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Citation: Chaos **27**, 083112 (2017); doi: 10.1063/1.4998141
View online: https://doi.org/10.1063/1.4998141
View Table of Contents: http://aip.scitation.org/toc/cha/27/8
Published by the American Institute of Physics

Distinguishability notion based on Wootters statistical distance: Application to discrete maps

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(Received 18 March 2017; accepted 28 July 2017; published online 15 August 2017)

We study the distinguishability notion given by Wootters for states represented by probability density functions. This presents the particularity that it can also be used for defining a statistical distance in chaotic unidimensional maps. Based on that definition, we provide a metric $\bar{d}$ for an arbitrary discrete map. Moreover, from $\bar{d}$, we associate a metric space with each invariant density of a given map, which results to be the set of all distinguished points when the number of iterations of the map tends to infinity. Also, we give a characterization of the wandering set of a map in terms of the metric $\bar{d}$, which allows us to identify the dissipative regions in the phase space. We illustrate the results in the case of the logistic and the circle maps numerically and analytically, and we obtain $\bar{d}$ and the wandering set for some characteristic values of their parameters. Finally, an extension of the metric space associated for arbitrary probability distributions (not necessarily invariant densities) is given along with some consequences. The statistical properties of distributions given by histograms are characterized in terms of the cardinal of the associated metric space. For two conjugate variables, the uncertainty principle is expressed in terms of the diameters of the associated metric space with those variables. *Published by AIP Publishing.*
[http://dx.doi.org/10.1063/1.4998141]

The use of statistical quantifiers in complex phenomena and chaos theory to characterize the underlying dynamics has proven to be one of the most powerful tools. One of them focuses on defining statistical distances between probability distributions to provide a notion of distinguishability between states, classical and quantum. Other approach is that given by the description based on the invariant density which characterizes a discrete system in the asymptotic limit of large iterations of the map that supplies the dynamics. Beyond some achievements by introducing statistical distances, a fully geometric notion of distinguishability in discrete maps seems to be absent. In this paper, we address the distinguishability notion based on the Wootters statistical distance and extend it as a metric for a given discrete map. Our study provides a metric space characterizing the map dynamics and presenting an intimate relationship with the set of dissipative points, i.e., the wandering set. As a consequence of this study, we extend the metric space associated for arbitrary probability distributions and explore some relations with regard to statistical properties of distributions given by histograms and the uncertainty principle for two conjugate variables by the Fourier transform.

I. INTRODUCTION

The concept of distance constitutes, not only in mathematics but also in physics, a measure of how apart two "objects" are. Depending on the context, a distance may refer to a physical length or to a metric function which gives place to the theory of metric spaces.[1–3] In physics, it is well known that the information about a system is contained in the state function, the evolution of which accounts for the features of the dynamics. When there is a limitation or uncertainty about the knowledge of the system (classical or quantum), it is common to consider the states as represented by probability density functions. Distances between probability density functions give the so-called *statistical measures of distinguishability* between states.[4] Many statistical distances have been defined for several purposes such as Kullback divergence,[5] Wootters[6] and Monge[7] distances in quantum mechanics, or the metric distance given by the Fisher–Rao tensor in information geometry.[8]

In the context of discrete maps, one of the most important statistical features of underlying dynamics is given by the probability distribution that is a fixed point of the corresponding Frobenius–Perron operator associated with the map.[9,10] This is the so-called invariant density which plays an important role in the ergodic theory, i.e., the study of the measures that are preserved by some function.[11] Relevant properties that lie at the foundations of statistical mechanics such as ergodicity and mixing are described in terms of invariant densities by means of the corresponding levels of the ergodic hierarchy,[11] with quantum extensions[12–16] that allow us to characterize the aspects of quantum chaos.[17] The relevance of discrete maps lies in the fact that they serve as simple but useful models in biology, physics, economics, etc.[18–20] Specifically, they have proven to be powerful tools for testing features of chaotic and complex phenomena.[21–24]

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In the case of the logistic map,[18,19] from the invariant density, Johal[25] proposed a statistical distance which serves to characterize the chaotic regime. In this paper, we explore, analytically and numerically, this idea by redefining the statistical distance given in Ref. 25 as a metric, and we apply this to characterize two emblematic chaotic maps: the logistic map and the circle map. We focus on some values of their parameters which are in correspondence to the characteristic regimes of the dynamics. Moreover, we define a metric space associated with values of the parameters of each map that takes into account the topology of the phase space according to dynamics.

Then, we give an extension of the metric space associated for arbitrary probability distributions and explore some consequences. For probability distributions given by histograms, we characterize the maximal ignorance, certainty, and number of bins in terms of the cardinal of the metric space associated. With regard to two probability distributions whose variables are conjugated by the Fourier transform, we express the uncertainty principle in terms of the diameters of their associated metric spaces.

This work is organized as follows: In Sec. II, we introduce the notions used throughout this paper. Section III is devoted to recalling the proposal of a statistical distance made by Johal,[25] and using equivalence classes, we redefine it as a metric. Next, we propose a metric space, induced by this metric, composed of all the phase space points that can be distinguished in the limit of a large number of iterations of the map. From the metric, we give a characterization of the wandering set of a map that allows one to classify the dissipative regions of the dynamics in phase space. Considering some characteristic values of their parameters, in Sec. IV, we illustrate the results for the logistic map and for the circle map. Then, in Sec. V, we provide new insights by extending the metric space associated with arbitrary probability distributions. Here, we apply this extension to distributions given by histograms, and we give an expression of the uncertainty principle for two distributions, whose variables are conjugated by the Fourier transform, in terms of the diameters of the metric spaces associated with the distributions. Finally, in Sec. VI, we draw some conclusions and outline future research directions.

II. PRELIMINARIES

We recall some notions from the probability theory, discrete maps, and metric spaces used throughout this paper.

A. Probability density and cumulative distribution functions

A probability density function defined over an abstract space Γ (typically, a subset of $\mathbb{R}^m$) is any nonnegative function $p:\Gamma\rightarrow\mathbb{R}_+$ such that

$$\int_\Gamma p(x)dx = 1, \tag{1}$$

where Γ is called the *space of events*. For instance, in an experiment, Γ represents the set of all possible outcomes of the phenomenon being observed. If Γ is composed of a discrete, say N, the number of results, then one has a discrete probability distribution which can be represented by a column vector $(p_1, p_2, \ldots, p_N)^t$, where p_i stands for the probability that the ith result occurs, and the normalization condition (1) now reads $\sum_{i=1}^N p_i = 1$.

Given the probability distribution p with $\Gamma\subseteq\mathbb{R}$, the so-called cumulative distribution function associated with $p(x)$ is defined as

$$C(x) = \int_{\{t\leq x\}} p(t)dt, \tag{2}$$

$C(x)$ is the probability that the variable takes a value less than or equal to x. When Γ is composed of a number N of results, the definition (2) reads as $C_i = \sum_{j=1}^i p_j$. The cumulative distribution function is frequently used in statistical analysis since an estimation of $C(x)$ can be given directly from the empirical distribution function in terms of the experimental data.

B. Discrete maps and invariant densities

Given a set Γ and a continuous function f: $\Gamma\rightarrow\Gamma$, it is said that the sequence $\{x_n\}_{n\in\mathbb{N}_0}\subseteq\Gamma$ such that

$$x_{n+1} = f(x_n) \quad \forall\, n\in\mathbb{N}_0 \;,\; x_0\in\Gamma, \tag{3}$$

defines a *discrete map*. More generally, one can allow f to have finite discontinuities. From the physical viewpoint, a discrete map models a system, the dynamics of which is given by iterating Eq. (3) where each iteration corresponds to a time step. That is, if the system is initially in a state x_0, then x_n represents the state after n time steps. If there exists an element $c\in\Gamma$ such that $f(c)=c$, it is said that c is a *fixed point* of f. Fixed points are physically interpreted as stationary states of the system subject to the map.

The dynamics of a discrete map can be characterized in terms of probability density functions in the following way. The Frobenius–Perron operator $P:\mathbb{L}^1(\Gamma)\rightarrow\mathbb{L}^1(\Gamma)$ associated with the map (3) is given by[9]

$$\int_A P\phi(x)dx = \int_{f^{-1}(A)} \phi(x)dx,$$

for all $\phi\in\mathbb{L}^1(\Gamma)$ and $A\subseteq\Gamma$, where $f^{-1}(A)$ is the preimage of A. Any nonnegative function $\rho\in\mathbb{L}^1(\Gamma)$, normalized to 1 in Γ, and such that

$$P\rho(x) = \rho(x) \quad \forall\, x\in\Gamma,$$

is called the *invariant density* of the map. Mathematically, invariant densities are a special case of invariant measures.[9] In the dynamical system theory, the invariant densities represent stationary states, which allows one to study the system in the asymptotic limit of large times.[26] As an example, in the case of the logistic map in its chaotic regime, one has $f(x) = 4x(1-x)$ and $\Gamma = [0, 1]$. It is found that (Ref. 9, p. 7)

$$Pf(x) = \frac{1}{4\sqrt{1-x}}\left\{f\left(\frac{1}{2}-\frac{1}{2}\sqrt{1-x}\right)+f\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-x}\right)\right\},$$

for all $f \in \mathbb{L}^1([0,1])$. By successively applying the operator P over any initial distribution f (i.e., $Pf, P^2f = P(Pf)$, $P^3f = P(P(Pf)), \ldots$), one obtains an analytical expression for the invariant density $\rho(x)$ in the limit as (Ref. 9, p. 7)

$$\rho(x) = \lim_{n\to\infty} P^n f(x) = \frac{1}{\pi\sqrt{x(1-x)}}, \quad x \in (0,1).$$

In general, except for some particular maps, the invariant density has no analytical expression, and therefore, one has to compute it numerically.[24]

The method consists of constructing a histogram to show the frequency with which states along a sequence $\{x_n\}$ fall into given regions of phase space Γ. In order to illustrate the procedure, let us consider $\Gamma = [0, 1]$. Then, we simply divide the interval [0, 1] into M discrete nonintersecting intervals of the form

$$\left[\frac{i-1}{M}, \frac{i}{M}\right) \quad i = 1, \ldots, M.$$

The next step is to consider an initial state x_0 and calculate the trajectory

$$x_0, \quad x_1 = f(x_0), \quad x_2 = f(f(x_0)), \ldots, x_T = f^T(x_0)$$

of length T with $T \gg M$. The fraction ρ_i of the T states visiting the *ith* interval is

$$\rho_i = \frac{\sharp\left\{x_\tau \in \left[\frac{i-1}{M}, \frac{i}{M}\right) \mid \tau = 0, 1, \ldots, T\right\}}{T}, \; i = 1, \ldots, M.$$

Finally, the invariant density ρ is given by

$$\rho(x) = \sum_{i=1}^{M} \rho_i \, \chi_{\left[\frac{i-1}{M}, \frac{i}{M}\right)}(x),$$

where $\chi_{\left[\frac{i-1}{M}, \frac{i}{M}\right)}(x)$ denotes the characteristic function of the interval $\left[\frac{i-1}{M}, \frac{i}{M}\right)$. It should be noted that in the limit $M, T \to \infty$, one has that $\rho(x)$ does not depend on the starting point. The cumulative distribution associated with ρ results as follows:

$$C(x) = \sum_{i=1}^{M} \left(\sum_{j=1}^{i} \rho_j\right) \chi_{\left[\frac{i-1}{M}, \frac{i}{M}\right)}(x).$$

C. Metric spaces

We consider a set Γ and a nonnegative function $d : \Gamma \times \Gamma \to \mathbb{R}$, satisfying the following axioms:

(a) Non–negativity: $d(x, y) \geq 0$
(b) Distinguishability: $d(x, y) = 0$ iff $x = y$
(c) Symmetry: $d(x, y) = d(y, x)$
(d) Triangle inequality: $d(x, y) + d(y, z) \geq d(x, z)$

The function d is called a *metric* or *distance*, and the pair (Γ, d) defines a *metric space*. In what follows, we will call metric or distance indistinctly, as is used in mathematical sense. The standard example of metric space is $\mathbb{R}^m$ with the usual Euclidean metric. Axiom (*a*) is the so–called non–negativity or separation axiom. Axiom (*b*) refers to the intuitive idea of how to distinguish two points when they are at a nonzero distance from each other. Axiom (*c*) expresses that any legitimate distance must be symmetric with respect to the pair of points. Axiom (*d*) is the well-known Pythagoras triangle inequality and is crucial to extend many theorems and properties out of $\mathbb{R}^m$. Relaxing only (*b*) defines a *pseudometric*, and relaxing only (*c*) gives a *quasi-metric*. Also, relaxing (*b*), (*c*), and (*d*) defines a *premetric*. It should be noted that premetrics are also called as *statistical distances* in the context of the information theory. Except in the cases when a clarification becomes relevant or necessary, from now on, we adopt this terminology since it is widely used in the literature.

In the context of the information theory, an example of statistical distance is the well-known *Kullback–Leibler divergence* as a relative entropy between two probability density functions.[5] In quantum mechanics, the *Jensen–Shannon divergence* is an example of distance in Hilbert space and also can be used to define a measure of distinguishability and entanglement between quantum states.[27–29] Also, using notions of entropy and purification, one can define metrics in quantum state spaces.[30]

The concept of metric space arises as a generalization of the Euclidean space, many of its relevant properties and results concerning completeness, convexity, etc., have been extended for abstract spaces in general.[1–3] In this work, we consider the state space Γ as a subset of some Euclidean space $\mathbb{R}^m$.

III. METRICS BASED ON INVARIANT DENSITIES

Based on the work by Johal,[25] we recall the motivation to define a statistical distance for discrete maps. From this, we provide a metric in a mathematically strict sense, i.e., obeying axioms (*a*)–(*d*). Then, we define a "natural" metric space associated with a discrete map, induced by the metric.

A. A metric for discrete maps

Wootters proposed a statistical distance between two probability distributions, which gives a notion of distinguishability between states (classical or quantum).[6] A typical example is to consider two weighted coins represented by the corresponding discrete probability distributions $p = (p_1, p_2)$ and $q = (q_1, q_2)$. The Wootters statistical distance D between p and q is defined as

$$D(p,q) = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \times [\text{maximum number of mutually} \times \text{distinguishable intermediate probablities in } n \text{ trials}].$$

The relevant conclusion of Wootters' contribution has a double value: on the one hand, this statistical distance can be defined on any probability space. On the other hand, and maybe more important, is the fact that $D(p, q)$ is involved in determining the geometry of the curved manifold of all distinguishable probability distributions.

For unidimensional discrete maps showing chaotic dynamics, inspired by Wootters ideas, a notion of statistical distance d between two points x^A and x^B of state space Γ was introduced in Ref. 25, which is given by

$$d(x^A,x^B) = \lim_{n\to\infty} [\text{probability of visiting the interval} \\ \times \text{ between } x^A \text{ and } x^B \text{ after } n \text{ time steps}].$$

Now, if $\rho(x)$ is the invariant density of the discrete map under study, Johal argues that $d(x^A, x^B)$ can be expressed as an integral of the invariant density on the interval. From Ref. 25,

$$d(x^A,x^B) = \left| \int_{x^A}^{x^B} \rho(x)dx \right|. \tag{4}$$

Using the properties of the integral, it is straightforward to check that $d(x^A, x^B)$ satisfies axioms (a), (c), and (d). Concerning axiom (b), there exist situations for which it is not fulfilled. For instance, if $\rho(x) = 0$ in almost every point of some interval $[x^A, x^B]$, then from (4), it follows that $d(x^A, x^B) = 0$ even when $x^A \neq x^B$. Strictly speaking, formula (4) does not define a metric unless one can guarantee that there is no interval $[x^A, x^B]$ for which $\rho(x) \equiv 0$ a.e. As we shall see, this condition cannot be guaranteed for all the regimes of the dynamics of a given map. Then, formula (4) only defines a pseudometric.

In order to solve this problem, we propose to use equivalence classes instead of points in Γ. More precisely, we define the following relation $\sim$ in $\Gamma \times \Gamma$:

$$x \sim x' \iff d(x,x') = 0. \tag{5}$$

Let us show that $\sim$ is an equivalence relation. By (4), one has $d(x^A,x^A) = 0$ and then $x^A \sim x^A$ for all $x^A \in \Gamma$. Due to symmetry of formula (4), it follows that $x^A \sim x^B$ if and only if $x^B \sim x^A$ for all x^A, $x^B \in \Gamma$. If x^A, x^B, $x^C \in \Gamma$ with $x^A \sim x^B$ and $x^B \sim x^C$, then $d(x^A,x^B) = 0 = d(x^B,x^C)$, and then, by the nonnegativity of d and the triangle inequality, one has $0 \leq d(x^A,x^C) \leq d(x^A,x^B) + d(x^B,x^C) = 0$, so one has $d(x^A,x^C) = 0$, i.e., $x^A \sim x^C$.

Considering the set of classes $\Gamma/\sim = \{\bar{x} : x \in \Gamma\}$, where $\bar{x} = \{x' \in \Gamma : x' \sim x\}$, one can define rigorously a metric $\bar{d}$ between elements of $\Gamma/\sim$ in the following way. Let us consider two points x^A and x^B, then

$$\bar{d} : \Gamma/\sim \times \Gamma/\sim \to \mathbb{R}_+ \bar{d}(\overline{x^A},\overline{x^B}) := d(x^A,x^B), \tag{6}$$

with $\overline{x^A},\overline{x^B}$ being the classes of x^A, and x^B respectively. Let us show that $\bar{d}$ is well defined in the sense that it is independent of the representative elements chosen for each class. For arbitrary $x \in \overline{x^A}, x' \in \overline{x^B}$, and then, one has $d(x,x^A) = d(x',x^B) = 0$. By applying the triangle inequality and the symmetry property

$$d(x,x') \leq d(x,x^A) + d(x^A,x^B) + d(x^B,x') = d(x^A,x^B)$$

and

$$d(x^A,x^B) \leq d(x^A,x) + d(x,x') + d(x',x^B) = d(x,x').$$

Thus, $d(x,x') = d(x^A,x^B)$, and by the definition (6), it follows that $\bar{d}(\bar{x},\bar{x'}) = \bar{d}(\overline{x^A},\overline{x^B})$. Using (6) and since d satisfies the axioms (b) and (c), then $\bar{d}$ also satisfies them. Now, we can see that $\bar{d}$ satisfies the distinguishability axiom (a). From definitions (5) and (6), one has

$$\bar{d}(\bar{x}^A,\bar{x}^B) = 0 \iff d(x^A,x^B) = 0 \\ \iff x^A \sim x^B \iff \bar{x}^A = \bar{x}^B.$$

Therefore, $\bar{d}$ is a metric on $\Gamma/\sim$.

B. A metric space associated with the map dynamics

Given a unidimensional discrete map characterized by f: $\Gamma \to \Gamma$, with $\Gamma \subseteq \mathbb{R}$, and the metric $\bar{d}$ induced by the invariant density $\rho : \Gamma \to \mathbb{R}_+$, we say that $(\Gamma/\sim, \bar{d})$ is the *metric space associated with the map dynamics* generated by f. In this way, given the pair $\{\Gamma, f\}$, one has the following "canonical" association between discrete maps and metric spaces:

$$\{\Gamma, f\} \to \rho \to d \to (\Gamma/\sim, \bar{d}). \tag{7}$$

In general, a discrete map can be given by a function $f_{\mathbf{r}}$: $\Gamma \to \Gamma$ depending on a set of real parameters $\mathbf{r} = (r_1, \ldots, r_k)$ which play the role of controlling the type of dynamics generated. Thus, for a set of parameters $\mathbf{r}$, the map is given in the following form:

$$x_{n+1} = f_{\mathbf{r}}(x_n) \quad \forall n \in \mathbb{N}_0 \ , \ x_0 \in \Gamma. \tag{8}$$

Therefore, one has a parameterized family of correspondences of the form (7), i.e.,

$$\{\Gamma, f_{\mathbf{r}}\} \to \rho_{\mathbf{r}} \to d_{\mathbf{r}} \to (\Gamma/\sim_{\mathbf{r}}, \bar{d}_{\mathbf{r}}), \tag{9}$$

where $d_{\mathbf{r}}$, $\bar{d}_{\mathbf{r}}$, and $\sim_{\mathbf{r}}$ are given as in Eqs. (4), (5), and (6), respectively. Varying the parameter $\mathbf{r}$, one can study transitions in the dynamics of the map in terms of the corresponding changes in the topology of $(\Gamma/\sim_{\mathbf{r}}, \bar{d}_{\mathbf{r}})$.

C. Wandering set of a map in terms of the metric

According to definitions (2) and (4) and by the properties of the integral, it follows that

$$d(x^A,x^B) = |C(x^B) - C(x^A)|, \tag{10}$$

where $C(x)$ is the cumulative distribution function of the invariant density ρ. This expression gives another characterization of the metric associated with a map that can be useful for studying the structure of the metric space. In fact, the so-called *wandering set* can be characterized in terms of the metric as follows: We recall the concept of *wandering point* of a map.

Given a map generated by a continuous function $f : \Gamma \to \Gamma$, a point $w \in \Gamma$ is said to be a *wandering point* if there is a neighbourhood U of w and a positive integer N such that for all $n > N$, one has

$$\mu(f^n(U) \cap U) = 0,$$

where μ is a measure defined over a σ–algebra Σ of Γ and $f^n(U)$ is the set $\{f^n(u) : u \in U\}$. The *wandering set*, which we denote as $\mathcal{W}(\Gamma)$, is defined as the set of all wandering points.

The following result relates the metric space associated with a map with its wandering set.

Theorem III.1. *Let $\Gamma = [a, b]$ be the phase space of the map generated by $f : [a,b] \to [a,b]$ and let $[x^A, x^B] \subseteq [a,b]$ be a subinterval of [a, b]. Then, considering the measure $\mu_\rho(A) = \int_A \rho(x)dx$ for all subset $A \subseteq [a,b]$ with $\rho(x)$ the invariant density of the map, the following propositions are equivalent:*

(a) $\rho(x) = 0$ a.e. in (x^A, x^B).
(b) $\bar{x}^A = \bar{x}^B$.
(c) $d(x^A, x^B) = 0$.
(d) $C(x)$ is constant in $[x^A, x^B]$.
(e) All $x \in (x^A, x^B)$ is a wandering point.

Proof. $(a) \Rightarrow (b)$: If $\rho(x) = 0$ a.e. in (x^A, x^B), then it is clear from (4) that $d(x^A, x^B) = 0$, which means by definition that $\bar{x}^A = \bar{x}^B$.

$(b) \Rightarrow (c)$: It follows by the definition (5).

$(c) \Rightarrow (d)$: Let us assume that $d(x^A, x^B) = 0$. Then, by (10), one obtains $C(x^A) = C(x^B)$. Now, since $C(x)$ is an increasing function of x, then $C(x)$ is constant in $[x^A, x^B]$.

$(d) \Rightarrow (e)$: Let x be a point of (x^A, x^B). Since (x^A, x^B) is a neighbourhood of x and $C(x)$ is constant in $[x^A, x^B]$, one has $C(x^B) - C(x^A) = \int_{x^A}^{x^B} \rho(x)dx = \mu_\rho((x^A, x^B)) = 0$. In particular, $f^n(x^A, x^B) \cap (x^A, x^B) \subseteq (x^A, x^B)$ for all $n \in \mathbb{N}$ from which follows that $\mu_\rho(f^n(x^A, x^B) \cap (x^A, x^B)) = 0$, i.e., x is a wandering point.

$(e) \Rightarrow (a)$: Let x be a point of (x^A, x^B). By hypothesis, there exists a neighbourhood U of x and a positive integer N such that $\mu_\rho(f^n(U) \cap U) = 0$ for all $n > N$. In particular, one can repeat the histogram construction for ρ by dividing the interval $[a, b]$ in M subintervals $I_1,\ldots, I_M$ of equal length in a such way that $x, x_0 \in I_{j_0} \subset U$ for some $j_0 \in \{1,\ldots, M\}$. That is, one has the invariant density $\rho(x) = \sum_{i=1}^M \rho_i \chi_{I_i}(x)$, with $\chi_{I_i}(x)$ the characteristic function of I_i for all $i = 1,\ldots, M$. Then, one has two situations that are mutually excluding.

If $f^k(x_0) \notin I_{j_0}$ for all $k > N$, then the subinterval I_{j_0} is only finitely visited in the limit of large iterations $T \gg M$, which means that $\rho_{j_0} = 0$, i.e., $\rho = 0$ a.e. in I_{j_0}.

On the contrary, if $f^{k_0}(x_0) \in I_{j_0}$ for some $k_0 > N$, then $f^{k_0}(I_{j_0}) \cap I_{j_0}$ is a nonempty set, and therefore, since f is assumed to be a continuous function, it follows that $f^{k_0}(I_{j_0}) \cap I_{j_0}$ is a nonempty interval. Thus, $|f^{k_0}(I_{j_0}) \cap I_{j_0}| > 0$ with $|\,.\,|$ the Lebesgue measure in $\mathbb{R}$. Moreover, since $\mu_\rho(f^{k_0}(I_{j_0}) \cap I_{j_0}) = 0$, one has

$$\int_{f^{k_0}(I_{j_0})\cap I_{j_0}} \rho(x)dx = \int_{f^{k_0}(I_{j_0})\cap I_{j_0}} \rho_{j_0}\chi_{I_{j_0}}(x) = \rho_{j_0}|f^{k_0}(I_{j_0}) \cap I_{j_0}| = 0,$$

with $|f^{k_0}(I_{j_0}) \cap I_{j_0}| > 0$. This implies that $\rho_{j_0} = 0$ and then $\rho = 0$, a.e. in I_{j_0}. In both cases, one obtains $\rho = 0$, a.e. in I_{j_0}, with $x \in I_{j_0} \subset (x^A, x^B)$. Therefore, one has that $\rho = 0$ a.e. in (x^A, x^B). □

Physically, the wandering set is an important concept since when a dynamical system has a wandering set of non-zero measure, then the system is *dissipative*. Thus, the wandering set characterizes dissipation in dynamical systems and in maps. Theorem III.1 states that dissipation in discrete maps can be expressed by means of the metric or alternatively by using the cumulative distribution function.

IV. MODELS AND RESULTS

In order to illustrate the ideas introduced in Sec. III, we consider two examples of chaotic maps: the logistic map and the circle map. The interest on the logistic map is, basically, that it contains all the features of the chaotic dynamics and the onset of chaos.[21] The circle map describes a simplified model of the phase–locked loop in electronics and also has been used to study the dynamical behavior of a beating heart,[32] among other applications.

A. The logistic map

The logistic map is defined by the sequence[18,19]

$$x_{n+1} = rx_n(1 - x_n) \quad \forall n \in \mathbb{N}_0,$$

i.e., in the form (8) where f_r: [0, 1] $\to$ [0, 1] is given by $f_r(x) = rx(1 - x)$, depending on a unique parameter $r \in (0, 4)$. The dynamics has been characterized for all r. Here, we focus on some special characteristic values which give rise to different relevant dynamics. Specifically, we consider r equal to 1, 2, 3.56995…, 3.82843, and 4.

For $r = 1$, one has a regular behavior where for any initial condition x_0, the sequence $\{x_n\}$ goes to zero, i.e., $x = 0$ is an attractor for all points of [0, 1].

For $r = 2$, the dynamics is also regular and all sequences $\{x_n\}$ tend to the fixed point $x = 1/2$.

At $r = r_c = 3.56995$, the onset of chaos occurs where from almost all initial conditions oscillations of finite period are not observed.

The value $r^* = 3.82843$ corresponds to the region known as the "Pomeau–Manneville scenario"[31] characterized by a periodic (laminar) phase interrupted by bursts of aperiodic behavior, with a mixed dynamics composed of chaotic trajectories and *islands of stability* that show non–chaotic behavior.

Finally, for $r = 4$, the behavior is fully chaotic and the dynamics satisfies the mixing property, with no oscillations, and the interval (0, 1) is an attractor for any initial condition x_0.

We calculate the invariant density for the logistic map. For the regular cases $r = 1$ and $r = 2$, since all the sequences tend to only one fixed point, it follows that the invariant density is a delta function; this fact can also be argued by using the Frobenius–Perron operator and Eq. (4). Indeed, for $r = 1$ and $r = 2$, it is satisfied that $\rho_1(x) = \delta(x)$ and $\rho_2(x) = \delta(x - \frac{1}{2})$, i.e., $x = 0$ and $x = \frac{1}{2}$ are attractors, respectively, for all initial conditions; the only point that can be distinguished in each case is precisely the attractor. For $r = 1$, all the interval [0, 1] collapses into two classes: $\bar{0}$ and $\bar{1}$, and for $r = 2$, the classes are $\bar{0}$, $\bar{\frac{1}{2}}$ and $\bar{1}$. For the fully chaotic case $r = 4$, as mentioned in Sec. II, an analytic result is available; indeed, the invariant density has the form $\rho_4(x) = \frac{1}{\pi\sqrt{x(1-x)}}$.

For the cases $r = r_c$ and r^*, and also for $r = 4$, we computed numerically the invariant density following the method described in Sec. II. We carried out this procedure for an initial state $x_0 = 0.1$ and taking $M = 10^3$ and $T = 10^5$. The process was repeated for other initial states, leading to the same result. This means that despite the sensitivity of trajectories to initial states, the invariant density remains the same when a large number of iterations are considered. The results are given in Fig. 1. In the fully chaotic case, one can see the agreement of the numerical computation with the analytical result for the invariant density ρ_4.

Now, we focus on the study of the metric space $([0,1]/\sim_r, \bar{d}_r)$, the wandering set $\mathcal{W}_r([0,1])$, and the cumulative distribution function $C_r(x)$. For $r = 1$, 2, and 4, analytical results can be provided from Eqs. (4)–(6) and Theorem III.1.

We obtain

- Case r = 1:
  $[0,1]/\sim_1 = \{\bar{0}\,,\ \bar{1}\}$, with

$$\bar{d}_1(\bar{0}, \bar{1}) = 1,$$

$$C_1(x) = 1 \quad \forall x \in [0,1]$$

  and

$$\mathcal{W}_1([0,1]) = (0,1].$$

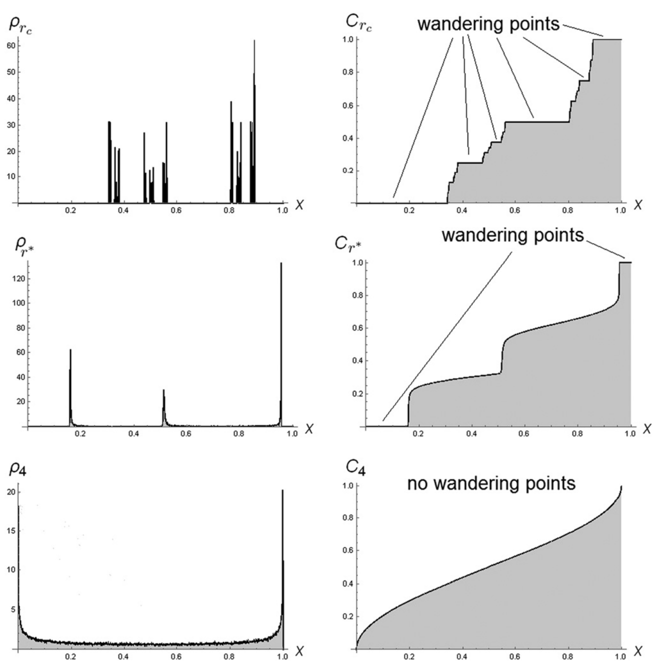

FIG. 1. Invariant densities (left column) and their cumulative distribution functions (right column) of the logistic map for $r = r_c$, $r = r_*^*$ and $r = 4$ with $T = 10^5$ and $M = 10^3$. From Theorem III.1, one has that the plateaus of $C_r(x)$ allow us to identify the dissipative regions of the dynamics, i.e., the wandering points. The transition to a fully chaotic dynamics implies the total suppression of the wandering set as a consequence of the emergence of the chaotic sea.

- Case $r = 2$:
  $[0,1]/\sim_2 = \left\{\bar{0}\,,\ \bar{\tfrac{1}{2}}\,,\ \bar{1}\right\}$, with

$$\bar{d}_2(\bar{0}, \bar{1}) = \bar{d}_2\left(\bar{0}, \bar{\frac{1}{2}}\right) = \bar{d}_2\left(\bar{\frac{1}{2}}, \bar{1}\right) = 1,$$

$$C_2(x) = 2\,\Theta(x - 1/2) \quad \forall x \in [0,1]$$

  and

$$\mathcal{W}_2([0,1]) = \left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right].$$

- Case $r = 4$:
  $[0,1]/\sim_4 = \{\bar{x} : x \in [0,1]\}$, with

$$\bar{d}_4(\bar{x}^A, \bar{x}^B) = |x^B - x^A|,$$

$$C_4(x) = \frac{2}{\pi}\arccos(x) \quad \forall x \in [0,1]$$

  and

$$\mathcal{W}_4([0,1]) = \varnothing.$$

For $r = 4$, we notice that making the change of variables $x = \sin^2\left(\frac{\pi y}{2}\right)$, which is a diffeomorphism of [0, 1] on itself, one has $dx = \pi \cos\frac{\pi y}{2} \sin\frac{\pi y}{2}\, dy$. Therefore,

$$\rho_4(x)\,dx = \frac{1}{\pi\sqrt{\sin^2\frac{\pi y}{2}\left(1 - \sin^2\frac{\pi y}{2}\right)}}\,\pi \cos\frac{\pi y}{2} \sin\frac{\pi y}{2}\, dy = dy.$$

This expresses that the distance d_4 is equivalent to

$$d_4'(y^A, y^B) = \left|\int_{y^A}^{y^B} dy\right| = |y^B - y^A|,$$

which is nothing but the Euclidean metric for the set of real numbers. Therefore, one can use d_4' instead of the metric that results from integrating the invariant density $\rho_4(x)$. The wandering set has been totally suppressed giving rise to the chaotic sea along the whole interval (0, 1), in agreement with Theorem III.1.

Now, for the cases $r = r_c$ and $r = r^*$, we perform a qualitative analysis from the numerical results depicted in Fig. 1.

- Case $r = r_c$:
  The complexity of dynamics is manifested by the substructures in the profile of the cumulative distribution $C_{r_c}(x)$, as shown in Fig. 1. From Theorem III.1, one can see that the plateaus of $C_{r_c}(x)$ account for the dissipative regions, i.e., the wandering points. As a consequence of the onset of chaos, the dynamics is mixed and composed of some stability islands represented by the increasing intervals of $C_{r_c}(x)$ immersed in the emergent chaotic sea. As soon as the parameter r increases slightly, the plateaus of $C_{r_c}(x)$ tend to disappear, meaning that the transition to the fully chaotic regime is associated with a suppression of the wandering set.

- Case $r = r^*$:

  The only stability islands are given by the few jump discontinuities observed in $C_{r^*}(x)$ while the wandering set has been reduced to [0, 0.15) ∪ (0.95, 1], as one can see from Fig. 1. The plateaus observed for r_c now are part of the chaotic sea representing increasing intervals of the cumulative distribution $C_{r^*}(x)$.

B. The circle map

With regard to the dynamics on the unitary circle S^1, Kolmogorov proposed a family of simplified models for driven mechanical rotors whose discretized equations define a map, called the *circle map*. This is defined by the sequence

$$\theta_{n+1} = \theta_n + \Omega - \frac{K}{2\pi}\sin(2\pi\theta_n) \quad \forall n \in \mathbb{N}_0,$$

which has the form (8) where $f_{(\Omega,K)} : [0,1] \to [0,1]$ is given by $f_{(\Omega,K)}(\theta) = \theta + \Omega - \frac{K}{2\pi}\sin(2\pi\theta)$, depending on two parameters, the coupling strength K and the driving phase Ω. Also, S^1 is represented by the interval (0, 1), i.e., the angle θ is expressed in units of 2π.

When $K=0$, the map is a rotation in an angle Ω. If one also chooses Ω irrational, the map reduces to an irrational rotation. The dynamics has been studied for multiple combinations of the values Ω and K exhibiting a very rich structure such as Cantor functions by plotting sections, subaharmonic routes of chaos, and the phenomena of phase locking. The complex structure of the dynamics has proved its usefulness in several applications.[32–34] Here, we focus on some special characteristic values where the dynamics can be well distinguished. We consider the values $(\frac{1}{8}, 0)$, $(\frac{\sqrt{2}}{2\pi}, 0)$, and (0, 3) for the pair (Ω, K).

For $\Omega = \frac{1}{8}$ and $K=0$, one has a rotation in an angle $\frac{\pi}{4}$ with a regular behavior for any initial condition θ_0. The sequence $\{\theta_n\}$ only takes the values θ_0, $\theta_0 + i\frac{1}{8}$ with $i=0,\ldots,7$, i.e., these ones are the only attractors for all initial conditions.

For $\Omega = \frac{\sqrt{2}}{2\pi}$ and $K=0$, one has the irrational rotation which corresponds to an ergodic dynamics. In fact, for any initial condition θ_0, the sequence $\{\theta_n\}$ travels around the entire phase space.

Finally, for $\Omega=0$ and $K=3$, the behavior is fully chaotic and the dynamics is mixing. As in the case of the irrational rotation, every sequence $\{\theta_n\}$ fills all the phase space but randomly.

Carrying out the histogram method described in Sec. II for an initial state $\theta_0=0.1$ and considering $M=10^3$ and $T=10^5$, we computed the invariant densities and their cumulative distribution functions for $(\Omega,K)=(\frac{1}{8},0)$, $(\Omega,K) = (\frac{\sqrt{2}}{2\pi},0)$, and $(\Omega, K)=(0, 3)$. The results are shown in Fig. 2. By applying (4)–(9) and the Theorem III.1, we obtain the metric space $(\Gamma/\!\sim_{(\Omega,K)}, \bar{d}_{(\Omega,K)})$, the wandering set $\mathcal{W}_{(\Omega,K)}([0,1))$, and the cumulative distribution function $C_{(\Omega,K)}(\theta)$ for each one of the cases studied

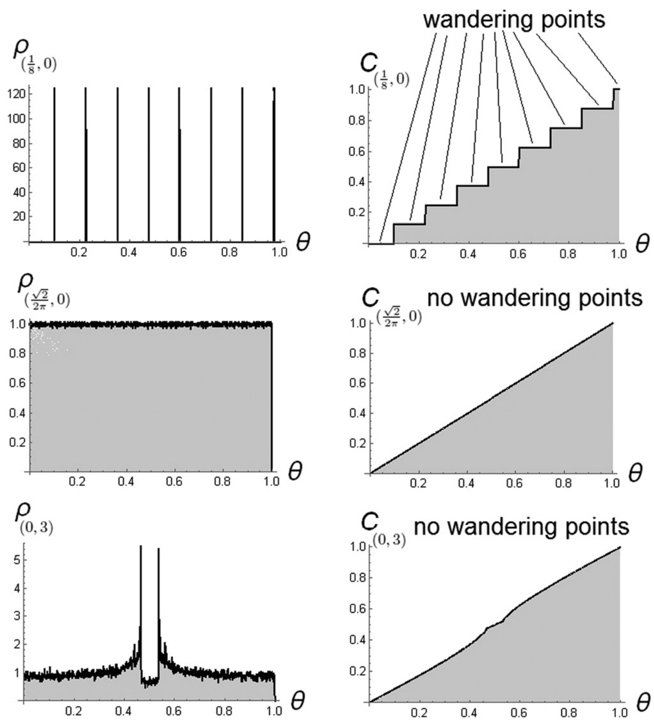

FIG. 2. Invariant densities (left column) and their cumulative distribution functions (right column) of the circle map for $(\Omega,K)=(\frac{1}{8},0)$, $(\Omega,K)=(\frac{\sqrt{2}}{2\pi},0)$, and $(\Omega,K)=(0,3)$ with $T=10^5$ and $M=10^3$. Again, Theorem III.1 allows us to identify the plateaus of $C_{(\Omega,K)}(\theta)$ with the dissipative regions of the dynamics. As in the logistic case, the transition from a regular dynamics to a chaotic one is associated with the suppression of the wandering set.

- Case $(\Omega,K)=(\frac{1}{8},0)$:
  $\Gamma/\!\sim_{(\frac{1}{8},0)} = \left\{\overline{\frac{i}{8}} : i=1,\ldots,8\right\} \cup \left\{\overline{\frac{2j+1}{16}} : j=0,\ldots,7\right\}$, with

$$\bar{d}_{(\frac{1}{8},0)}\left(\overline{\frac{i_1}{8}},\overline{\frac{i_2}{8}}\right) = \frac{1}{8}\left(|i_1-i_2| + \chi_{(0,+\infty)}(|i_1-i_2|)\right)$$
$$\forall\, i_1,i_2 = 1,\ldots,8$$
$$\bar{d}_{(\frac{1}{8},0)}\left(\overline{\frac{2j_1+1}{16}},\overline{\frac{2j_2+1}{16}}\right) = \frac{1}{8}|j_1-j_2|$$
$$\forall\, j_1,j_2 = 0,\ldots,7$$
$$\bar{d}_{(\frac{1}{8},0)}\left(\overline{\frac{i_1}{8}},\overline{\frac{2j_2+1}{16}}\right) = \frac{1}{8}\left(|i_1-j_2| + \delta_{|i_1-j_2|,0}\right)$$
$$\forall\, i_1=1,\ldots,8,\; j_2=0,\ldots,7$$
$$C_{(\frac{1}{8},0)}(\theta) = \frac{i}{8} \;\; \forall\,\theta\in\left[\frac{i}{8},\frac{i+1}{8}\right)\;,\;\; i=0,\ldots,7$$

  and

$$\mathcal{W}_{(\frac{1}{8},0)}([0,1)) = \{\bar{\theta} : \theta\in[0,1)\} - \left\{\overline{\frac{i}{8}} : i=1,\ldots,8\right\}$$

- Cases $(\Omega,K)=(\frac{\sqrt{2}}{2\pi},0)$ and $(\Omega,K)=(0,3)$:
  $\Gamma/\!\sim_{(\frac{\sqrt{2}}{2\pi},0)} = \Gamma/\!\sim_{(0,3)} = \{\bar{\theta} : \theta\in[0,1)\}$, with

$$\bar{d}_{(\frac{\sqrt{2}}{2\pi},0)}(\bar{\theta}_1,\bar{\theta}_2) = \bar{d}_{(0,3)}(\bar{\theta}_1,\bar{\theta}_2) = |\theta_1-\theta_2|$$
$$C_{(\frac{\sqrt{2}}{2\pi},0)}(\theta) = C_{(0,3)}(\theta) = \theta\;,\;\; \forall\,\theta\in[0,1)$$

and

$$\mathcal{W}_{\left(\frac{\sqrt{2}}{2\pi},0\right)}([0,1)) = \mathcal{W}_{(0,3)}([0,1)) = \varnothing.$$

As in the logistic case, one can see that the transition from a regular dynamics (the rotation in an angle $\frac{\pi}{4}$ for $\Omega = \frac{1}{8}$ and $K=0$) to a chaotic one (the ergodic irrational rotation for $\Omega = \frac{\sqrt{2}}{2\pi}, K = 0$ and the mixing regime $\Omega = 0, K = 3$) is associated with a suppression of the wandering set.

Moreover, from Fig. 2, one finds that the two levels of chaos involved in the cases, ergodic for $\Omega = \frac{\sqrt{2}}{2\pi}, K = 0$ and mixing for $\Omega = 0$ and $K = 3$, can be only distinguished by means of the profile of their respective invariant densities since the cumulative distributions, the wandering sets, and the metric spaces are the same in both cases. An explanation of this fact based on the statistical information of the dynamics of the map contained in its invariant density can be given as follows. The signatures of chaos in the invariant density are expressed as statistical fluctuations observed in its profile, which are more pronounced in the mixing case than in the ergodic case, as one can see from Fig. 2. By integrating the invariant density, one averages these fluctuations and then they disappear. Thus, one obtains the same cumulative distribution in both cases, and therefore, the wandering set and the metric space must also remain the same.

Our analytical results about the characterization of the map dynamics of the logistic and the circle maps are summarized in Table I.

TABLE I. Relationship between the metric space and the wandering set for the logistic and the circle maps. In both cases, it is observed that the dissipation (measured by the wandering set) and the chaotic behavior are contrary properties, i.e., when one increases, the other decreases and vice versa, as a consequence of Theorem III.1.

Parameters	Metric space $\Gamma/\sim_{\bar{r}}$	Wandering set $\mathcal{W}(\Gamma)$
r = 1	$\{\bar{0}, \bar{1}\}$	(0,1]
r = 2	$\left\{\bar{0}, \bar{\tfrac{1}{2}}, \bar{1}\right\}$	$[0,\frac{1}{2}) \cup (\frac{1}{2}, 0]$
r = 4	$\{\bar{x} : x \in [0,1]\}$	$\varnothing$
$\Omega = \frac{1}{8}, K = 0$	$\left\{\bar{\tfrac{i}{8}} : i = 1,\ldots,8\right\} \cup \left\{\bar{\tfrac{2j+1}{16}} : j = 0,\ldots,7\right\}$	$\{\bar{\theta} : \theta \in [0,1)\} - \left\{\bar{\tfrac{i}{8}} : i = 1,\ldots,8\right\}$
$\Omega = \frac{\sqrt{2}}{2\pi}, K = 0$	$\{\bar{\theta} : \theta \in [0,1)\}$	$\varnothing$
$\Omega = 0, K = 3$	$\{\bar{\theta} : \theta \in [0,1)\}$	$\varnothing$

V. SCOPE AND NEW INSIGHTS

Here, we provide extensions of the concept of metric space associated with a map which gives innovative tools for reinterpreting some topics.

A. A metric space associated with a probability distribution

The construction of the metric space of Sec. III only involves the invariant density (which in particular is a probability distribution) of the discrete map, so one can generalize it for an arbitrary probability distribution. Given a probability distribution $\varphi(x)$ on a unidimensional variable $x \in X$ (here we assumed that $X \subseteq \mathbb{R}$) and repeating the steps of Sec. III, one obtains

$$\bar{d}_\varphi : X/\sim_\varphi \times X/\sim_\varphi \to \mathbb{R}_+$$
$$\bar{d}_\varphi(\overline{x^A}, \overline{x^B}) := d_\varphi(x^A, x^B) = \left| \int_{x^A}^{x^B} \varphi(x)dx \right| = |C_\varphi(x^A) - C_\varphi(x^B)|, \tag{11}$$

for all x^A, $x^B \in X$, where $C_\varphi(x)$ is the cumulative distribution associated with $\varphi(x)$ and $\sim_\varphi$ is the equivalence relation defined in Eq. (5) with φ instead of ρ. Moreover, one can also recast Theorem III.1 in the following form.

Theorem V.1. *Let $\varphi(x)$ be a probability distribution on $x \in X \subseteq \mathbb{R}$ and let $[x^A, x^B] \subseteq X$ be a subinterval of X. Then, the following propositions are equivalent:*

(a) $\varphi(x) = 0$ a.e. in (x^A, x^B).
(b) $\bar{x}^A = \bar{x}^B$.
(c) $d_\varphi(x^A, x^B) = 0$.
(d) $C_\varphi(x)$ is constant in $[x^A, x^B]$.

Proof. $(a) \Rightarrow (b)$, $(b) \Rightarrow (c)$, and $(c) \Rightarrow (d)$ are the same as in Theorem III. The implication $(d) \Rightarrow (a)$ follows from Eqs. (4) and (10) applied to $\varphi(x)$.

Note that in the general case when $\varphi(x)$ is not an invariant density of a map but only a probability distribution, then one cannot associate a dynamics with $\varphi(x)$. In other words, the wandering point set does not exist for all $\varphi(x)$ necessarily. This explains why the item (e) is absent in Theorem V.1.

The physical interpretation of $X/\sim_\varphi$ is as follows: Suppose that one deals with N data (results) $x_1, x_2,\ldots, x_N \in X = [a, b]$ of an experiment. Consider that $M \leq N$ is the number of bins in $[a, b]$ and let $\varphi(x) = \sum_{i=1}^{M} \varphi_i \chi_{I_i}(x)$ be the histogram distribution of $x_1, x_2,\ldots, x_N$ with φ_i the fraction of results lying in $I_i = [a + \frac{(i-1)(b-a)}{M}, a + \frac{i(b-a)}{M})$ and $\chi_{I_i}(x)$ the characteristic function of I_i. Then, it is clear that in each subinterval (bin) I_i, the distribution $\varphi(x)$ is constant and equal to φ_i. Therefore, due to Theorem V.1., all points in I_i belong to the same class, namely, $\overline{a + \frac{(i-1)(b-a)}{M}}$. Now, let $\varphi_{i_1}, \varphi_{i_2}, \ldots, \varphi_{i_P} = \mathrm{Im}\varphi$ the image of $\varphi(x)$. Then, one has the following relationship between the number of bins and the metric space $X/\sim_\varphi$:

(i) If $P = M = N$, each bin I_i contains only a point, i.e., $a + \frac{(i-1)(b-a)}{M}$. In such a case, $\varphi(x)$ results in the uniform distribution, and thus, all the points $x_1, x_2,\ldots, x_N$ belong to the same class. This implies that $X/\sim_\varphi$ is composed of only an element, $\sharp(X/\sim_\varphi) = 1$, where $\sharp$ stands for the cardinal.

(ii) If $1 = P < M$, all the points are lying in only one bin, say I_i. In this case, $\varphi(x)$ is equal to 1 in I_i and zero outside. This corresponds to the opposite case of the uniform distribution, and it follows by the arguments used in Sec. IV that $\sharp(X/\sim_\varphi) = 2$ or 3.

(iii) If $P = M$, each bin I_i contains different amounts of points and then $\sharp(X/\sim_\varphi) = M$.

Now, recalling that $\varphi(x)$ represents the available information about the variable x, one can interpret the above mentioned as follows: Due to (i), one knows nothing about x

which corresponds to the uniform distribution, and in such a case, $X/\sim_\varphi$ contains only one class. That is, one can associate ignorance with having only one class in $X/\sim_\varphi$. On the other hand, in the case of (*ii*), one has certainty in some bin I_i, and then, the number of classes is 2 or 3. For the remaining cases (*iii*), the cardinal of $X/\sim_\varphi$ results to be equal to the number of bins. In this way, one can see that $X/\sim_\varphi$ characterizes the loss or gain of information about the variable x and is related to the number of bins from a statistical viewpoint.

B. Conjugated variables by Fourier transform

In order to explore the properties of $X/\sim_\varphi$, now we consider the case of having two probability distributions $\varphi(x) = |f(x)|^2$ and $\phi(y) = |g(y)|^2$, where $g(y)$ is the Fourier transform of $f(x)$, i.e.,

$$g(y) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x y} dx. \tag{12}$$

In this case, we consider that $X = Y = \mathbb{R}$ with $x \in X$ and $y \in Y$. Recall that the diameter $\mathrm{diam}(\Gamma)$ of a metric space (Γ, d) is defined as

$$\mathrm{diam}(\Gamma) = \sup\{d(x^A, x^B) : x^A, x^B \in X\}, \tag{13}$$

where sup stands for the supreme. From the definition (4) and due to the Plancherel Theorem Eq. (13), for the case in which f and g are integrable and $\Gamma = X = Y = \mathbb{R}$, we get

$$\begin{aligned}\mathrm{diam}(X/\sim_\varphi) &= \int_{-\infty}^{\infty} \varphi(x) dx = \int_{-\infty}^{\infty} |f(x)|^2 dx \\ &= \int_{-\infty}^{\infty} |g(y)|^2 dy = \int_{-\infty}^{\infty} \phi(y) dy = \mathrm{diam}(Y/\sim_\phi).\end{aligned}$$

In other words, the metric spaces $X/\sim_\varphi$ and $Y/\sim_\phi$ have the same diameter, where $x \in X$ and $p \in Y$ are conjugated variables by the Fourier transform. This can be considered a consequence of expressing the metric in terms of a probability distribution along with the Fourier transform.

C. Uncertainty principle in terms of metric spaces

In order to characterize the uncertainty principle by means of properties about metric spaces induced by probability distributions, it is appropriate to consider the following functions:

$$D_\varphi(x^A, x^B) = \left| \int_{x^A}^{x^B} x^2 \varphi(x) dx \right| \quad \forall x^A, x^B \in X$$

$$D_\varphi(y^A, y^B) = \left| \int_{y^A}^{y^B} p^2 \phi(y) dy \right| \quad \forall y^A, y^B \in Y.$$

Also, let us assume that $\varphi(x)$ is par in x. Then, $\phi(y)$ results par in y also. Performing the same steps of Sec. III A, it is not difficult to prove that D_φ and D_ϕ induce metric spaces in $X/\sim_\varphi \times X\sim_\varphi$ and $Y\sim_\phi \times Y\sim_\phi$. More precisely,

$$\begin{aligned}&\bar{D}_\varphi : X/\sim_\varphi \times X/\sim_\varphi \to \mathbb{R}_+ \\ &\bar{D}_\varphi(\overline{x^A}, \overline{x^B}) := D_\varphi(x^A, x^B)\end{aligned} \tag{14}$$

and

$$\begin{aligned}&\bar{D}_\phi : Y/\sim_\phi \times Y/\sim_\phi \to \mathbb{R}_+ \\ &\bar{D}_\phi(\overline{y^A}, \overline{y^B}) := D_\phi(y^A, y^B).\end{aligned} \tag{15}$$

On the other hand, the uncertainty principle states that (in dimensionless units)

$$\left(\int_{-\infty}^{\infty} x^2 \varphi(x) dx\right)\left(\int_{-\infty}^{\infty} y^2 \phi(y) dy\right) \geq \frac{1}{16\pi^2}. \tag{16}$$

Now, by the definition of the diameter of a metric space (13) and Eqs. (14) and (15), one has that

$$\begin{aligned}\mathrm{diam}(X/\sim_\varphi) &= \lim_{x^A \to -\infty, x^B \to \infty} D_\varphi(x^A, x^B) = \int_{-\infty}^{\infty} x^2 \varphi(x) dx \\ \mathrm{diam}(Y/\sim_\phi) &= \lim_{y^A \to -\infty, y^B \to \infty} D_\varphi(y^A, y^B) = \int_{-\infty}^{\infty} y^2 \phi(y) dy.\end{aligned} \tag{17}$$

Joining Eqs. (16) and (17), one obtains

$$\mathrm{diam}\big(X/\sim_\varphi\big)\mathrm{diam}\big(Y/\sim_\phi\big) \geq \frac{1}{16\pi^2}, \tag{18}$$

which is an expression of the uncertainty principle in terms of the diameters of the metric spaces associated with the probability distributions of x and its conjugated variable y. The physical meaning of (18) can be interpreted as follows: For instance, assume that one has a particle moving in one dimension where X and Y represent the position and the momentum spaces. When one knows about x, this implies a localized distribution $\varphi(x)$ and then a delocalized or big ignorance about p takes place. In terms of metric spaces, this can be expressed as a small diameter of $X/\sim_\varphi$ [a localized $X/\sim_\varphi$ increases regions having low values of $\varphi(x)$] and a large one for $Y/\sim_\varphi$ (increase in regions with nonzero values of $\phi(x)$) and vice versa.

VI. CONCLUSIONS

We proposed a notion of distinguishability for points of discrete maps of arbitrary dimension in the limit where the number of iterations tends to infinity. Based on a statistical distance for unidimensional maps defined by means of the invariant density and using equivalence classes instead of points, we redefined it as a metric satisfying the axioms of distinguishability, symmetry, and the triangular inequality. Furthermore, from this metric, we defined a metric space associated with the map which is composed of all the points that can be distinguished among the map dynamics in the limit of a large number of iterations. The importance of the metric space is that by means of changes in its structure, it allows one to study the transitions of the dynamics, as the map parameters vary. Complementarily, we proved Theorem III.1 that characterizes the wandering set of a map, which

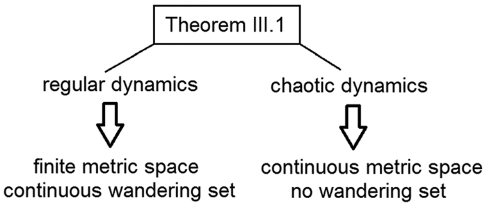

FIG. 3. A schematic picture of Theorem III.1 illustrating its consequences with regard to the dynamics of the maps studied, i.e., the logistic and the circle maps.

measures the dissipation present in the dynamics, in terms of the metric space and the cumulative distribution function of the invariant density. As a consequence of Theorem III.1, we obtained that the size of the associated metric space increases as the wandering set decreases and vice versa. An illustrating scheme is shown in Fig. 3.

In addition, we extended the metric space to any probability distribution, not necessarily an invariant density, and we explored some consequences with regard to statistical properties, distributions related to Fourier transform, and the uncertainty principle. Also, we generalized Theorem III.1 for an arbitrary one–dimensional probability distribution which is the content of Theorem V.1.

The analysis of all the above mentioned leads to the following facts:

- In both maps, when the metric space is finite, the dynamics is regular and the wandering set is the whole phase space except for some discrete points that correspond to the classes composed of a single element. These single classes are precisely the only attractors of the dynamics.
- In both maps, the chaotic regimes studied are characterized by a maximal size of the metric space and a minimal size of the wandering set. In their fully chaotic cases, i.e., $r=4$ in the logistic map and $\Omega=0$ and $K=3$ in the circle map, the metric space is all the entire phase space and the wandering set is empty, corresponding to the absence of dissipation in the dynamics.
- In the case of the logistic map, the complex structure in the onset of chaos when $r=r_c$ is expressed by substructures in the profile cumulative distribution function of the invariant density.
- In both maps, it is observed that the transition from a less chaotic dynamics to a more chaotic one is associated with a suppression of the wandering set, making plausible to consider this fact as a signature of chaos of the dynamics of a map.
- In the case of the circle map, the impossibility of distinguishing an ergodic dynamics ($\Omega=\frac{\sqrt{2}}{2\pi}, K=0$) from a mixing one ($\Omega=0$, $K=3$) is explained on the basis that by integrating statistical fluctuations of different invariant densities can lead to the same cumulative distribution function and consequently to the same metric space and wandering set associated with the map dynamics.
- The extension of the concept of metric space (associated with a discrete map) for an arbitrary probability distribution provides a unified metric framework of several topics: discrete maps, statistics properties of distributions, and quantum mechanical features using the Fourier transform. A summarized scheme is depicted in Fig. 4.
- Theorem V.1 implies that the information contained in a probability distribution is characterized by the cardinality of its metric space associated. Maximal ignorance corresponds to metric space having only a single class, while certainty corresponds to a cardinal equals to 2 or 3. Moreover, when each of the all the bins contains different amounts of points, then the cardinal of the metric space results to be equal to the number of bins.
- Uncertainty principle for two conjugated variables whose probability distributions are related by the Fourier transform can be expressed in terms of the diameters of the corresponding metric spaces associated.

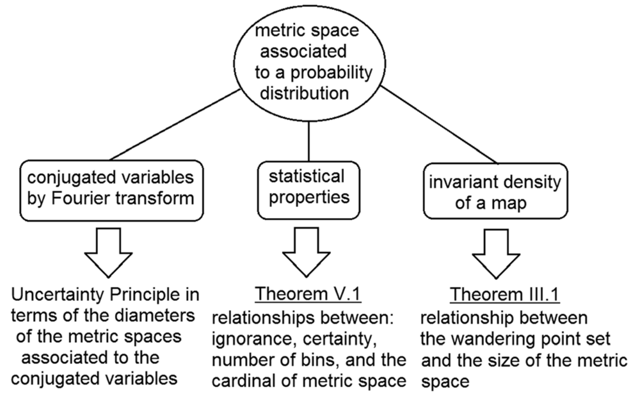

FIG. 4. Scope and new insights achieved by the concept of the metric space associated with a probability distribution.

From all these facts, we can consider that the proposed metric space associated with a probability distribution is a good and novel tool along with Theorems III.1 and V.1 for characterizing several topics: dynamics in discrete maps, statistical properties of distributions given by histograms of a series data, and uncertainty relations. We hope that these conclusions can be analyzed with more examples, and for more dimensions than one, in future researches.

ACKNOWLEDGMENTS

This work was partially supported by CONICET, Universidad Nacional de La Plata, and Universidad Nacional de Córdoba, Argentina.

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