

An extrapolation theorem with applications to weighted estimates for singular integrals

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Abstract

We prove an extrapolation theorem saying that the weighted weak type $(1, 1)$ inequality for A_1 weights implies the strong $L^p(w)$ bound in terms of the $L^p(w)$ operator norm of the maximal operator M . The weak Muckenhoupt–Wheeden conjecture along with this result allows us to conjecture that the following estimate holds for a Calderón–Zygmund operator T for any $p > 1$:

$$\|T\|_{L^p(w)} \leq c \|M\|_{L^p(w)}^p.$$

The latter conjecture would yield the sharp estimates for $\|T\|_{L^p(w)}$ in terms of the A_q characteristic of w for any $1 < q < p$. In this paper we get a weaker inequality

$$\|T\|_{L^p(w)} \leq c \|M\|_{L^p(w)}^p \log(1 + \|M\|_{L^p(w)})$$

with the corresponding estimates for $\|w\|_{A_q}$ when $1 < q < p$.

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1. Introduction

In this paper we continue to study the sharp weighted inequalities for singular integrals T in terms of the A_p characteristic of the weight:

$$\|w\|_{A_p} \equiv \sup_Q \left(\frac{1}{|Q|} \int_Q w \, dx \right) \left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{p-1}} \, dx \right)^{p-1} \quad (1 < p < \infty).$$

For $p = 1$ we set

$$\|w\|_{A_1} \equiv \sup_x \frac{Mw(x)}{w(x)},$$

where M is the Hardy–Littlewood maximal function.

The main conjecture concerning the behavior of T on $L^p(w)$ says that

$$\|T\|_{L^p(w)} \leq c(T, n, p) \|w\|_{A_p}^{\max(1, \frac{1}{p-1})} \quad (1 < p < \infty). \tag{1.1}$$

Currently this conjecture is proved by Petermichl for the Hilbert transform [12] and Riesz transforms [13] and by Petermichl and Volberg for the Ahlfors–Beurling operator [14]. The proofs in [12–14] are based on the so-called Haar shift operators combined with the Bellman function technique. Recently, new approaches to these proofs have been found in [3,8].

We consider several questions related to conjecture (1.1), and which are of independent interest. Suppose T is a general Calderón–Zygmund operator (see Section 2 below for its precise definition). The first question is about the sharp relation between the $L^p(w)$ operator norms of T and M . Observe that for any $p > 1$,

$$\|w\|_{S_p} \leq \|M\|_{L^p(w)} \leq c(p, n) \|w\|_{S_p}, \tag{1.2}$$

where

$$\|w\|_{S_p} = \sup_Q \left(\frac{\int_Q (M(\sigma \chi_Q))^p w}{\int_Q \sigma} \right)^{1/p} \quad (\sigma = w^{-\frac{1}{p-1}}).$$

The left-hand side of (1.2) is trivial, while the right-hand side is a recent interesting result by Moen [11]. Its proof is based on a close examination of Sawyer’s two weighted characterization for M [16] applied to the case of equal weights. Taking into account (1.2), our first question can be interpreted as the question about the sharp estimates for T in terms of the S_p characteristic of the weight.

Our second question is about the sharp estimates for T in terms of the A_q characteristic of the weight for $1 < q < p$. The case when $q = 1$ was recently solved by Lerner, Ombrosi and Pérez in [10]: for any Calderón–Zygmund operator T ,

$$\|T\|_{L^p(w)} \leq c(T, n) \frac{p^2}{p-1} \|w\|_{A_1} \quad (1 < p < \infty). \tag{1.3}$$

Note that (1.3) in the case $p \geq 2$ for classical convolution singular integrals was proved previously by Fefferman and Pipher [5] by means of different methods. However, the main difficulty was in establishing (1.3) for $1 < p < 2$ with the sharp dependence both on $\|w\|_{A_1}$ and on p . Such estimates are motivated by the following so-called weak Muckenhoupt–Wheeden conjecture:

$$\|Tf\|_{L^{1,\infty}(w)} \leq c(T, n)\|w\|_{A_1}\|f\|_{L^1(w)}. \tag{1.4}$$

Observe that the question whether (1.4) holds is still open even for the Hilbert transform. Using (1.3), it was shown in [10] that for any Calderón–Zygmund operator T ,

$$\|Tf\|_{L^{1,\infty}(w)} \leq c(T, n)\|w\|_{A_1} \log(1 + \|w\|_{A_1})\|f\|_{L^1(w)}. \tag{1.5}$$

As we shall see below, both our questions could be solved under the assumption that the weak Muckenhoupt–Wheeden conjecture is true. This follows from the next theorem, which is the main result of this paper.

Theorem 1.1. *Let T be a linear operator satisfying*

$$\|T^*f\|_{L^{1,\infty}(w)} \leq c(T, n)\varphi(\|w\|_{A_1})\|f\|_{L^1(w)}, \tag{1.6}$$

where T^* is a formal adjoint of T , and φ is a non-decreasing function on $[1, \infty)$ such that $\varphi(2t) \leq c\varphi(t)$ for $t \geq 1$. Then for any $1 < p < \infty$,

$$\|T\|_{L^p(w)} \leq c(T, n, p)\|M\|_{L^p(w)}^{p-1}\varphi(\|M\|_{L^p(w)}).$$

Since for a given Calderón–Zygmund operator T , its adjoint is also a Calderón–Zygmund operator, the weak Muckenhoupt–Wheeden conjecture (1.4) along with Theorem 1.1 immediately leads to the following.

Conjecture 1.2. *Let T be a Calderón–Zygmund operator. Then*

$$\|T\|_{L^p(w)} \leq c(T, n, p)\|M\|_{L^p(w)}^p \quad (1 < p < \infty). \tag{1.7}$$

As it was observed by Buckley (see [1, Remark 2.8]), $\|M\|_{L^p(w)} \leq c\|w\|_{A_q}^{1/p}$ for $q < p$ (when $q = 1$ this follows from the Fefferman–Stein inequality [4]). This along with (1.7) leads to the following.

Conjecture 1.3. *Let T be a Calderón–Zygmund operator. Then*

$$\|T\|_{L^p(w)} \leq c(T, n, p, q)\|w\|_{A_q} \quad (1 < q < p < \infty). \tag{1.8}$$

Since $\|w\|_{A_q} \leq \|w\|_{A_1}$, and (1.3) is best possible in terms of $\|w\|_{A_1}$, we clearly have that (1.8) is best possible with respect to $\|w\|_{A_q}$ and the exponent p on the right-hand side of (1.7) is sharp.

Note that (1.1) in the case $p \geq 2$ implies Conjecture 1.3. Therefore, in this case we have that (1.8) holds for the Hilbert, Riesz and Ahlfors–Beurling transforms. However, in the case $1 < q < p < 2$ Conjecture 1.3 is open even for the Hilbert transform.

By the same reasons as above, inequality (1.5) combined with Theorem 1.1 yields the following particular results related to our questions and to Conjectures 1.2 and 1.3.

Theorem 1.4. *Let T be a Calderón–Zygmund operator. Then*

$$\|T\|_{L^p(w)} \leq c(T, n, p) \|M\|_{L^p(w)}^p \log(1 + \|M\|_{L^p(w)}) \quad (1 < p < \infty)$$

and

$$\|T\|_{L^p(w)} \leq c(T, n, p, q) \|w\|_{A_q} \log(1 + \|w\|_{A_q}) \quad (1 < q < p < \infty).$$

Some words about the proof of Theorem 1.1. Suppose that we use (1.6) directly with T instead of T^* . Then, by the Rubio de Francia extrapolation argument, this implies

$$\|Tf\|_{L^{p,\infty}(w)} \leq c(T, n, p) \varphi(\|M\|_{L^{p'}(\sigma)}) \|f\|_{L^p(w)} \quad (1 < p < \infty). \tag{1.9}$$

Now, the standard approach to (1.9) is based on Buckley’s work [1]. First, by [1], $\|M\|_{L^{p'}(\sigma)} \leq c\|w\|_{A_p}$. Second, as in [1], applying (1.9) with $\|w\|_{A_p}$ on the right-hand side to $p - \varepsilon$ and $p + \varepsilon$, where $\varepsilon = c\|w\|_{A_p}^{1-p'}$, and using that $\|w\|_{A_{p-\varepsilon}} \leq c\|w\|_{A_p}$ along with the Marcinkiewicz interpolation theorem, we obtain

$$\|T\|_{L^p(w)} \leq c(T, n, p) \|w\|_{A_p}^{\frac{1}{p(p-1)}} \varphi(\|w\|_{A_p}).$$

However, it is easy to see that if, for example, $\varphi(t) = t$, the latter estimate does not yield (1.8).

In our approach we do not pass to $\|w\|_{A_p}$ in (1.9). Instead of this, we apply (1.9) to $p - \varepsilon$ and $p + \varepsilon$ but with $\varepsilon = c\|M\|_{L^{p'}(\sigma)}^{-p'}$. The most complicated part of the proof was to show that for such a choice of ε we have properties similar to $\|w\|_{A_{p-\varepsilon}} \leq c\|w\|_{A_p}$ but for the corresponding $L^{(p-\varepsilon)'(\sigma)}$ and $L^{(p+\varepsilon)'(\sigma)}$ operator norms of M . Here we use essentially Moen’s recent estimate (1.2) along with several other ingredients. We get

$$\|T\|_{L^p(w)} \leq c(T, p, n) \|M\|_{L^{p'}(\sigma)}^{p'/p} \varphi(\|M\|_{L^{p'}(\sigma)}).$$

In order to have the same operator norms on both sides of this inequality, we use the initial assumption (1.6) with T^* along with the dual relation $\|T\|_{L^{p'}(\sigma)} = \|T^*\|_{L^p(w)}$. Finally, replacing p' by p and σ by w , we obtain the desired inequality.

The paper is organized as follows. Section 2 contains some preliminaries along with the standard ingredients used in the proof. In Section 3 we prove Theorem 1.1.

2. Preliminaries

Throughout the paper we use the standard notations: $p' = \frac{p}{p-1}$, $\sigma = w^{-\frac{1}{p-1}}$, $w_Q = \frac{1}{|Q|} \int_Q w \, dx$ and $w(Q) = \int_Q w \, dx$.

By a Calderón–Zygmund operator we mean a continuous linear operator $T : C_0^\infty(\mathbb{R}^n) \rightarrow \mathcal{D}'(\mathbb{R}^n)$ that extends to a bounded operator on $L^2(\mathbb{R}^n)$, and whose distributional kernel K coincides away from the diagonal $x = y$ in $\mathbb{R}^n \times \mathbb{R}^n$ with a function K satisfying the size estimate

$$|K(x, y)| \leq \frac{c}{|x - y|^n}$$

and the regularity condition: for some $\varepsilon > 0$,

$$|K(x, y) - K(z, y)| + |K(y, x) - K(y, z)| \leq c \frac{|x - z|^\varepsilon}{|x - y|^{n+\varepsilon}},$$

whenever $2|x - z| < |x - y|$, and so that

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy,$$

whenever $f \in C_0^\infty(\mathbb{R}^n)$ and $x \notin \text{supp}(f)$.

Recall that the Hardy–Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

where the supremum is taken over all cubes Q containing the point x .

We shall need the following particular case of the Marcinkiewicz interpolation theorem (see, e.g., [6, p. 31]).

Lemma 2.1. *Let T be a sublinear operator such that*

$$\|Tf\|_{L^{p-\varepsilon, \infty}(w)} \leq A \|f\|_{L^{p-\varepsilon}(w)} \quad \text{for all } f \in L^{p-\varepsilon}(w),$$

where $0 < \varepsilon < p$, and

$$\|Tf\|_{L^{p+\varepsilon, \infty}(w)} \leq A \|f\|_{L^{p+\varepsilon}(w)} \quad \text{for all } f \in L^{p+\varepsilon}(w).$$

Then for any $f \in L^p(w)$ we have

$$\|Tf\|_{L^p(w)} \leq 2(2p)^{1/p} \frac{A}{\varepsilon^{1/p}} \|f\|_{L^p(w)}.$$

The proof of the next statement is well known, and we give it for the sake of completeness.

Lemma 2.2. *Let f and g be measurable functions such that for any $w \in A_1$,*

$$\|f\|_{L^{1, \infty}(w)} \leq \varphi(\|w\|_{A_1}) \|g\|_{L^1(w)}.$$

Then for any $1 < p < \infty$ and for all $w \in A_p$,

$$\|f\|_{L^{p, \infty}(w)} \leq 2c\varphi(\|M\|_{L^{p'}(\sigma)}) \|g\|_{L^p(w)},$$

where c is the doubling constant of φ : $c = \sup_{t \geq 1} \varphi(2t)/\varphi(t)$.

Proof. Given $\psi \geq 0$ with $\|\psi\|_{L^{p'}(\sigma)} = 1$, following Rubio de Francia’s method [15], set

$$\mathcal{R}\psi(x) = \sum_{k=0}^{\infty} \frac{M^k \psi(x)}{(2\|M\|_{L^{p'}(\sigma)})^k}.$$

Then $\psi(x) \leq \mathcal{R}\psi(x)$, $\|\mathcal{R}\psi\|_{L^{p'}(\sigma)} \leq 2$, and $\mathcal{R}\psi \in A_1$ with

$$\|\mathcal{R}\psi\|_{A_1} \leq 2\|M\|_{L^{p'}(\sigma)}.$$

Therefore,

$$\begin{aligned} \lambda \int_{\{|f|>\lambda\}} \psi &\leq \lambda \int_{\{|f|>\lambda\}} \mathcal{R}\psi \leq \varphi(\|\mathcal{R}\psi\|_{A_1}) \int_{\mathbb{R}^n} |g| \mathcal{R}\psi \, dx \\ &\leq c\varphi(\|M\|_{L^{p'}(\sigma)}) \|g\|_{L^p(w)} \|\mathcal{R}\psi\|_{L^{p'}(\sigma)} \leq 2c\varphi(\|M\|_{L^{p'}(\sigma)}) \|g\|_{L^p(w)}. \end{aligned}$$

Taking the supremum over all $\psi \geq 0$ with $\|\psi\|_{L^{p'}(\sigma)} = 1$ completes the proof. \square

It is a classical result that the A_p weight satisfies the reverse Hölder inequality (see, e.g., [2]). We will use the following version of this result.

Lemma 2.3. *If $w \in A_p$ and $\delta = \frac{1}{2^{n+2}\|M\|_{L^{p'}(\sigma)}}$, then for any cube Q ,*

$$\left(\frac{1}{|Q|} \int_Q w^{1+\delta} \, dx \right)^{\frac{1}{1+\delta}} \leq 2 \frac{1}{|Q|} \int_Q w \, dx.$$

Proof. Let M_Q^d denote the dyadic maximal operator restricted to a cube Q . It was shown in the proving of [9, Lemma 3.1] that

$$\int_Q (M_Q^d w)^\delta w \, dx \leq (w_Q)^\delta \int_Q w \, dx + \frac{2^n \delta}{\delta + 1} \int_Q (M_Q^d w)^{1+\delta} \, dx. \tag{2.1}$$

Next, by Hölder’s inequality,

$$\begin{aligned} \int_Q (M_Q^d w)^{1+\delta} \, dx &= \int_Q (M_Q^d w)^{\delta/p} w^{1/p} (M_Q^d w)^{1+\delta/p'} w^{-1/p} \, dx \\ &\leq \left(\int_Q (M_Q^d w)^\delta w \, dx \right)^{1/p} \left(\int_Q (M_Q^d w)^{p'+\delta} \sigma \, dx \right)^{1/p'} \\ &\leq \|M\|_{L^{p'+\delta}(\sigma)}^{1+\delta/p'} \left(\int_Q w^{1+\delta} \, dx \right)^{1/p'} \left(\int_Q (M_Q^d w)^\delta w \, dx \right)^{1/p} \\ &\leq \|M\|_{L^{p'}(\sigma)}^{1+\delta/p'} \int_Q (M_Q^d w)^\delta w \, dx \end{aligned}$$

(we have used here an obvious fact that $\|M\|_{L^{p_1}(\mu)} \leq \|M\|_{L^{p_2}(\mu)}$ if $p_1 \geq p_2$). Combining this with (2.1) yields

$$\int_Q (M_Q^d w)^\delta w \, dx \leq (w_Q)^\delta \int_Q w \, dx + \frac{2^n \delta}{\delta + 1} \|M\|_{L^{p'}(\sigma)}^{1+\delta/p'} \int_Q (M_Q^d w)^\delta w \, dx.$$

Letting $\delta = \frac{1}{2^{n+2} \|M\|_{L^{p'}(\sigma)}}$, we get

$$\frac{2^n \delta}{\delta + 1} \|M\|_{L^{p'}(\sigma)}^{1+\delta/p'} \leq \frac{1}{4} \|M\|_{L^{p'}(\sigma)}^{\frac{1}{2^{n+2} p' \|M\|_{L^{p'}(\sigma)}}} \leq \frac{1}{4} e^{\frac{1}{2^{n+2} p' e}} \leq \frac{1}{2}.$$

This along with the previous inequality implies

$$\int_Q (M_Q^d w)^\delta w \, dx \leq 2(w_Q)^\delta \int_Q w \, dx,$$

which, by Lebesgue’s differentiation theorem, completes the proof. \square

Remark 2.4. Lemma 2.3 can be restated in a dual form as follows: if $w \in A_p$ and $\delta = \frac{1}{2^{n+2} \|M\|_{L^p(w)}}$, then for any cube Q ,

$$\left(\frac{1}{|Q|} \int_Q \sigma^{1+\delta} \, dx \right)^{\frac{1}{1+\delta}} \leq 2 \frac{1}{|Q|} \int_Q \sigma \, dx.$$

Lemma 2.5. For any $w \in A_p$,

$$\|w\|_{A_p}^{\frac{1}{p}} \leq \|M\|_{L^p(w)} \leq c(n, p) \|w\|_{A_p}^{\frac{1}{p-1}}. \tag{2.2}$$

The right-hand side of (2.2) was proved by Buckley [1]. The left-hand side follows easily from $Mf \geq (\frac{1}{|Q|} \int_Q |f|) \chi_Q$ applied to $f = \sigma \chi_Q$.

3. Proof of the main result

The key ingredient in our proof is the following lemma.

Lemma 3.1. Let $v \in A_r, r > 1$, and $\varepsilon = \frac{r-1}{c(n,r)(1+2^{n+2})} \frac{1}{\|M\|_{L^{r'}(v_r)}^{r'}}$, where $c(n, r)$ is the constant from (2.2) and $v_r = v^{-\frac{1}{r-1}}$. Then

$$\|M\|_{L^{(r+\varepsilon)'}(v^{-\frac{1}{r+\varepsilon-1}})} \leq c_1(n, r) \|M\|_{L^{r'}(v_r)} \tag{3.1}$$

and

$$\|M\|_{L^{(r-\varepsilon)'}(v^{-\frac{1}{r-\varepsilon-1}})} \leq c_2(n, r) \|M\|_{L^{r'}(v_r)}. \tag{3.2}$$

Observe that ε is chosen so that $r - \varepsilon > 1$. Before proving the lemma, let us show how the proof of Theorem 1.1 follows.

Proof of Theorem 1.1. Take ε as in Lemma 3.1 and apply Lemma 2.2 with $p = r - \varepsilon$ and $p = r + \varepsilon$. Using (3.1) and (3.2), we get

$$\|T^* f\|_{L^{r-\varepsilon,\infty}(v)} \leq c\varphi\left(\|M\|_{L^{(r-\varepsilon)'}(v^{-\frac{1}{r-\varepsilon-1}})}\right) \|f\|_{L^{r-\varepsilon}(v)} \leq c\varphi\left(\|M\|_{L^{r'}(v_r)}\right) \|f\|_{L^{r-\varepsilon}(v)}$$

and

$$\|T^* f\|_{L^{r+\varepsilon,\infty}(v)} \leq c\varphi\left(\|M\|_{L^{(r+\varepsilon)'}(v^{-\frac{1}{r+\varepsilon-1}})}\right) \|f\|_{L^{r+\varepsilon}(v)} \leq c\varphi\left(\|M\|_{L^{r'}(v_r)}\right) \|f\|_{L^{r+\varepsilon}(v)}.$$

From this and from Lemma 2.1,

$$\|T\|_{L^{r'}(v_r)} = \|T^*\|_{L^r(v)} \leq c\|M\|_{L^{r'/r}(v_r)}^{r'/r} \varphi\left(\|M\|_{L^{r'}(v_r)}\right).$$

Taking here $r = p'$ and $v = w^{-\frac{1}{p-1}}$ completes the proof. \square

We turn now to the proof of Lemma 3.1.

Proof of (3.1). By Moen’s estimate (1.2),

$$\|M\|_{L^{(r+\varepsilon)'}(v^{-\frac{1}{r+\varepsilon-1}})} \leq c \sup_Q \left(\frac{\int_Q M(v\chi_Q)^{(r+\varepsilon)'} v^{-\frac{1}{r+\varepsilon-1}} dx}{v(Q)} \right)^{1/(r+\varepsilon)'}$$

Further, by Hölder’s inequality and by Lemma 2.3,

$$\begin{aligned} \frac{1}{|Q|} \int_Q M(v\chi_Q)^{(r+\varepsilon)'} v^{-\frac{1}{r+\varepsilon-1}} dx &\leq \left(\frac{1}{|Q|} \int_Q M(v\chi_Q)^{\frac{r+\varepsilon}{r-1}} v_r dx \right)^{\frac{r-1}{r+\varepsilon-1}} \\ &\leq \|M\|_{L^{\frac{r+\varepsilon}{r-1}}(v_r)}^{(r+\varepsilon)'} \left(\frac{1}{|Q|} \int_Q v^{1+\varepsilon/(r-1)} dx \right)^{\frac{1}{1+\varepsilon/(r-1)}} \\ &\leq 2\|M\|_{L^{r'}(v_r)}^{(r+\varepsilon)'} \frac{1}{|Q|} \int_Q v dx. \end{aligned}$$

Combining this with the previous estimate gives (3.1) \square

It turns out that the proof of (3.2) is more complicated. We shall need the following covering lemma.

Lemma 3.2. Let f be a non-negative integrable function on a cube Q . Assume that $f_Q < \lambda$ and $\Omega_\lambda = \{x \in Q: M(f\chi_Q)(x) > \lambda\}$ is not empty. Then there exists a sequence $\{Q_j\}$ of cubes such that $(f\chi_Q)_{Q_j} = \lambda/2^n$ and

- (i) $\Omega_\lambda \subset \bigcup_{k=1}^{B_n} \bigcup_{j \in F_k} Q_j$, where each of the family $\{Q_j\}_{j \in F_k}$ is formed by pairwise disjoint cubes and a constant B_n depends only on n ;
- (ii) the ratio of any two sidelengths of rectangles $Q \cap Q_j$ is bounded by 2;
- (iii) for any j , $|Q_j| \leq 2^n |Q \cap Q_j|$.

Proof. Let $x \in Q$ and Q' be an arbitrary cube centered at x and such that $\ell(Q') < 2\ell(Q)$, where $\ell(Q)$ denotes the sidelength of Q . It is a simple geometric observation that $Q' \cap Q$ is a rectangle where the ratio of any two sidelengths is bounded by 2 and $|Q'| \leq 2^n |Q' \cap Q|$.

Further, since $Mf(x) \leq 2^n M^c f(x)$, where $M^c f$ is the centered maximal function, we have that if $x \in \Omega_\lambda$, then $M^c(f\chi_Q)(x) > \lambda/2^n$. Hence, there exists a cube Q' centered at x such that $(f\chi_Q)_{Q'} > \lambda/2^n$. Setting $\psi(r) = (f\chi_Q)_{Q(x,r)}$, where $Q(x,r)$ denotes the cube centered at x with sidelength equal to r , we have that $\psi(r)$ is a continuous function and $\psi(r) \leq f_Q/2^n < \lambda/2^n$ for $r \geq 2\ell(Q)$. Therefore, there exists $r' = r'(x)$ such that $0 < r' < 2\ell(Q)$ and $(f\chi_Q)_{Q(x,r')} = \lambda/2^n$.

Applying to the family $\bigcup_{x \in \Omega_\lambda} \{Q(x, r'(x))\}$ the Besicovitch covering theorem [7], we get the required sequence of cubes $\{Q_j\}$. \square

Lemma 3.3. *Let P be a rectangle satisfying property (ii) of Lemma 3.2, and let $f \in L^1(P)$. Then there exists a cube $Q \subset P$ such that $|P| \leq 2^n |Q|$ and*

$$\int_P |f| dx \leq 2^n \int_Q |f| dx. \tag{3.3}$$

Proof. Subdividing each side of P into two equal parts, we get 2^n pairwise disjoint rectangles $P_k \subset P$ such that $P = \bigcup_{k=1}^{2^n} P_k$ and $|P_k| = |P|/2^n$. Hence, there is k_0 such that

$$\int_P |f| dx \leq 2^n \int_{P_{k_0}} |f| dx.$$

Since P satisfies property (ii), we get that the biggest side of P_{k_0} is less or equal than the smallest side of P . Therefore, there is a cube Q such that $P_{k_0} \subset Q \subset P$. From this we have (3.3). Also, $|P| = 2^n |P_{k_0}| \leq 2^n |Q|$, and hence the proof is complete. \square

Proof of (3.2). First, using again (1.2), we get

$$\|M\|_{L^{(r-\varepsilon)'}(v^{-\frac{1}{r-\varepsilon-1}})} \leq c \sup_Q \left(\frac{\int_Q M(v\chi_Q)^{(r-\varepsilon)'} v^{-\frac{1}{r-\varepsilon-1}} dx}{v(Q)} \right)^{1/(r-\varepsilon)'}$$

Fix a cube Q and set $\Omega_\lambda = \{x \in Q: M(v\chi_Q)(x) > \lambda\}$. Write

$$\begin{aligned} \int_Q M(v\chi_Q)^{(r-\varepsilon)'} v^{-\frac{1}{r-\varepsilon-1}} dx &= (r-\varepsilon)' \int_0^{v_Q} \lambda^{(r-\varepsilon)'-1} \int_{\Omega_\lambda} v^{-\frac{1}{r-\varepsilon-1}} dx d\lambda \\ &+ (r-\varepsilon)' \int_{v_Q}^\infty \lambda^{(r-\varepsilon)'-1} \int_{\Omega_\lambda} v^{-\frac{1}{r-\varepsilon-1}} dx d\lambda \equiv I_1 + I_2. \end{aligned}$$

In order to estimate I_1 , we use a simple argument. It is easy to see that

$$I_1 \leq (v_Q)^{(r-\varepsilon)'} \int_Q v^{-\frac{1}{r-\varepsilon-1}} dx = (v_Q)^{(r-\varepsilon)'} \int_Q (v_r)^{1+\frac{\varepsilon}{r-\varepsilon-1}} dx.$$

Further, by Lemma 2.5,

$$\|M\|_{L^{r'}(v_r)}^{r'} \geq \|v_r\|_{A_{r'}} = \|v\|_{A_r}^{\frac{1}{r-1}} \geq \frac{1}{c(n, r)} \|M\|_{L^r(v)}, \tag{3.4}$$

and hence,

$$\frac{\varepsilon}{r - \varepsilon - 1} \leq \frac{1}{2^{n+2} \|M\|_{L^r(v)}}.$$

Therefore, by Remark 2.4 and by the left-hand side of (3.4) we obtain

$$\begin{aligned} I_1 &\leq 3|Q|(v_Q)^{(r-\varepsilon)'} \left(\frac{1}{|Q|} \int_Q v_r dx \right)^{\frac{r-1}{r-1-\varepsilon}} \\ &= 3|Q|v_Q \left(v_Q \left(\frac{1}{|Q|} \int_Q v_r dx \right)^{r-1} \right)^{\frac{1}{r-1-\varepsilon}} \\ &\leq 3v(Q) \|v\|_{A_r}^{\frac{1}{r-1-\varepsilon}} \leq 3v(Q) \|M\|_{L^{r'}(v_r)}^{\frac{r}{r-1-\varepsilon}}. \end{aligned}$$

Now we estimate I_2 . We are going to prove that for any $\lambda > v_Q$,

$$\int_{\Omega_\lambda} v^{-\frac{1}{r-\varepsilon-1}} dx \leq c(r, n) \left(\frac{1}{\lambda} \right)^{\frac{\varepsilon}{(r-1)(r-\varepsilon-1)}} \int_{\Omega_{\lambda/2^n}} v_r dx. \tag{3.5}$$

Assuming for a moment (3.5) to be true, let us show how to finish the proof of (3.2). Using (3.5), we get

$$\begin{aligned} I_2 &\leq c \int_0^\infty \lambda^{\frac{1}{r-\varepsilon-1} - \frac{\varepsilon}{(r-1)(r-\varepsilon-1)}} \int_{\Omega_{\lambda/2^n}} v_r dx d\lambda \\ &\leq c \int_0^\infty \lambda^{r'-1} \int_{\Omega_\lambda} v_r dx d\lambda \\ &\leq c \int_Q M(v\chi_Q)^{r'} v_r dx \leq cv(Q) \|M\|_{L^{r'}(v_r)}^{r'}. \end{aligned}$$

Combining the estimates for I_1 and I_2 yields

$$I_1 + I_2 \leq c v(Q) \|M\|_{L^{r'}(v_r)}^{\frac{r}{r-1-\varepsilon}},$$

and thus,

$$\left(\frac{\int_Q M(v\chi_Q)^{(r-\varepsilon)'} v^{-\frac{1}{r-\varepsilon-1}} dx}{v(Q)} \right)^{1/(r-\varepsilon)'} \leq c \|M\|_{L^{r'}(v_r)}^{1+\frac{\varepsilon}{r-\varepsilon}}.$$

But since $\|M\|_{L^{r'}(v_r)} \geq 1$, we easily have that

$$\|M\|_{L^{r'}(v_r)}^{\frac{\varepsilon}{r-\varepsilon}} \leq \|M\|_{L^{r'}(v_r)}^{c\|M\|_{L^{r'}(v_r)}^{-r'}} \leq c,$$

which combined with the previous estimate completes the proof of (3.2).

It remains to prove (3.5). Let $\lambda > v_Q$. Applying Lemma 3.2 to the set Ω_λ , we get a sequence of cubes $\{Q_j\}$ satisfying properties (i)–(iii) of the lemma and such that $(v\chi_Q)_{Q_j} = \lambda/2^n$. Set $P_j = Q \cap Q_j$. By Lemma 3.3 choose a cube \tilde{Q}_j corresponding to P_j and $f = v^{-\frac{1}{r-\varepsilon-1}}$. Using (3.3) and arguing exactly as in the above argument for I_1 , we get

$$\begin{aligned} \int_{P_j} v^{-\frac{1}{r-\varepsilon-1}} &= \int_{P_j} (v_r)^{1+\frac{\varepsilon}{r-\varepsilon-1}} \leq 2^n \int_{\tilde{Q}_j} (v_r)^{1+\frac{\varepsilon}{r-\varepsilon-1}} \\ &\leq 3 \cdot 2^n |\tilde{Q}_j| \left(\frac{1}{|\tilde{Q}_j|} \int_{\tilde{Q}_j} v_r dx \right)^{1+\frac{\varepsilon}{r-\varepsilon-1}} \\ &= 3 \cdot 2^n v_r(\tilde{Q}_j) \left(\frac{1}{|\tilde{Q}_j|} \int_{\tilde{Q}_j} v_r dx \right)^{\frac{\varepsilon}{r-\varepsilon-1}}. \end{aligned} \tag{3.6}$$

Next, observe that by (iii) of Lemma 3.2 and by Lemma 3.3,

$$|Q_j| \leq 2^n |P_j| \leq 4^n |\tilde{Q}_j|.$$

Also, by the left-hand side of (3.4),

$$\varepsilon \leq c \|v\|_{A_r}^{-\frac{1}{r-1}}.$$

Using these estimates and the fact that $\|v\|_{A_r} \geq 1$, we get

$$\begin{aligned} ((v_r)_{\tilde{Q}_j})^{\frac{\varepsilon}{r-\varepsilon-1}} &= (2^n/\lambda)^{\frac{\varepsilon}{(r-1)(r-\varepsilon-1)}} ((v\chi_Q)_{Q_j})^{\frac{\varepsilon}{(r-1)(r-\varepsilon-1)}} ((v_r)_{\tilde{Q}_j})^{\frac{\varepsilon}{r-\varepsilon-1}} \\ &\leq c \left(\frac{1}{\lambda} (v_{Q_j}) ((v_r)_{Q_j})^{r-1} \right)^{\frac{\varepsilon}{(r-1)(r-\varepsilon-1)}} \end{aligned}$$

$$\begin{aligned} &\leq c \left(\frac{1}{\lambda} \|v\|_{A_r} \right)^{\frac{\varepsilon}{(r-1)(r-\varepsilon-1)}} \leq c \left(\frac{1}{\lambda} \right)^{\frac{\varepsilon}{(r-1)(r-\varepsilon-1)}} \|v\|_{A_r}^{c\|v\|_{A_r}^{-\frac{1}{r-1}}} \\ &\leq c \left(\frac{1}{\lambda} \right)^{\frac{\varepsilon}{(r-1)(r-\varepsilon-1)}}, \end{aligned}$$

where the constant c depends only on n and r . Combining this with (3.6) yields

$$\int_{P_j} v^{-\frac{1}{r-\varepsilon-1}} \leq c v_r(\tilde{Q}_j) \left(\frac{1}{\lambda} \right)^{\frac{\varepsilon}{(r-1)(r-\varepsilon-1)}}. \tag{3.7}$$

Since $\tilde{Q}_j \subset Q_j$, we have that $M(v\chi_Q)(x) \geq \lambda/2^n$ for any $x \in \tilde{Q}_j$. Also, for each $k, 1 \leq k \leq B_n$ the cubes $\{\tilde{Q}_j\}_{j \in F_k}$ are pairwise disjoint. From this and from (3.7),

$$\sum_{j \in F_k} \int_{P_j} v^{-\frac{1}{r-\varepsilon-1}} dx \leq c \left(\frac{1}{\lambda} \right)^{\frac{\varepsilon}{(r-1)(r-\varepsilon-1)}} \int_{\Omega_{\lambda/2^n}} v_r dx.$$

Therefore,

$$\int_{\Omega_\lambda} v^{-\frac{1}{r-\varepsilon-1}} dx \leq \sum_{k=1}^{B_n} \sum_{j \in F_k} \int_{P_j} v^{-\frac{1}{r-\varepsilon-1}} dx \leq c B_n \left(\frac{1}{\lambda} \right)^{\frac{\varepsilon}{(r-1)(r-\varepsilon-1)}} \int_{\Omega_{\lambda/2^n}} v_r dx.$$

We have proved (3.5), and therefore the proof is complete. \square

Added in proof. Since the paper was submitted for publication, a number of important results have been appeared in this subject. We mention briefly only some of them.

First, conjecture (1.1) has been completely proved for any Calderón–Zygmund operator. We refer to

T.P. Hytönen, The sharp weighted bound for general Calderón–Zygmund operators, <http://arxiv.org/abs/1007.4330>

and

T.P. Hytönen, C. Pérez, S. Treil, A. Volberg, Sharp weighted estimates for dyadic shifts and the A_2 conjecture, <http://arxiv.org/abs/1010.0755>.

Second, Conjecture 1.3 has been proved in

J. Duoandikoetxea, Extrapolation of weights revisited: new proofs and sharp bounds, J. Funct. Anal. 260 (2011) 1886–1901.

Third, the weak Muckenhoupt–Wheeden conjecture (1.4) has been disproved (F. Nazarov, A. Reznikov, V. Vasyunin, A. Volberg, personal communication).

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