ of *kernel merge* ($\circ_\sigma : 2^{\mathcal{L}} \times 2^{\mathcal{L}} \Rightarrow 2^{\mathcal{L}}$) is defined as $K \circ_\sigma A = (K \cup A) \setminus \sigma((K \cup A) \perp\!\!\!\perp)$.

An operator \circ is a kernel merge for K and A if and only if there is a consolidated incision function σ such that for all sets K and A , $K \circ A = K \circ_\sigma A$.

Example 7 Let p, q be logically independent propositional sentences. Suppose that $K = \{p, p \rightarrow q\}$ and $A = \{p, \neg q\}$. Then, the set of minimal inconsistent subsets of $K \cup A$ is:

$$(K \cup A) \perp\!\!\!\perp = \{\{p, p \rightarrow q, \neg q\}\}$$

and, the possible results for a general incision function for K and A , and for its associated kernel merge are the following:

$$\sigma_1((K \cup A) \perp \perp) = \{p\} \text{ and } K \circ_{\sigma_1} A = \{p \rightarrow q, \neg q\}.$$

$$\sigma_2((K \cup A) \perp \perp) = \{p \rightarrow q\} \text{ and } K \circ_{\sigma_2} A = \{p, \neg q\}.$$

$$\sigma_3((K \cup A) \perp \perp) = \{\neg q\} \text{ and } K \circ_{\sigma_3} A = \{p, p \rightarrow q\}.$$

$$\sigma_4((K \cup A) \perp \perp) = \{p, p \rightarrow q\} \text{ and } K \circ_{\sigma_4} A = \{\neg q\}.$$

$$\sigma_5((K \cup A) \perp \perp) = \{p, \neg q\} \text{ and } K \circ_{\sigma_5} A = \{p \rightarrow q\}.$$

$$\sigma_6((K \cup A) \perp \perp) = \{p \rightarrow q, \neg q\} \text{ and } K \circ_{\sigma_6} A = \{p\}.$$

$$\sigma_7((K \cup A) \perp \perp) = \{p, p \rightarrow q, \neg q\} \text{ and } K \circ_{\sigma_7} A = \emptyset.$$

The mechanism of this operator is to join K and A and then eliminate from the result all possible inconsistencies by means of an incision function that makes a “cut” over each minimally inconsistent subset of $K \cup A$. Since this operator uses an incision function and the set of \perp -kernels, we call it kernel merge.

Theorem 3 *Let K and A be two belief bases. The operator \circ_{σ} is a kernel merge of K and A if and only if it satisfies inclusion, strong consistency, global core-retainment and reversion.*

4.3 Partial Meet Merge

In order to define a partial meet version of a merge operator, we need a general selection function.

Definition 15 A general selection function for \mathcal{L} is a function γ ($\gamma : \mathbf{2}^{2^{\mathcal{L}}} \Rightarrow \mathbf{2}^{2^{\mathcal{L}}}$) such that for any $X \subseteq \mathcal{L}$, it holds that:

- 1) $\gamma(X \perp \perp) \subseteq X \perp \perp$.
- 2) $\gamma(X \perp \perp) \neq \emptyset$.

Since every set $X \subseteq \mathcal{L}$ contains a consistent subset, $X \perp \perp$ is always non-empty.

Definition 16 Let γ be a general selection function. Then γ is an *equitable selection function* if, for any $X, X' \subseteq \mathcal{L}$, $X \perp \perp = X' \perp \perp$ implies that $X \setminus \bigcap \gamma(X \perp \perp) = X' \setminus \bigcap \gamma(X' \perp \perp)$.

The intuition behind this definition is that, if the set of minimally inconsistent subsets of $K \cup A$ is equal to the set of minimal inconsistent subsets of $K' \cup A'$ then α is erased in the selection of \perp -remainders of $K \cup A$ if and only if it is erased in the selection of \perp -remainders of $K' \cup A'$ [12].

Example 8 Let p, q, r, s, t and u be logically independent sentences. For example, let $K = \{p, q, \neg r\}$, $A = \{\neg q, r, s, t\}$ and $B = \{\neg q, r, u\}$. Then:

$$(K \cup A)\underline{\perp}\underline{\perp} = \{\{p, q, r, s, t\}, \{p, \neg q, r, s, t\}, \{p, q, \neg r, s, t\}, \{p, \neg q, \neg r, s, t\}\}$$

$$(K \cup B)\underline{\perp}\underline{\perp} = \{\{p, q, r, u\}, \{p, \neg q, r, u\}, \{p, q, \neg r, u\}, \{p, \neg q, \neg r, u\}\}$$

We have that $(K \cup A)\underline{\perp}\underline{\perp} = (K \cup B)\underline{\perp}\underline{\perp} = \{\{q, \neg q\}, \{r, \neg r\}\}$. Suppose that $\gamma((K \cup A)\underline{\perp}\underline{\perp}) = \{\{p, q, r, s, t\}, \{p, \neg q, r, s, t\}\}$. Then:

$$(K \cup A) \setminus \bigcap \gamma((K \cup A)\underline{\perp}\underline{\perp}) = \{q, \neg r, \neg q\}$$

Hence, if γ is an equitable selection function then $\gamma((K \cup B)\underline{\perp}\underline{\perp})$ must be equal to $\{\{p, q, r, u\}, \{p, \neg q, r, u\}\}$ (given that $\gamma((K \cup B)\underline{\perp}\underline{\perp})$ must be such that $(K \cup B) \setminus \bigcap \gamma((K \cup B)\underline{\perp}\underline{\perp}) = \{q, \neg r, \neg q\}$).

Definition 17 Let K and A be two belief bases and γ an equitable selection function. The operator \circ_γ of *partial meet merge* ($\circ_\gamma : 2^{\mathcal{L}} \times 2^{\mathcal{L}} \Rightarrow 2^{\mathcal{L}}$) is defined as $K \circ_\gamma A = \bigcap \gamma((K \cup A)\underline{\perp}\underline{\perp})$.

An operator \circ is a partial meet merge for K and A if and only if there is an equitable selection function γ such that for all sets K and A , $K \circ A = K \circ_\gamma A$.

The mechanism of this operator is to join K and A and then eliminate from the result all possible inconsistencies by means of an equitable selection function that makes a choice among the maximally consistent subsets of $K \cup A$ and intersect them. Since this operator uses a selection function and the remainder set, we call it partial meet merge.

Example 9 As in Example 7, let p, q be logically independent propositional sentences, $K = \{p, p \rightarrow q\}$ and $A = \{p, \neg q\}$. Then, the set of maximal consistent subsets of $K \cup A$ is:

$$(K \cup A)\underline{\perp}\underline{\perp} = \{\{p, p \rightarrow q\}, \{p, \neg q\}, \{p \rightarrow q, \neg q\}\}$$

and, the possible results for an equitable selection function and for its associated partial meet merge for K and A are the following:

$$\gamma_1((K \cup A)\underline{\perp}\underline{\perp}) = \{\{p, p \rightarrow q\}\} \text{ and } K \circ_{\gamma_1} A = \{p, p \rightarrow q\}.$$

$$\gamma_2((K \cup A)\underline{\perp}\underline{\perp}) = \{\{p, \neg q\}\} \text{ and } K \circ_{\gamma_2} A = \{p, \neg q\}.$$

$$\gamma_3((K \cup A)\underline{\perp}\underline{\perp}) = \{\{p \rightarrow q, \neg q\}\} \text{ and } K \circ_{\gamma_3} A = \{p \rightarrow q, \neg q\}.$$

$$\gamma_4((K \cup A)\underline{\perp}\underline{\perp}) = \{\{p, p \rightarrow q\}, \{p, \neg q\}\} \text{ and } K \circ_{\gamma_4} A = \{p\}.$$

$$\gamma_5((K \cup A)\underline{\perp}\underline{\perp}) = \{\{p, p \rightarrow q\}, \{p \rightarrow q, \neg q\}\} \text{ and } K \circ_{\gamma_5} A = \{p \rightarrow q\}.$$

$$\gamma_6((K \cup A)\underline{\perp}\underline{\perp}) = \{\{p, \neg q\}, \{p \rightarrow q, \neg q\}\} \text{ and } K \circ_{\gamma_6} A = \{\neg q\}.$$

$$\gamma_7((K \cup A)\underline{\perp}\underline{\perp}) = \{\{p, p \rightarrow q\}, \{p, \neg q\}, \{p \rightarrow q, \neg q\}\} \text{ and } K \circ_{\gamma_7} A = \emptyset.$$

Fuhrmann presented the following representation theorem, including the postulate of *congruence*, for this kind of operators.

Theorem 4 (Fuhrmann [18]) *Let K and A be two belief bases. The operator \circ_γ is a partial meet merge of K and A if and only if it satisfies inclusion, strong consistency, global relevance and congruence.*

The next result consists of an alternative axiomatic characterization which makes use of *reversion* (cf. item ‘f’ of Lemma 1).

Theorem 5 *Let K and A be two belief bases. The operator \circ_γ is a partial meet merge of K and A if and only if it satisfies inclusion, strong consistency, global relevance and reversion.*

Since, according to Lemma 1, if global relevance is satisfied then global core-retainment is satisfied, it follows immediately from Theorems 3 and 5 that every partial meet merge is a kernel merge. On the other hand, by looking at the constructions, is easy to conclude that every kernel (or partial meet) merge satisfies *vacuity* and *symmetry*. These facts are formally stated in the following corollary.

Corollary 2 *Let K and A be two belief bases:*

- a) *If \circ is a partial meet merge for K and H then \circ is a kernel merge for K and H .*
- b) *If \circ is a kernel merge for K and H then it satisfies vacuity and symmetry.*

More relations between partial meet and kernel operators can be found in [10].

4.4 Connection with Other Merging Operators

Konieczny and Pino Pérez proposed merging operators without integrity constraints, called *pure merging operators* [32]. With the same preliminaries as in Section 3.5, the operator Δ merges a belief set Ψ to a belief base $\Delta(\Psi)$. Δ is said to be a pure belief merging operator if and only if it satisfies the following postulates:

- (A1) $\Delta(\Psi)$ is consistent.
- (A2) If $\bigwedge \Psi$ is consistent then $\Delta(\Psi) = \bigwedge \Psi$.
- (A3) If $\Psi_1 \leftrightarrow \Psi_2$ then $\Delta(\Psi_1) = \Delta(\Psi_2)$.
- (A4) If $\psi_1 \wedge \psi_2$ is not consistent then $\Delta(\psi_1 \sqcup \psi_2) \not\vdash \psi_1$.
- (A5) $\Delta(\Psi_1) \wedge \Delta(\Psi_2) \vdash \Delta(\Psi_1 \sqcup \Psi_2)$.
- (A6) If $\Delta(\Psi_1) \wedge \Delta(\Psi_2)$ is consistent then $\Delta(\Psi_1 \sqcup \Psi_2) \vdash \Delta(\Psi_1) \wedge \Delta(\Psi_2)$.

(A1) and (A2) correspond to our postulates *strong consistency* and *vacuity* respectively. (A3) is the principle of the irrelevance of syntax, i.e., if two belief sets are equivalent then the belief bases resulting from merging each of the

two sets will be logically equivalent; such a postulate is not present in our framework in which the result of merging depends on the belief bases' syntax. (A4) establishes that, if two belief bases are inconsistent then some belief of each of them will be discarded in their merging; this postulate can be approximated by *global core-retainment* and *global relevance*. (A5) and (A6) state that if one can find two subgroups which agree in at least one alternative, then the result of global merging will be those alternatives the two groups agree on. These last two postulates are not present in our framework.

Konieczny and Pino Pérez further observed that there are syntactically merging operators which do not satisfy the principle of irrelevance of syntax (AC3), similar to our approach of merging and the partial meet merge proposed in [18]. However, we proposed an alternative construction, called kernel merge, giving an axiomatic characterization of it and an alternative axiomatic characterization of partial meet merge.

Finally, since our merging operator satisfies congruence and, therefore, also satisfies symmetry (see Lemma 1), we can conclude that kernel and partial meet merge are commutative operators, as are the *arbitration operators* (different from arbitration operators formulated by Konieczny and Pino Pérez) presented in [34, 35]. A deep comparison with arbitration operators of Liberatore and Schaerf is not presented here because it has been show that these operators can be characterized by IC merging operators [33].

4.5 Connection with Consolidation

Hansson proposed an operator, called *consolidation*, to make a belief base consistent [24]. A possible way to perform a consolidation it is to contract by *falsum* (contradiction). Let K be a belief base and $!$ be a consolidation operator for K . The following postulates have been proposed for consolidation [24, 28]:

- (C-Consistency) $K!$ is consistent.
- (C-Inclusion) $K! \subseteq K$.
- (C-Core-Retainment) If $\alpha \in (K \setminus K!)$ then there is some K' with $K' \subseteq K$ such that K' is consistent but $K' \cup \{\alpha\}$ is inconsistent.
- (C-Relevance) If $\alpha \in (K \setminus K!)$ then there is some K' with $K! \subseteq K' \subseteq K$ such that K' is consistent but $K' \cup \{\alpha\}$ is inconsistent.

C-Consistency states that a consolidated belief base must be consistent. C-Inclusion determines that nothing is added in a consolidation process. C-Core-Retainment and C-Relevance state that if some belief is excluded when a belief base is consolidated, then it must have contributed to making the original belief base inconsistent.

In a later work, Hansson identified two kind of consolidation operators: kernel consolidations and partial meet consolidations. In particular, kernel consolidations are based on the following observation: a subset of K implies

a sentence α if and only if it contains some minimal α -implying subset of K . In order to remove α from K , it is necessary and sufficient to remove at least one element from each minimal α -implying subset of K . A kernel consolidation of K removes at least one sentence of every minimal inconsistent subset of K .

Definition 18 An operation $!$ is a *kernel consolidation* for K if there exists an incision function σ for K such that:

$$K! = K \setminus \sigma(K \perp \perp)$$

Theorem 6 (Hansson [28]) *Let K be a belief base. An operation $!$ is a kernel consolidation for K if and only if it satisfies C-consistency, C-inclusion and C-core-retainment.*

Partial meet consolidations are based on a partial meet contraction by *falsum*. The partial meet consolidation of K is the intersection of the most preferred maximal consistent subsets of K .

Definition 19 An operation $!$ is a *partial meet consolidation* for K if there exists a selection function γ for K such that:

$$K! = \bigcap \gamma(K \perp \perp)$$

Theorem 7 (Hansson [28]) *Let K be a belief base. An operation $!$ is a partial meet consolidation for K if and only if it satisfies C-consistency, C-inclusion and C-relevance.*

We want to characterize a particular class of partial meet consolidations based on equitable selection functions.

Definition 20 An operation $!$ is an *equitable partial meet consolidation* for K if there exists an equitable selection function γ such that:

$$K! = \bigcap \gamma(K \perp \perp)$$

Fuhrmann proposed a way to define merge from consolidation and *vice versa* [18]. However, he proposed the definition only for partial meet merge and partial meet consolidation. Here we extend the proposal for the case of kernel merge and kernel consolidation.

Definition 21 Let K_1, K_2 be sets of sentences and $!$ be a consolidation operator for $K_1 \cup K_2$. Then, the *merge* of K_1 and K_2 , noted by $K_1 \circ K_2$, is defined as:

$$K_1 \circ K_2 = (K_1 \cup K_2)!$$

Having in mind Definitions 14, 17, 18 and 20 it is easy to show that if $!$ is a kernel (equitable partial meet) consolidation operator then \circ , defined according to Definition 21, is a kernel (partial meet) merge operator.

Proposition 2 *Let K_1, K_2 be sets of sentences and $!$ be a consolidation operator for $K_1 \cup K_2$. Let \circ be the merge of K_1 and K_2 defined according to Definition 21. Then the following statements hold:*

1. *If $!$ is a kernel consolidation then \circ is a kernel merge.*
2. *If $!$ is an equitable partial meet consolidation then \circ is a partial meet merge.*

Definition 22 *Let K be a set of sentences and \circ be a merge operator for K and K . Then, the consolidation of K , noted by $K!$, is defined as:*

$$K! = K \circ K$$

Now, taking into account Definitions 14, 17, 18 and 19 it is easy to show that if \circ is a kernel (partial meet) merge operator then $!$, defined according to Definition 22, is a kernel (partial meet) consolidation operator.

Proposition 3 *Let K be a set of sentences and \circ be a merge operator for K and K . Let $!$ be the consolidation of K defined according to Definition 22. Then the following statements hold:*

1. *If \circ is a kernel merge then $!$ is a kernel consolidation.*
2. *If \circ is a partial meet merge then $!$ is a partial meet consolidation.*

5 Related Works

There are some other relevant works related to multiple change. We begin by recalling works on multiple revision. Peppas [40] defines a multiple revision operator with respect to a (possibly infinite) set of sentences using Grove's sphere system. Contrary to our approach focusing on sets of sentences, Peppas proposed ways to reduce multiple revision to classical AGM sentence revision, devising a particular condition which is shown to be necessary and sufficient for such a reduction.

Revision and expansion operators of logic programs by answer set semantics have been proposed in [7]. They gave properties for their operators also considering their complexity. Expansion is characterized in terms of intersections of models, and revision is characterized in terms of minimal distance between models. Here, we consider multiple revision in a more general classical-logical environment.

Furthermore, and at first sight less related to our work, some work has been done on multiple contraction. Niederée [38] introduced the concept of multiple or simultaneous contraction, showing that recovery cannot be satisfied in such cases. van der Hoek and de Rijke [43] study an approach to *concurrent*

contraction. They assume the existence of a set of agents simultaneously contracting their beliefs. The epistemic input is a vector of sentences that represents the sentences that should be contracted from each agent's epistemic state. Zhang and Foo [45] extend the AGM theory to deal with *infinitary belief change*, representing the epistemic states by belief sets. They generalize both axiomatization and modeling of the AGM theory by extending to the case in which the epistemic input is a set of sentences. They show that most properties of AGM model are inherited.

In [19] two ways of multiple partial meet contraction are defined: *package contraction*, where all the sentences of the epistemic input must be removed from a belief set, and *choice contraction*, where it is sufficient that at least one of the sentences is removed. Fermé et al. [15] proposed *kernel* versions of package and choice contractions for belief bases. Comparing the results of these last two papers and our definitions of multiple kernel revision (Definition 5) and of multiple partial meet revision (Definition 7) it seems that multiple partial meet revision could be defined from partial meet package contraction [19], and similarly, multiple kernel revision from kernel package contraction [15] via a Generalized Levi identity. However, the confirmation of this conjecture requests for a more detailed investigation which we leave as a subject of future work.

Suppose that \div is a package contraction (partial meet or kernel) for a belief base K . Given a set $A = \{\alpha_1, \dots, \alpha_n\}$, then we can define $\neg A = \{\neg\alpha_1, \dots, \neg\alpha_n\}$. Then we can define a prioritized multiple revision for K from the package contraction by $K * A = (K \div \neg A) \cup A$. Moreover, since choice contraction allows to select the beliefs to be discarded, we could define a prioritized multiple revision for K from a choice contraction giving place to a *multiple selective revision* operator. This operator, similar to *selective revision* by a single sentence [14], is an operator that can accept a subset of the new information. However, it should be noted that the notions of package remainders and package kernels defined in those papers do not correspond to our definitions of A -consistent remainders and A -inconsistent kernels, so a more thorough investigation has to be done.

The significance of our work can be summarized in the following items (one concerning operators of belief revision and the other operators of belief merge):

- We have defined two new kinds of prioritized multiple revision operators on belief bases and obtained axiomatic characterizations for them: prioritized multiple kernel revision and prioritized multiple partial meet revision. These operators of multiple change are based in two new concepts: A -inconsistent-kernels and A -consistent-remainders, respectively. Thus, we have presented a way of constructing multiple revision operators without needing to make use of the Generalized Levi's identity nor of a previously defined multiple contraction operator.
- We have proposed an alternative axiomatization for partial meet merge (which was originally proposed and axiomatically characterized by

Fuhrmann) and, additionally, we have defined a new kind of merge operators called kernel merge which constitute a more general class than that of partial meet merge. Also for such operators, we presented representation theorems to characterize them by axioms.

6 Conclusions and Future Work

In this paper we presented different approaches to multiple change. We proposed a set of postulates and constructed two main families of change operators: prioritized multiple change operators, and merge or symmetric change operators. In either case, we presented different constructions via kernel change and partial meet change techniques. For prioritized changes, we found axiomatic characterizations for multiple kernel revision and multiple partial meet revision.

One of the contribution of our work is the construction of the prioritized revision operators. They are based on minimal inconsistent kernels (for prioritized multiple kernel revision) and maximal consistent remainders (for prioritized multiple partial meet revision). We proposed a set of postulates and some relations among them.

Analogously, we proved representation theorems for kernel merge and partial meet merge. From the representation theorems, its clear that each partial meet operator is a kernel operator. As part of our future work, we will study selective multiple revision, that is, revisions in which only a subset of the epistemic input is accepted. Also, we want to define multiple semi-revision operators from merge operators. Finally, we want to investigate the interrelations between operators of prioritized multiple partial meet revision and of prioritized multiple kernel revision, on the one hand side, multiple partial meet contraction [19] and multiple kernel contraction [15], on the other.

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Appendix

Proof of Lemma 1

The items (a), (b), (c) and (g) are trivially shown. We will prove the items (d), (e) and (f).

(d) **Uniformity 1 implies Uniformity 2:** We must show that:

$$\text{if } K \setminus (K * A) = K \setminus (K * B) \text{ then } K \cap (K * A) = K \cap (K * B)$$

By set properties, it holds that:

$$K = (K \cap (K * A)) \cup (K \setminus (K * A))$$

where $(K \cap (K * A)) \cap (K \setminus (K * A)) = \emptyset$. Similarly, it holds that:

$$K = (K \cap (K * B)) \cup (K \setminus (K * B))$$

where $(K \cap (K * B)) \cap (K \setminus (K * B)) = \emptyset$. Therefore, we can conclude that $K \cap (K * A) = K \cap (K * B)$.

Uniformity 2 implies Uniformity 1: We must show that:

$$\text{if } K \cap (K * A) = K \cap (K * B) \text{ then } K \setminus (K * A) = K \setminus (K * B)$$

Using set properties, the proof is similar to the above case.

- (e) Let \circ be an operator that satisfies **inclusion**, **weak success** and **core-retainment**, and assume that $K \cup A \not\perp$. We must show that $K * A = K \cup A$.
 - (\subseteq) By **inclusion** it holds that $K * A \subseteq K \cup A$.
 - (\supseteq) Since $K \cup A \not\perp$ then A is consistent and by **weak success** it holds that $A \subseteq K * A$. It remains to show that $K \subseteq K * A$. By *reductio ad absurdum*, suppose that $K \not\subseteq K * A$. Therefore, there is some $\alpha \in K \setminus K * A$. Due to **core-retainment**, there is a set C such that $C \subseteq (K \cup A)$, C is consistent with A but $C \cup \{\alpha\}$ is inconsistent with A . But this is absurd because $K \cup A \not\perp$. Therefore $K \subseteq K * A$ and $K \cup A \subseteq K * A$.
- (f) Let \circ be an operator that satisfies **reversion** and **inclusion**, and assume that $K \cup A = H \cup B$. Due to reversion, $(K \cup A) \setminus (K \circ A)$ is equal to $(H \cup B) \setminus (H \circ B)$. Therefore, by **inclusion** it holds that $K \circ A = H \circ B$ and **congruence** is satisfied.

Proof of Observation 1

Let K, A be sets of sentences, A be consistent. Suppose that $\alpha \in K$ and $\alpha \in A$. We must show that $\alpha \notin \bigcup(K \perp \perp A)$.

By *reductio ad absurdum*, suppose that $\alpha \in \bigcup(K \perp \perp A)$. Then, there is some $X \in K \perp \perp A$ such that $\alpha \in X$. By Definition 2 then:

- (1) $X \subseteq K$.
- (2) $X \cup A$ is inconsistent.
- (3) For any $X' \subset X \subseteq K$ then $X' \cup A$ is consistent.

But this is absurd. Indeed, condition (3) is not satisfied because $X'' = X \setminus \{\alpha\}$ is such that $X'' \cup A$ is inconsistent due to (2) and the fact that $X'' \cup A$ is equal to $X \cup A$. Therefore, $\alpha \notin \bigcup(K \perp \perp A)$.

Proof of Observation 2

Let K, A be sets of sentences, A be consistent. Suppose that $\alpha \in K$ and $\alpha \in A$. We must show that $\alpha \in \bigcap(K \perp_{\top} A)$.

By *reductio ad absurdum*, suppose that $\alpha \notin \bigcap(K \perp_{\top} A)$. Then, there is some $X \in K \perp_{\top} A$ such that $\alpha \notin X$. By Definition 3 then:

- (1) $X \subseteq K$.
- (2) $X \cup A$ is consistent.
- (3) For any $X \subset X' \subseteq K$ then $X' \cup A$ is inconsistent.

But this is absurd. Indeed, condition (3) is not satisfied because $X'' = X \cup \{\alpha\}$ is such that $X'' \cup A$ is consistent due to (2) and the fact that $X'' \cup A$ is equal to $X \cup A$. Therefore, $\alpha \in \bigcap(K \perp_{\top} A)$.

Proof of Theorem 1

- *Construction to Postulates.* Let σ be an arbitrary consolidated incision function for K and $*_{\sigma}$ be the prioritized multiple kernel revision on K that is generated by σ . Then, for all sets A :

$$K *_{\sigma} A = \begin{cases} (K \setminus \sigma(K \perp_{\perp} A)) \cup A & \text{if } A \text{ is consistent} \\ K & \text{otherwise} \end{cases}$$

We must show that $*_{\sigma}$ satisfies all the enumerated postulates.

- Inclusion Trivial by definition.
- Consistency Suppose that A is consistent. Since $\sigma(K \perp_{\perp} A)$ cuts every subset of K inconsistent with A then $K \setminus \sigma(K \perp_{\perp} A)$ is consistent with A . Therefore, $K \setminus \sigma(K \perp_{\perp} A) \cup A$ is consistent.
- Weak Success Suppose that A is consistent. Then it is trivial by definition that $A \subseteq K *_{\sigma} A$.
- Vacuity 1 Trivial by definition.
- Uniformity 1 Given A and B two consistent sets, suppose that for all subset X of K , $(X \cup A) \vdash \perp$ if and only if $(X \cup B) \vdash \perp$. Then $K \perp_{\perp} A = K \perp_{\perp} B$ and since σ is a well defined function then $\sigma(K \perp_{\perp} A) = \sigma(K \perp_{\perp} B)$. Therefore, $K \setminus (K *_{\sigma} A) = K \setminus (K *_{\sigma} B)$.
- Core-Retainment Let $\beta \in K$ and $\beta \notin K *_{\sigma} A$. Then $K *_{\sigma} A \neq K$ and, from the definition of $*_{\sigma}$, it follows that:

$$K *_{\sigma} A = (K \setminus \sigma(K \perp_{\perp} A)) \cup A$$

Therefore, from $\beta \notin (K \setminus \sigma(K \perp_{\perp} A)) \cup A$ and $\beta \in K$, we can conclude that $\beta \in \sigma(K \perp_{\perp} A)$. By definition

$\sigma(K \perp\!\!\!\perp A) \subseteq \bigcup K \perp\!\!\!\perp A$, and it follows that there is some $X \in K \perp\!\!\!\perp A$ such that $\beta \in X$. X is a minimal K -subset inconsistent with A . Let $Y = X \setminus \{\beta\}$. Then Y is such that $Y \subset X \subseteq K \subseteq K \cup A$, Y is consistent with A but $Y \cup \{\beta\}$ is inconsistent with A .

- *Postulates to Construction.* Let K be a set of sentences and $*$ be an operator on K that satisfies the enumerated postulates. We must show that $*$ is a prioritized multiple kernel revision. Let σ be a function such that, for every set K and for every consistent set A , it holds that:

$$\sigma(K \perp\!\!\!\perp A) = \{\alpha : \alpha \in (K \setminus (K * A))\}$$

We must show:

Part A.

1. σ is a well defined function. Given A and B two consistent sets, suppose that $K \perp\!\!\!\perp A = K \perp\!\!\!\perp B$. It follows from **uniformity 1** that $K \setminus (K * A) = K \setminus (K * B)$. Therefore:

$$\sigma(K \perp\!\!\!\perp A) = \sigma(K \perp\!\!\!\perp B)$$

2. $\sigma(K \perp\!\!\!\perp A) \subseteq \bigcup K \perp\!\!\!\perp A$. Let $\alpha \in \sigma(K \perp\!\!\!\perp A)$. Then $\alpha \in K \setminus K * A$. Due to **core-retainment** there is some C such that $C \subseteq (K \cup A)$, $C \cup A \not\vdash \perp$ but $C \cup A \cup \{\alpha\} \vdash \perp$. Now let X be an arbitrary element of $C \cup \{\alpha\} \perp\!\!\!\perp A$ (notice that it follows from $C \cup \{\alpha\} \cup A \vdash \perp$ and $C \cup A \not\vdash \perp$ that $C \cup \{\alpha\} \perp\!\!\!\perp A \neq \emptyset$). Then X is such that $X \subseteq C \cup \{\alpha\}$, $X \cup A \vdash \perp$ and, for any X' such that $X' \subset X$, it holds that $X' \cup A \not\vdash \perp$. From all the above facts we can conclude that $\alpha \in X$ and that $X \cap A = \emptyset$. Hence, combining the latter equality with $X \subseteq C \cup \{\alpha\}$ and $C \subseteq (K \cup A)$ we obtain that $X \subseteq K$. Therefore, $X \in K \perp\!\!\!\perp A$ and, since $\alpha \in X$, we can conclude that $\alpha \in \bigcup (K \perp\!\!\!\perp A)$.
3. If $X \in K \perp\!\!\!\perp A$ then $X \cap (\sigma(K \perp\!\!\!\perp A)) \neq \emptyset$. Let $X \in K \perp\!\!\!\perp A$. We need to show that $X \cap \sigma(K \perp\!\!\!\perp A) \neq \emptyset$. Suppose that A is consistent. By **consistency** then $K * A$ is consistent. Since X is inconsistent with A then $X \not\subseteq K * A$ by **weak success**. This means that there is some $\beta \in X$ and $\beta \notin K * A$. Therefore, $\beta \in \sigma(K \perp\!\!\!\perp A)$.

Part B. $*_{\sigma}$ is equal to $*$, that is, $K *_{\sigma} A = K * A$, for any set A . **Vacuity 1** covers the limit case in which A is inconsistent. Suppose that A is consistent.

- (\supseteq) Let $\alpha \in K * A$. It follows from the definition of $\sigma(K \perp\!\!\!\perp A)$ that $\alpha \notin \sigma(K \perp\!\!\!\perp A)$. By **inclusion** it holds that $K * A \subseteq K \cup A$ and $\alpha \in K \cup A$. Therefore, by definition of $*_{\sigma}$, $\alpha \in K *_{\sigma} A$.
- (\subseteq) Let $\alpha \notin K * A$. We must show that $\alpha \notin K *_{\sigma} A$. By **weak success** we have that $\alpha \notin A$. Since by definition $K *_{\sigma} A$ is equal to

$(K \setminus \sigma(K \perp_{\perp} A)) \cup A$ then it only remains to show that $\alpha \notin K \setminus \sigma(K \perp_{\perp} A)$. To do that we consider two cases:

- $\alpha \notin K$. Then it follows trivially that $\alpha \notin K \setminus \sigma(K \perp_{\perp} A)$.
- $\alpha \in K$. Then it follows from the definition of $\sigma(K \perp_{\perp} A)$ that $\alpha \in \sigma(K \perp_{\perp} A)$ and, therefore, $\alpha \notin K \setminus \sigma(K \perp_{\perp} A)$.

Proof of Theorem 2

- *Construction to Postulates.* Let γ be an arbitrary consolidated selection function for K and $*_{\gamma}$ be the prioritized multiple partial meet revision on K that is generated by γ . Then, for all sets A :

$$K *_{\gamma} A = \begin{cases} \bigcap \gamma(K \perp_{\top} A) \cup A & \text{if } A \text{ is consistent} \\ K & \text{otherwise} \end{cases}$$

We must show that $*_{\gamma}$ satisfies all the enumerated postulates.

- Inclusion** Since every $X \in K \perp_{\top} A$ is such that $X \subseteq K$ then this postulate is trivially shown.
- Consistency** Suppose that A is consistent. Then $K \perp_{\top} A \neq \emptyset$ and by definition, every $X \in K \perp_{\top} A$ is consistent with A . Therefore, the intersection of any subset of $K \perp_{\top} A$ is consistent with A . Finally, $K *_{\gamma} A$ is consistent and we are done.
- Weak Success** Suppose that A is consistent. Then it is trivial by definition that $A \subseteq K *_{\gamma} A$.
- Vacuity 1** Trivial by definition.
- Uniformity 2** Given two consistent sets A and B , suppose that for all subsets X of K , $(X \cup A) \vdash \perp$ if and only if $(X \cup B) \vdash \perp$. We must show that $K \cap (\bigcap \gamma(K \perp_{\top} A) \cup A) = K \cap (\bigcap \gamma(K \perp_{\top} B) \cup B)$. We start by observing that:

$$\begin{aligned} & K \cap \left(\bigcap \gamma(K \perp_{\top} A) \cup A \right) \\ &= \left(K \cap \left(\bigcap \gamma(K \perp_{\top} A) \right) \right) \cup (K \cap A) \\ &= \left(\bigcap \gamma(K \perp_{\top} A) \right) \cup (K \cap A) \\ &= \bigcap \gamma(K \perp_{\top} A) \end{aligned}$$

where the last equality above is due to the fact that $K \cap A \subseteq X$ for all $X \in K \perp_{\top} A$ (see Observation 2) and, consequently, $K \cap A \subseteq \bigcap \gamma(K \perp_{\top} A)$. Analogously:

$$K \cap \left(\bigcap \gamma(K \perp_{\top} B) \cup B \right) = \bigcap \gamma(K \perp_{\top} B).$$

Since γ is a well defined function then $\gamma(K \perp_{\top} A) = \gamma(K \perp_{\top} B)$ (notice that $K \perp_{\top} A = K \perp_{\top} B$ follows from our assumption that for all subsets X of K , $(X \cup A) \vdash \perp$

if and only if $(X \cup B) \vdash \perp$. Therefore, $\bigcap \gamma(K \perp_{\top} A) = \bigcap \gamma(K \perp_{\top} B)$. Hence:

$$K \cap \left(\bigcap \gamma(K \perp_{\top} A) \cup A \right) = K \cap \left(\bigcap \gamma(K \perp_{\top} B) \cup B \right)$$

Relevance Suppose that $K \perp_{\top} A \neq \emptyset$. Let $\beta \in K$ and $\beta \notin K *_{\gamma} A$. Then there is some $X \in K \perp_{\top} A$ such that $\beta \notin X$. Therefore, there is some X such that $\beta \notin X \subseteq K$, $X \cup A$ is consistent but $X \cup A \cup \{\beta\}$ is inconsistent and we are done.

Suppose that $K \perp_{\top} A = \emptyset$ in which case A is inconsistent. By definition, $K *_{\gamma} A = K$ and relevance is vacuously satisfied.

- *Postulates to Construction.* Let K be a set of sentences and $*$ be an operator on K that satisfies the enumerated postulates. We must show that $*$ is a prioritized multiple partial meet revision. Let γ be a function such that, for every set A , it holds that:

- $\gamma(K \perp_{\top} A) = \{K\}$ whenever $K \perp_{\top} A = \emptyset$.
- $\gamma(K \perp_{\top} A) = \{X \in K \perp_{\top} A : (K \cap (K * A)) \subseteq X\}$ whenever $K \perp_{\top} A \neq \emptyset$.

We must show:

Part A.

1. γ is a well defined function. Suppose that $K \perp_{\top} A = K \perp_{\top} B$ for A and B two consistent sets. It follows from **uniformity 2** that $K \cap (K * A) = K \cap (K * B)$. Hence $\gamma(K \perp_{\top} A) = \gamma(K \perp_{\top} B)$.
2. If $K \perp_{\top} A = \emptyset$ we must show that $\gamma(K \perp_{\top} A) = \{K\}$. It follows from the definition of γ .
3. If $K \perp_{\top} A \neq \emptyset$ we must show that $\emptyset \neq \gamma(K \perp_{\top} A) \subseteq K \perp_{\top} A$. It follows from the definition of γ that $\gamma(K \perp_{\top} A) \subseteq K \perp_{\top} A$. Now we prove that $\gamma(K \perp_{\top} A) \neq \emptyset$. By **weak success** it holds that $A \subseteq K * A$ and, consequently $(K \cap (K * A)) \cup A \subseteq K * A$. On the other hand, it follows from **consistency** that $K * A \not\vdash \perp$ and, therefore, we can conclude that $(K \cap K * A) \cup A \not\vdash \perp$. Hence, there must be some set $X \in K \perp_{\top} A$ such that $K \cap K * A \subseteq X$. Thus $\gamma(K \perp_{\top} A) \neq \emptyset$.

Part B. $*_{\gamma}$ is equal to $*$, that is, $K *_{\gamma} A = K * A$, for any set A . **Vacuity 1** covers the limit case in which A is inconsistent. Suppose that A is consistent.

(\supseteq) Let $\alpha \in K * A$. We must show that $\alpha \in K *_{\gamma} A$. We have two cases:

- $\alpha \in A$ then it is trivial that $\alpha \in K *_{\gamma} A$.
- $\alpha \notin A$. By **inclusion** it holds that $K * A \subseteq K \cup A$ and we can conclude that $\alpha \in K$. Therefore, $\alpha \in K \cap K * A$. Hence, it follows from the definition of γ that $\alpha \in X$ for all $X \in \gamma(K \perp_{\top} A)$ and we are done.

- (\subseteq) Let $\alpha \notin K * A$. We must show that $\alpha \notin K *_{\gamma} A$. If $\alpha \notin K * A$ we need to find some $X \in \gamma(K \perp_{\top} A)$ such that $\alpha \notin X$. We have two cases:
 - $\alpha \in K$. By **relevance** there exists some C such that $K * A \subseteq C \subseteq (K \cup A)$, $C \cup A \not\vdash \perp$ and $C \cup A \cup \{\alpha\} \vdash \perp$. Then we may infer that $\alpha \notin A$ and $\alpha \notin C$. Let $H = K \cap C$. Then $\alpha \notin H$, $(K \cap (K * A)) \subseteq H \subseteq K$, $H \cup A \not\vdash \perp$ and $H \cup A \cup \{\alpha\} \vdash \perp$. Then we may extend H to a maximal subset X of K consistent with A such that $(K \cap (K * A)) \subseteq X$ and $\alpha \notin X$. Hence, $X \in \gamma(K \perp_{\top} A)$ and $\alpha \notin X$.
 - $\alpha \notin K$. Then no set in $K \perp_{\top} A$ will contain α .

Proof of Proposition 1

Let K be a belief base and $*$ be either a prioritized multiple kernel revision or a prioritized multiple partial meet revision for K . It follows from Theorems 1 and 2 and Lemma 1, items (a) and (d), that $*$ satisfies *inclusion*, *consistency*, *weak success*, *vacuity 1*, *uniformity 1*, *uniformity 2* and *core-retainment*. Let $\mathbf{K} = Cn(K)$ and \otimes be a revision operator for \mathbf{K} defined, for any sentence α , by:

$$\mathbf{K} \otimes \alpha = \begin{cases} Cn(K * A) & \text{if } A \text{ is consistent} \\ Cn(K \cup A) & \text{otherwise} \end{cases}$$

where A is any set of sentences such that $\alpha = \bigwedge A$.

- AGM-Closure Straightforward from definition.
- AGM-Success Straightforward from definition and **-weak success*.
- AGM-Inclusion Straightforward from definition and **-inclusion*.
- AGM-Vacuity If $\mathbf{K} \not\vdash \neg\alpha$ then $\mathbf{K} \cup \{\alpha\} \not\vdash \perp$. From definition $K \cup A \not\vdash \perp$. From item (e) of Lemma 1, since $*$ satisfies *inclusion*, *weak success* and *core-retainment* then $*$ satisfies *vacuity 2*. Therefore, from definition and **-vacuity 2* then *AGM-vacuity* is satisfied.
- AGM-Consistency Straightforward from **-consistency*.
- AGM-Extensionality Let A and B two sets of sentences such that $\alpha = \bigwedge A$ and $\beta = \bigwedge B$ and suppose that $\vdash \alpha \leftrightarrow \beta$. It follows immediately from the respective definitions that if $*$ is a prioritized multiple kernel revision or a prioritized multiple partial meet revision then it holds that, for any consistent set A , $K * A = (K \cap K * A) \cup A$ (i). By Observation 1.13 of [30], we have that $Cn(X \cup Y) = Cn(X \cup Cn(Y))$ (ii). By hypothesis, we

know that $Cn(A) = Cn(B)$ (iii) and, by *uniformity 2*, $K \cap (K * A) = K \cap (K * B)$ (iv). Therefore:

$$\begin{aligned} Cn(K * A) &= Cn((K \cap K * A) \cup A) \text{ from (i)} \\ &= Cn((K \cap K * A) \cup Cn(A)) \text{ from (ii)} \\ &= Cn((K \cap K * B) \cup Cn(B)) \text{ from (iii) and (iv)} \\ &= Cn((K \cap K * B) \cup B) \text{ from (ii)} \\ &= Cn(K * B) \text{ from (ii)} \end{aligned}$$

Therefore, AGM-extensionality is satisfied.

Proof of Theorem 3

- *Construction to Postulates* Let \circ be a kernel merge for K . We must show that \circ satisfies the postulates enumerated in the theorem. Let $K \circ A = (K \cup A) \setminus (\sigma((K \cup A) \perp \perp))$.

Inclusion Straightforward from the definition.

Strong Consistency Since all sets in $(K \cup A) \perp \perp$ are minimally inconsistent, and σ cuts every set in it, then $(K \cup A) \setminus \sigma((K \cup A) \perp \perp)$ is consistent.

Global Core-Retainment Suppose that $\alpha \in (K \cup A) \setminus (K \circ A)$. That is, $\alpha \in K \cup A$ and $\alpha \notin K \circ A$. Then $\alpha \in \sigma((K \cup A) \perp \perp)$. Since $\sigma((K \cup A) \perp \perp) \subseteq \bigcup ((K \cup A) \perp \perp)$ there is some X such that $\alpha \in X$ and $X \in (K \cup A) \perp \perp$. Let $Y = X \setminus \{\alpha\}$. Then Y is such that $Y \subseteq (K \cup A)$, $Y \not\vdash \perp$ but $Y \cup \{\alpha\} \vdash \perp$. Therefore, global core-retainment is satisfied.

Reversion Suppose that $K \cup A$ and $K' \cup A'$ have the same minimally inconsistent subsets. That means that $(K \cup A) \perp \perp = (K' \cup A') \perp \perp$. Since σ is a well defined function then:

$$\sigma((K \cup A) \perp \perp) = \sigma((K' \cup A') \perp \perp)$$

We need to show that $(K \cup A) \setminus (K \circ A) = (K' \cup A') \setminus (K' \circ A')$.

- (\subseteq) If $\alpha \in (K \cup A) \setminus (K \circ A)$ then, by definition of \circ , it holds that $\alpha \in \sigma((K \cup A) \perp \perp)$. From the hypothesis we know that $\sigma((K \cup A) \perp \perp) = \sigma((K' \cup A') \perp \perp)$. Then $\alpha \in K' \cup A'$ and $\alpha \notin K' \circ A'$. Therefore:

$$(K \cup A) \setminus (K \circ A) \subseteq (K' \cup A') \setminus (K' \circ A')$$

- (\supseteq) By symmetry, it is the same proof as the one for (\subseteq).

- *Postulates to Construction* Let σ be a function such that, for every pair of sets K and A , it holds that:

$$\sigma((K \cup A)\perp\perp) = \{\alpha : \alpha \in (K \cup A) \setminus (K \circ A)\}$$

We must show:

Part A.

1. σ is a well defined function.
 If K' and A' are sets of sentences such that $(K \cup A)\perp\perp$ is equal to $(K' \cup A')\perp\perp$, then we must show that:

$$\sigma((K \cup A)\perp\perp) = \sigma((K' \cup A')\perp\perp)$$

From the hypothesis we have that $K \cup A$ and $K' \cup A'$ have the same minimally inconsistent subsets. It follows from **reversion** that $(K \cup A) \setminus (K \circ A) = (K' \cup A') \setminus (K' \circ A')$:

$$\begin{aligned} \sigma((K \cup A)\perp\perp) &= \{\alpha : \alpha \in (K \cup A) \setminus (K \circ A)\} \\ &= \{\alpha : \alpha \in (K' \cup A') \setminus (K' \circ A')\} \\ &= \sigma((K' \cup A')\perp\perp) \end{aligned}$$

Therefore, σ is well defined.

2. $\sigma((K \cup A)\perp\perp) \subseteq \cup((K \cup A)\perp\perp)$.
 Let $\alpha \in \sigma((K \cup A)\perp\perp)$. Then $\alpha \in (K \cup A) \setminus (K \circ A)$. Due to **global core-retainment** there is some E such that $E \subseteq (K \cup A)$, $E \not\vdash \perp$ but $E \cup \{\alpha\} \vdash \perp$. Since $\alpha \in K \cup A$ then there is a \perp -kernel X in $(K \cup A)$ (i.e., there is a minimally inconsistent subset of $K \cup A$) such that $X \subseteq E \cup \{\alpha\}$ and $\alpha \in X$. Therefore, $\alpha \in \bigcup((K \cup A)\perp\perp)$.
3. If $X \in (K \cup A)\perp\perp$ then $(X \cap \sigma((K \cup A)\perp\perp)) \neq \emptyset$.
 Let $X \in ((K \cup A)\perp\perp)$. We need to show that:

$$X \cap \sigma((K \cup A)\perp\perp) \neq \emptyset$$

Due to **strong consistency** $K \circ A \not\vdash \perp$. Since $X \vdash \perp$ we may conclude that $X \not\subseteq K \circ A$. This means that there is some β such that $\beta \in X$ and $\beta \notin K \circ A$. Since $X \subseteq (K \cup A)$ then $\beta \in (K \cup A) \setminus (K \circ A)$, i.e., $\beta \in \sigma((K \cup A)\perp\perp)$. So $\beta \in (X \cap \sigma((K \cup A)\perp\perp))$. Therefore, $(X \cap \sigma((K \cup A)\perp\perp)) \neq \emptyset$.

Part B: \circ_σ is equal to \circ .

Due to **inclusion** and from the definition of $\sigma((K \cup A)\perp\perp)$ we conclude that $K \circ A = K \circ_\sigma A$.

Proof of Theorem 5

- *Construction to Postulates* Let \circ be a partial meet merge for K . We must show that \circ satisfies the postulates enumerated in the theorem. Let $K \circ A = \bigcap \gamma((K \cup A)_{\perp\perp})$, for all sets A .

Inclusion Straightforward from the definition.

Strong Consistency Since all sets in $(K \cup A)_{\perp\perp}$ are consistent, so is their intersection.

Global Relevance Suppose that $\alpha \in (K \cup A) \setminus (K \circ A)$. That is, $\alpha \in K \cup A$ and $\alpha \notin K \circ A$. Then $\alpha \notin \bigcap \gamma((K \cup A)_{\perp\perp})$. Hence, there is some X such that $\alpha \notin X$ and $X \in \gamma((K \cup A)_{\perp\perp})$. Then, since $\gamma((K \cup A)_{\perp\perp}) \subseteq (K \cup A)_{\perp\perp}$, we can conclude that $X \subseteq K \cup A$, $X \not\vdash \perp$ but $X \cup \{\alpha\} \vdash \perp$. On the other hand, it follows from $\bigcap \gamma((K \cup A)_{\perp\perp}) \subseteq X$ that $K \circ A \subseteq X$. Therefore, relevance is satisfied.

Reversion Suppose that $K \cup A$ and $K \cup B$ have the same minimal inconsistent subsets, that is, $(K \cup A)_{\perp\perp} = (K \cup B)_{\perp\perp}$. We need to show that $(K \cup A) \setminus (K \circ A) = (K \cup B) \setminus (K \circ B)$. Straightforward since γ is an equitable selection function.

- *Postulates to Construction* We will show that if an operator \circ satisfies the enumerated postulates then it is possible to build an operator \circ_γ of partial meet merge such that \circ is equal to \circ_γ . Let γ be a function such that, for all pairs of sets K and A , it holds that:

$$\gamma((K \cup A)_{\perp\perp}) = \{X \in (K \cup A)_{\perp\perp} : K \circ A \subseteq X\}$$

Let \circ_γ be a merge operator defined as follows:

$$K \circ_\gamma A = \bigcap \gamma((K \cup A)_{\perp\perp})$$

We must show that:

Part A: \circ_γ is equal to \circ , i.e., $\bigcap \gamma((K \cup A)_{\perp\perp}) = K \circ A$, for all sets A .

(\supseteq) It follows from the definition.

(\subseteq) Let $\alpha \notin K \circ A$. We must prove that $\alpha \notin \bigcap \gamma((K \cup A)_{\perp\perp})$.

That is, we need to find some $X \in \gamma((K \cup A)_{\perp\perp})$ such that $\alpha \notin X$. We have two cases:

1. $\alpha \in K \cup A$: by **global relevance** we have that there is some H such that $K \circ A \subseteq H \subseteq K \cup A$, $H \not\vdash \perp$ and $H \cup \{\alpha\} \vdash \perp$. From this we have that $H \not\vdash \alpha$. It is clear that we may extend the set H to a maximally consistent set H' such that $H' \in (K \cup A)_{\perp\perp}$ and $\alpha \notin H'$. Since $K \circ A \subseteq H'$ then $H' \in \gamma((K \cup A)_{\perp\perp})$.
2. $\alpha \notin K \cup A$: then no set in $(K \cup A)_{\perp\perp}$ will contain α .

Part B.

1. γ is a well defined function.
 Let K, K', A and B be sets of sentences such that:

$$(K \cup A)\underline{\perp\perp} = (K' \cup B)\underline{\perp\perp}$$

We must show that $\gamma((K \cup A)\underline{\perp\perp}) = \gamma((K' \cup B)\underline{\perp\perp})$. If $(K \cup A)\underline{\perp\perp} = (K' \cup B)\underline{\perp\perp}$ then it follows that $K \cup A = K' \cup B$ [12]. From item (f) of Lemma 1 it follows that if **inclusion** and **reversion** hold then congruence holds. Therefore, $K \circ A = K' \circ B$ and:

$$\begin{aligned} \gamma((K \cup A)\underline{\perp\perp}) &= \{X \in (K \cup A)\underline{\perp\perp} : K \circ A \subseteq X\} \\ &= \{X \in (K' \cup B)\underline{\perp\perp} : K' \circ B \subseteq X\} \\ &= \gamma((K' \cup B)\underline{\perp\perp}) \end{aligned}$$

That means that the function γ is well defined.

2. γ is an equitable selection function.
 First we will show that γ is a general selection function. That is to say that $\emptyset \neq \gamma((K \cup A)\underline{\perp\perp}) \subseteq (K \cup A)\underline{\perp\perp}$. That $\gamma((K \cup A)\underline{\perp\perp}) \subseteq (K \cup A)\underline{\perp\perp}$ follows immediately from the definition of γ . Now we prove that $\gamma((K \cup A)\underline{\perp\perp}) \neq \emptyset$. By **inclusion** $(K \circ A) \subseteq (K \cup A)$. Due to **strong consistency** $K \circ A \not\subseteq \perp$. Hence, there must exist a set H between $K \circ A$ and $K \cup A$ which is maximally consistent. Then $H \in (K \cup A)\underline{\perp\perp}$ and $K \circ A \subseteq H$. Therefore, $\gamma((K \cup A)\underline{\perp\perp}) \neq \emptyset$ and γ is a general selection function.

It remains to show that γ is an equitable selection function. Suppose that $K \cup A$ and $K' \cup B$ have the same minimal inconsistent subsets. It follows from **reversion** that:

$$(K \cup A) \setminus (K \circ A) = (K' \cup B) \setminus (K' \circ B)$$

It follows from Part A that $K \circ A = \bigcap \gamma((K \cup A)\underline{\perp\perp})$ and $K' \circ B = \bigcap \gamma((K' \cup B)\underline{\perp\perp})$. Therefore, γ is an equitable selection function.

Proof of Proposition 2

Let K_1, K_2 be sets of sentences and $!$ be a consolidation operator for $K_1 \cup K_2$. Let \circ be the merge of K_1 and K_2 defined according to Definition 21.

1. Suppose that $!$ is a *kernel consolidation* for $K_1 \cup K_2$. We must show that \circ is a *kernel meet merge* for K_1 and K_2 .
 Since $!$ is a kernel consolidation for $K_1 \cup K_2$, then (according to Definition 18) there exists an incision function σ for $K_1 \cup K_2$ such that $K_1 \circ K_2 = (K_1 \cup K_2)! = (K_1 \cup K_2) \setminus \sigma(K_1 \cup K_2)\underline{\perp\perp}$ and it follows immediately from Definition 14 that \circ is a kernel merge for K_1 and K_2 .
2. Suppose that $!$ is a *equitable partial meet consolidation* for $K_1 \cup K_2$. We must show that \circ is a *partial meet merge* for K_1 and K_2 .

Since $!$ is an equitable partial meet consolidation for $K_1 \cup K_2$, then (according to Definition 20) there exists an equitable selection function γ such that: $K_1 \circ K_2 = (K_1 \cup K_2)! = \bigcap \gamma((K_1 \cup K_2)\underline{\perp}\underline{\perp})$. Therefore, it follows immediately from Definition 17 that \circ is a partial meet merge for K_1 and K_2 .

Proof of Proposition 3

Let K be a set of sentences and \circ be a merge operator for K and K . Let $!$ be the consolidation of K defined according to Definition 22. Then:

1. If \circ is a *kernel merge* then $!$ is a *kernel consolidation*.
Straightforward from Definitions 14 and 18.
2. If \circ is a *partial meet merge* then $!$ is a *partial meet consolidation*.
Straightforward from Definitions 17 and 19.

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