Adaptive Control of Local Errors for Elliptic Problems Using Weighted Sobolev Norms

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We develop an a posteriori error estimator which focuses on the local H^1 error on a region of interest. The estimator bounds a weighted Sobolev norm of the error and is efficient up to oscillation terms. The new idea is very simple and applies to a large class of problems. An adaptive method guided by this estimator is implemented and compared to other local estimators, showing an excellent performance. Numer Methods Partial Differential Eq 000: 000–000, 2017

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I. INTRODUCTION

In many practical problems, the *region of interest* is much smaller than the whole domain where a differential equation must be solved to obtain meaningful results. The goal of this article is to design a new adaptive method for the approximation of general second order elliptic problems, with the adaptive strategy focused on controlling the error in a localized region of interest. Throughout this article, *local error* stands for the $H^1(\Omega_0)$ error and Ω_0 is an open subset of the whole domain Ω which we refer to as *region of interest*. Adaptive methods and a posteriori error estimators for controlling local errors in elliptic problems have already been developed and studied [1–4]. In these papers the local error is bounded by the sum of two terms. The H^1-type estimators on a region slightly larger than Ω_0 and a weaker norm of the error on the whole domain; due to the so-called *pollution effect*. Xu and Zhou [1] bound the latter by the $L^2(\Omega)$ error, which in turn can be bounded a posteriori when the domain Ω is convex. On the other hand, Demlow [3] considers Poisson's equation $-\Delta u = f$ and bounds the $L^2(\Omega)$ error by the $L^p(\Omega)$ estimators

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for some $p>2$ —when the $L^2(\Omega)$ estimators are not available due to geometry constraints (e.g., reentrant corners). Recall that all known L^2-type a posteriori error estimators are not a guaranteed upper bound for the $L^2(\Omega)$ error unless the problem is H^2-regular, because the only available proofs are based on duality techniques. Liao and Nochetto [2], instead, resort to estimators for the error in a weighted $L^2(\Omega)$ norm, with weights that are singular at the reentrant corners; their result is valid for the equation $-\nabla \cdot (\mathcal{A}\nabla u) = f$ with $\mathcal{A}$ smooth. Very recently, Demlow [4] defined adaptive methods which control the local H^1 error and also the pollution error in L^2, and proved optimality for Poisson's equation on convex polyhedral domains of any dimension.

In this work, we bound the local H^1 error by a weighted H^1 error on the whole domain, and then by a posteriori error estimators of residual type. The weight is chosen such that the global weighted error is an upper bound of the local H^1 error, but does not overestimate it by too much. This is a simple idea, motivated by the need to use weighted norms when working with point sources, and has the advantage of allowing us to work with the general second order linear elliptic PDE on Lipschitz polygonal/polyhedral domains in 2D/3D without convexity constraints, allowing also discontinuous coefficients. We consider the following problem on a Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n=2, 3$, with a polygonal/polyhedral boundary $\partial\Omega$:

$$\begin{cases} -\nabla \cdot (\mathcal{A}\nabla u) + \boldsymbol{b} \cdot \nabla u + cu = g, & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\mathcal{A} \in L^\infty(\Omega; \mathbb{R}^{n\times n})$ is symmetric, piecewise $W^{1,\infty}$ and uniformly positive definite over Ω, i.e., there exist constants $0 < \gamma_1 \le \gamma_2$ such that

$$\gamma_1|\xi|^2 \le \xi^T \mathcal{A}(x)\xi \le \gamma_2|\xi|^2, \quad \forall x \in \Omega,\ \xi \in \mathbb{R}^n;$$

$\boldsymbol{b} \in W^{1,\infty}(\Omega; \mathbb{R}^n)$, and $c \in L^\infty(\Omega)$ with $c - \frac{1}{2}\text{div}(\boldsymbol{b}) \ge 0$

We consider both the case of regular sources, i.e., $g \in L^2(\Omega)$, and the case of a singular point source assuming that $g = f + \nu\delta_{x_0}$, where $f \in L^2(\Omega)$, $\nu \in \mathbb{R}$ and δ_{x_0} is the Dirac delta distribution supported at an inner point x_0 of Ω. Applications arise in different areas, such as in the study of pollutant diffusion in aquatic media [5], in the mathematical modeling of electromagnetic fields [6], or in optimal control of elliptic problems with state constraints [7]. Other applications involve the coupling between reaction-diffusion problems taking place in domains of different dimension, which arise in tissue perfusion models [8].

When $\nu=0$ ($g = f \in L^2(\Omega)$) we say that $u \in H_0^1(\Omega) := W_0^{1,2}(\Omega)$ is a weak solution of (1) if

$$\mathcal{B}[u, v] = F(v), \quad \forall v \in H_0^1(\Omega),$$

where $\mathcal{B}$ is the bilinear form given by

$$\mathcal{B}[u, v] = \int_\Omega \mathcal{A}\nabla u \cdot \nabla v + \boldsymbol{b} \cdot \nabla u v + cuv, \tag{2}$$

and $F(v) := \int_\Omega fv$. If $\nu \ne 0$, the solution u of (1) does not belong to $H_0^1(\Omega)$, but defining the weight $\omega(x) \cong |x - x_0|^{2\alpha}$ for certain values of $\alpha > 0$, the following weak formulation is well-posed [9, Theorem 2.3]

$$u \in H_0^1(\Omega, \omega) : \qquad \mathcal{B}[u, v] = F(v), \quad \forall v \in H_0^1(\Omega, \omega^{-1}),$$

where $F(v) = \int_\Omega f v + \nu\delta_{x_0}(v)$ and $\mathcal{B}[\cdot,\cdot]$ is given by Eq. (2). Here, $H_0^1(\Omega,\omega)$ and $H_0^1(\Omega,\omega^{-1})$ denote weighted Sobolev spaces which will be explicitely defined in the next section.

The main results of this article are presented in Section III(B) where a posteriori error estimators are presented for the weighted error, and the reliability and efficiency are discussed. The numerical results of Section IV show an excellent performance of an adaptive method guided by these a posteriori estimators.

The rest of the article is organized as follows. In Section II we discuss the formulation of elliptic problems in weighted Sobolev spaces, including the design of weights which *localize* the error and cope with the singularities due to the point source. In Section III we state the discrete formulation of the problem and present a posteriori error estimators for the weighted norm of the error with their reliability and efficiency. In Section IV we present numerical experiments.

II. LINEAR ELLIPTIC PROBLEMS IN WEIGHTED SPACES

We start this section briefly introducing some notions about weighted Sobolev spaces, which are useful for the purpose of this article.

A. Weighted Sobolev spaces

We consider weights belonging to the Muckenhoupt class A_2, which are defined as the set of positive functions $\omega \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that their A_2-constant

$$\sup_B \left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega(x)^{-1}dx\right)$$

is finite, where the supremum is taken over all balls B in $\mathbb{R}^n$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. If $\omega \in A_2$, we denote by $L^2(\Omega,\omega)$ the space of measurable functions u such that

$$\|u\|_{L^2(\Omega,\omega)} := \left(\int_\Omega |u(x)|^2\omega(x)dx\right)^{\frac{1}{2}} < \infty.$$

Notice that $L^2(\Omega,\omega)$ is a Hilbert space equipped with the scalar product

$$\langle u, v\rangle_{\Omega,\omega} := \int_\Omega u(x)v(x)\omega(x)dx.$$

We also define the weighted Sobolev space $H^1(\Omega,\omega)$ of weakly differentiable functions u such that $\|u\|_{H^1(\Omega,\omega)} < \infty$, where

$$\|u\|_{H^1(\Omega,\omega)} := \|u\|_{L^2(\Omega,\omega)} + \|\nabla u\|_{L^2(\Omega,\omega)}.$$

Finally, $H_0^1(\Omega,\omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega,\omega)$. By [10, Theorem 1.3] we have that a Poincaré inequality holds in $H_0^1(\Omega,\omega)$, and thus,

$$\|u\|_{H_0^1(\Omega,\omega)} := \|\nabla u\|_{L^2(\Omega,\omega)}$$

is a norm in $H_0^1(\Omega,\omega)$ equivalent to the inherited norm $\|u\|_{H^1(\Omega,\omega)}$. More precisely, there exists a constant $C_P > 0$, depending on n, the diameter of Ω, and the A_2-constant of ω such that

$$\|u\|_{H_0^1(\Omega,\omega)} \le \|u\|_{H^1(\Omega,\omega)} \le C_P\|u\|_{H_0^1(\Omega,\omega)}, \qquad u \in H_0^1(\Omega,\omega). \tag{3}$$

B. Well-Posedness

The next result generalizes those given by [9, Theorem 2.3] and [11, Lemma 7.7]. When considering the full elliptic equation, we will need to assume that the weight function ω satisfies the following:

Assumption 2.1. The positive weight $\omega \in L^1_{\text{loc}}(\mathbb{R}^n)$ satisfies that $H^1_0(\Omega) \subset L^2(\Omega, \omega^{-1})$, i.e., there exists a constant $C_E > 0$ such that

$$\|v\|_{L^2(\Omega,\omega^{-1})} \leq C_E \|\nabla v\|_{L^2(\Omega)}, \qquad \text{for } v \in H^1_0(\Omega).$$

Remark 2.2. By [11, Theorem 6.1] (see also [12]) we have that a weight $\omega \in A_2$ fulfills $H^1_0(\Omega) \subset L^2(\Omega, \omega^{-1})$ if there exists a constant $C_\omega > 0$ such that the following *compatibility condition* holds:

$$\left(\int_{B(x,r)} \omega^{-1}\right)\left(\int_{B(x,R)} \omega^{-1}\right)^{-1} \leq C_\omega \left(\frac{r}{R}\right)^{n-2}, \qquad \text{for all } x \in \Omega \text{ and } r \leq R. \tag{4}$$

It can be easily checked that if $\omega(x) \cong |x - x_0|^{2\alpha}$, where x_0 is an inner point of Ω, then (4) holds for $\alpha < 1$.

Theorem 2.3. *Let $\omega \in A_2$ and $\omega \in L^\infty(\Omega)$. In addition, if $\boldsymbol{b} \not\equiv 0$ or $c \not\equiv 0$, let us suppose that Assumption 2.1 holds. Let $\mathcal{B}[\cdot,\cdot] : H^1_0(\Omega,\omega) \times H^1_0(\Omega,\omega^{-1}) \to \mathbb{R}$ be as in (2). Then, for each $F \in (H^1_0(\Omega,\omega^{-1}))'$, there exists a unique solution $u \in H^1_0(\Omega,\omega)$ of*

$$\mathcal{B}[u, v] = F(v), \quad \forall v \in H^1_0(\Omega, \omega^{-1}),$$

which satisfies

$$\|u\|_{H^1_0(\Omega,\omega)} \leq C_* \|F\|_{(H^1_0(\Omega,\omega^{-1}))'},$$

where the constant $C_* = \frac{2}{\gamma_1}(1 + \|\omega\|^{\frac{1}{2}}_{L^\infty(\Omega)} \frac{C_P C_E}{\gamma_1} \max\left\{\|\boldsymbol{b}\|_{L^\infty(\Omega)}, \|c\|_{L^\infty(\Omega)}\right\})$

Moreover, the following inf-sup condition holds:

$$\inf_{w \in H^1_0(\Omega,\omega)} \sup_{v \in H^1_0(\Omega,\omega^{-1})} \frac{\mathcal{B}[w, v]}{\|w\|_{H^1_0(\Omega,\omega)} \|v\|_{H^1_0(\Omega,\omega^{-1})}} \geq \frac{1}{C_*}. \tag{5}$$

The proof of this theorem is identical to that of [9, Theorem 2.3] and will thus be omitted in this article.

C. Global Weighted Norms to Localize the Energy Norm

Let the *region of interest* Ω_0 be a fixed open subset of Ω. We are interested in estimating $\|e\|_{H^1(\Omega_0)}$, where e is the error between the weak solution of problem (1) and its finite element approximation. We will develop a posteriori error estimators and propose adaptive methods oriented toward reducing the local error $\|e\|_{H^1(\Omega_0)}$ with the least amount of degrees of freedom (DOF).

The weight in charge of *localizing* the H^1-norm is φ_0 which we only assume to be in $L^1_{\text{loc}}(\mathbb{R}^n)$ and satisfy the following properties:

(a) $\varphi_0(x) > 0$ for a.e. $x \in \mathbb{R}^n$ and $\varphi_0 \in A_2$.
(b) $\varphi_0(x) \leq 1$, for a.e. $x \in \Omega$ and $\varphi_0(x) = 1$, for a.e. $x \in \Omega_0$.
(c) There exists a constant $C_{\varphi_0} > 0$ such that

$$\left(\int_{B(x,r)} \varphi_0^{-1}\right)\left(\int_{B(x,R)} \varphi_0^{-1}\right)^{-1} \leq C_{\varphi_0}\left(\frac{r}{R}\right)^{n-2}, \qquad \text{for all } x \in \Omega \text{ and } r \leq R. \tag{6}$$

A simple way to construct such a weight is to let $\varphi_0(x) := \varphi(\mathrm{dist}(x, \Omega_0))$, with $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ a decreasing positive function such that $\varphi(0) = 1$. In Section IV we make some particular choices of φ and φ_0.

Recall that the source term of problem (1) is $g = f + \nu\delta_{x_0}$, with $f \in L^2(\Omega)$, $\nu \in \mathbb{R}$ and δ_{x_0} is the Dirac delta distribution supported at an inner point x_0 of Ω. From now on, we assume that x_0 does not belong to the region of interest, that is, $x_0 \notin \Omega_0$.

The use of a weight will also allow us to overcome the difficulty produced by a Dirac delta source term. We thus consider two cases depending on the actual presence of a point source:

- Case 1: *There is no point source*. If $\nu = 0$, let

$$\omega(x) = \varphi_0(x), \quad x \in \mathbb{R}^n. \tag{7a}$$

- Case 2: *There is a point source*. If $\nu \neq 0$, let

$$\omega(x) = \min\left(\left(\frac{\mathrm{d}_{x_0}(x)}{D_0}\right)^{2\alpha}, \varphi_0(x)\right), \quad x \in \mathbb{R}^n. \tag{7b}$$

Here, $D_0 := \mathrm{dist}(x_0, \Omega_0) = \inf_{x \in \Omega_0} |x - x_0| > 0$, $\mathrm{d}_{x_0}(x) := |x - x_0|$, for $x \in \mathbb{R}^n$, and α is fixed, with $\alpha \in \mathbb{I}$, where $\mathbb{I} = (n/2 - 1, n/2)$ if $\boldsymbol{b} \equiv 0$ and $c \equiv 0$ and $\mathbb{I} = (n/2 - 1, 1)$ otherwise.

We claim that the weight ω defined by Eq. (7) satisfies the hypoteses of Theorem 2.3. Indeed, since $\mathrm{d}_{x_0}^{2\alpha} \in A_2$ if and only if $-n/2 < \alpha < n/2$ [9], and since $\varphi_0 \in A_2$ we conclude that in both cases $\omega \in A_2$, because—as can be easily checked—the minimum of two A_2 weights also belongs to A_2. Additionally, we have to check that Assumption 2.1 holds when $\boldsymbol{b} \not\equiv 0$ or $c \not\equiv 0$. In view of Remark 2.2, it is enough to check that ω satisfies the compatibility condition (4). We notice that this is indeed the case, due to assumption (6) on φ_0 and the fact that (4) also holds for the weight $\mathrm{d}_{x_0}^{2\alpha}$, for $\alpha < 1$. The requirement $\alpha > n/2 - 1$ is necessary to have δ_{x_0} well defined in $H_0^1(\Omega, \omega^{-1})$; this is discussed in the next section.

Furthermore, since $\omega = 1$ in Ω_0,

$$\|u\|_{H^1(\Omega_0)} \leq \|u\|_{H^1(\Omega,\omega)},$$

i.e., the usual local energy $\|u\|_{H^1(\Omega_0)}$ is bounded above by the global weighted norm $\|u\|_{H^1(\Omega,\omega)}$. This bound becomes sharper when φ_0 is chosen smaller on $\Omega \setminus \Omega_0$.

In Section III(B) we develop a posteriori estimators for the error $\|e\|_{H^1(\Omega,\omega)}$, they will be used to guide an adaptive method, its goal is to reduce the desired quantity $\|e\|_{H^1(\Omega_0)}$.

D. Weak Formulation

To state a variational formulation for the linear elliptic problem (1), we first note that, due to the definition of the weight ω given by (7), if $\nu \neq 0$, $H_0^1(\Omega, \omega^{-1}) \subset H_0^1(\Omega, \mathrm{d}_{x_0}^{-2\alpha})$. Thus, in view of [13, Lema 7.1.3] we have that $\langle g, v\rangle := \int_\Omega f v + \nu\langle\delta_{x_0}, v\rangle$ is a bounded linear functional on $H_0^1(\Omega, \omega^{-1})$, for $f \in L^2(\Omega)$, whenever $\frac{n}{2} - 1 < \alpha < \frac{n}{2}$. We consider the following weak formulation of problem (1):

$$u \in H_0^1(\Omega, \omega): \qquad \mathcal{B}[u, v] = \int_\Omega f v + \nu\langle\delta_{x_0}, v\rangle, \quad \forall v \in H_0^1(\Omega, \omega^{-1}). \tag{8}$$

Recall that $\mathcal{B}$ is the bilinear form given by Eq. (2), which is clearly well-defined and bounded in $H_0^1(\Omega, \omega) \times H_0^1(\Omega, \omega^{-1})$ due to Hölder inequality. As a consequence of Theorem 2.3, the bilinear form $\mathcal{B}[\cdot, \cdot]$ satisfies the inf-sup condition (5), which yields the existence and uniqueness of solution to the variational problem (8).

III. DISCRETE PROBLEM AND A POSTERIORI ERROR ANALYSIS

A. Finite Element Discretization

Let $\mathcal{T}$ be a conforming triangulation of the domain $\Omega \subset \mathbb{R}^n$. That is, a partition of Ω into n-simplexes such that if two elements intersect, they do so at a full vertex/edge/face of both elements. We define the mesh regularity constant

$$\kappa := \sup_{T \in \mathcal{T}} \frac{\operatorname{diam}(T)}{\rho_T},$$

where $\operatorname{diam}(T)$ is the diameter of T, and ρ_T is the radius of the largest ball contained in it. Also, the diameter of any element $T \in \mathcal{T}$ is equivalent to the local mesh-size $h_T := |T|^{1/n}$, with equivalence constants depending on κ.

On the other hand, we denote the subset of $\mathcal{T}$ consisting of an element T and its neighbors by $\mathcal{N}_T$, and the union of the elements in $\mathcal{N}_T$ by S_T. More precisely, for $T \in \mathcal{T}$,

$$\mathcal{N}_T := \left\{T' \in \mathcal{T} \,|\, T \cap T' \neq \emptyset\right\}, \qquad S_T := \bigcup_{T' \in \mathcal{N}_T} T'.$$

We denote by $\mathcal{E}_\Omega$ the set of sides (edges for $n=2$ and faces for $n=3$) of the elements in $\mathcal{T}$ which are inside Ω, and by $\mathcal{E}_{\partial\Omega}$ the set of sides which lie on the boundary of Ω. We define S_E as the union of the two elements sharing E, if $E \in \mathcal{E}_\Omega$, and as the unique element T_E satisfying $E \subset \partial T_E$ if $E \in \mathcal{E}_{\partial\Omega}$.

For the discretization, we consider piecewise polynomial Lagrange finite elements, more precisely, we let

$$\mathbb{V}_{\mathcal{T}}^{\ell} := \left\{V \in H_0^1(\Omega) |\, V_{|T} \in \mathcal{P}_\ell(T),\, \forall T \in \mathcal{T}\right\}.$$

where $\ell \in \mathbb{N}$ is a fixed polynomial degree. The discrete counterpart of Eq. (8) reads:

$$U \in \mathbb{V}_{\mathcal{T}}^{\ell}: \qquad \mathcal{B}[U, V] = \int_\Omega f V + \nu\langle\delta_{x_0}, V\rangle, \quad \forall V \in \mathbb{V}_{\mathcal{T}}^{\ell}. \tag{9}$$

Notice that it is the standard finite element discretization. The weighted norms have no influence in the formulation of the discrete problem.

B. A Posteriori Estimation of the Local Error

In this section, we derive computable bounds for the error measured in the weighted norm $\|\cdot\|_{H^1(\Omega,\omega)}$, where the weight function ω is given by Eq. (7).

Let u be the solution of Eq. (8) and let $U \in \mathbb{V}_{\mathcal{T}}$ be the solution of the discrete problem (9). Integrating by parts on each $T \in \mathcal{T}$ we have that

$$\mathcal{B}[U-u,v] = \sum_{T\in\mathcal{T}} \left(\int_T Rv + \int_{\partial T} Jv \right) - \nu\langle \delta_{x_0}, v\rangle, \qquad \forall v \in H_0^1(\Omega,\omega^{-1}),$$

where R denotes the *element residual* given by

$$R_{|T} := -\nabla\cdot(\mathcal{A}\nabla U) + \boldsymbol{b}\cdot\nabla U + cU - f, \qquad \forall T \in \mathcal{T},$$

and J is the *jump residual* given by

$$J_{|E} := \frac{1}{2}\left[\mathcal{A}\nabla U_{|T_1}\cdot\vec{n}_1 + \mathcal{A}\nabla U_{|T_2}\cdot\vec{n}_2\right], \text{ if } E \in \mathcal{E}_\Omega, \quad J_{|E} = 0, \text{ if } E \in \mathcal{E}_{\partial\Omega}.$$

Here, T_1 and T_2 denote the elements of $\mathcal{T}$ sharing E, and $\vec{n}_1$ and $\vec{n}_2$ are the outward unit normals of T_1 and T_2 on E, respectively.

Let $\omega_T := \sup_{x\in S_T} \omega(x)$. We define *a posteriori local error estimator* η_T by

$$\eta_T^2 := \begin{cases} h_T^2\omega_T\|R\|_{L^2(T)}^2 + h_T\omega_T\|J\|_{L^2(\partial T)}^2 + \nu^2 D_0^{-2\alpha} h_T^{2\alpha+2-n}, & \text{if } x_0 \in T \\ h_T^2\omega_T\|R\|_{L^2(T)}^2 + h_T\omega_T\|J\|_{L^2(\partial T)}^2, & \text{if } x_0 \notin T \end{cases} \tag{10}$$

and the *global error estimator* η by $\eta := \left(\sum_{T\in\mathcal{T}} \eta_T^2\right)^{\frac{1}{2}}$.

Notice that for elements T such that $S_T \subset \overline{\Omega_0}$, $\omega_T = 1$ so that the local estimator η_T coincides with the usual local H^1-estimator, i.e.,

$$\eta_T = \left(h_T^2\|R\|_{L^2(T)}^2 + h_T\|J\|_{L^2(\partial T)}^2\right)^{\frac{1}{2}},$$

whereas for the others elements the usual local H^1-estimator is weakened by the constant ω_T.

The exact same proof of [9, Theorem 5.1] allows us to conclude the assertion of the following theorem.

Theorem 3.1. (Reliability of the global error estimator). *Let ω be defined by Eq. (7). Let $u \in H_0^1(\Omega,\omega)$ be the solution of problem (8) and let $U \in \mathbb{V}_{\mathcal{T}}^{\ell}$ be the solution of the discrete problem (9). Then, there exists a constant $c_1 = c_1(\kappa,\alpha)$ such that*

$$\|U-u\|_{H^1(\Omega,\omega)} \le c_1 C_* C_P \eta,$$

where C_* is the continuous inf-sup constant from Eq. (5) and C_P is the Poincaré constant from Eq. (3).

We now discuss the *efficiency* of the local error estimators. In this context, the boundedness of the bilinear form yields the following: if $C_{\mathcal{B}} := \max\{\gamma_2, \|\boldsymbol{b}\|_{L^\infty}, \|c\|_{L^\infty}\}$, then

$$\left|\mathcal{B}[U,v] - \left(\int_\Omega fv + \nu\langle\delta_{x_0},v\rangle\right)\right| = |\mathcal{B}[U,v] - \mathcal{B}[u,v]| \le C_{\mathcal{B}}\|U-u\|_{H^1(\Theta,\omega)}\|v\|_{H^1(\Theta,\omega^{-1})},$$

for all $\Theta \subset \overline{\Omega}$ and all $v \in H_0^1(\Omega, \omega^{-1})$ with $\text{supp}(v) \subset \Theta$. This bound and the usual steps with scaled bubble functions, as first introduced in [14, Section 4] (see also Ref. [15]), allow us to conclude the following result, related to the elements which are *far away* from x_0; the detailed proof is omitted.

Theorem 3.2. (Efficiency of local estimators 1). *Let $u \in H_0^1(\Omega, \omega)$ be the solution of problem (8) and let $U \in \mathbb{V}_{\mathcal{T}}^{\ell}$ be the solution of the discrete problem (9). Then, there exists a constant $c_2 = c_2(\kappa)$ such that*

$$c_2 \eta_T \leq C_\omega^T C_{\mathcal{B}} \|U - u\|_{H^1(S_T, \omega)} + osc_T, \tag{11}$$

for all $T \in \mathcal{T}$ such that $x_0 \notin S_T$, where $C_\omega^T := \left(\frac{\max_{S_T} \omega}{\min_{S_T} \omega}\right)^{\frac{1}{2}}$. Here, the local oscillation osc_T is given by

$$osc_T := \left(h_T^2 \omega_T \|R - \bar{R}\|_{L^2(S_T)}^2 + h_T \omega_T \|J - \bar{J}\|_{L^2(\partial T)}^2\right)^{\frac{1}{2}}, \tag{12}$$

where $\bar{R}_{|T'}$ denotes the L^2 projection of R on $\mathcal{P}_{\ell-1}(T')$, for all $T' \in \mathcal{T}$, and for each side E, $\bar{J}_{|E}$ denotes the L^2 projection of J on $\mathcal{P}_{\ell-1}(E)$

As it usually happens for residual based error estimators, the lower bound is local, and holds up to some oscillation terms. Notice that for elements T such that $S_T \subset \overline{\Omega_0}$, this result coincides with the usual H^1-local error estimation, because $\omega_{|S_T} \equiv 1$.

Remark 3.3. We notice that the constants in the estimation (11), depend on the weight ω only through the quotient $C_\omega^T = \left(\frac{\max_{S_T} \omega}{\min_{S_T} \omega}\right)^{\frac{1}{2}}$, which tends to one when the meshsize tends to zero, provided ω is continuous.

Now we consider the efficiency of the local estimators asociated to the elements T which are near x_0, more precisely, elements T such that $x_0 \in S_T$. As a consequence of [9, Theorem 5.3] we obtain the following result, which holds if we assume that for such T's, $\omega_{|S_T} = \left(\frac{d_{x_0}}{D_0}\right)^{2\alpha}$. Taking into account the definition of ω given in Eq. (7), we notice that this will be the case as soon as the mesh around x_0 is fairly refined, because $\text{dist}(x_0, \Omega_0) > 0$.

Theorem 3.4. (Efficiency of local estimators 2). *Let $u \in H_0^1(\Omega, \omega)$ be the solution of problem (8) and let $U \in \mathbb{V}_{\mathcal{T}}^{\ell}$ be the solution of the discrete problem (9). Then, there exists a constant $c_3 = c_3(\kappa, \alpha) > 0$ such that*

$$c_3 \eta_T \leq C_{\mathcal{B}} \|U - u\|_{H^1(S_T, \omega)} + osc_T,$$

for all $T \in \mathcal{T}$ such that $x_0 \in S_T$.[1]

[1] The oscillation osc_T when $x_0 \notin T$ is given by (12). In the case that $x_0 \in T$, the jump oscillations in Eq. (12) are considered over all $E \in \mathcal{E}_\Omega$ that touch T, not only those contained in ∂T (Cf. Ref. [9])

IV. NUMERICAL EXPERIMENTS AND APPLICATIONS

In this section we perform some numerical experiments in two-dimensional (2D) domains with linear elements ($\ell = 1$), to illustrate the performance of our estimators and compare with other already known estimators [2, 3, 9]. We consider a standard adaptive loop of the form

$$\textsc{Solve} \quad \to \quad \textsc{Estimate/Mark} \quad \to \quad \textsc{Refine}.$$

The step SOLVE consists in solving the discrete system (9) for the current mesh $\mathcal{T}$. For the step ESTIMATE/MARK we consider the computation of different alternative a posteriori error estimators and select in $\mathcal{M}$, for refinement, some elements of $\mathcal{T}$ according to different marking strategies. This gives rise to particular adaptive algorithms that we describe in detail below. The last step REFINE consists in performing two bisections to each marked element, and refining some extra elements to keep conformity of the meshes, using the *newest-vertex bisection*. We used a custom implementation in MATLAB.

We now describe in detail the three ESTIMATE/MARK alternatives to be considered in this article. We start with our proposal.

Localized Weighted Estimators (LWE). We consider two possible choices for the function φ in Section II(C), that we call φ_1 and φ_2, respectively.

- $\varphi_1(x) := \left(1 + a_1 \frac{x}{L}\right)^{-1}$, where $L := \max_{x\in\Omega}\text{dist}(x, \Omega_0)$ and a_1 is a parameter to be fixed.
- $\varphi_2(x) := \begin{cases} a_2, & x>0, \\ 1, & x=0, \end{cases}$ where a_2 is a small parameter to be fixed.

Then, for $j = 1, 2$, we denote ω_j the corresponding weight given by Eq. (7), where $\varphi_0(x) = \varphi_j(\text{dist}(x, \Omega_0))$ and $\alpha = \frac{1}{2}$.

The local a posteriori error estimator given by Eq. (10) when using $\omega = \omega_2$ is almost identical to the one proposed by Bank and Holst in Ref. [16], in the context of a parallel adaptive meshing algorithm. Their idea was to multiply the estimators corresponding to elements outside the region of interest by a very small number (10^{-6}), so that those elements are not selected for refinement. We found that the choice $a_2 = 10^{-4}$ yields the best performance in the experiments reported in Section IV. This choice favors the refinement inside the region of interest, but also performs some significant refinement outside, to control the pollution error and lead to convergence.

We compute the local estimators η_T given by Eq. (10) and apply the Dörfler strategy with parameter $\theta = 0.5$ for marking, i.e., we collect in $\mathcal{M}$ those elements $T \in \mathcal{T}$ with largest estimators η_T until

$$\sum_{T\in\mathcal{M}} \eta_T^2 \geq \theta^2 \sum_{T\in\mathcal{T}} \eta_T^2.$$

Liao-Nochetto's Estimators (LNE). In [2], Liao and Nochetto considered the equation $-\text{div}(\mathcal{A}(x)\nabla u) = f$ in 2D with homogeneous Dirichlet boundary conditions for a smooth coefficient matrix $\mathcal{A}(x)$, and proved that

$$\|u - U\|^2_{H^1(\Omega_0)} \leq C_{LN} \sum_{T\in\mathcal{T}} \eta^2_{\text{LN},T},$$

with

$$\eta^2_{\text{LN},T} = \begin{cases} \eta_1^2(T) & \text{if } T \subset \Omega_1, \\ \frac{1}{d^2}\max_i |\log h_i| \eta^2_{0,-\beta}(T) & \text{if } T \not\subset \Omega_1, \end{cases}$$

where h_i denotes the meshsize at the i-th reentrant corner of Ω, $d = \text{dist}(\Omega \setminus \Omega_1, \Omega_0)$, $\Omega_1 \supset \Omega_0$ and

$$\eta_1^2(T) = h_T^2 \| -\text{div}(\mathcal{A}\nabla U) - f\|_{L^2(T)}^2 + h_T \|[\mathcal{A}\nabla U]\|_{L^2(\partial T)}^2,$$
$$\eta_{0,-\beta}^2(T) = h_T^4 \|(-\text{div}(\mathcal{A}\nabla U) - f)\sigma_{-\beta}\|_{L^2(T)}^2 + h_T^3 \|[\mathcal{A}\nabla U]\sigma_{-\beta}\|_{L^2(\partial T)}^2.$$

Here $\sigma_{-\beta} : \Omega \to \mathbb{R}$ is a mesh-dependent weight defined as

$$\sigma_{-\beta}(x) = \min_i \left(r_i^2(x) + h(x)^2\right)^{-\beta_i/2},$$

with r_i the distance to the i-th corner of Ω, $h(x) = h_T$ if $x \in T$, and $\beta_i = \max\{0, 1 - \pi/w_i\}$ where w_i is the size of the inner angle at the i-th corner of Ω.

Notice that the estimators depend on the choice of the set Ω_1, which is slightly larger than Ω_0, or in other words, on the parameter $d = \text{dist}(\Omega \setminus \Omega_1, \Omega_0)$.

They also propose to use Dörfler's strategy, i.e., in each step of the adaptive loop, the set of marked elements $\mathcal{M}$ is chosen to satisfy

$$\sum_{T\in\mathcal{M}} \eta_{\text{LN},T}^2 \geq \theta^2 \sum_{T\in\mathcal{T}} \eta_{\text{LN},T}^2,$$

for some $\theta \in (0, 1)$. We chose $\theta = 0.5$ in our experiments below.

It is worth noticing that even though the Liao–Nochetto's estimator (LNE) was developed for source terms $g = f \in L^2(\Omega)$, the same estimator constitutes an upper bound for the error when $g = f + \nu\delta_{x_0}$, if x_0 is a vertex of the triangulation. A proof of this fact can be obtained following the steps in Ref. [17], where L^p and $W^{1,p}$ estimators were developed for Poisson's equation with a point source, and the authors first observed the surprising fact that the estimator is not explicitely influenced by the point source, when it is located at a vertex of the triangulation. Notice also that $\eta_1^2(T)$ is computed only for elements T such that $x_0 \notin T$.

Demlow's Estimators (DE). In Ref. [3], Demlow considered Poisson equation $-\Delta u = f$ and proved that

$$\|u - U\|_{H^1(\Omega_0)} \leq C_D \left[\left(\sum_{\substack{T\in\mathcal{T}\\ T\subset\Omega_1}} \eta_1^2(T) \right)^{1/2} + \frac{1}{d^{\frac{1}{2}+\frac{1}{p}}} \left(\sum_{T\in\mathcal{T}} \eta_{L^p}^p(T) \right)^{1/p} \right],$$

for $p = 2$ if Ω is convex and $4 < p < \infty$ otherwise, where $\eta_1(T)$ is defined as in the previous paragraph (with $\mathcal{A} = I$), and

$$\eta_{L^p}^p(T) = h_T^{2p} \| -\Delta U - f\|_{L^p(T)}^p + h_T^{p+1} \|[\nabla U]\|_{L^p(\partial T)}^p.$$

Demlow proved convergence of an AFEM with adaptive pollution control, which is based on the following marking strategy [3, Section 4.3]: Given $\zeta > 0$,

(a) if $\left(\sum_{\substack{T\in\mathcal{T}\\ T\subset\Omega_1}} \eta_1^2(T)\right)^{1/2} > \zeta \left(\sum_{T\in\mathcal{T}} \eta_{L^p}^p(T)\right)^{1/p}$, then

$$\text{take } \mathcal{M} \subset \{T \in \mathcal{T} : T \subset \Omega_1\} \text{ such that } \quad \sum_{T\in\mathcal{L}} \eta_1^2(T) \geq \theta^2 \sum_{\substack{T\in\mathcal{T}\\ T\subset\Omega_1}} \eta_1^2(T),$$

(b) otherwise,

$$\text{take } \mathcal{M} \subset \mathcal{T} \text{ such that } \sum_{T\in\mathcal{M}} \eta_{L^p}^p(T) \geq \theta^p \sum_{T\in\mathcal{T}} \eta_{L^p}^p(T).$$

Demlow also proved optimality of this strategy in Ref. [4], where he calls it the *alternating marking strategy*. We chose $\theta = 0.5$ in our experiments below.

Notice that in this case, not only the parameter d has to be chosen, but also the power p and the coefficient ζ for the separate marking.

We present three examples. The first one with a known solution to Poisson's equation on an L-shaped domain with a point source, the second one with a known solution to a diffusion problem with discontinuous coefficient $\mathcal{A}$, piecewise constant on a *checkerboard* pattern. The third one is a diffusion-advection-reaction problem, with variable coefficients, simulating a wiggling flow on a canal. The goal of the first two examples is to compare the three proposals stated above. We have run the algorithms considering different choices of the parameters a_1 and a_2 for the localized weighted estimators (LWE), the parameter d for LNE and the parameters d and ζ for Demlow's estimators (DE), but we only report the results corresponding to the best performance of each proposal. The goal of the third example is to show the performance of the newly proposed algorithm in a similar to real life problem, with variable convection coefficient, which simulates the transport, decay and diffusion of a pollutant from a point source.

A. Point Source on L-Shaped Domain

We consider the equation $-\Delta u = \delta_{x_0}$ in $\Omega = (-1,1)^2/[0,1]^2$ with Dirichlet boundary values on $\partial\Omega$ and $x_0 = (0.5, 0.5)$. The boundary values were taken such that the exact solution is $u(x) = -\frac{1}{2\pi}\log|x - x_0| + |x|^{2/3}\sin(2\phi/3)$, with ϕ the angle between x and the positive x_1-axis. This solution exhibits two different singularities, due to the reentrant corner and the point source, respectively. We considered Ω_0 a strip on the left side of the domain, namely the set $\Omega_0 = (-1,-0.5)\times(-1,1)$. To compare our proposal LWE with those by [2, 3] (LNE and DE, respectively), we show in Fig. 1 the local error decay (i.e., $\|u - U\|_{H^1(\Omega_0)}$) versus DOFs obtained with the different strategies. We have considered $a_1 = 10^5$ and $a_2 = 10^{-4}$ in the weights ω_1 and ω_2 for LWE, respectively; $d = 0.25$ for LNE; and $d = 0.25$, $p = 5$ and $\zeta = \frac{1}{2}d^{-\frac{1}{2}-\frac{1}{p}}$ for DE. We have also considered the algorithm guided by the *global weighted estimators* ($H^1_\alpha(\Omega)$-estimators, with $\alpha = \frac{1}{2}$) from Ref. [9], because

$$\|u - U\|_{H^1(\Omega_0)} \leq C_{\Omega_0,x_0}\|u - U\|_{H^1_\alpha(\Omega)} \leq C_{AGM}\left(\sum_{T\in\mathcal{T}} \eta^2_{AGM,T}\right)^{\frac{1}{2}}, \tag{13}$$

where $\|\cdot\|_{H^1_\alpha(\Omega)} := \|\cdot\|_{L^2(\Omega,\mathrm{d}^{2\alpha}_{x_0})} + \|\nabla\cdot\|_{L^2(\Omega,\mathrm{d}^{2\alpha}_{x_0})}$.

We notice that the new proposal (LWE) behaves a bit better than the one by Liao–Nochetto (LNE) and is very similar to the one by Demlow (DE).

The meshes obtained after 5, 10, 15, 20, 25, and 30 iterations of the adaptive algorithm guided by the $H^1(\Omega,\omega_1)$-estimators (LWE) are shown in Fig. 2. It is worth noticing how the refinement concentrates around x_0, the reentrant corner and the region of interest Ω_0.

Finally, to compare the behavior of the LWE for different parameters we present in Fig. 3 the local error decay for different parameters defining the weights ω_1 (left) and ω_2 (right). We notice that the behavior of the adaptive algorithm is similar, and quasi-optimal, for all the parameters

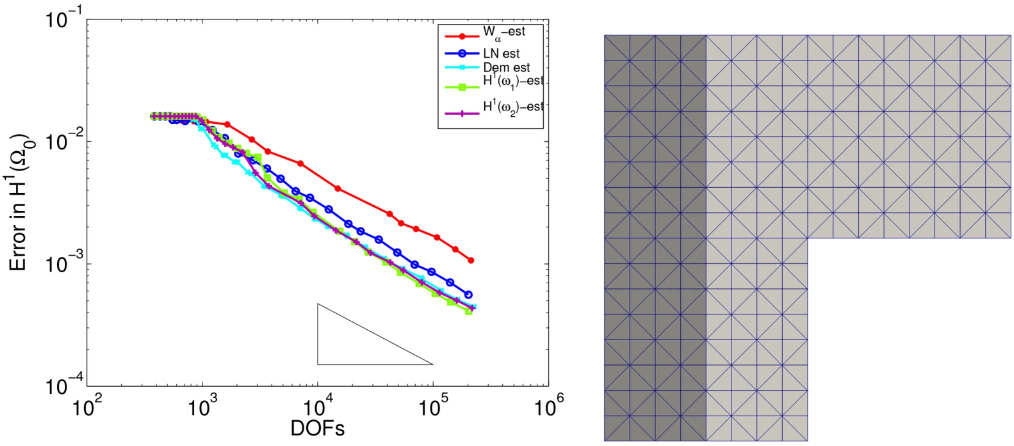

FIG. 1. Error decay and initial mesh with Ω_0 shaded (225 DOFs). We plot the $H^1(\Omega_0)$-error versus the number of DOFs, obtained with an adaptive procedure guided by the newly proposed estimators taking $\alpha = \frac{1}{2}$; and $a_1 = 10^5$ and $a_2 = 10^{-4}$ in the weights ω_1 and ω_2 for LWE, respectively; and also by the estimators proposed by Liao–Nochetto (LNE) with $d = 0.25$, by Demlow (DE) with $d = 0.25$, $p = 5$ and $\zeta = \frac{1}{2} d^{-\frac{1}{2}-\frac{1}{p}}$, and the (global) $H^1_\alpha(\Omega)$-estimators from Eq. (13), with $\alpha = \frac{1}{2}$. The new proposal behaves a bit better than the one by Liao–Nochetto and is very similar to the one by Demlow. [Color figure can be viewed at wileyonlinelibrary.com]

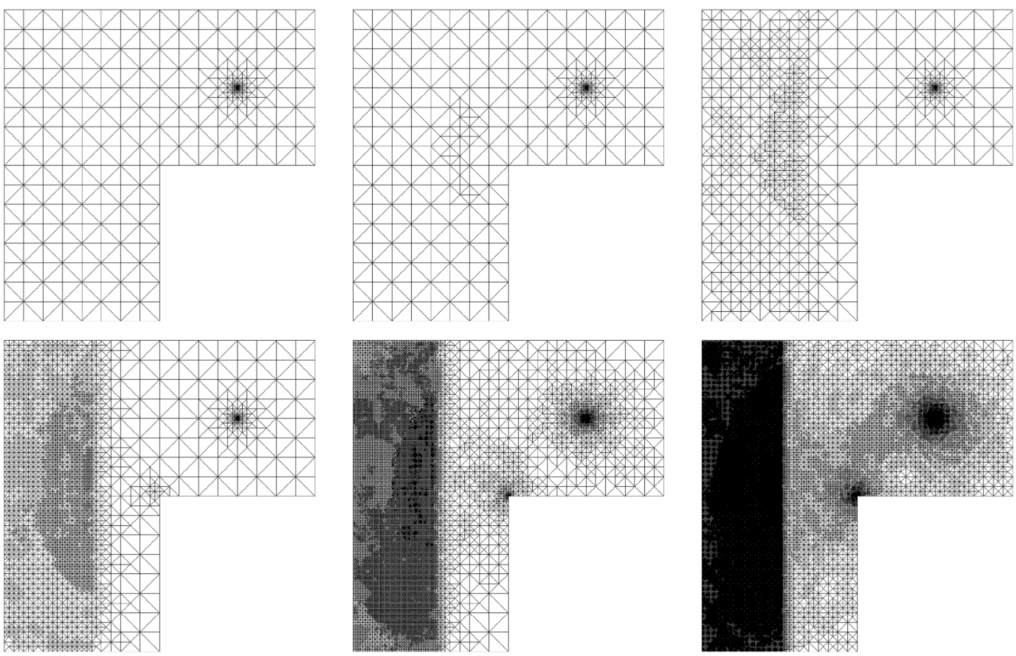

FIG. 2. Meshes after 5, 10, 15, 20, 25, and 30 iterations of the adaptive algorithm guided by the $H^1(\Omega, \omega_1)$-estimators (LWE). They have 345, 486, 1052, 3523, 19715, and 71403 degrees of freedom, respectively. It is worth noticing how the refinement concentrates around x_0, the reentrant corner and the region of interest Ω_0. [Color figure can be viewed at wileyonlinelibrary.com]

tested, in a wide range of possibilities. The differences between the curves with respect to the lowest error curve are significant when the parameters differ from the best one by several orders of magnitude.

We remark that we have chosen $\alpha = \frac{1}{2}$ for simplicity, taking into account that any $\alpha \in (0, 1)$ is admissible and the fact that the results should not be sensitive to the particular choice of α, unless it is very close to zero. This issue has been studied and reported in [9, Example 6.1].

B. Discontinuous Coefficients

We now consider the following diffusion equation with discontinuous diffusion coefficient

$$\begin{cases} -\nabla \cdot (a\nabla u) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

with $\Omega = (-1, 1)^2$, and $\Omega_0 = (-1, 1) \times (-1, -0.75)$, a band at the lower side of the square. We consider two situations, one with $a(x_1, x_2) = 25.2741423690882$ if $x_1x_2 > 0$ and 1 otherwise, and another one with $a(x_1, x_2) = 161.447638797588$ if $x_1x_2 > 0$ and 1 otherwise. The boundary values were taken so that the exact solution is, in polar coordinates, $u(r, \phi) = r^\gamma \mu(\phi)$, where

$$\mu(\phi) = \begin{cases} \cos((\pi/2 - \sigma)\gamma) \cdot \cos((\phi - \pi/2 + \rho)\gamma) & \text{if } 0 \le \phi \le \pi/2 \\ \cos(\rho\gamma) \cdot \cos((\phi - \pi + \sigma)\gamma) & \text{if } \pi/2 \le \phi \le \pi \\ \cos(\sigma\gamma) \cdot \cos((\phi - \pi - \rho)\gamma) & \text{if } \pi \le \phi < 3\pi/2 \\ \cos((\pi/2 - \rho)\gamma) \cdot \cos((\phi - 3\pi/2 - \sigma)\gamma) & \text{if } 3\pi/2 \le \phi \le 2\pi. \end{cases}$$

The constants take the values $\rho = 0.785398163397448$, $\gamma = 0.25$, $\sigma = -5.49778714378214$ in the first case and $\rho = 0.785398163397448$, $\gamma = 0.1$, $\sigma = -14.9225651045515$ in the second. These solutions have a singularity like $|x|^{0.25}$ and $|x|^{0.1}$, respectively, around the origin.

The estimators from Refs. [2, 3], which need a duality argument for the lower order error, do not carry over immediately to this situation due to the lack of corresponding regularity results with discontinuous coefficients. To make a comparison, we assume that the upper bound for the LNE holds with

$$\sigma_{-\beta}(x) = \left(|x|^2 + h(x)^2\right)^{-\beta/2},$$

where $\beta = 1 - \gamma$, and for the DE, we assume the upper bound holds for $p = 9$ in the first case and $p = 21$ in the second case $\left(p = \left\lceil \frac{2}{\gamma} \right\rceil\right)$.

We believe this is reasonable under the assumption that the precise singularity $|x|^\gamma$ is the worst one for such coefficients. The chosen values of p satisfy that the exact solution u belongs to $W^{2,p'}(\Omega)$ with $1/p + 1/p' = 1$. However, we emphasize that the equivalences between error and estimator for the Liao–Nochetto and DE have not been rigorously proved, we infer the possible form of the estimators by analogy after looking at the worst singularity.

To compare the behavior of our method with those of Liao-Nochetto and Demlow we plot in Fig. 4 the local error (i.e., $\|u - U\|_{H^1(\Omega_0)}$) versus the number of DOFs. For the first case (left), we have taken $a_1 = 10^4$ and $a_2 = 10^{-4}$ in the weights ω_1 and ω_2 for LWE, respectively; $d = 0.25$ for LNE; and $d = 0.125$, $p = 9$ and $\zeta = \frac{1}{4}d^{-\frac{1}{2}-\frac{1}{p}}$ for DE. For the second case (right), we have considered $a_1 = 10^5$ and $a_2 = 10^{-4}$ in the weights ω_1 and ω_2 for LWE, respectively; $d = 0.75$

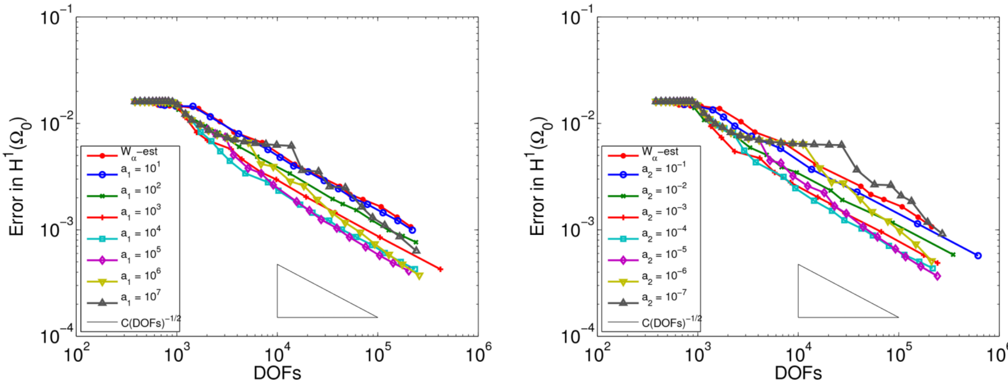

FIG. 3. We plot the $H^1(\Omega_0)$-error versus the number of degrees of freedom (DOFs), obtained with an adaptive procedure guided by the (global) $H^1_\alpha(\Omega)$-estimators from Eq. (13), with $\alpha = \frac{1}{2}$; and the newly proposed LWE taking $a_1 = 10^1, 10^2, 10^3, 10^4, 10^5, 10^6, 10^7$ in the weight ω_1 (left) and $a_2 = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}, 10^{-7}$ in the weight ω_2 (right). We notice that the behavior of the adaptive algorithm is similar, and quasi-optimal, for all the parameters tested, in a wide range of possibilities. The differences between the curves with respect to the lowest error curve are significant when the parameters differ from the best one by several orders of magnitude. [Color figure can be viewed at wileyonlinelibrary.com]

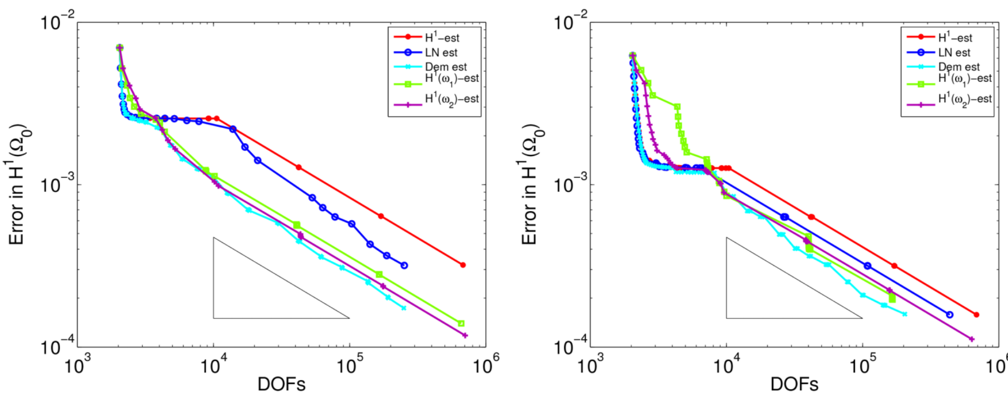

FIG. 4. Local error versus DOFs for the discontinuous coefficient examples. The situation is similar to the one depicted for the L-shaped domain. Our approach is a bit better than the one by Liao and Nochetto (LNE) and very similar to the one by Demlow (DE). We recall that our estimators are indeed an upper bound for the error, but there is no rigorous proof (yet) that the others bound the local error, due to the lack of a corresponding regularity result for the duality approach. [Color figure can be viewed at wileyonlinelibrary.com]

for LNE; and $d = 0.125$, $p = 21$ and $\zeta = \frac{1}{4} d^{-\frac{1}{2} - \frac{1}{p}}$ for DE. We have also considered the algorithm guided by the standard global $H^1(\Omega)$-estimators.

A sequence of meshes for the more singular case is shown in Fig. 5.

C. Diffusion–Advection–Reaction with a Point Source

We end this section by showing how the adaptive method behaves on a diffusion–advection–reaction equation. This is another case where the duality theory fails and our approach provides a simple a posteriori estimator for the local error, by just using an appropriate weight. We consider the

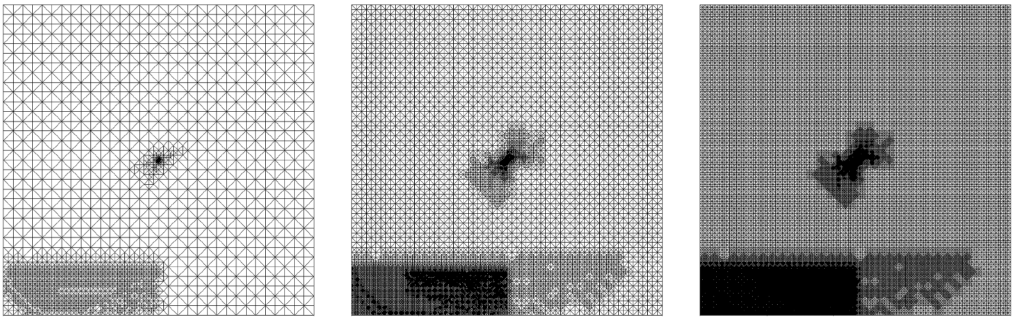

FIG. 5. Meshes after 10, 20, and 30 iterations of the adaptive algorithm guided by the $H^1(\Omega, \omega_1)$-estimators (LWE) for the more singular case.

equation

$$\begin{cases} -0.02\Delta u + \begin{bmatrix} 2 \\ \sin(5x_1) \end{bmatrix} \cdot \nabla u + 0.1u = \delta_{(0.2,0.4)} & \text{in } \Omega = (0,3) \times (0,1), \\ u = 0 & \text{on } \partial\Omega \cap \{x_1 < 3\}, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \cap \{x_1 = 3\}, \end{cases}$$

and the region of interest is $\Omega_0 = (0,3) \times (0, 0.25)$. An approximate solution obtained with an adaptive method tailored to the W_α error has been presented in Ref. [9]. We now present the sequence of meshes (Fig. 6) obtained by the new LWE taking $\alpha = \frac{1}{2}$ and $a_1 = 10^5$ in the definition of the weight ω_1. Notice that the refinement focuses on the region of interest Ω_0 which is now a narrow band at the bottom of the domain. This emulates the situation where there is a pollutant discharge in a river or canal, and we are interested in the amount of pollutant at the coast.

Remark 4.1. (3D situation). It is known that edge singularities on polyhedral domains are not always resolvable with the expected rate $DOF^{-\ell/3}$ in the H^1-norm when using AFEM with shape-regular (isotropic) elements of degree ℓ. Our algorithm suffers from the same limitation, even when Ω_0 is at a positive distance from the edges. Indeed, consider the case where Ω is a polyhedral domain in 3D with edges that lead to singularities which can only be adaptively resolved with rate $DOF^{-\ell/3+\sigma}$ in the energy norm (for some $\sigma > 0$). If no point source is present, the weighted norms presented here are in fact equivalent to the energy norm, albeit with potentially quite large constants. Thus, our algorithm will eventually lead to an error decrease with rate $DOF^{-\ell/3+\sigma}$. On the other hand, the theory of Ref. [4] essentially ensures that the algorithms of Demlow or a slight modification of the one by Liao and Nochetto will yield a convergence rate $DOF^{-\ell/3}$ whenever the exact solution can be approximated in the norm $\|\cdot\|_{H^1(\Omega_0)} + \|\cdot\|_{L^2(\Omega)}$ with such a rate. This will be the case if Ω_0 is at positive distance from the edges, for some degrees ℓ. To be able to obtain such rates with our algorithm, one could try to use weights that vanish at the edges; the study of this idea falls beyond the scope of this article, and will be subject of future research.

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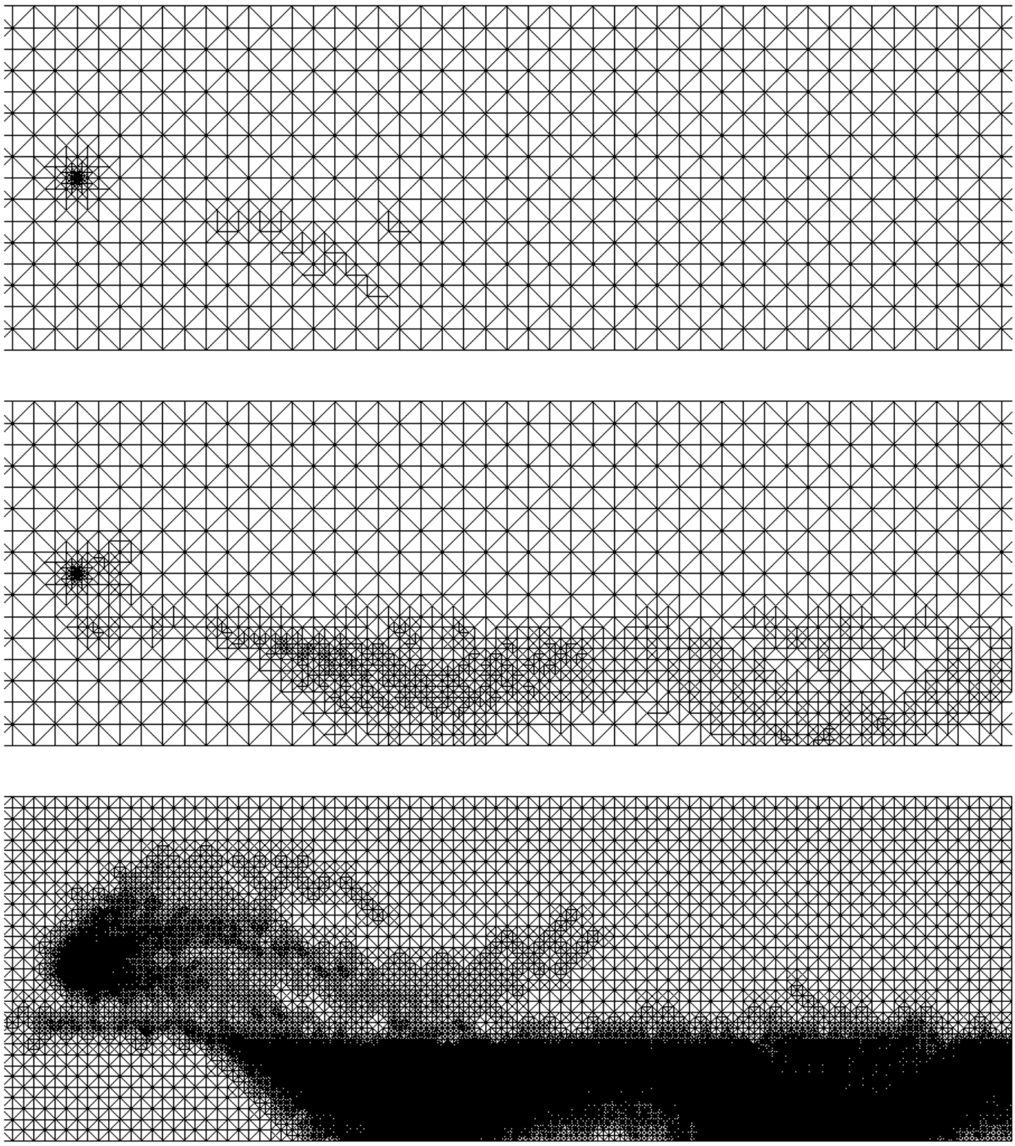

FIG. 6. Sequence of meshes for the diffusion-advection-reaction experiment. Meshes obtained after 15 (2338 elements/1234 DOFs), 20 (4799 elements/2473 DOFs) and 25 (251744 elements/126181 DOFs) iterations of the adaptive algorithm guided by the LWE. During the first 14 adaptive steps the refinement was focused on solving the singularity due to the point source. Starting at step 15, the meshes get refined at the region of interest, and a combination of refinement around the singularity and the region of interest occurs after step 20.

References

1. J. Xu and A. Zhou, Local and parallel finite element algorithms based on two-grid discretizations, Math Comp 69 (2000), 881–909.

2. X. Liao and R. H. Nochetto, Local a posteriori error estimates and adaptive control of pollution effects, Numer Methods Partial Differential Equations 19 (2003), 421–442.
3. A. Demlow, Convergence of an adaptive finite element method for controlling local energy errors, SIAM J Numer Anal 48 (2010), 470–497.
4. A. Demlow, Quasi-optimality of adaptive finite element methods for controlling local energy errors, Numer Math 134 (2016), 27–60.
5. R. Araya, E. Behrens and R. Rodríguez, An adaptive stabilized finite element scheme for a water quality model, Comput Methods Appl Mech Engrg 196 (2007), 2800–2812.
6. J. D. Jackson, Classical electrodynamics, 2nd Ed., Wiley, New York-London-Sydney, 1975.
7. E. Casas, Control of an elliptic problem with pointwise state constraints, SIAM J Control Optim 24 (1986), 1309–1318.
8. C. D'Angelo and A. Quarteroni, On the coupling of 1D and 3D diffusion-reaction equations, Application to tissue perfusion problems, Math Models Methods Appl Sci 18 (2008), 1481–1504.
9. J. P. Agnelli, E. M. Garau and P. Morin, A posteriori error estimates for elliptic problems with Dirac measure terms in weighted spaces, ESAIM Math Model Numer Anal 48 (2014), 1557–1581.
10. E. Fabes, C. Kenig and R. Serapioni, The local regularity of solutions of degenerate elliptic equations, Comm Partial Differential Equations 7 (1982), 77–116.
11. R. H. Nochetto, E. Otárola, A. J. Salgado, Piecewise polynomial interpolation in Muckenhoupt weighted Sobolev spaces and applications, Numer Math 132 (2016), 85–130.
12. S. Chanillo and R. L. Wheeden, Weighted Poincar and Sobolev inequalities and estimates for weighted Peano maximal functions, Amer J Math 107 (1985), 1191–1226.
13. V. A. Kozlov, V. G. Maz'ya, J. Rossmann, Elliptic boundary value problems in domains with point singularities, Mathematical surveys and monographs, vol. 52, American Mathematical Society, Providence, RI, 1997.
14. R. Verfürth, A posteriori error estimation and adaptive mesh-refinement techniques, J Comput Appl Math 50 (1994), 67–83.
15. R. Verfürth, A review of a posteriori error estimation and adaptive mesh-refinement technique, Wiley-Teubner, Chichester, 1996.
16. R. E. Bank and M. Holst, A new paradigm for parallel adaptive meshing algorithms, SIAM J Sci Comput 22 (2000), 1411–1443.
17. R. Araya, E. Behrens and R. Rodríguez, A posteriori error estimates for elliptic problems with Dirac delta source terms, Numer Math 105 (2006), 193–216.