RADIAL-TYPE MUCKENHOUPT WEIGHTS

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Abstract. Given a space of homogeneous type (X, d, μ) and $1 < p < \infty$, the main purpose of this note is to find sufficient conditions on a function w and on a subset F of X, such that $w(d(x, F))$ belongs to the Muckenhoupt class $A_p(X, d, \mu)$. Here $d(x, F)$ denotes the distance between $x \in X$ and F.

Introduction

The class of Muckenhoupt weights are extensively used in real and harmonic analysis, as well as, in the theory of partial differential equations. For example, the behavior of the source near the boundary of the domain of a Dirichlet boundary value problem, may cause non-solvability in a non-weighted Sobolev space. Nevertheless, the problem can be solved in an adequate weighted Sobolev space, in which the difficulties might be avoided. If the source has an unbounded growth near the boundary F of the domain, we should search for a weight which vanishes there. This is the case of the power-type weights, which are of the form $d^\beta(x, F)$, where $d(x, F)$ is the distance from the point x to F.

Weighted Sobolev spaces with Muckenhoupt weights are of particular interest since weighted imbedding theorems and Poincaré type inequalities hold.

In the general framework of metric measure spaces (X, d, μ), in [ACDT14] the authors give sufficient conditions on a closed set $F \subseteq X$ and on a real number β in such a way that $d(x, F)^\beta$ becomes a Muckenhoupt weight.

On the other hand, Kokilashvili and Samko study in [KS08] under which conditions $w(d(x, x_0))$ belongs to the Muckenhoupt class $A_p(X, d, \mu)$, where $x_0 \in X$ and $w(t)$ is a function generalizing the powers t^β.

In this note we obtain sufficient conditions on $w(t)$ and on a subset F of a space of homogeneous type (X, d, μ), such that $w(d(x, F)) \in A_p(X, d, \mu)$. This result is contained in Theorem 1, and state a joint condition on the function w, the set F and the measure μ, that for the case of $F = \{x_0\}$, is the same that the required in [KS08].

In Theorem 4 we study when this condition is satisfied, in the particular case that (X, d, μ) is a g-Ahlfors space and F is a h-set, where g and h are dimension functions. In this setting, we also consider the specific function $w(t) = h(t)/g(t)$ and we obtain a condition in terms on the upper and lower indices of h and g respectively. Precisely, in Theorem 7 we prove that if $I(h) < i(g)$, then $w(d(x, F))^\beta \in A_p(X, d, \mu)$ for every $(1 - p) < \beta < 1$. When $g(t) = t^\alpha$ and $h(t) = t^s$ we recover the result given in [ACDT14].

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We finish the note exhibit an example of a Muckenhoupt weight obtained as consequence of our results.

1. Notation, definitions and basic results

Let X be a set. A ***quasi-distance*** on X is a non-negative symmetric function d defined on $X \times X$ such that $d(x,y) = 0$ if and only if $x = y$, and there exists a constant $K \geq 1$ such that the inequality

$$d(x,y) \leq K(d(x,z) + d(z,y))$$

holds for every $x, y, z \in X$. We will refer to K as the triangle constant for d. A quasi-distance d on X induces a topology through the neighborhood system given by the family of all subsets of X containing a d-ball $B(x,r) = \{y \in X : d(x,y) < r\}$, $r > 0$ (see [CW71]). In a quasi-metric space (X,d) the ***diameter*** of a subset E is defined as

$$\operatorname{diam}(E) = \sup\{d(x,y) : x, y \in E\}.$$

Throughout this paper (X,d) shall be a quasi-metric space such that the d-balls are open sets.

We shall say that (X,d,μ) is a ***space of homogeneous type*** if μ is a non-negative Borel measure μ satisfying the ***doubling condition***

$$0 < \mu(B(x,2r)) \leq A\mu(B(x,r)) < \infty$$

for some constant $A \geq 1$, for every $x \in X$ and every $r > 0$. We will refer to A as the doubling constant for μ. We shall assume that $\mu(\{x\}) = 0$ for every $x \in X$, or in other words, (X,d,μ) is a ***non-atomic*** space of homogeneous type.

First we shall recall a basic property of spaces of homogeneous type that we shall need. This property is actually contained in [CW71], and reflects the fact that spaces of homogeneous type have finite metric (or Assouad) dimension (see [Ass79]). The expression ***finite metric dimension*** means that there exists a constant $N \in \mathbb{N}$ such that no ball of radius r contains more than $N\delta^{-\log_2 N}$ points of any δr-disperse subset of X, for every $\delta \in (0,1)$. A set U is said to be ***r-disperse*** if $d(x,y) \geq r$ for every $x, y \in U$, $x \neq y$. An ***r-net*** in X is a maximal r-disperse set. It is easy to check that U is an r-net in X if and only if U is an r-disperse and r-dense set in X, where ***r-dense*** means that for every $x \in X$ there exists $u \in U$ with $d(x,u) < r$. It is well known that if a quasi-metric space (X,d) has finite metric dimension, then every bounded subset F of X is totally bounded, so that for every $r > 0$ there exists a finite r-net on F, whose cardinal depends on $\operatorname{diam}(F)$ and on r.

Let us recall that a ***weight*** ρ on (X,d,μ) is a locally integrable non-negative function defined on X. By locally we mean integrable over balls, i.e. $\int_B \rho\, d\mu < \infty$ for every d-ball B in X. For $1 < p < \infty$ the ***Muckenhoupt class*** $\boldsymbol{A_p(X,d,\mu)}$ is defined as the set of all weights ρ defined on X for which there exists a constant C such that the inequality

$$\left(\frac{1}{\mu(B)}\int_B \rho\, d\mu\right)\left(\frac{1}{\mu(B)}\int_B \rho^{-\frac{1}{p-1}}\, d\mu\right)^{p-1} \leq C$$

holds for every d-ball B in X. For $p = 1$, we say that $\rho \in A_1(X, d, \mu)$ if there exists a constant C such that

$$\frac{1}{\mu(B)} \int_B \rho \, d\mu \leq C \, \rho(x)$$

holds for every d-ball B in X and μ-almost every $x \in B$. The classical reference for the basic theory of Muckenhoupt weights is Chapter IV in [GCRdF85].

We now introduce some notions which are of interest in studying the order of growth of functions. Let $0 < \ell \leq \infty$, and let $w : [0, \ell] \to [0, +\infty]$. We shall say that $w(t)$ is ***almost increasing*** (a.i.) if there is a constant $C \geq 1$ such that $w(t_1) \leq C w(t_2)$ if $t_1 < t_2$. Analogously, $w(t)$ is ***almost decreasing*** (a.d.) if there is a constant $C \geq 1$ such that if $t_1 < t_2$ then $w(t_2) \leq C w(t_1)$. Following the notation in [KS08], we shall write

$$W([0, \ell]) = \{w \in C([0, \ell]) : w(0) = 0, \; w(t) > 0 \text{ for } t > 0 \text{ and } w(t) \text{ is a.i.}\},$$

and

$$\tilde{W}([0, \ell]) = \{\varphi : x^\nu \varphi(x) \in W([0, \ell]) \text{ for some } \nu \in \mathbb{R}\}.$$

For $w \in \tilde{W}([0, \ell])$, the ***lower and upper indices*** is defined by

$$m(w) = \sup\{\nu \in \mathbb{R} : t^{-\nu} w(t) \text{ is a.i. in } [0, \ell]\},$$
$$M(w) = \inf\{\nu \in \mathbb{R} : t^{-\nu} w(t) \text{ is a.d. in } [0, \ell]\}.$$

Remark 1. For a large class of functions w, the indices $m(w)$ and $M(w)$ coincide with the lower and upper Matuszewska-Orlicz indices (see for example [Mal85] and [KS04]).

Remark 2. It is easy to check that if $t^{-\nu} w(t)$ is a.i. on $[0, \ell]$, then the same is true for $t^{-\beta} w(t)$ for every $\beta < \nu$. Analogously, if $t^{-\nu} w(t)$ is a.d. on $[0, \ell]$, then the same holds for $t^{-\beta} w(t)$ for every $\beta > \nu$. Then, if $m(w) > -\infty$ and $\nu < m(w)$, then the function $w_\nu(t) := t^{-\nu} w(t)$ is a.i. on $[0, \ell]$. Similarly, if $M(w) < \infty$ and $\nu > M(w)$, then $w_\nu(t)$ is a.d. on $[0, \ell]$ and then, from the definition of w_ν we have that it is a ***doubling function*** on $[0, \ell]$. This means that there exists a positive constant C such that $w_\nu(2t) \leq C w_\nu(t)$, for every $0 < t \leq \ell/2$.

Remark 3. It is not difficult to check that $M(w) = -m(w^{-1})$, and that for each $\alpha > 0$, we have $m(w^\alpha) = \alpha m(w)$ and $M(w^\alpha) = \alpha M(w)$.

Given a space of homogeneous type (X, d, μ) with diameter ℓ, the main purpose of this note is to find sufficient conditions on a function $w \in \tilde{W}([0, \ell])$ and on a subset F of X, such that $\rho(x) := w(d(x, F))$ belongs to the Muckenhoupt class $A_p(X, d, \mu)$. Here $d(x, F)$ denotes the distance between $x \in X$ and F, defined in the classical way by $d(x, F) = \inf\{d(x, y) : y \in F\}$. In this paper we extend the results in [KS08], where is considered the particular case $F = \{x_0\}$ for some $x_0 \in X$.

Since functions in the Muckenhoupt class are positive almost everywhere and $\rho(x) = w(d(x, F)) = w(0) = 0$ for every $x \in F$, we shall assume from now on that $\mu(F) = 0$.

In what follows the letter C will denote a generic constant but not always the same at each occurrence, if is necessary we refer about the dependencies.

2. The main result

Let (X, d, μ) be a given space of homogeneous type and set $\ell = \operatorname{diam}(X)$. Given a subset F of X and $\varepsilon > 0$, the ***ε-enlargement*** $[F]_\varepsilon$ of F is defined by

$$[F]_\varepsilon = \{x \in X : d(x, F) < \varepsilon\} = \bigcup_{y \in F} B(y, \varepsilon).$$

Let $w \in \tilde{W}([0, \ell])$. We shall say that w belongs to $\mathcal{W}^F$ if there exists a constant C such that

$$\int_0^r \frac{\mu([F]_t \cap B(x, r))}{t w(t)}\, dt \le C\, \frac{\mu(B(x, r))}{w(r)}, \tag{2.1}$$

holds for every $0 < r < \ell$ and every $x \in F$.

We can now state our main result, which give us sufficient conditions on a weight w and on a given set F, in such a way that $w(d(x, F))$ belongs to a Muckenhoupt class.

Theorem 1. *Let $1 < p < \infty$ and $F \subseteq X$ given, and let $w \in \tilde{W}([0, \ell])$ with both $m(w)$ and $M(w)$ finite. If w^{-1} and $w^{\frac{1}{p-1}}$ belong to $\mathcal{W}^F$, then $w(d(x, F)) \in A_p(X, d, \mu)$.*

Remark 4. Let us observe that for the particular case $F = \{x_0\}$ for some $x_0 \in X$, the condition $w^{-1} \in \mathcal{W}^{\{x_0\}}$ says

$$\int_0^r \frac{\mu(B(x_0, t) \cap B(x_0, r)) w(t)}{t}\, dt = \int_0^r \frac{\mu(B(x_0, t) w(t)}{t}\, dt \le C\, \mu(B(x_0, r)) w(r).$$

Analogously, $w^{\frac{1}{p-1}} \in \mathcal{W}^{\{x_0\}}$ is equivalent to

$$\int_0^r \frac{\mu(B(x_0, t)}{t\, w^{\frac{1}{p-1}}(t)}\, dt \le C\, \frac{\mu(B(x_0, r))}{w^{\frac{1}{p-1}}(r)}.$$

The above two conditions are required in [KS08] *uniformly* in x in order to obtain $w(d(x, x_0)) \in A_p(X, d, \mu)$. Nevertheless, following the proof there, one can see that it is sufficient that both conditions hold just for x_0. Then, for the case of $F = \{x_0\}$, hypotheses in our theorem are the same as in [KS08].

In order to show the above theorem, we shall need the following result that will be proved at the end of this section.

Lemma 2. *Let (X, d, μ) be a space of homogeneous type and let $w \in \tilde{W}([0, \ell])$ with both $m(w)$ and $M(w)$ finite. If $w \in \mathcal{W}^F$, there exists a constant C such that for every $x \in X$ and $0 < r < \ell$ we have*

(i) *if $d(x, F) \ge 2Kr$,*

$$\frac{w(d(x, F))}{\mu(B(x, r))} \int_{B(x, r)} \frac{d\mu(y)}{w(d(y, F))} \le C; \tag{2.2}$$

(ii) *if $d(x, F) \le 2Kr$,*

$$\frac{w(r)}{\mu(B(x, r))} \int_{B(x, r)} \frac{d\mu(y)}{w(d(y, F))} \le C. \tag{2.3}$$

Here K denotes the triangular constant for d.

Proof of Theorem 1. Let $q = \frac{1}{p-1}$. We shall apply Lemma 2 to w^{-1} and w^q. If $d(x,F) \geq 2Kr$, from (2.2) we obtain

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} w(d(y,F))d\mu(y) \leq Cw(d(x,F))$$

and

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} \frac{d\mu(y)}{[w(d(y,F))]^q} \leq \frac{C}{[w(d(x,F))]^q}.$$

If $d(x,F) \leq 2Kr$ we proceed in the same way but applying (2.3). Then we have that $w(d(y,F)) \in A_p(X,d,\mu)$.

□

In order to prove Lemma 2, we shall use the following result.

Lemma 3. *Let $w \in \mathcal{W}^F$ with both $m(w)$ and $M(w)$ finite. Then there exists a constant C such that*

$$\int_{B(x,r)} \frac{d\mu(y)}{w(d(y,F))} \leq C\,\frac{\mu(B(x,r))}{w(r)},$$

for every $x \in F$ and $0 < r < \ell$.

Proof. Fix $x \in F$ and $0 < r < \ell$. For each non-negative integer k we define

$$X_k = X_k(x,r) := \{y \in B(x,r) : 2^{-k-1}r \leq d(y,F) < 2^{-k}r\}.$$

Since $d(y,F) \leq d(y,x)$ for every y, we have $B(x,r) = \bigcup_{k=0}^{\infty} X_k$. Also for every $y \in X_k$ it holds that $d(y,F)^{-\nu} \leq c(2^{-k}r)^{-\nu}$, with $c = 2^{\max\{\nu,0\}}$. Then, for a fixed $\nu < m(w)$, since w_ν is almost increasing we obtain

$$\begin{aligned}
\int_{B(x,r)} \frac{d\mu(y)}{w(d(y,F))} &= \sum_{k=0}^{\infty} \int_{X_k} \frac{d(y,F)^{-\nu}d\mu(y)}{w_\nu(d(y,F))} \\
&\leq c \sum_{k=0}^{\infty} \int_{X_k} \frac{(2^{-k}r)^{-\nu}d\mu(y)}{w_\nu(2^{-k-1}r)} \\
&= c \sum_{k=0}^{\infty} \frac{(2^{-k}r)^{-\nu}(2^{-k-1}r)^{\nu}}{w(2^{-k-1}r)}\,\mu(X_k) \\
&= c \sum_{k=0}^{\infty} 2^{-\nu}\frac{\mu(X_k)}{w(2^{-k-1}r)} \\
&= c\,2^{-\nu}\left(\frac{\mu(X_0)}{w(r/2)} + \sum_{k=0}^{\infty} \frac{\mu(X_{k+1})}{w(2^{-k-2}r)}\right) \\
&\leq c\,2^{-\nu}4^{\beta}\left(\frac{\mu(X_0)}{w(r)} + \sum_{k=0}^{\infty} \frac{\mu(X_{k+1})}{w(2^{-k}r)}\right),
\end{aligned}$$

where β is any fixed real number satisfying $\beta > M(w)$ fixed, since w_β is almost decreasing.

We claim that

$$\frac{\mu(X_{k+1})}{w(2^{-k}r)} \leq C \int_{2^{-k-1}}^{2^{-k}} \frac{\mu([F]_{tr} \cap B(x,r))}{tw(tr)}dt$$

Assuming that the above claim holds, we have that

$$\begin{aligned}\int_{B(x,r)} \frac{d\mu(y)}{w(d(y,F))} &\leq C\,\frac{\mu(B(x,r))}{w(r)} + C\int_0^1 \frac{\mu([F]_{tr}\cap B(x,r))}{tw(tr)}dt\\ &= C\,\frac{\mu(B(x,r))}{w(r)} + C\int_0^r \frac{\mu([F]_{s}\cap B(x,r))}{sw(s)}ds,\end{aligned}$$

and the result is proved since $w \in \mathcal{W}^F$.

Then it only remains to prove the claim. In order to do this, notice that from the definition we have that $X_{k+1} \subseteq [F]_{2^{-k-1}r} \cap B(x,r)$. Also, since we can assume $\nu \neq 1$, we have that

$$\int_{2^{-k-1}}^{2^{-k}} (tr)^{-\nu}\, dt = C2^{-k}(2^{-k}r)^{-\nu}.$$

Then, since w_ν is almost increasing we have

$$\begin{aligned}\frac{\mu(X_{k+1})}{w(2^{-k}r)} &\leq \frac{\mu([F]_{2^{-k-1}r}\cap B(x,r))}{w(2^{-k}r)}\\ &= C\,\frac{\mu([F]_{2^{-k-1}r}\cap B(x,r))}{w_\nu(2^{-k}r)}\,2^k\int_{2^{-k-1}}^{2^{-k}}(tr)^{-\nu}\,dt\\ &\leq C\int_{2^{-k-1}}^{2^{-k}}\frac{\mu([F]_{tr}\cap B(x,r))}{w_\nu(tr)}2^k(tr)^{-\nu}\,dt\\ &\leq C\int_{2^{-k-1}}^{2^{-k}}\frac{\mu([F]_{tr}\cap B(x,r))}{tw(tr)}\,dt.\end{aligned}$$

□

Proof of Lemma 2. Fix $x \in X$ and $0 < r < \ell$. In order to prove (i), fix $y \in B(x,r)$ and $x_0 \in F$. Since we are assuming $d(x,F) \geq 2Kr$, then $d(x,x_0) \geq 2Kr$ and

$$d(y,x_0) \geq \frac{d(x,x_0)}{K} - d(x,y) \geq \frac{d(x,x_0)}{2K}.$$

Hence, taking supremum on $x_0 \in F$,

$$2Kd(y,F) \geq d(x,F). \tag{2.4}$$

Then, for a fixed $\nu < m(w)$, since w_ν is almost increasing we have $w_\nu\left(\frac{1}{2K}d(x,F)\right) \leq Cw_\nu(d(y,F))$. On the other hand, fix $\beta > M(w)$. Due to w_β is almost decreasing we obtain

$$\frac{w(d(x,F))}{d(x,F)^\beta} \leq C\,\frac{w\left(\frac{1}{2K}d(x,F)\right)}{\left(\frac{1}{2K}d(x,F)\right)^\beta},$$

so that $w(d(x,F)) \leq C_\beta w(\frac{1}{2K}d(x,F))$. Hence we can conclude that

$$w_\nu(d(x,F)) \leq Cw_\nu(d(y,F)).$$

Then

$$\begin{aligned}\frac{w(d(x,F))}{\mu(B(x,r))}\int_{B(x,r)}\frac{d\mu(y)}{w(d(y,F))} &= \frac{d(x,F)^\nu}{\mu(B(x,r))}\int_{B(x,r)}\frac{w_\nu(d(x,F))d\mu(y)}{w_\nu(d(y,F))d(y,F)^\nu}\\ &\leq C\frac{d(x,F)^\nu}{\mu(B(x,r))}\int_{B(x,r)}\frac{d\mu(y)}{d(y,F)^\nu}.\end{aligned}$$

So that the result is proved from (2.4) if we can choose $\nu > 0$. Otherwise, if $\nu < 0$ we use that

$$d(y,x_0) \leq K(d(y,x)+d(x,x_0)) \leq K\left(\frac{1}{2K}d(x,F)+d(x,x_0)\right)$$

for every $x_0 \in F$, to obtain again $d(y,F) \leq Cd(x,F)$ and then (i) is proved.

We shall now prove (ii). Let us assume first $r < \frac{\ell}{4K^2}$. Since $d(x,F) \leq 2Kr$, there exists $x_0 \in F$ such that $d(x,x_0) < 3Kr$. Then, for every $z \in B(x,r)$ we have

$$d(z,x_0) \leq K(d(z,x)+d(x,x_0)) < 4K^2r.$$

In a similar way we can see that $B(x_0,4K^2r) \subseteq B(x,7K^3r)$. So that from the doubling property of μ we have that

$$\mu(B(x_0,4K^2r)) \leq C\mu(B(x,r)).$$

Then, since in this case we have $4K^2r < \ell$, from Lemma 3 we obtain

$$\begin{aligned}\int_{B(x,r)} \frac{d\mu(y)}{w(d(y,F))} &\leq \int_{B(x_0,4K^2r)} \frac{d\mu(y)}{w(d(y,F))}\\ &\leq C\,\frac{\mu(B(x_0,4K^2r))}{w(4K^2r)}\\ &\leq C\,\frac{\mu(B(x,r))}{w(4K^2r)}.\end{aligned}$$

So that the result is proved if $w(r) \leq Cw(4K^2r)$, which follows from the fact that w_ν is almost increasing for every fixed $\nu < m(w)$.

Finally, we shall consider the case $\frac{\ell}{4K^2} \leq r < \ell$. Define $t = \frac{\ell}{8K^2}$. Since (X,d) has finite metric dimension, there exists a finite t-net U in $B(x,r)$, let us say $U = \{x_1,x_2,\dots,x_{\tilde{N}}\}$, where $\tilde{N}$ is a constant which does not depend on x and r (see [CW71]). Hence $\{B(x_i,t) : i = 1,\dots,\tilde{N}\}$ is a cover of $B(x,r)$, and $B(x_i,t) \subseteq B(x,2Kr)$ for every i, so that

$$\begin{aligned}\int_{B(x,r)} \frac{d\mu(y)}{w(d(y,F))} &\leq \int_{\bigcup_{i=1}^{\tilde{N}} B(x_i,t)} \frac{d\mu(y)}{w(d(y,F))}\\ &\leq \sum_{i=1}^{\tilde{N}} \int_{B(x_i,t)} \frac{d\mu(y)}{w(d(y,F))}\\ &\leq C\sum_{i=1}^{\tilde{N}} \frac{\mu(B(x_i,t))}{w(t)}\\ &\leq C\tilde{N}\,\frac{\mu(B(x,r))}{w(t)},\end{aligned}$$

where we have applied the previous case since $t < \ell/(4K^2)$ and the doubling property of μ. Finally, from the facts that w_ν is almost increasing for every fixed $\nu < m(w)$ and that w_β is doubling for every fixed $\beta > M(w)$, we can conclude that $w(r) \leq Cw(\ell) \leq Cw(t)$, and the result is proved. $\square$

3. Particular cases and examples

In this section we explore sufficient conditions under which w^{-1} and $w^{\frac{1}{p-1}}$ belong to $\mathcal{W}^F$ in some particular cases of the underlying space X and of sets F.

We shall denote $\mathbb{H}$ the class of all right continuous monotone increasing doubling functions $h : [0,+\infty] \to [0,+\infty]$ such that $h(u) > 0$ if $u > 0$. We refer to $\mathbb{H}$ as to the set of all ***gauge functions*** or ***dimension functions***. Given $g \in \mathbb{H}$, we shall say that (X, d, μ) is an ***g-Ahlfors space*** if there exists a constant $C \geq 1$ satisfying the inequalities

$$C^{-1}g(r) \leq \mu(B(x,r)) \leq Cg(r),$$

for every $x \in X$ and every $0 < r < \operatorname{diam}(X)$. Notice that in the particular case $g(r) = r^\alpha$, we have the classical α-regular Ahlfors spaces. It is easy to see that if (X, d, μ) is an g-Ahlfors space, then it is a non-atomic space of homogeneous type with doubling constant for μ which only depends on C and g.

Also, given $h \in \mathbb{H}$, a subset F of X is said to be ***h-set with measure $\boldsymbol{\nu}$*** if ν is a Borel measure supported in F such that there exists a constant $C \geq 1$ satisfying the inequalities

$$C^{-1}h(r) \leq \nu(B(x,r)) \leq Ch(r),$$

for every $x \in F$ and every $0 < r < \operatorname{diam}(F)$. In the particular case $h(r) = r^s$, we deal with the class of sets which are known as ***s-sets*** in some references related to problems of harmonic analysis and partial differential equations (see for example [Sjö97]).

Following again [KS08], we shall denote $\mathcal{Z}^0$ the class of functions $f(t)$ satisfying the condition

$$\int_0^r \frac{f(t)}{t}\,dt \leq Cf(r),$$

for a constant $C > 0$ which does not depend on $r > 0$.

In this particular case of space X and set F, we obtain a direct relationship between the behavior of the functions g and h and the condition $w \in \mathcal{W}^F$. This result is contained in the next theorem, and allows us obtain a large class of examples.

Theorem 4. *Let (X, d, μ) be an g-Ahlfors space and let F be an h-set. If $\frac{g(t)}{w(t)h(t)} \in \mathcal{Z}^0$, then $w \in \mathcal{W}^F$.*

In order to proof the above result, we shall use the following lemma.

Lemma 5. *Let F be an h-Ahlfors set in (X, d, μ). Then there exist a finite subset $\{x_1, x_2, \dots, x_{I_{t,r}}\}$ of F and a constant C such that*

$$\mu([F]_t \cap B(x,r)) \leq C \sum_{i=1}^{I_{t,r}} \mu(B(x_i,t)),$$

with $I_{t,r} \leq C\frac{h(r)}{h(t)}$ for every $x \in F$, $r > 0$ and $0 < t \leq r$.

Proof. Fix $x \in F$, $r > 0$ and $0 < t \leq r$. Since (X, d) has finite metric dimension and $F \cap B(x, 2Kr)$ is bounded, there exists a finite t-net U in $F \cap B(x, 2Kr)$, let us say $U = \{x_1, x_2, \ldots, x_{I_{t,r}}\}$. Hence $\{B(x_i, 2Kt) : i = 1, \ldots, I_{t,r}\}$ is a cover of $[F]_t \cap B(x, r)$, so that

$$\mu([F]_t \cap B(x,r)) \leq \sum_{i=1}^{I_{t,r}} \mu(B(x_i, 2Kt)) \leq C \sum_{i=1}^{I_{t,r}} \mu(B(x_i, t)).$$

To estimate $I_{t,r}$, notice that $B\left(x_i, \frac{t}{2K}\right) \cap B\left(x_j, \frac{t}{2K}\right) = \emptyset$ for $i \neq j$. Then, since F is h-set with a measure ν and h is doubling, we have

$$\begin{aligned} I_{t,r} h(t) &\leq C \sum_{i=1}^{I_{t,r}} \nu\left(B\left(x_i, \frac{t}{2K}\right)\right) \\ &= C\nu\left(\bigcup_{i=1}^{I_{t,r}} B\left(x_i, \frac{t}{2K}\right)\right) \\ &\leq C\nu(B(x, 3K^2 r)) \\ &\leq Ch(r). \end{aligned}$$

So that $I_{t,r} \leq Ch(r)/h(t)$. □

Proof of Theorem 4. Fix $x \in F$ and $r > 0$. For each $0 < t \leq r$, let $\{x_1, x_2, \ldots, x_{I_{t,r}}\} \subseteq F$ as in Lemma 5. Then

$$\begin{aligned} \int_0^r \frac{\mu([F]_t \cap B(x,r))}{t w(t)}\, dt &\leq C \int_0^r \sum_{i=1}^{I_{t,r}} \frac{\mu(B(x_i, t))}{t w(t)}\, dt \\ &\leq Ch(r) \int_0^r \frac{g(t)}{t w(t) h(t)}\, dt \\ &\leq Ch(r) \frac{g(r)}{w(r) h(r)} \\ &\leq C \frac{\mu(B(x,r))}{w(r)}. \end{aligned}$$

□

Remark 5. Observe that if we take $h(t) = t^s$, $g(t) = t^\alpha$ and $w(t) = t^\beta$ we recover the result given in [ACDT14], that is, if (X, d, μ) is an α-Ahlfors regular space and F is an s-set with $0 \leq s < \alpha$, then $d(x, F)^\beta \in A_p(X, d, \mu)$ for every $(s - \alpha) < \beta < (\alpha - s)(p - 1)$. Indeed, in this case it is easy to see that $t^{\alpha - s + \beta}$ and $t^{\alpha - s - \frac{\beta}{p-1}}$ belong to $\mathcal{Z}^0$. This implies that $w^{-1}, w^{\frac{1}{p-1}} \in \mathcal{W}^F$, so that $w(d(x, F)) \in A_p(X, d, \mu)$ from Theorem 1.

The class of function that satisfies the hypothesis of Theorem 4 are not only the power functions. In fact, we can use more general functions g and h. To prove this we shall need the following definitions.

Let $\phi : [0, +\infty] \to [0, +\infty]$. The function $\phi(t)$ is said to be of ***lower type $\boldsymbol{\alpha}$***, $0 \leq \alpha < \infty$, if there exists a constant $C > 0$ such that

$$\phi(\lambda t) \leq C\lambda^\alpha \phi(t),$$

for every $0 \leq \lambda < 1$. Similarly, $\phi(t)$ is of ***upper type*** $\boldsymbol{\alpha}$, $0 \leq \alpha < \infty$ if there exists a constant $C > 0$ such that

$$\phi(\lambda t) \leq C\lambda^{\alpha}\phi(t),$$

for every $\lambda \geq 1$.

It is clear that if ϕ is of lower type α and $0 \leq \beta < \alpha$, then ϕ is of lower type β. Analogously, if ϕ is of upper type α and $\beta > \alpha$, then ϕ is of upper type β. Then we define the ***lower and upper indices*** respectively by

$$i(\phi) = \sup\{\alpha : \phi \text{ is of lower type } \alpha\},$$

and

$$I(\phi) = \inf\{\alpha : \phi \text{ is of upper type } \alpha\}.$$

Next lemma summarizes some useful and known facts about the above functions that we will need in the following.

Lemma 6. *Let $\phi : [0, +\infty] \to [0, +\infty]$ given. Then:*

(i) *ϕ is of lower type α for some $0 \leq \alpha < \infty$ if and only if it is quasi-increasing;*
(ii) *ϕ is of upper type α if and only if the function $\phi(t)/t^{\alpha}$ is quasi-decreasing;*
(iii) *ϕ is of upper type α for some $0 \leq \alpha < \infty$ if and only if it satisfies the doubling condition.*

Theorem 7. *Let (X, d, μ) be an g-Ahlfors regular space, F an h-set and $w(t) = h(t)/g(t)$. If $I(h) < i(g)$, then $w(d(x, F))^{\beta} \in A_p(X, d, \mu)$ for every $(1 - p) < \beta < 1$.*

Proof. Fix $(1 - p) < \beta < 1$. From Theorem 4 and 1, it is sufficient to see that the functions $(g/h)^{1-\beta}$ and $(g/h)^{1+\frac{\beta}{p-1}}$ belong to $\mathcal{Z}^0$.

Notice that g is of lower type α for every $\alpha < i(g)$ and h is of upper type γ for every $\gamma > I(h)$. We shall fix $\varepsilon > 0$ and define $\alpha = i(g) - \varepsilon$ and $\gamma = i(g)$. For each $0 \leq t \leq r$ we have

$$g(t) \leq C\left(\frac{t}{r}\right)^{\alpha} g(r), \qquad h(r) \leq C\left(\frac{r}{t}\right)^{\gamma} h(t).$$

Then, if $\delta > 0$, we have

$$\begin{aligned}
\int_0^r \left(\frac{g(t)}{h(t)}\right)^{\delta} \frac{dt}{t} &\leq C\left(\frac{g(r)}{h(r)}\right)^{\delta} \int_0^r \left(\frac{t}{r}\right)^{\alpha\delta} \left(\frac{t}{r}\right)^{-\gamma\delta} \frac{dt}{t} \\
&= C\left(\frac{g(r)}{h(r)}\right)^{\delta} r^{\varepsilon} \int_0^r \frac{1}{t^{1-\varepsilon\delta}}\, dt \\
&= C\left(\frac{g(r)}{h(r)}\right)^{\delta}.
\end{aligned}$$

In other words, $(g/h)^{\delta} \in \mathcal{Z}^0$ for every $\delta > 0$. Since $(1 - p) < \beta < 1$, we obtain $(g/h)^{1-\beta}, (g/h)^{1+\frac{\beta}{p-1}} \in \mathcal{Z}^0$, as desired. $\square$

We shall finish this note with an example of Muckenhoupt weight obtained as a consequence of Theorem 7.

Let us consider $\mathbb{R}^n$ equipped with the usual distance d and the n-dimensional Lebesgue measure μ. So that $(\mathbb{R}^n, d, \mu)$ is an g-Ahlfors space, with $g(t) = t^n$.

We shall fix constants s and γ, and we shall consider the function h defined on $(0, 1)$ by

$$h(t) = t^s e^{|\log t|^\gamma}.$$

From [Bri02, Prop. 7.5], there exists a compact h-set F for each $0 < s < n$ and $0 < \gamma < \frac{1}{2}$ given. Since $e^{|\log t|^\gamma}$ is decreasing, for each $\lambda \geq 1$ we have

$$h(\lambda t) \leq \lambda^s h(t),$$

so that $I(h) \leq s < n = i(g)$. Then, from Theorem 7 we have that

$$d(x, F)^{\beta(s-n)} e^{\beta|\log d(x,F)|^\gamma} \in A_p(\mathbb{R}^n),$$

for every $(1 - p) < \beta < 1$.

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