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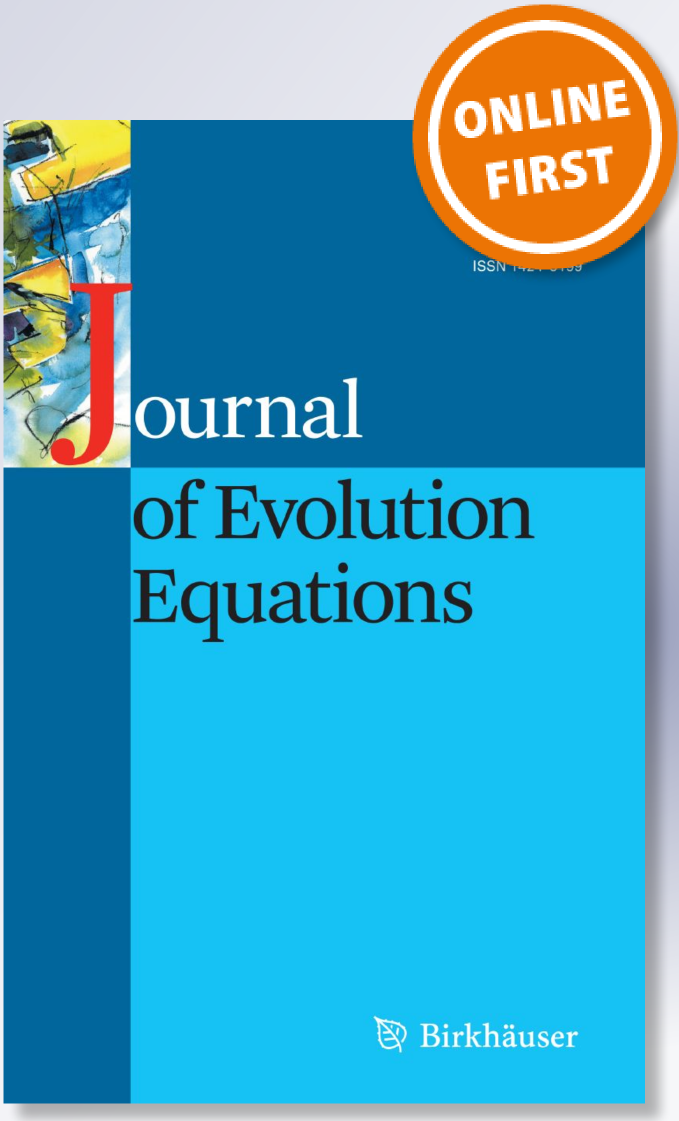

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Approximation of solutions of fractional diffusions in compact metric measure spaces

MARCELO ACTIS AND HUGO AIMAR

Abstract. In this note we prove that the solutions to diffusions associated with fractional powers of the Laplacian in compact metric measure spaces can be obtained as limits of the solutions to particular rescalings of some non-local diffusions with integrable kernels. The abstract approach considered here has several particular and interesting instances. As an illustration of our results, we present the case of dyadic metric measure spaces where the existence of solutions was already been proven in Actis and Aimar (Fract Calc Appl Anal 18(3):762–788, 2015).

1. Introduction

The Cauchy problem for the heat equation in $\mathbb{R}^n$, i.e., $u_t = \Delta u$ in $\mathbb{R}^{n+1}_+$, with $u(x,0) = u_0$ in $\mathbb{R}^n$, admits an immediate generalization to the case of non-local diffusions. In this case, the Laplacian in the space variables is replaced by the fractional Laplacian operator of order s with $0 < s < 2$, which is given by

$$-(-\Delta)^{s/2} f(x) = c_{n,s}\, v.p. \int \frac{f(x) - f(y)}{|x-y|^{n+s}}\, dy, \tag{1.1}$$

and is a representation of the generalized *Dirichlet to Neumann operator* (see [7]). For the semigroup approach to the theory see also [16]. The standard linear evolution equation $u_t = -(-\Delta)^{s/2} u$ involving the fractional Laplacian has been widely studied and usually used in modeling processes like anomalous diffusion (see [17] and the references therein).

The aim of this paper is to approach the study of fractional diffusions in metric measure spaces where despite of the lack of differential structure in these contexts, some problems associated with non-local operators can be considered. As it is explicitly observed in [5], usually the solutions to non-local evolution equations with integrable kernels approximate solutions of some classical local evolution problems such as the heat equation (see [10]). What we do here is to extend this basic principle both to non-local and to non-Euclidean settings. For a related approach in the Euclidean case, see [11,12].

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Let us observe that there are several settings where the application of our main result, contained in Theorem 9, can provide good approximation of solutions: Euclidean domains, the n-dimensional torus, classical fractals, etc. These seemingly diverse settings can be unified in their approach by noticing that all of them have the same structural form in Ahlfors regular spaces. We say that a metric measure space (X, d, μ) is an Ahlfors α-regular space if there exists two positive constants, say c_1 and c_2, such that

$$c_1 r^\alpha \le \mu(B(x,r)) \le c_2 r^\alpha,$$

for all $x \in X$ and for all real number $r < 2\,\mathrm{diam}(X)$, where $B(x,r)$ denotes the d-ball centered in x of radius r and $\mathrm{diam}(X) = \sup\{\mathrm{d}(x,y) : x, y \in X\}$ denotes the diameter of the whole space X.

In this generalized context, the fractional diffusion problem takes the form

$$\begin{cases} u_t(x,t) = -D^s u(x,t), & x \in X, t \in (0,T), \\ u(x,0) = u_0(x), & x \in X, \end{cases} \tag{1.2}$$

where D^s is the natural extension given by (1.1) of the fractional Laplacian to Ahlfors α-regular spaces, i.e.,

$$D^s f(x) = \int_X \frac{f(x) - f(y)}{\mathrm{d}(x,y)^{\alpha+s}} \mathrm{d}\mu(y).$$

The basic difficulty in this problem is the (local) non-integrability of the kernel $k_s = \mathrm{d}(x,y)^{-\alpha-s}$, so that the methods in [9] are not directly applicable. Nevertheless, in this note we prove that problem (1.2) can be regarded as the limit of a family of problems, each of them built on a adequate rescaling of an integrable kernel J.

As a first elementary step, in order to approximate (1.2), we use the Banach fixed point method to solve the problems of the type

$$\begin{cases} u_t(x,t) = \displaystyle\int_X J(x,y)[u(y,t) - u(x,t)]\mathrm{d}\mu(y), & x \in X, t \in (0,T), \\ u(x,0) = u_0(x), & x \in X. \end{cases} \tag{1.3}$$

where the kernels $J : X \times X \to \mathbb{R}^+$ are integrable and Lipschitz continuous on each variable uniformly in the other. Some previous work in this direction can be found in [6,15], nevertheless we shall deal with different function spaces. Then we obtain approximations of k_s by a family $\{J_\epsilon : \epsilon > 0\}$ of kernels with the above described properties. We prove that the solution u of (1.2), provided its existence, is the uniform limit of the solutions u^ϵ of (1.3) associated with the kernels J_ϵ.

The paper is organized in six sections. In the second one we give the description of the setting, we prove existence of solutions of (1.3) and we state a comparison principle. In Sect. 3 we consider the approximation of D^s by non-local operators with integrable kernels. In Sect. 4 we state and prove our main result, where we show how the solutions of the scaled problems with integrable kernels approximates solutions of the fractional diffusion. In the Sect. 5 we provide an application of our result to a non-classic context: dyadic metric measure spaces. Finally in Sect. 6 we present some conclusion and remarks.

2. Setup and preliminary results

The results of this section are quite general and do not depend on any doubling or homogeneity property of the underlying metric measure space (X, d, μ). Let us start by introducing some basic notation and definitions. The homogeneous Lipschitz continuous space of order $r > 0$, denoted by $\Lambda_r(X, d, \mu)$, is the space of functions in $L^\infty(X, d, \mu)$ such that for some positive constant C

$$|f(x) - f(y)| \leq C\mathrm{d}(x, y)^r, \tag{2.1}$$

for all $x, y \in X$. Actually, $\Lambda_r(X, d, \mu)$ can be equipped with the norm

$$\|f\|_{\Lambda_r} := \|f\|_\infty + [f]_{\Lambda_r},$$

where $[f]_{\Lambda_r}$ is the seminorm given by

$$[f]_{\Lambda_r} := \sup_{x \in X} \sup_{y \in X \setminus \{x\}} \frac{|f(x) - f(y)|}{\mathrm{d}(x, y)^r}.$$

Let $J : X \times X \to \mathbb{R}^+$ be a nonnegative measurable function with respect to the product σ-algebra in $X \times X$ satisfying the following properties

(i) $J(x, y) = J(y, x)$, for all $x, y \in X$;
(ii) J is integrable in each variable uniformly in the other, i.e.

$$\int_X J(x, y)\mathrm{d}\mu(y) \leq \beta, \quad \text{for every } x \in X.$$

(iii) $J(\cdot, y)$ is Lipschitz continuous of order $r > 0$ uniformly in $y \in X$, i.e.,

$$[J(\cdot, y)]_{\Lambda_r} \leq \lambda, \quad \text{for every } y \in X.$$

Note that in our case X is compact, so the L^∞-norm is controlled by the Lipschitz seminorm. Then property (iii) implies (ii).

Given $T \in \mathbb{R}^+$ fixed and $u_0 \in \Lambda_r(X, d, \mu)$ we consider the non-local problem

$$\begin{cases} u_t(x, t) = \int_X J(x, y)[u(y, t) - u(x, t)]\mathrm{d}\mu(y), & x \in X, t \in [0, T], \\ u(x, 0) = u_0(x), & x \in X. \end{cases} \tag{2.2}$$

We say that a function u is a *solution* of (2.2) if u belongs to mixed space

$$\mathbb{B}_{\Lambda_r} = C^1((0, T); \Lambda_r(X, d, \mu)) \cap C(([0, T]; \Lambda_r(X, d, \mu))$$

and satisfies

$$u(x,t) = u_0(x) + \int_0^t \int_X J(x,y)(u(y,s) - u(x,s))\,\mathrm{d}\mu(y)\,\mathrm{d}s,$$

where the integral in the right hand side is formally understood as a Bochner integral. Existence and uniqueness of solutions of problem (2.2) are consequences of the Banach fixed point theorem and will be proved in Theorem 3. Before, let us state two auxiliary lemmas.

Given $t_0 > 0$, let $\mathbb{X}_{t_0}$ be the space of continuous functions from $[0, t_0]$ to $\Lambda_r(X, d, \mu)$, i.e.,

$$\mathbb{X}_{t_0} = C([0, t_0]; \Lambda_r(X, d, \mu)),$$

which is a Banach space equipped with the norm

$$|||w|||_r = \max_{t\in[0,t_0]} \|w(\cdot, t)\|_{\Lambda_r}.$$

For any $w_0 \in \Lambda_r(X, d, \mu)$, let T_{w_0} be the operator defined on $\mathbb{X}_{t_0}$ by

$$T_{w_0}(w)(x,t) = w_0(x) + \int_0^t \int_X J(x,y)(w(y,s) - w(x,s))\,\mathrm{d}\mu(y)\,\mathrm{d}s. \tag{2.3}$$

LEMMA 1. *The operator T_{w_0} maps $\mathbb{X}_{t_0}$ into $\mathbb{X}_{t_0}$.*

Proof. Note that for any $t \in \mathbb{R}^+$ and any $x, z \in X$ we have that

$$\begin{aligned}
|T_{w_0}(w)(x,t) - T_{w_0}(w)(z,t)| \le & \left| \int_0^t \int_X [J(x,y) - J(z,y)] w(y,s)\,\mathrm{d}\mu(y)\,\mathrm{d}s \right| \\
& + \left| \int_0^t \int_X [J(x,y) - J(z,y)] w(z,s)\,\mathrm{d}\mu(y)\,\mathrm{d}s \right| \\
& + \left| \int_0^t \int_X J(x,y)[w(x,s) - w(z,s)]\,\mathrm{d}\mu(y)\,\mathrm{d}s \right| \\
\le & \, 2t\mu(X) \sup_{y\in X} [J(\cdot, y)]_{\Lambda_r} \sup_{t\in[0,t_0]} \|w(\cdot,t)\|_\infty \mathrm{d}(x,z)^r \\
& + t\mu(X) \int_X J(x,y)\mathrm{d}\mu(y) \sup_{t\in[0,t_0]} [w(\cdot,t)]_{\Lambda_r} \mathrm{d}(x,z)^r
\end{aligned}$$

Hence, from properties (ii) and (iii) of the kernel J, we obtain

$$|T_{w_0}(w)(x,t) - T_{w_0}(w)(z,t)| \le Ct\, |||w|||_r\, \mathrm{d}(x,z)^r. \tag{2.4}$$

Then $[T_{w_0}(w)(\cdot,t) - w_0]_{\Lambda_r} \le Ct$, which proves the continuity at $t = 0$. In a similar way, if $t_1, t_2 \in \mathbb{R}^+$ such that $0 < t_1 < t_2 \le 0$ we see that

$$[T_{w_0}(w)(\cdot, t_1) - T_{w_0}(w)(\cdot, t_2)]_{\Lambda_r} \le C(t_1 - t_2).$$

Hence $T_{w_0}(w) \in \mathbb{X}_{t_0}$. □

LEMMA 2. *Let $w, v \in \mathbb{X}_{t_0}$ then*

$$\left|\left|\left|T_{w_0}(w) - T_{w_0}(v)\right|\right|\right|_r \leq Ct_0 \, |||w - v|||_r \, .$$

Proof. Let $0 < t < t_0$ and $u := w - v$. Note that $[T_{w_0}(w) - T_{w_0}(v)]_{\Lambda_r} = [T_{w_0}(u)]_{\Lambda_r}$. Since by (2.4)

$$|T_{w_0}(u)(x, t) - T_{w_0}(u)(z, t)| \leq Ct \, |||u|||_r \, \mathrm{d}(x, z)^r,$$

then $[T_{w_0}(u)]_{\Lambda_r} \leq Ct \, |||u|||_r$. Therefore $[T_{w_0}(w) - T_{w_0}(v)]_{\Lambda_r} \leq Ct_0 \, |||w - v|||_r$ as desired. □

THEOREM 3. (Existence and uniqueness) *Let $u_0 \in \Lambda_r(X, d, \mu)$ and J satisfying (i), (ii) and (iii). Then there exists a unique solution $u \in \mathbb{B}_{\Lambda_r}$ of* (2.2).

Proof. Taking t_0 in Lemma 2 satisfying $Ct_0 < 1$ we obtain that T_{u_0} is a contractive operator on $\mathbb{X}_{t_0}$. Then the existence and uniqueness of a solution satisfying (2.2) follows from the Banach fixed point theorem on the interval $[0, t_0]$.

To extend the solution to $[0, T]$, we take as initial data $u(x, t_0) \in \Lambda_r(X, d, \mu)$ to obtain a solution up to $[0, 2t_0]$. Iterating this process we get a solution defined on $[0, T]$. □

Finally, we present our last preliminary result before dealing with the approximation of D^s. It is a comparison principle which shall be useful at proving our main result.

We say that a function $u \in \mathbb{B}_C = C^1((0, T), C(X)) \cap C([0, T], C(X))$ is a *supersolution* of (2.2) if

$$\begin{cases} u_t(x, t) \geq \displaystyle\int_X J(x, y)[u(y, t) - u(x, t)] \, \mathrm{d}\mu(y), & x \in X, t \in (0, T), \\ u(x, 0) \geq u_0(x), & x \in X. \end{cases}$$

LEMMA 4. (Comparison principle) *Let $u \in \mathbb{B}_C$ be a supersolution of* (2.2) *with initial datum $u_0 \in C(X)$ and such that $u_0 \geq 0$. Then $u \geq 0$.*

Proof. Suppose that u is negative somewhere. Let $v(x, t) = u(x, t) + \epsilon t$, with ϵ small enough to make v negative at some point. So, if (x_0, t_0) is the point where v reaches its minimum, then $t_0 > 0$ (since $v(x, 0) = u(x, 0) \geq 0$). Further

$$\begin{aligned} v_t(x_0, t_0) = u_t(x_0, t_0) + \epsilon &> \int_X J(x, y)[u(y, t_0) - u(x, t_0)] \mathrm{d}\mu(y) \\ &= \int_X J(x, y)[v(y, t_0) - v(x, t_0)] \mathrm{d}\mu(y) \geq 0. \end{aligned}$$

Therefore, $v_t(x_0, t_0) > 0$ which contradicts the fact that (x_0, t_0) is a point where v reaches its minimum. Then, $u \geq 0$. □

3. Approximation of D^s by rescaling kernels

In this section (X, d, μ) is Ahlfors α-regular space. Precisely, (X, d) is a metric space and measure and metric are related by

$$c_1 r^\alpha \leq \mu(B(x, r)) \leq c_2 r^\alpha, \tag{3.1}$$

for some positive constants c_1 and c_2 and for every $x \in X$ and $r < 2\,\mathrm{diam}(X)$. This situation, although restrictive, is natural in several classic geometric contexts such as Riemannian manifolds and even fractals coming from iterated function systems like the Cantor set or the Sierpinski gasket (see [14]).

The first result in this section is an elementary lemma which reflects the α dimensional character of X under the assumption (3.1). The result is well known (see for example [1]), but we include it for the sake of completeness. For notational simplicity we shall write $A \simeq B$ when the quotient A/B is bounded above and below by positive and finite constants. In a similar way, we write $A \lesssim B$ when A/B is bounded above.

LEMMA 5. *Let (X, d, μ) be an Ahlfors α-regular space. Then for any $\delta > 0$ and any $\epsilon > 0$ we have that*

$$\int_{B(x,\epsilon)} \frac{d\mu(y)}{d(x, y)^{\alpha-\delta}} \simeq \epsilon^\delta,$$

and

$$\int_{X\setminus B(x,\epsilon)} \frac{d\mu(y)}{d(x, y)^{\alpha+\delta}} \simeq \epsilon^{-\delta}$$

where the hidden constants only depend on α and δ.

Proof. In order to prove the first estimate, let us rewrite $B(x, \epsilon)$ as the union of annuli of the form $A_j = B(x, 2^{-(j-1)}\epsilon)\setminus B(x, 2^{-j}\epsilon)$, with $j \in \mathbb{N}$. Hence

$$\begin{aligned}
\int_{B(x,\epsilon)} \frac{\mathrm{d}\mu(y)}{\mathrm{d}(x, y)^{\alpha-\delta}} &= \sum_{j=1}^{\infty} \int_{A_j} \frac{\mathrm{d}\mu(y)}{\mathrm{d}(x, y)^{\alpha-\delta}} \\
&\leq \sum_{j=1}^{\infty} \left(2^{-j}\epsilon\right)^{-\alpha+\delta} \int_{A_j} \mathrm{d}\mu(y) \\
&\leq \sum_{j=1}^{\infty} \left(2^{-j}\epsilon\right)^{-\alpha+\delta} \mu(B(x, 2^{-(j-1)}\epsilon)).
\end{aligned}$$

Therefore, from the upper bound in (3.1) we obtain

$$\int_{B(x,\epsilon)} \frac{\mathrm{d}\mu(y)}{\mathrm{d}(x, y)^{\alpha-\delta}} \leq c_2 \sum_{j=1}^{\infty} \left(2^{-j}\epsilon\right)^{-\alpha+\delta} (2^{-(j-1)}\epsilon)^\alpha \leq c_2 \frac{2^\alpha}{2^\delta - 1}\epsilon^\delta.$$

In an analogous way, it can be proved the lower bound and also the second estimate over $X\setminus B(x, \epsilon)$. □

LEMMA 6. *For $f \in \Lambda_r(X, d, \mu)$ with $s < r \leq 1$, the integral*

$$\int_X \frac{f(x) - f(y)}{d(x, y)^{\alpha+s}} d\mu(y)$$

is absolutely convergent and bounded by the Λ_r norm of f.

Proof. Setting $B := B(x, 1)$, then

$$\int_X \frac{f(x) - f(y)}{\mathrm{d}(x, y)^{\alpha+s}} \mathrm{d}\mu(y) = \int_B \frac{f(x) - f(y)}{\mathrm{d}(x, y)^{\alpha+s}} \mathrm{d}\mu(y) + \int_{X\setminus B} \frac{f(x) - f(y)}{\mathrm{d}(x, y)^{\alpha+s}} \mathrm{d}\mu(y). \tag{3.2}$$

Since f satisfies (2.1) then

$$\int_B \frac{f(x) - f(y)}{\mathrm{d}(x, y)^{\alpha+s}} \mathrm{d}\mu(y) \leq [f]_{\Lambda_r} \int_B \frac{\mathrm{d}\mu(y)}{\mathrm{d}(x, y)^{\alpha-(r-s)}}.$$

Then from Lemma 5 we obtain

$$\int_B \frac{f(x) - f(y)}{\mathrm{d}(x, y)^{\alpha+s}} \mathrm{d}\mu(y) \lesssim [f]_{\Lambda_r}. \tag{3.3}$$

To estimate the integral over $X\setminus B$ of (3.2) we use the fact that f is bounded and again the Lemma 5,

$$\int_{X\setminus B} \frac{f(x) - f(y)}{\mathrm{d}(x, y)^{\alpha+s}} \mathrm{d}\mu(y) \leq 2\|f\|_{L^\infty} \int_{X\setminus B} \frac{\mathrm{d}\mu(y)}{\mathrm{d}(x, y)^{\alpha+s}} \lesssim \|f\|_{L^\infty}. \tag{3.4}$$

Therefore, from (3.2), (3.3) and (3.4) we get that

$$\left| \int_X \frac{f(x) - f(y)}{\mathrm{d}(x, y)^{\alpha+s}} \mathrm{d}\mu(y) \right| \lesssim \|f\|_{\Lambda_r},$$

so the proof is completed. □

The operator that assigns to every $f \in \Lambda_r(X, d, \mu)$ the function

$$D^s f(x) = \int_X \frac{f(x) - f(y)}{\mathrm{d}(x, y)^{\alpha+s}} \mathrm{d}\mu(y),$$

is called the fractional derivative operator of order s. Note that D^s has a (local) non-integrable kernel $k_s = \mathrm{d}(x, y)^{-\alpha-s}$. However, k_s can be regarded as the limit of a family of integrable kernels. In order to build this kernels, take $\psi : \mathbb{R}^+ \to \mathbb{R}_0^+$ defined by

$$\psi(t) = \begin{cases} 1, & \text{si } t < 1, \\ t^{-\alpha-s}, & \text{si } t \geq 1. \end{cases}$$

For each $0 < \epsilon \leq 1$ we define the kernels J_ϵ in the following way,

$$J_\epsilon(x, y) := \frac{1}{\epsilon^\alpha} \psi\left(\frac{\mathrm{d}(x, y)}{\epsilon}\right). \tag{3.5}$$

LEMMA 7. *The kernels* J_ϵ *defined by* (3.5) *are symmetric and positive. Moreover,*

$$\int_X J_\epsilon(x, y)\, d\mu(y) \simeq C, \tag{3.6}$$

where C *is a constant independent of* ϵ, *and for any* $r \in (0, 1]$ *we have*

$$[J_\epsilon(\cdot, y)]_{\Lambda_r} \lesssim \epsilon^{-(\alpha+r)}. \tag{3.7}$$

Proof. The symmetry and the positivity are inherited from the distance d and the function ψ, respectively. Besides,

$$\begin{aligned}\int_X J_\epsilon(x, y)\, \mathrm{d}\mu(y) &= \frac{1}{\epsilon^\alpha}\int_X \psi\left(\frac{\mathrm{d}(x, y)}{\epsilon}\right) \mathrm{d}\mu(y)\\ &= \frac{1}{\epsilon^\alpha}\left[\int_{B(x,\epsilon)} \mathrm{d}\mu(y) + \epsilon^{\alpha+s}\int_{X\setminus B(x,\epsilon)} \frac{\mathrm{d}\mu(y)}{\mathrm{d}(x, y)^{\alpha+s}}\right].\end{aligned}$$

Hence, by (3.1) and Lemma 5 we obtain (3.6).

On the other hand, since $|\psi'| \le \alpha + s$ we have

$$\begin{aligned}|J_\epsilon(x, y) - J_\epsilon(z, y)| &= \frac{1}{\epsilon^\alpha}\left|\psi\left(\frac{\mathrm{d}(x, y)}{\epsilon}\right) - \psi\left(\frac{\mathrm{d}(z, y)}{\epsilon}\right)\right|\\ &\le \frac{[\psi]_{\Lambda_r}}{\epsilon^\alpha}\left|\frac{\mathrm{d}(x, y)}{\epsilon} - \frac{\mathrm{d}(z, y)}{\epsilon}\right|^r\\ &\le \frac{\alpha + s}{\epsilon^{\alpha+r}}\mathrm{d}(x, y)^r,\end{aligned}$$

which implies (3.7), so the proof is completed. □

Finally let L_ϵ be an operator given by

$$L_\epsilon f(x) = \frac{1}{\epsilon^s}\int_X J_\epsilon(x, y)[f(x) - f(y)]\, \mathrm{d}\mu(y).$$

The next statement shows that L_ϵ converge weakly to D^s as $\epsilon \to 0$.

THEOREM 8. (Weak approximation) *Let* $f \in \Lambda_r(X, d, \mu)$ *then*

$$\sup_{x\in X} |L_\epsilon f(x) - D^s f(x)| \lesssim [f]_{\Lambda_r}\epsilon^{r-s}.$$

Proof. Since $f \in \Lambda_r(X, d, \mu)$ then $D^s f$ and $L_\epsilon f$ are well defined. Furthermore note that

$$\left|\frac{1}{\epsilon^s}J_\epsilon(x, y) - k_s(x, y)\right| \le \frac{\chi_{\{\mathrm{d}(x,y)<\epsilon\}}}{\mathrm{d}(x, y)^{\alpha+s}}.$$

Hence we obtain that

$$\begin{aligned}|L_\epsilon f(x) - D^s f(x)| &= \left| \int_X \left[\frac{1}{\epsilon^s} J_\epsilon(x, y) - k_s(x, y) \right] [f(x) - f(y)]\, \mathrm{d}\mu(y) \right| \\ &\leq \int_{B(x,\epsilon)} \frac{|f(x) - f(y)|}{\mathrm{d}(x, y)^{\alpha+s}}\, \mathrm{d}\mu(y) \\ &\leq [f]_{\Lambda_r} \int_{B(x,\epsilon)} \frac{\mathrm{d}\mu(y)}{\mathrm{d}(x, y)^{\alpha-(r-s)}}.\end{aligned}$$

Thus, by Lemma 5 we get that

$$|L_\epsilon f(x) - D^s f(x)| \lesssim [f]_{\Lambda_r} \epsilon^{r-s}$$

and so the result is immediate. □

4. Main result

Since by Lemma 7 for each $\epsilon \in (0, 1)$ the kernel J_ϵ defined in (3.5) satisfies (i), (ii) and (iii), then from Theorem 3 the problem

$$\begin{cases} u_t(x, t) = -L_\epsilon u(x, t), & x \in X, t \in (0, T) \\ u(x, 0) = u_0(x), & x \in X. \end{cases} \tag{4.1}$$

has a unique solution $u^\epsilon \in \mathbb{B}_{\Lambda_r}$.

The next theorem shows that, provided the existence of a solution u of problem 1.2 then the solutions u^ϵ of the problems (4.1) converge to u as $\epsilon \to 0^+$.

THEOREM 9. *Let $u_0 \in \Lambda_r(X, d, \mu)$ and let $s, r \in \mathbb{R}$ be such that* $0 < s < r \leq 1$. *Suppose there exists a solution $u(x, t) \in \mathbb{B}_{\Lambda_r}$ of the problem* (1.2). *Then the solutions u^ϵ of the problems* (4.1) *satisfy*

$$\sup_{t\in[0,T]} \sup_{x\in X} |u(x, t) - u^\epsilon(x, t)| \lesssim T\epsilon^{r-s}.$$

Proof. Let $w^\epsilon = u - u^\epsilon$. Observe that

$$\begin{cases} w_t^\epsilon(x, t) = -L_\epsilon w^\epsilon(x, t) + F_\epsilon(x, t), & x \in X, t \in (0, T), \\ w^\epsilon(x, 0) = 0, & x \in X, \end{cases}$$

where $F_\epsilon(x, t) = L_\epsilon u(x, t) - D^s u(x, t)$.

Define $\overline{z} = k\epsilon^{r-s} t - w^\epsilon$, where k is a arbitrary constant. Observe that

$$\overline{z}_t(x, t) = k\epsilon^{r-s} - w_t^\epsilon(x, t) = k\epsilon^{r-s} - (F_\epsilon(x, t) - L_\epsilon w^\epsilon(x, t)).$$

We already know by Theorem 8 that $|F_\epsilon(x,t)| \lesssim \epsilon^{r-s}$. Thus, choosing k large enough we have that $k\epsilon^\theta - F_\epsilon(x,t) \geq 0$. Then

$$\overline{z}_t(x,t) \geq L_\epsilon w^\epsilon(x,t) = -L_\epsilon \overline{z}(x,t).$$

Therefore $\overline{z}$ is a supersolution of the problem (4.1). Since $\overline{z}(x,0) = 0$ then by Lemma 4 it turns out that $\overline{z}(x,t) \geq 0$. Hence $w^\epsilon(x,t) \leq k\epsilon^{r-s}t$. In a similar way, if we define $\underline{z}(x,t) = k\epsilon^{r-s}t + w^\epsilon(x,t)$ we can prove that $\underline{z}(x,t) \geq 0$ and so $w^\epsilon(x,t) \geq k\epsilon^{r-s}t$. Thereby

$$|v(x,t) - u^\epsilon(x,t)| = |w^\epsilon(x,t)| \leq k\epsilon^{r-s}t$$

which implies that

$$\sup_{t\in[0,T]} \sup_{x\in X} |v(x,t) - u^\epsilon(x,t)| \lesssim T\epsilon^{r-s}.$$

□

One main disadvantage of the abstract approach considered here is that we required the a priori existence of solution of problem (1.2), which is not known in plenty of contexts. However, in the next section, we will exemplify how this theorem works in a context where we already know the existence and, even more, the structure of a solution for (1.2).

5. Application to dyadic operators

Let us provide a non-classic context in which our main result applies and where the approximation process can be appreciated. Our example deals with a dyadic version of the operator D^s considered in [1] in the case of spaces of homogeneous type. Let us briefly introduce the setting. For a more detailed approach, see [1].

Let (X, d, μ) be a compact space of homogeneous type (see [13]). Let $\mathscr{D}$ be a dyadic family in X as constructed by Christ in [8]. Let $\mathscr{H}$ be a Haar system for $\mathbb{L}^p(X,\mu) = \{f \in L^p(X,\mu) : \int_X f\,\mathrm{d}\mu = 0\}$ associated with $\mathscr{D}$ as built in [4]. The system $\mathscr{H}$ is an unconditional basis for $\mathbb{L}^p(X,\mu)$, for $1 < p < \infty$ (see [4]). By $Q(h)$ we denote the dyadic cube on which h *is based*, i.e., the smallest member of $\mathscr{D}$ containing the set $\{x \in X : h(x) \neq 0\}$.

A distance in X associated with $\mathscr{D}$ can be defined by $\delta(x,y) = \min\{\mu(Q) : Q \in \mathscr{D}$ such that $x, y \in Q\}$ when $x \neq y$ and $\delta(x,x) = 0$. The space X equipped with δ and μ turns out to be a 1-Ahlfors regular space. In this context, the fractional differential operator of order s, with $0 < s < 1$, is given by

$$D^s f(x) = \int_X \frac{f(x) - f(y)}{\delta(x,y)^{1+s}} \mathrm{d}\mu(y).$$

Since the Haar functions are Lipschitz functions of order 1 with respect to δ, the operator D^s is well defined on the Haar system. Moreover, each $h \in \mathscr{H}$ is an eigenfunction of D^s, indeed

$$D^s h(x) = m_h \mu(Q(h))^{-s} h(x),$$

where m_h are real numbers bounded above and below by positive constants (see Theorem 3.1 in [1]). Hence, in this context D^s also takes the form

$$D^s f(x) = \sum_{h \in \mathscr{H}} m_h \mu(Q(h))^{-s} \langle f, h \rangle h(x).$$

The solution of problem (1.2) is given by

$$u(x, t) = \sum_{h \in \mathscr{H}} e^{-m_h \mu(Q(h))^{-s} t} \langle u_0, h \rangle h(x) \tag{5.1}$$

(see Theorem 4.2 in [1]). If the initial datum u_0 belongs to $\Lambda_r(X, \delta, \mu)$ it can easily be prove that u also belongs to $\Lambda_r(X, \delta, \mu)$ for every $t > 0$ (see Theorem 1.2 in [3]). Therefore, all the hypothesis of Theorem 9 are fulfilled, and it can be applied in this case. However, following the same techniques developed in [1] it can also be shown that the operator L_ϵ would take the form

$$L_\epsilon f(x) = \sum_{h \in \mathscr{H} \setminus \mathscr{H}_\epsilon} m_h \mu(Q(h))^{-s} \langle f, h \rangle h(x) + \sum_{h \in \mathscr{H}_\epsilon} m_h^\epsilon \mu(Q^\epsilon(h))^{-s} \langle f, h \rangle h(x),$$

where $\mathscr{H}_\epsilon = \{h \in \mathscr{H} : \mu(Q(h)) < \epsilon\}$, $Q^\epsilon(h)$ is the biggest cube containing $Q(h)$ such $\mu(Q^\epsilon(h)) < \epsilon$ and m_h^ϵ are real numbers bounded above and below by positive constants. Even more the solution of problem (4.1) is given by

$$\begin{aligned} u^\epsilon(x, t) = &\sum_{h \in \mathscr{H} \setminus \mathscr{H}_\epsilon} e^{-m_h \mu(Q(h))^{-s} t} \langle u_0, h \rangle h(x) \\ &+ \sum_{h \in \mathscr{H}_\epsilon} e^{-m_h^\epsilon \mu(Q^\epsilon(h))^{-s} t} \langle u_0, h \rangle h(x). \end{aligned} \tag{5.2}$$

Knowing the explicit form of the solutions u^ϵ and u, we can estimate $|u(x, t) - u^\epsilon(x, t)|$ directly as follows.

$$\begin{aligned} |u(x, t) - u^\epsilon(x, t)| &\leq \sum_{h \in \mathscr{H}_\epsilon} |e^{-m_h \mu(Q(h))^{-s} t} - e^{-m_h^\epsilon \mu(Q^\epsilon(h))^{-s} t}| |\langle u_0, h \rangle| |h(x)| \\ &\leq 2 \sum_{h \in \mathscr{H}_\epsilon} |\langle u_0, h \rangle| |h(x)| \\ &\leq 2 \|u_0\|_{\Lambda_r} \sum_{h \in \mathscr{H}_\epsilon} \mu(Q(h))^r \mu(Q(h))^{1/2} \frac{\chi_{Q(h)}(x)}{\mu(Q(h))^{1/2}}, \end{aligned}$$

where the last inequality is consequence of Theorem 1.1 in [3]. Set $\mathscr{H}_\epsilon^x = \{h \in \mathscr{H}_\epsilon : x \in Q(h)\}$, then

$$\begin{aligned}|u(x,t) - u^\epsilon(x,t)| &\leq 2\|u_0\|_{\Lambda_r} \sum_{h \in \mathscr{H}_\epsilon^x} \mu(Q(h))^r \\ &\leq 2\|u_0\|_{\Lambda_r} \epsilon^r \sum_{k \in \mathbb{N}} (C^r)^k \\ &\lesssim \epsilon^r.\end{aligned}$$

Note that the last estimate gives a better order of convergence than our main theorem. This is because we have the exact solutions of problems (4.1) and (1.2) in this case, allowing us to compute explicitly the estimate. In the general context the better we have is order ϵ^{r-s}.

6. Conclusions and remarks

We have presented a new approach to approximate the solution of diffusions associated with fractional powers of the Laplacian as limits of the solutions to particular rescalings of some non-local diffusions with integrable kernels. The theory is valid in a general setting of compact metric measure spaces, which include fractals, manifolds and domains of $\mathbb{R}^n$ as particular cases. We proved error estimates in $L^\infty([0,T]; L^\infty(X,\mu))$ whenever the initial datum belongs to a Lipschitz spaces with regularity greater than the order of the fractional derivative.

We also have studied some existence theorems for non-local diffusions associated with integrable and Lipschitz kernels and a comparison principle.

The abstract approach considered here has several particular and interesting instances. One main disadvantage is that we required the a priori existence of solution of problem (1.2). However, we believe that is possible to prove the Cauchy character of the approximant sequence even when no assumption on the existence of solution for the fractional diffusion is made. To make this possible, we must prove some stability of the Lipschitz norm of the solutions of (4.1) in terms of the Lipschitz norm of the initial data independently of the norm of the kernel. Some numerical experiments made in [2] support empirically this conjecture. However we did not dwell on this matter in this article, but rather on the proposal of a first approximation method for fractional diffusions on Ahlfors regular spaces, and the proof of error estimates. We think this work can be approach in a future paper.

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Marcelo Actis and Hugo Aimar
Instituto de Matemática Aplicada del Litoral (UNL,CONICET)
IMAL-CCT-CONICET
Santa Fe S3007ABA
Argentina
E-mail: mactis@santafe-conicet.gov.ar
URL: http://www.imal.santefe-conicet.gov.ar

Hugo Aimar
E-mail: haimar@santafe-conicet.gov.ar