Free-decomposability in Varieties of Semi-Heyting Algebras

M. Abad[*1], **J. M. Cornejo**[**1], and **J. P. Díaz Varela**[***1,2]

[1] Departamento de Matemática
Universidad Nacional del Sur
8000 Bahía Blanca, Argentina
[2] Instituto de Matemática (INMABB) - UNS - CONICET

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In this paper we prove that the free algebras in a subvariety $\mathcal{V}$ of the variety $\mathcal{SH}$ of semi-Heyting algebras are directly indecomposable if and only if $\mathcal{V}$ satisfies the Stone identity.

1 Introduction and preliminaries

The variety $\mathcal{SH}$ of semi-Heyting algebras was introduced by Sankappanavar in [11] as an abstraction of Heyting algebras. This variety includes Heyting algebras and share with them some rather strong properties. For example, the variety of semi-Heyting algebras is arithmetical and their congruences are determined by filters. Also, semi-Heyting algebras are pseudocomplemented distributive lattices, with the pseudocomplement given by $x^* = x \to 0$ (see [11]).

But at the same time, semi-Heyting algebras present remarkable differences from Heyting algebras. For example, the implication operation on a semi-Heyting algebra $\mathbf{A}$ is not determined by the lattice order of $\mathbf{A}$; in fact, we can have many semi-Heyting operations on an given distributive lattice, being the greatest of then, the Heyting implication.

It is known that Heyting algebras play a fundamental role in the study of Intuitionistic Logic. In [7], J. M. Cornejo defines a new logic $\mathcal{SI}$ called Semi-intuitionistic Logic such that the semi-Heyting algebras are the semantics for $\mathcal{SI}$, and the Intuitionistic Logic is an axiomatic extension of $\mathcal{SI}$. As Sankappanavar states in [11], we believe that semi-Heyting algebras will be of interest from the point of view of Many Valued Logic. For example, there are ten non-isomorphic semi-Heyting algebras on a 3-element chain, only one of which, of course, is a Heyting algebra. Each of the other nine algebras can provide a new interpretation for the implication connective: for instance, if T, F, U stand respectively for "true", "false" and "unsure", it is reasonable to have $F \to T = U$, $F \to U = U$ and $U \to T = U$.

The set of regular elements of a pseudocomplemented distributive lattice $\mathbf{L}$ forms a subuniverse of a subalgebra of $\mathbf{L}$ if and only if $\mathbf{L}$ satisfies the Stone condition $x^* \vee x^{**} \approx 1$ (see [3]). This result easily extends to Heyting algebras, that is, the regular elements of a Heyting algebra $\mathbf{A}$ form a subalgebra of $\mathbf{A}$ if and only if $\mathbf{A}$ satisfies the Stone condition [8, 9]. Nevertheless, this Stone condition is not longer sufficient in the case of semi-Heyting algebras. We shall prove that the regular elements of a semi-Heyting algebra $\mathbf{A}$ form a subalgebra of $\mathbf{A}$ if and only if, in addition to the Stone condition, $\mathbf{A}$ satisfies the identity $(0 \to 1) \vee (0 \to 1)^* \approx 1$.

In this paper we derive a Glivenko style theorem for the variety of semi-Heyting algebras and we prove that the class of Boolean semi-Heyting algebras (algebras with an underlying structure of Boolean lattice) constitutes a reflective subcategory of $\mathcal{SH}$.

* Corresponding author: e-mail: imabad@criba.edu.ar
** E-mail: ?????@uns.edu.ar
*** E-mail: usdiavar@criba.edu.ar

Finally we obtain a characterization of the decomposability of free semi-Heyting algebras, the main result of this paper. In fact, we prove that the free algebras in a subvariety $\mathcal{V}$ of $\mathcal{SH}$ are directly indecomposable if and only if $\mathcal{V}$ satisfies the Stone identity.

A *semi-Heyting algebra* is an algebra $\mathbf{A} = \langle A, \vee, \wedge, \to, 0, 1\rangle$ such that $\langle A, \vee, \wedge, 0, 1\rangle$ is a lattice with 0 and 1 and the following equations hold:

(a) $x \wedge (x \to y) \approx x \wedge y$,

(b) $x \wedge (y \to z) \approx x \wedge [(x \wedge y) \to (x \wedge z)]$,

(c) $x \to x \approx 1$.

For basic notation and results, the reader is referred to [3], [4], [5] and [11]. We will denote $\mathcal{SH}$ the class of semi-Heyting algebras.

Sankappanavar obtained the following characterization of subdirectly irreducible semi-Heyting algebras.

Theorem 1.1 [11] *Let $\mathbf{A} \in \mathcal{SH}$ with $|\mathbf{A}| \geq 2$. The following are equivalent:*

(a) $\mathbf{A}$ *is subdirectly irreducible.*

(b) $\mathbf{A}$ *has a unique coatom.*

Observe that as a consequence of this theorem, if $\mathbf{A}$ is subdirectly irreducible, then $1 \in A$ is join irreducible.

Since semi-Heyting algebras are pseudocomplemented lattices, the following properties hold.

Lemma 1.2 (a) *If $a \leq b$ then $b^* \leq a^*$.*

(b) $a \leq a^{**}$.

(c) $a \wedge b = 0$ *if and only if* $a^{**} \wedge b = 0$.

(d) *If* $b \wedge a^* = 0$ *then* $b \leq a^{**}$.

(e) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$.

(f) $a^{***} = a^*$.

(g) *If* $a \wedge b = 0$ *then* $a \leq b^*$.

(h) $(a \vee b)^* = a^* \wedge b^*$.

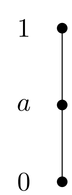
The set of *regular* elements of a semi-Heyting algebra $\mathbf{A}$ is $Reg(\mathbf{A}) = \{a \in A : \ a^{**} = a\}$, and the set of its *dense* elements is $D(\mathbf{A}) = \{a \in A : \ a^* = 0\}$. It is easy to see that $D(\mathbf{A})$ is a filter of $\mathbf{A}$.

An element $a \in A$ is said to be *complemented* (*Boolean*) if there exists $b \in A$ such that $a \wedge b = 0$ and $a \vee b = 1$; the element b is called the *complement* of a. If $a \in A$ has a complement, it is unique and it is a^*. If $B(\mathbf{A})$ denotes the set of complemented elements of A, then $B(\mathbf{A}) = \{a \in A : \ a \vee a^* = 1\}$ and, consequently, $B(\mathbf{A}) \subseteq Reg(\mathbf{A})$.

If $\mathbf{A} \in \mathcal{SH}$, then $Reg(\mathbf{A})$ is not, in general, a subalgebra of $\mathbf{A}$, as the following example shows.

Consider the three-element semi-Heyting algebra $\mathbf{A} = \langle \{0, a, 1\}, \wedge, \vee, \to, 0, 1\rangle$, whose lattice order and whose operation $\to$ is given below.

$\mathbf{A}$

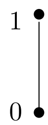


$\to$	0	a	1
0	1	a	a
a	0	1	a
1	0	a	1

We have that $B(\mathbf{A}) = \{0, 1\}$ and $0 \to 1 = a \notin B(\mathbf{A})$, so $B(\mathbf{A})$ is not a subalgebra of $\mathbf{A}$.

The algebras $\mathbf{2}$ and $\overline{\mathbf{2}}$, which have the two-element chain as their lattice reduct and whose $\to$ operation is defined in the following figure, are two important examples of semi-Heyting algebras. One easily verifies that $\mathbf{2}$ is a Heyting algebra while $\overline{\mathbf{2}}$ is not.

$\mathbf{2}$:



$\to$	0	1
0	1	1
1	0	1

$\overline{\mathbf{2}}$:

$\to$	0	1
0	1	0
1	0	1

The varieties generated by $\mathbf{2}$ and $\overline{\mathbf{2}}$, denoted by $\mathcal{V}(\mathbf{2})$ and $\mathcal{V}(\overline{\mathbf{2}})$ respectively, are the only atoms in the lattice of subvarieties of $\mathcal{SH}$. Let $\mathcal{V}(\mathbf{2}, \overline{\mathbf{2}})$ denote the subvariety of $\mathcal{SH}$ generated by $\mathbf{2}$ and $\overline{\mathbf{2}}$. Then we have the following lemma.

Lemma 1.3 [11] *$\mathcal{V}(\mathbf{2}, \overline{\mathbf{2}})$ is characterized within $\mathcal{SH}$ by the equation $x \vee x^* \approx 1$.*

Observe that if $\mathbf{A} \in \mathcal{V}(\mathbf{2}, \overline{\mathbf{2}})$. then its $\{\wedge, \vee, *, 0, 1\}$-reduct is a Boolean algebra. We say that a semi-Heyting algebra $\mathbf{A}$ is *a Boolean semi-Heyting algebra* if $\mathbf{A} \in \mathcal{V}(\mathbf{2}, \overline{\mathbf{2}})$.

2 Glivenko's theorem

In this section we prove a Glivenko style theorem for semi-Heyting algebras, as a generalization of similar results for Heyting algebras as presented, e.g., in [9]. We also prove that the category of Boolean semi-Heyting algebras constitutes a reflective subcategory of the category of semi-Heyting algebras and homomorphisms.

In Heyting algebras, as well as in pseudocomplemented distributive lattices, the regular elements form a Boolean algebra. The following lemma provides a similar result for semi-Heyting algebras, namely $Reg(\mathbf{A}) \in \mathcal{V}(\mathbf{2}, \overline{\mathbf{2}})$, that is, $Reg(\mathbf{A})$ has an underlying structure of complemented distributive lattice, although we must emphasize that the implication on $Reg(\mathbf{A})$ is not the classical implication.

Lemma 2.1 *Let $\mathbf{A}$ be a semi-Heyting algebra. If we define the following operations on $Reg(\mathbf{A})$*

$$x \wedge^R y = x \wedge y, \quad x \vee^R y = (x \vee y)^{**}, \quad 0_R = 0, \quad 1_R = 1 \quad \textit{and} \quad x \Rightarrow y = (x \to y)^{**}$$

then $\langle Reg(\mathbf{A}), \wedge^R, \vee^R, \Rightarrow, 0_R, 1_R \rangle \in \mathcal{SH}$ and satisfies the equation $x \vee^R x^* \approx 1$, or equivalently, $\langle Reg(\mathbf{A}), \wedge^R, \vee^R, \Rightarrow, 0_R, 1_R \rangle \in \mathcal{V}(\mathbf{2}, \overline{\mathbf{2}})$.

Proof. From Lema 1.2, $a \wedge^R b, a \vee^R b, a \Rightarrow b \in Reg(\mathbf{A})$ whenever $a, b \in Reg(\mathbf{A})$. In addition, $0_R^{**} = 0^{**} = 1^* = 0 = 0_R$ and $1_R^{**} = 1^{**} = 0^* = 1 = 1_R$, so $Reg(\mathbf{A})$ with the above defined operations is a bounded lattice. Let us see that $\Rightarrow$ is a semi-Heyting implication.

$$\begin{aligned}
a \wedge (a \Rightarrow b) &= a \wedge (a \to b)^{**} = a^{**} \wedge (a \to b)^{**} \\
&= [a \wedge (a \to b)]^{**} = (a \wedge b)^{**} = a^{**} \wedge b^{**} = a \wedge b; \\
a \wedge (b \Rightarrow c) &= a \wedge (b \to c)^{**} = a^{**} \wedge (b \to c)^{**} = [a \wedge (b \to c)]^{**} \\
&= [a \wedge ((a \wedge b) \to (a \wedge c))]^{**} = a^{**} \wedge ((a \wedge b) \to (a \wedge c))^{**} \\
&= a^{**} \wedge ((a \wedge b) \Rightarrow (a \wedge c)) = a \wedge ((a \wedge b) \Rightarrow (a \wedge c)); \\
a \Rightarrow a &= (a \to a)^{**} = 1^{**} = 1 = 1_R.
\end{aligned}$$

Thus $\langle Reg(\mathbf{A}), \wedge^R, \vee^R, \Rightarrow, 0_R, 1_R \rangle$ is a semi-Heyting algebra.

Finally, $Reg(\mathbf{A})$ satisfies the equation $x \vee^R x^* \approx 1$. Indeed, by Lema 1.2 (h), $(a \vee a^*)^{**} = (a^* \wedge a^{**})^*$. Then $a \vee^R a^* = (a \vee a^*)^{**} = (a^* \wedge a^{**})^* = 0^* = 1 = 1_R$. □

From this lemma, there exists an embedding $\alpha : Reg(\mathbf{A}) \to \prod \mathbf{2}^I \times \overline{\mathbf{2}}^J$ for some subsets I, J. Observe that in the semi-Heyting algebra $\mathbf{2}$, $a \to b = a^* \vee b$, while in $\overline{\mathbf{2}}$, $a \to b = (a^* \vee b) \wedge (b^* \vee a)$. Hence, if $a, b \in Reg(\mathbf{A})$ and π_i is the i-th projection of $\prod \mathbf{2}^I \times \overline{\mathbf{2}}^J$, then

$$\pi_i(\alpha(a \Rightarrow b)) = \alpha(a \Rightarrow b)(i) = \alpha(a)(i) \to \alpha(b)(i) =$$

$$\begin{cases} \alpha(a)(i)^* \vee \alpha(b)(i) & \text{if} \quad \pi_i(\alpha(Reg(\mathbf{A}))) = \mathbf{2} \\ (\alpha(a)(i)^* \vee \alpha(b)(i)) \wedge (\alpha(b)(i)^* \vee \alpha(a)(i)) & \text{if} \quad \pi_i(\alpha(Reg(\mathbf{A}))) = \overline{\mathbf{2}} \end{cases}$$

Lemma 2.2 *The mapping $r_{\mathbf{A}} : \mathbf{A} \to Reg(\mathbf{A})$ defined by $r_A(a) = a^{**}$ is a homomorphism of semi-Heyting algebras.*

Proof. The mapping $r_{\mathbf{A}}$ preserves $\wedge$, $\vee$, 0 and 1, as in the case of psudocomplemented distributive lattices.
Observe that in order to prove that $r_{\mathbf{A}}(a \to b) = r_{\mathbf{A}}(a) \Rightarrow r_{\mathbf{A}}(b)$ we can show that $\alpha(r_{\mathbf{A}}(a \to b)) = \alpha(r_{\mathbf{A}}(a) \Rightarrow r_{\mathbf{A}}(b))$, and then we may consider the cases in which $\pi_i(Reg(\mathbf{A})) = \mathbf{2}$ and $\pi_i(Reg(\mathbf{A})) = \overline{\mathbf{2}}$. □

The previous results can be stated as the Glivenko's Theorem for semi-Heyting algebras.

Theorem 2.3 (Glivenko's Theorem) *Let $\mathbf{A} \in \mathcal{SH}$. Then $\langle Reg(\mathbf{A}), \wedge^R, \vee^R, \to^R, 0_R, 1_R\rangle$ is a Boolean semi-Heyting algebra. Moreover, the mapping $r_{\mathbf{A}} : \mathbf{A} \to Reg(\mathbf{A})$ defined by $r_{\mathbf{A}}(a) = a^{**}$ is a homomorphism and $Reg(\mathbf{A}) \simeq \mathbf{A}/D(\mathbf{A})$.*

We will say that a subcategory $\mathcal{A}$ of the category $\mathcal{SH}$ is *reflective* if there is a functor $\mathcal{R} : \mathcal{SH} \to \mathcal{A}$, called a *reflector*, such that for each $\mathbf{A} \in Obj(\mathcal{SH})$ there exists a morphism $\Phi_{\mathcal{R}}(\mathbf{A}) : \mathbf{A} \to \mathcal{R}(\mathbf{A})$ of $\mathcal{SH}$ with the following properties:

a) If $f \in Hom(\mathcal{SH})$ with $f : \mathbf{A} \to \mathbf{A}'$ then $\Phi_{\mathcal{R}}(\mathbf{A}') \circ f = \mathcal{R}(f) \circ \Phi_{\mathcal{R}}(\mathbf{A})$.

b) If $\mathbf{A} \in Obj(\mathcal{A})$ and $f \in Hom(\mathcal{SH})$ with $f : \mathbf{A} \to \mathbf{A}$ then there exists a unique morphism $f' \in Hom(\mathcal{A})$ with $f' : \mathcal{R}(\mathbf{A}) \to \mathbf{A}$ such that $f' \circ \Phi_{\mathcal{R}}(\mathbf{A}) = f$.

Theorem 2.4 [3, Thm. I.18.2] *Let $\mathcal{A}$ be a subcategory of $\mathcal{SH}$. $\mathcal{A}$ is a reflective subcategory of $\mathcal{SH}$ if and only if there exists a function which assigns to every object $\mathbf{A}$ of $\mathcal{SH}$ an object $\mathcal{R}(\mathbf{A})$ of $\mathcal{A}$ and a function which assigns to every object $\mathbf{A}$ of $\mathcal{SH}$ a morphism $\Phi_{\mathcal{R}}(\mathbf{A}) : \mathbf{A} \to \mathcal{R}(\mathbf{A})$ of $\mathcal{SH}$ such that for every $\mathbf{A} \in Obj(\mathcal{A})$ and $f \in Hom(\mathcal{SH})$ with $f : \mathbf{A} \to \mathbf{A}$ there exists a unique morphism $f' \in Hom(\mathcal{A})$ with $f' : \mathcal{R}(\mathbf{A}) \to \mathbf{A}$ such that $f' \circ \Phi_{\mathcal{R}}(\mathbf{A}) = f$.*

Let us prove now that the class of Boolean semi-Heyting algebras constitutes a reflective subcategory of $\mathcal{SH}$.

Lemma 2.5 *$\mathcal{V}(\mathbf{2}, \overline{\mathbf{2}})$ is a reflective subcategory of $\mathcal{SH}$.*

Proof. Define $\mathcal{R} : Obj(\mathcal{SH}) \to Obj(\mathcal{V}(\mathbf{2}, \overline{\mathbf{2}}))$ by $\mathcal{R}(\mathbf{A}) = Reg(\mathbf{A})$. For $\mathbf{A} \in Obj(\mathcal{SH})$ we define $\Phi_{\mathcal{R}}(\mathbf{A}) : \mathbf{A} \to \mathcal{R}(\mathbf{A})$ by $\Phi_{\mathcal{R}}(\mathbf{A})(a) = r_A(a)$. From lemma 2.2, $\Phi_{\mathcal{R}}(\mathbf{A}) = r_{\mathbf{A}}$ is well defined. Let $\mathbf{B} \in Obj(\mathcal{V}(\mathbf{2}, \overline{\mathbf{2}}))$ and let $f : \mathbf{A} \to \mathcal{V}(\mathbf{2}, \overline{\mathbf{2}})$ a semi-Heyting homomorphism. We want to prove that there exists a unique morphism in the category $\mathcal{V}(\mathbf{2}, \overline{\mathbf{2}})$, $f' : Reg(\mathbf{A}) \to \mathbf{B}$ such that $f' \circ \Phi_{\mathcal{R}}(\mathbf{A}) = f$. Define $f' : Reg(\mathbf{A}) \to \mathbf{B}$ by $f' = f|_{Reg(\mathbf{A})}$. Let us see that $(f' \circ \Phi_{\mathcal{R}}(\mathbf{A}))(c) = f(c)$ for any $c \in Reg(\mathbf{A})$. Since $c = c^{**}$, $(f' \circ \Phi_{\mathcal{R}}(\mathbf{A}))(c) = f'(r_A(c)) = f'(c^{**}) = f'(c) = f|_{Reg(\mathbf{A})}(c) = f(c)$. For the uniqueness, let $f'' : Reg(\mathbf{A}) \to \mathbf{B}$ such that $(f'' \circ \Phi_{\mathcal{R}}(\mathbf{A}))(c) = f(c)$. Then $f''(c) = f''(c^{**}) = f''(r_A(c)) = f''(\Phi_{\mathcal{R}}(\mathbf{A})(c)) = f(c) = f'(\Phi_{\mathcal{R}}(\mathbf{A})(c)) = f'(r_A(c)) = f'(c^{**}) = f'(c)$. By theorem 2.4, we have that $\mathcal{V}(\mathbf{2}, \overline{\mathbf{2}})$ is a reflective subcategory of $\mathcal{SH}$. □

3 Decomposability of free semi-Heyting algebras

It is known that for a pseudocomplemented distributive lattice $\mathbf{L}$, $Reg(\mathbf{L})$is a sublattice of $\mathbf{L}$ if and only if $\mathbf{L}$ satisfies the Stone identity $x^* \vee x^{**} \approx 1$ (see [3]). In the case of a Heyting algebra $\mathbf{A}$, $Reg(\mathbf{A})$ is closed under the operation of implication, so we also have that $Reg(\mathbf{A})$ is a subalgebra of $\mathbf{A}$ if and only if $\mathbf{A}$ satisfies the Stone condition [8, 9]. This result is no longer true in the general case of semi-Heyting algebras, as it is shown by the example of Section 1, where we have an algebra $\mathbf{A}$ that satisfies the Stone equation but $Reg(\mathbf{A})$ is not a subalgebra of $\mathbf{A}$. In what follows we denote by $\mathcal{SH}^S$ the subvariety of Stone semi-Heyting algebras, that is the subvariety of $\mathcal{SH}$ defined by the Stone identity $x^* \vee x^{**} \approx 1$.

Let $\mathcal{D}$ denote the subvariety of $\mathcal{SH}$ that satisfies the identities $x^* \vee x^{**} \approx 1$ and $(0 \to 1) \vee (0 \to 1)^* \approx 1$. The next lemma gives a necessary and sufficient condition on an algebra $\mathbf{A} \in \mathcal{SH}$ for $Reg(\mathbf{A})$ to be a subalgebra of $\mathbf{A}$.

Lemma 3.1 *Let $\mathbf{A} \in \mathcal{SH}$. Then $Reg(\mathbf{A})$ is a subalgebra of $\mathbf{A}$ if and only if $\mathbf{A} \in \mathcal{D}$.*

P r o o f. Suppose that $Reg(\mathbf{A})$ is a subalgebra of $\mathbf{A}$. Observe that the fact that $Reg(\mathbf{A})$ is a subalgebra of $\mathbf{A}$ implies that $\vee^R = \vee$ and $a \vee a^* = 1$ for each $a \in Reg(\mathbf{A})$. So for any $a \in A$, since $a^* \in Reg(\mathbf{A})$ we have that $a^* \vee a^{**} = 1$. On the other hand, $0, 1 \in Reg(\mathbf{A})$, and then $0 \to 1 \in Reg(\mathbf{A})$. So $(0 \to 1) \vee (0 \to 1)^* = 1$.

For the converse, suppose that $\mathbf{A} \in \mathcal{D}$. Let $a, b \in Reg(\mathbf{A})$ and let us prove that $a \vee b = a \vee^R b$. From $a^* \vee a^{**} = 1$ and $a = a^{**}$ we have $a^* \vee a = 1$, and similarly, $b^* \vee b = 1$. Observe that $(a \vee b)^* \vee (a \vee b) = (a^* \wedge b^*) \vee (a \vee b) = (a^* \vee (a \vee b)) \wedge (b^* \vee (a \vee b)) = 1 \vee 1 = 1$. So $a \vee^R b = (a \vee b)^{**} = (a \vee b)^{**} \wedge 1 = (a \vee b)^{**} \wedge [(a \vee b)^* \vee (a \vee b)] = (a \vee b)^{**} \wedge (a \vee b) = (a \vee^R b) \wedge (a \vee b)$. Consequently, $a \vee^R b \leq a \vee b$. The inequality $a \vee b \leq (a \vee b)^{**} = a \vee^R b$ follows from lemma 1.2 (b). Let us now prove that $Reg(\mathbf{A})$ is closed under $\to$. Since $\mathbf{A} \in \mathcal{D}$, $\mathbf{A}$ is a subdirect product of a family $\{\mathbf{A}_i\}_{i \in I}$ of subdirectly irreducible algebras in $\mathcal{D}$. In addition, since $\mathbf{A}$ satisfies the Stone condition, it is easy to see that $Reg(\mathbf{A}) = B(\mathbf{A})$. So if $a, b \in Reg(\mathbf{A})$, a, b can be identified as sequences of $0'$s and $1'$s in $\prod_{i \in I} \mathbf{A}_i$. On the other hand, in a subdirectly irreducible algebra $\mathbf{A}_i$, the equation $(0 \to 1) \vee (0 \to 1)^* \approx 1$ is satisfied if and only if $\{0, 1\}$ is a subalgebra of $\mathbf{A}_i$. As $0 \to 1 = 1$ or $0 \to 1 = 0$ in each $\mathbf{A}_i$, then it is clear that $a \to b$ is a sequence of $0'$s and $1'$s, that is, $a \to b \in B(\mathbf{A}) = Reg(\mathbf{A})$. $\square$

For a given class $\mathcal{K}$ of algebras, let $\mathfrak{F}_{\mathcal{K}}(X)$ denote the free algebra in $\mathcal{K}$ over the set X of free generators.

Now we are going to give a description of $\mathfrak{F}_{\mathcal{V}(\mathbf{2},\bar{\mathbf{2}})}(X_n)$, the free algebra in the variety $\mathcal{V}(\mathbf{2}, \bar{\mathbf{2}})$, with $X_n = \{x_1, x_2, \ldots, x_n\}$. We will prove that the Boolean reduct of $\mathfrak{F}_{\mathcal{V}(\mathbf{2},\bar{\mathbf{2}})}(X_n)$ is isomorphic to the free Boolean algebra over $n+1$ free generators $\mathfrak{F}_{\mathcal{B}}(n+1)$.

Observe that $\mathcal{V}(\mathbf{2}, \bar{\mathbf{2}})$ is a discriminator variety [11, Theorem 7.3]. In addition, $\mathcal{V}(\mathbf{2}, \bar{\mathbf{2}})$ is a finitely generated variety, and so it is locally finite. Let us determine $\mathfrak{F}_{\mathcal{V}(\mathbf{2},\bar{\mathbf{2}})}(X_n)$ finite. Since $\mathcal{V}(\mathbf{2}, \bar{\mathbf{2}})$ is a discriminator variety, then $\mathfrak{F}_{\mathcal{V}(\mathbf{2},\bar{\mathbf{2}})}(X_n)$ is a Boolean product of the algebras $\mathbf{2}$ and $\bar{\mathbf{2}}$, that is, $\mathfrak{F}_{\mathcal{V}(\mathbf{2},\bar{\mathbf{2}})}(X_n)$ is isomorphic to a subalgebra of $\mathbf{2}^{\alpha_1} \times (\bar{\mathbf{2}})^{\alpha_2}$. We have that $\alpha_1 = |Hom(\mathfrak{F}_{\mathcal{V}(\mathbf{2},\bar{\mathbf{2}})}(X_n), \mathbf{2})| = |\{f : f : X_n \to \mathbf{2}|\} = 2^n$, and similarly, $\alpha_2 = 2^n$. Hence $\mathfrak{F}_{\mathcal{V}(\mathbf{2},\bar{\mathbf{2}})}(X_n) \cong \mathbf{2}^{2^n} \times (\bar{\mathbf{2}})^{2^n}$ and, consequently, $|\mathfrak{F}_{\mathcal{V}(\mathbf{2},\bar{\mathbf{2}})}(X_n)| = 2^{2^{n+1}}$.

Hence the generators $x_k \in X_n$, $1 \leq k \leq n$, can be represented in the following way: $x_k = (f_1, f_2)$ where $f_1 : 2^n \to \mathbf{2}$, $f_2 : 2^n \to \bar{\mathbf{2}}$ and $f_1(i) = f_2(i)$, $1 \leq i \leq 2^n$, that is, f_1 and f_2 are both equal 2^n-tuples of 0's and 1's. So there exists $I \subseteq \{1, 2, \ldots, 2^n\}$ such that

$$x_k(i) = \begin{cases} 1 & \text{si} \quad i \in I \\ 0 & \text{si} \quad i \notin I \end{cases}, \text{ and for } 1 \leq i \leq 2^n,\ x(2^n + i) = x(i).$$

Lemma 3.2 *Let $I_n = \{1, 2, \ldots, n\}$. If α is an atom of $\mathfrak{F}_{\mathcal{V}(\mathbf{2},\bar{\mathbf{2}})}(X_n)$ then , for some $J \subseteq I_n$*

$$\alpha = \bigwedge_{j \in J} x_j \wedge \bigwedge_{j \notin J} x_j^* \wedge (0 \to 1)$$

or

$$\alpha = \bigwedge_{j \in J} x_j \wedge \bigwedge_{j \notin J} x_j^* \wedge (0 \to 1)^*.$$

P r o o f. Let α be an atom of $\mathfrak{F}_{\mathcal{V}(\mathbf{2},\bar{\mathbf{2}})}(X_n)$. Then $\alpha = (\alpha(1), \alpha(2), \ldots, \alpha(2^{n+1}))$ where for some $k \in \{1, \ldots, 2^{n+1}\}$, $\alpha(k) = 1$ and $\alpha(i) = 0$ for $i \neq k$.

If $k < 2^n$, let $\widetilde{\alpha}$ be the element $(\alpha(1), \alpha(2), \ldots, \alpha(2^n), \alpha(1), \alpha(2), \ldots, \alpha(2^n))$ (observe that α and $\widetilde{\alpha}$ differ only in the coordinate $2^n + k$). Then, since the first half of $\widetilde{\alpha}$ is an atom of $\mathfrak{F}_{\mathcal{V}(\mathbf{2})}(X_n)$, there exists $J \subseteq X_n$ such that $\widetilde{\alpha} = \bigwedge_{j \in J} x_j \wedge \bigwedge_{j \notin J} x_j^*$. Now, in $\mathfrak{F}_{\mathcal{V}(\mathbf{2},\bar{\mathbf{2}})}(X_n)$, the element $0 \to 1$ is the 2^{n+1}-tuple $(1, 1, \ldots, 1, 0, 0, \ldots, 0)$, i.e, $(0 \to 1)(i) = 1$ for $0 \leq i \leq 2^n$ and $(0 \to 1)(i) = 0$, for $2^n + 1 \leq i \leq 2^{n+1}$, since $0 \to 1 \approx 1$ in $\mathbf{2}$ and $0 \to 1 \approx 0$ in $\bar{\mathbf{2}}$. So $\alpha = \widetilde{\alpha} \wedge (0 \to 1)$, that is $\alpha = \bigwedge_{j \in J} x_j \wedge \bigwedge_{j \notin J} x_j^* \wedge (0 \to 1)$.

If $k > 2^n$, then $\widetilde{\alpha} = (\alpha(2^n + 1), \alpha(2^n + 2), \ldots, \alpha(2^{n+1}), \alpha(2^n + 1), \alpha(2^n + 2), \ldots, \alpha(2^{n+1}))$ is such that $\widetilde{\alpha} = \bigwedge_{j \in J} x_j \wedge \bigwedge_{j \notin J} x_j^*$, and we have $\alpha = \widetilde{\alpha} \wedge (0 \to 1)^*$, that is, $\alpha = \bigwedge_{j \in J} x_j \wedge \bigwedge_{j \notin J} x_j^* \wedge (0 \to 1)^*$. $\square$

Corollary 3.3 $\{x_1, \ldots, x_n, 0 \to 1\}$ *is a generating set for the (Boolean) $\{\wedge, \vee, *, 0, 1\}$-reduct of $\mathfrak{F}_{\mathcal{V}(\mathbf{2},\bar{\mathbf{2}})}(X_n)$*

Lemma 3.4 *The $\{\wedge, \vee, *, 0, 1\}$-reduct of $\mathfrak{F}_{\mathcal{V}(\mathbf{2},\bar{\mathbf{2}})}(X_n)$ is isomorphic to the free Bolean algebra over $n+1$ free generators $\mathfrak{F}_{\mathcal{B}}(n+1)$.*

Proof. Let $Y_n = \{y_1, y_2, \ldots, y_{n+1}\}$ be a generating set for $\mathfrak{F}_{\mathcal{B}}(n+1)$. Let $h : Y_n \to X_n \cup \{0 \to 1\}$ be defined by $h(y_i) = x_i$ for $1 \leq i \leq n$ and $h(y_{n+1}) = 0 \to 1$. Let $\bar{h} : \mathfrak{F}_{\mathcal{B}}(n+1) \to \mathfrak{F}_{\mathcal{V}(\mathbf{2},\bar{\mathbf{2}})}(X_n)$ be the extension homomorphism of h. From lemma 3.2, $\bar{h}$ is onto. As $|\mathfrak{F}_{\mathcal{B}}(n+1)| = |\mathfrak{F}_{\mathcal{V}(\mathbf{2},\bar{\mathbf{2}})}(X_n)| = 2^{2^{n+1}}$, then h is an isomorphism. □

Since the varieties $\mathcal{V}(\mathbf{2})$ (Boolean algebras) and $\mathcal{V}(\bar{\mathbf{2}})$ are the only atoms in the lattice of subvarieties of $\mathcal{SH}$, then any non-trivial subvariety $\mathcal{V}$ of $\mathcal{SH}$ satisfies one of the following properties:

(I) $\mathcal{V}(\mathbf{2}) \subseteq \mathcal{V}$ and $\mathcal{V}(\bar{\mathbf{2}}) \nsubseteq \mathcal{V}$, or

(II) $\mathcal{V}(\bar{\mathbf{2}}) \subseteq \mathcal{V}$ and $\mathcal{V}(\mathbf{2}) \nsubseteq \mathcal{V}$, or

(III) $\mathcal{V}(\mathbf{2},\bar{\mathbf{2}}) \subseteq \mathcal{V}$.

Then we have the following.

Theorem 3.5 *For every subvariety $\mathcal{V} \subseteq \mathcal{SH}$, $Reg(\mathfrak{F}_{\mathcal{V}}(X))$ is isomorphic to $\mathfrak{F}_{\mathcal{V}(\mathbf{2})}(X^{**})$ if $\mathcal{V}$ satisfies* (I)*, is isomorphic to $\mathfrak{F}_{\mathcal{V}(\bar{\mathbf{2}})}(X^{**})$ if $\mathcal{V}$ satisfies* (II) *and is isomorphic to $\mathfrak{F}_{\mathcal{V}(\mathbf{2},\bar{\mathbf{2}})}(X^{**})$ if $\mathcal{V}$ satisfies* (III)*, with $X^{**} = \{x^{**} : x \in X\}$.*

Recall that $\mathcal{D}$ denote the subvariety of $\mathcal{SH}$ that satisfies the identities $x^* \vee x^{**} \approx 1$ and $(0 \to 1) \vee (0 \to 1)^* \approx 1$.

Corollary 3.6 *If $\mathcal{V}$ is a subvariety of $\mathcal{D}$, then $Reg(\mathfrak{F}_{\mathcal{V}}(X))$ is isomorphic to either $\mathfrak{F}_{\mathcal{V}(\mathbf{2})}(X^{**})$ or $\mathfrak{F}_{\mathcal{V}(\bar{\mathbf{2}})}(X^{**})$, or $\mathfrak{F}_{\mathcal{V}(\mathbf{2},\bar{\mathbf{2}})}(X^{**})$, and $Reg(\mathfrak{F}_{\mathcal{V}}(X))$ is a retract of $\mathfrak{F}_{\mathcal{V}}(X)$.*

In what follows we study the decomposability of $\mathfrak{F}_{\mathcal{V}}(X)$, for a given subvariety $\mathcal{V}$ of $\mathcal{SH}$.

Assume that $\mathcal{V}$ is a subvariety of $\mathcal{SH}^S$ that satisfies (I) or (II), and X is a set with $|X| > 0$. Since every term depends only on a finite set of variables, then we can assume, without loss of generality, that X is finite. Let $X_n = \{x_1, \ldots, x_n\}$ and $I_n = \{1, \ldots, n\}$. For any $I \subseteq I_n$, consider the element

$$\alpha_I(x_1, \ldots, x_n) = \bigwedge_{i \in I} x_i^{**} \wedge \bigwedge_{i \notin I} x_i^*.$$

The correspondence $I \mapsto \alpha_I(x_1, \ldots, x_n)$ gives a one-to-one map from $P(I_n)$, the power set of I_n, onto the set of all atoms of the free Boolean semi-Heyting algebra $Reg(\mathfrak{F}_{\mathcal{V}}(X_n)) \cong \mathfrak{F}_{\mathcal{V}(\mathbf{2})}(X^{**}) \cong \mathfrak{F}_{\mathcal{V}(\bar{\mathbf{2}})}(X^{**})$. Hence for any $b \in Reg(\mathfrak{F}_{\mathcal{V}}(X_n))$, there exists $N \subseteq P(I_n)$ such that

$$b = \left(\bigvee_{I \in N} \alpha_I(x_1, \ldots, x_n) \right)^{**}$$

where $N = \{I \in P(I_n) : \alpha_I \leq b\}$.

Lemma 3.7 [6] *For any $J \subseteq I_n$ and $x \in \mathfrak{F}_{\mathcal{V}}(X)$, consider the n-tuple $\overrightarrow{x}_J$ whose i-th component is x for $i \in J$, and 1 for $i \notin J$. For any $I \subseteq I_n$, we get*

$$\overline{\alpha}_I(\overrightarrow{x}_J) = \begin{cases} 1 & \text{if } I = I_n \text{ and } J = \emptyset \\ x^{**} & \text{if } I = I_n \text{ and } J \neq \emptyset \\ x^* & \text{if } I = I_n \setminus J \text{ and } J \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Now we prove that main result of the paper.

Theorem 3.8 *Let $\mathcal{V}$ be a non-trivial subvariety of $\mathcal{SH}$. Then $\mathfrak{F}_{\mathcal{V}}(X)$ is directly decomposable if and only if $\mathcal{V}$ satisfies the Stone identity.*

Proof. Suppose that $\mathfrak{F}_{\mathcal{V}}(X)$ is directly decomposable. Then there exists $\alpha \in \mathfrak{F}_{\mathcal{V}}(X)$ such that $\alpha \vee \alpha^* = 1$ and $\alpha \neq 0, 1$.

Suppose first that $\mathcal{V}$ satisfies either (I) or (II). We can assume that $\alpha = \alpha(x_1, \ldots, x_n) \in \mathfrak{F}_{\mathcal{V}}(X_n)$, as above. Since $\alpha \in Reg(\mathfrak{F}_{\mathcal{V}}(X_n))$, there exists $N \subseteq P(I_n)$, $N \neq \emptyset, P(I_n)$, such that

$$\alpha(x_1, \ldots, x_n) = \left(\bigvee_{I \in N} \alpha_I \right)^{**}.$$

Let us prove that $x^* \vee x^{**} = 1$, for any $x \in \mathfrak{F}_{\mathcal{V}}(X)$.

Suppose that $I_n \notin N$. Fix $K \in N$ and let $J = I_n \setminus K$. Since $K \neq I_n$, $J \neq \emptyset$ and the previous lemma implies that

$$\alpha_I(\overrightarrow{x}_J) = \begin{cases} x^* & \text{if } I = K \\ 0 & \text{if } I \in N, I \neq K. \end{cases}$$

It follows that $\alpha_I(\overrightarrow{x}_J) = (x^*)^{**} = x^*$. Therefore, as $\alpha \vee \alpha^* = 1$ we get $x^{**} \vee x^* = 1$, as desired. Now assume that $I_n \in N$. Choose $J \subseteq I_n$ such that $J \neq I_n \setminus I$ for every $I \in N$. Observe that this is possible since $N \neq P(I_n)$. By the previous lemma we get

$$\alpha_I(\overrightarrow{x}_J) = \begin{cases} x^{**} & \text{if } I = I_n \\ 0 & \text{if } I \in N, I \neq I_n. \end{cases}$$

Therefore, $\alpha_I(\overrightarrow{x}_J) = x^{**}$ and the equation $\alpha \vee \alpha^* = 1$ turns into Stone's equation $x^{**} \vee x^* = 1$. This shows that $\mathcal{V}$ satisfies the Stone identity.

Suppose now that $\mathcal{V}$ satisfies (III). Then $Reg(\mathfrak{F}_{\mathcal{V}}(X))$ is isomorphic to $\mathfrak{F}_{\mathcal{V}(\mathbf{2},\overline{\mathbf{2}})}(X^{**})$. Since $\alpha \in Reg(\mathfrak{F}_{\mathcal{V}}(X_n))$, there exists $N \subseteq P(I_n)$, $N \neq \emptyset, P(I_n)$, such that

$$\alpha(x_1, \ldots, x_n, z) = \left(\bigvee_{I \in N} (\alpha_I \wedge z) \right)^{**}$$

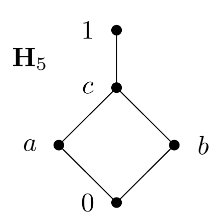
where $z \in \{(0 \to 1)^*, (0 \to 1)^{**}\}$, by lemma 3.2.

By Corollary 3.3 and lemma 3.4, z is a free generator of the Boolean reduct of $\mathfrak{F}_{\mathcal{V}(\mathbf{2},\overline{\mathbf{2}})}(X^{**})$, that is, z is a free generator of $\mathfrak{F}_{\mathcal{B}}(n+1) \cong Reg(\mathfrak{F}_{\mathcal{V}}(X)) \cong \mathfrak{F}_{\mathcal{V}(\mathbf{2},\overline{\mathbf{2}})}(X^{**})$. Then, from $\alpha(x_1, \ldots, x_n, z) \vee (\alpha(x_1, \ldots, x_n, z))^* = 1$ we get

$$\alpha(x_1, \ldots, x_n, x_n) \vee (\alpha(x_1, \ldots, x_n, x_n))^* = 1,$$

which evaluated in the same $(n+1)$-tuple of the cases (I) and (II) gives us $x^{**} \vee x^* = 1$, that is, $\mathcal{V}$ satisfies the Stone identity. □

Consider the following five-element pseudocomplemented distributive lattice $\mathbf{H}_5 = \langle \{0, a, b, c, 1\}, \wedge, \vee, \to, 0, 1 \rangle$:



Lemma 3.9 *A semi-Heyting algebra* $\mathbf{A}$ *does not satisfy the equation* $x^* \vee x^{**} \approx 1$ *if and only if* $\mathbf{A}$ *contains a pseudocomplemented sublattice isomorphic to* $\mathbf{H}_5$.

Proof. Let $\mathbf{A}$ be a semi-Heyting algebra and suppose that there exists $a \in A$ such that $a^* \vee a^{**} \neq 1$. Then the set $\{0, a^*, a^{**}, a^* \vee a^{**}, 1\}$ is the universe of a pseudocomplemented lattice isomorphic to $\mathbf{H}_5$. The converse is immediate. □

We can summarize the above results in the next corollary:

Corollary 3.10 *For any non-trivial variety $\mathcal{V}$ of $\mathcal{SH}$, the following conditions are equivalent:*

1. $\mathfrak{F}_{\mathcal{V}}(X)$ *is directly indecomposable,*
2. $\mathcal{V} \nsubseteq \mathcal{SH}^S$,
3. $\mathcal{V}$ *contains an algebra whose pseudocomplemented lattice reduct is isomorphic to* $\mathbf{H}_5$.

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