Implicative subreducts of MV-algebras: free and weakly projective objects

DIEGO N. CASTAÑO

ABSTRACT. In this article we explore in some detail the free and weakly projective objects of the variety of Łukasiewicz implication algebras (the implicative subreducts of MV-algebras). We review the two already known descriptions of finitely generated free algebras, giving new insights into their structure and their connection, as well as providing new proofs of the characterizations. We give a representation theorem for weakly projective algebras as algebras of certain McNaughton functions restricted to rational polyhedra and prove that finitely generated weakly projective algebras coincide with finitely presented ones. We also prove that finite chains are the only totally ordered weakly projective examples in this variety.

1. Introduction and preliminaries

Łukasiewicz implication algebras are the $\{\to, 1\}$-subreducts of MV-algebras and also constitute the equivalent algebraic semantics for the implicational fragment of Łukasiewicz infinite-valued logic [8]. We denote by $\mathbb{MV}$ and $\mathbb{L}$ the varieties of MV-algebras and Łukasiewicz implication algebras, respectively. A good introduction to MV-algebras may be found in [4] and the basic properties of Łukasiewicz implication algebras may be found, for example, in [6, 1]. We assume some familiarity with both MV-algebras and Łuaksiewicz implication algebras, but we start this article with a brief summary of the most important facts about both varieties, which also serves the purpose of setting the appropriate notation. Then, in Section 2, we will review the known descriptions of finitely generated free algebras (see [10] and [7]) and provide new proofs of them that allow us to better see their connection. In Section 3, based on the work of Cabrer and Mundici (see [2]) on projective MV-algebras and having a good understanding of the free objects, we will give a quite simple characterization of finitely generated weakly projective objects in the variety $\mathbb{L}$ as well as show that they coincide with finitely presented algebras. As a consequence, every finite algebra is weakly projective and, in particular, free algebras in the proper subvarieties of $\mathbb{L}$ are weakly projective in $\mathbb{L}$ (a direct proof of this is provided in the appendix). In Section 4, we will deal specifically with totally

Presented by ...
Received ...; accepted in final form ...
2010 *Mathematics Subject Classification*: Primary: 06D30; Secondary: 06D35, 08B20, 08B30, 57Q05.
Key words and phrases: Łukasiewicz logic; MV-algebra; implicative algebra; free algebra; projective; finitely presented.

ordered algebras. More precisely, we will show that the only totally ordered weakly projective objects in $\mathbb{L}$ are finite chains.

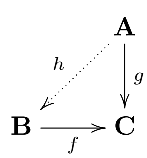
Throughout the article we write $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ for the sets of positive integers, integers, and rational numbers, respectively. We denote by $\mathbf{[0,1]} = \langle [0,1], \oplus, \neg, 0\rangle$ the standard MV-algebra on the real unit interval, where $x \oplus y := \min(1, x+y)$, $\neg x := 1-x$. For each $n \in \mathbb{N}$, let $\mathbf{S}_n$ be the subalgebra of $\mathbf{[0,1]}$ with universe $S_n = \{\frac{k}{n} : k \in \mathbb{Z}, 0 \leq k \leq n\}$. On any MV-algebra $\mathbf{A}$, an implication may be defined as $x \to y := \neg x \oplus y$. We denote the $\{\to, 1\}$-reduct of $\mathbf{A}$ by $\mathbf{A}^{\to}$. We also set $\mathbf{L}_n = \mathbf{S}_n^{\to}$ for convenience. Moreover, if $\mathbf{A}$ is any MV-algebra and $B \subseteq A$ is an increasing subset of A, then B is closed under the implication operation of $\mathbf{A}$, so we define a Łukasiewicz implication algebra $\mathbf{B}^{\to} = \langle B, \to, 1\rangle$. Conversely, if $\mathbf{A} \in \mathbb{L}$ has a least element 0, we define operations on A given by $x \oplus y := (x \to 0) \to y$, $\neg x := x \to 0$. It is known that $\langle A, \oplus, \neg, 0\rangle$ is an MV-algebra, which we denote by $\mathbf{A}_0$. Observe that $(\mathbf{A}_0)^{\to} = \mathbf{A}$. Another important Łukasiewicz implication algebra is $\mathbf{L}_\omega$, which may be defined on the universe $L_\omega = \{1 = a^0, a = a^1, a^2, a^3, \dots\}$ by setting $a^i \to a^j := a^{\max(0, j-i)}$. Note that $\mathbf{L}_\omega$ is isomorphic to the $\{\to, 1\}$-reduct of the radical of Chang's MV-algebra.

Unlike in the case of MV-algebras, $\mathbf{L}_n$ is embeddable in $\mathbf{L}_m$ if and only if $n \leq m$. Consequently, the lattice of subvarieties of $\mathbb{L}$ is much simpler than the lattice of subvarieties of $\mathbb{MV}$. It was completely described by Komori in [8] and consists of the following chain of subvarieties:

$$\mathbb{T} \subseteq V(\mathbf{L}_1) \subseteq V(\mathbf{L}_2) \subseteq \cdots \subseteq V(\mathbf{L}_n) \subseteq \cdots \subseteq V(\mathbf{L}_\omega) = \mathbb{L}.$$

Here $\mathbb{T}$ stands for the variety of trivial algebras and, in the sequel, we denote $\mathbb{L}_n = V(\mathbf{L}_n)$ for $n \geq 1$. Observe also that $\mathbb{L}_1$ is the variety of Tarski algebras or implication algebras.

We recall now the notions regarding projectivity. An algebra $\mathbf{A}$ in a variety $\mathbb{V}$ is *weakly projective* if whenever $f\colon \mathbf{B} \to \mathbf{C}$ is a surjective homomorphism between algebras $\mathbf{B}, \mathbf{C} \in \mathbb{V}$ and $g\colon \mathbf{A} \to \mathbf{C}$ is a homomorphism, there is a homomorphism $h\colon \mathbf{A} \to \mathbf{B}$ such that $g = f \circ h$; in other words, the following diagram commutes:



Remark 1.1. A note is in order to explain why we focus on weakly projective instead of projective algebras. Recall that in the definition of projective objects in category theory epimorphisms are considered instead of surjective homomorphisms. In this article we deal with the category associated to a variety of algebras $\mathbb{L}$, that is, the category whose objects are the algebras in $\mathbb{L}$ and whose morphisms are homomorphisms between algebras. It is known that in the category associated with any variety of algebras, monomorphisms

coincide with injective homomorphisms. However, although every surjective homomorphism is an epimorphism, the converse relation does not hold in general. In $\mathbb{L}$, for example, consider the algebra $\mathbf{A} = \mathbf{L}_2^2$ and the subalgebra $\mathbf{B}$ of $\mathbf{A}$ with universe $B = \{(1,1),(1,\frac{1}{2}),(\frac{1}{2},1)\}$. It is easy to show that the inclusion homomorphism $\iota\colon \mathbf{B} \to \mathbf{A}$ is a non-surjective epimorphism in $\mathbb{L}$. The key property here is the fact that homomorphisms in $\mathbb{L}$ preserve existent meets.

We claim that the only projective algebra in $\mathbb{L}$ is the trivial algebra. Indeed, let $\mathbf{A} \in \mathbb{L}$ be any non-trivial algebra and consider an algebra $\mathbf{B} \leq \mathbf{A}^2$ with universe $B = \{(x,1) : x \in A\} \cup \{(1,x) : x \in A\}$. As in the previous example, the inclusion homomorphism $\iota\colon \mathbf{B} \to \mathbf{A}^2$ is an epimorphism. However if $g\colon \mathbf{A} \to \mathbf{A}^2$ is the *diagonal* homomorphism given by $g(a) = (a,a)$, $a \in A$, there cannot exist a homomorphism $h\colon \mathbf{A} \to \mathbf{B}$ such that $g = \iota \circ h$. This shows that $\mathbf{A}$ is not projective in $\mathbb{L}$. It also shows that the usual notion of projectivity is of little use in the variety $\mathbb{L}$. On the contrary, we will see that weakly projective algebras abound in $\mathbb{L}$ and hence constitute much more interesting objects.

From now on, we will deal exclusively with *weakly* projective algebras, so we will henceforth omit the adverb *weakly*.

Recall that a *retract* of an algebra $\mathbf{A}$ is an algebra $\mathbf{B}$ for which there exists a pair of homomorphisms $\pi\colon \mathbf{A} \to \mathbf{B}$ and $\iota\colon \mathbf{B} \to \mathbf{A}$ such that $\pi \circ \iota$ is the identity map on B. It follows immediately that π is always surjective and ι is always injective. Thus $\mathbf{B}$ is both a homomorphic image of $\mathbf{A}$ and isomorphic to a subalgebra of $\mathbf{A}$.

The following proposition lists a few easy consequences of the definition of projective algebra.

Proposition 1.2. *Given a variety $\mathbb{V}$, we have:*

(*a*) *Free algebras in $\mathbb{V}$ are projective.*
(*b*) *$\mathbf{A} \in \mathbb{V}$ is projective if and only if it is the retract of a free algebra in $\mathbb{V}$.*
(*c*) *If $\mathbf{A} \in \mathbb{V}$ is projective, every retract of $\mathbf{A}$ is also projective in $\mathbb{V}$.*

2. Free Łukasiewicz implication algebras

There are two characterizations of finitely generated free Łukasiewicz implication algebras in the literature, one given by Rose [10] and another given by Díaz Varela [7]. In this section we will review these characterizations, give new proofs of them as well as give some insight into the connection between the two. A good understanding of free algebras will allow us to study projectivity in the next section.

We denote by $\mathbf{Free}_{\mathbb{MV}}(X)$ the free MV-algebra over a set X of free generators. Since Łukasiewicz implication algebras are the $\{\to, 1\}$-subreducts of MV-algebras, $\mathbf{Free}_{\mathbb{L}}(X)$ (the free Łukasiewicz implication algebra on X) is simply the subalgebra of the reduct $\mathbf{Free}_{\mathbb{MV}}(X)^{\to}$ generated by X.

In this article we will mainly deal with finitely generated free algebras. Recall that McNaughton's Theorem (see e.g. [4, Theorem 9.5.1]) characterizes free MV-algebras. More specifically, the n-generated free MV-algebra $\mathbf{Free}_{\mathbb{MV}}(n)$ is isomorphic to the algebra $\mathbf{M}([0,1]^n)$ of McNaughton functions on the real n-cube $[0,1]^n$. The generators are the projection functions $\pi_1, \ldots, \pi_n$, defined by $\pi_i(x_1, \ldots, x_n) = x_i$. In this article, it will be also useful to note that, given $S \subseteq [0,1]^n$, the restrictions to S of functions in $M([0,1]^n)$ also constitute an MV-algebra, which we denote by $\mathbf{M}(S)$. Moreover, in light of the facts mentioned in the previous paragraph, the n-generated free Łukasiewicz implication algebra may be characterized within the reduct $\mathbf{M}([0,1]^n)^{\rightarrow}$ as the implicative subalgebra generated by the projections $\pi_1, \ldots, \pi_n$. We denote this implicative subalgebra $\mathbf{R}([0,1]^n)$ and we recall the characterization of the members of $R([0,1]^n)$ given by A. Rose in [10].

Theorem 2.1 ([10])**.** *Given a McNaughton function $f \in M([0,1]^n)$, $f \in R([0,1]^n)$ if and only if the following conditions hold:*

(i) *$f(\lambda x_1 + 1 - \lambda, \ldots, \lambda x_n + 1 - \lambda) = \lambda f(x_1, \ldots, x_n) + 1 - \lambda$, for every $\lambda \in [0,1]$.*

(ii) *There exists $\ell \in \{1, \ldots, n\}$ such that, for every $(x_1, \ldots, x_n) \in [0,1]^n$, $f(x_1, \ldots, x_n) \geq x_\ell$.*

We will now analyze in detail the conditions of the previous theorem and their implications. While performing this task we will produce a complete proof of it.

The easiest part is the necessity of condition (ii). Since $x \rightarrow y \geq y$ holds in any Łukasiewicz implication algebra, for any nonconstant implicative term $t(x_1, \ldots, x_n)$, $t(x_1, \ldots, x_n) \geq x_\ell$ holds in any Łukasiewicz implication algebra, if we let x_ℓ be the rightmost variable appearing in t.

In order to understand condition (i), we first show the following proposition.

Proposition 2.2. *For any $\lambda \in [0,1]$, the function $\varphi_\lambda(x) = (1-\lambda)x + \lambda$ is an implicative homomorphism from $\mathbf{[0,1]}^{\rightarrow}$ onto its subalgebra $\boldsymbol{[\lambda,1]}^{\rightarrow}$. Moreover, φ_λ is an isomorphism iff $\lambda \neq 1$.*

Proof. It suffices to check that for any $x, y \in [0,1]$, $(1-\lambda)(x \rightarrow y) + \lambda = \min\{1, 1 - (1-\lambda)x + (1-\lambda)y\} = ((1-\lambda)x + \lambda) \rightarrow ((1-\lambda)y + \lambda)$. ☐

We may now reinterpret condition (i) in Rose's Theorem as stating that any implicative function f must be preserved by all endomorphisms $\varphi_{1-\lambda}$ with $\lambda \in [0,1]$, or equivalently, by all endomorphisms φ_λ with $\lambda \in [0,1]$, that is, $\varphi_\lambda(f(x_1, \ldots, x_n)) = f(\varphi_\lambda(x_1), \ldots, \varphi_\lambda(x_n))$.

Remark 2.3. The endomorphisms in Proposition 2.2 are *all* the endomorphisms of $\mathbf{[0,1]}^{\rightarrow}$. Indeed, if $\varphi \colon \mathbf{[0,1]}^{\rightarrow} \rightarrow \mathbf{[0,1]}^{\rightarrow}$ is any nontrivial endomorphism, then, letting $\lambda = \varphi(0) \neq 1$, $\varphi \colon \mathbf{[0,1]} \rightarrow \boldsymbol{[\lambda,1]}^{\rightarrow}_{\lambda}$ is an MV-homomorphism. But $\psi \colon \boldsymbol{[\lambda,1]}^{\rightarrow}_{\lambda} \rightarrow \mathbf{[0,1]}$ given by $\psi(x) = \dfrac{x-\lambda}{1-\lambda}$ is also an

MV-homomorphism. Thus $\psi \circ \varphi$ is an MV-endomorphism of $\mathbf{[0,1]}$. Since the only endomorphism of the MV-algebra $\mathbf{[0,1]}$ is the identity map, it follows that $\psi \circ \varphi = id$, which implies that $\varphi(x) = (1-\lambda)x + \lambda = \varphi_\lambda(x)$.

It may be checked by a direct computation that $\varphi_{(1-\lambda)} \circ \varphi_{(1-\mu)} = \varphi_{(1-\lambda\mu)}$. Thus, identifying each endomorphism φ_λ with the real number $1-\lambda \in [0,1]$, it follows that the endomorphism monoid of $\mathbf{[0,1]}^{\rightarrow}$ is isomorphic to the real unit interval $[0,1]$ with the standard product of real numbers.

Another important consideration is that condition (i) implies that functions $f \in R([0,1]^n)$ are completely determined by their values on

$$O^n := \{(x_1, \dots, x_n) \in [0,1]^n : x_i = 0 \text{ for some } i\}.$$

Indeed, let $\overline{1} := (1,1,\dots,1)$ and $\overline{x} = (x_1,\dots,x_n) \in [0,1]^n$, $\overline{x} \neq \overline{1}$. Then $\overline{x}$ is a convex combination of some $\overline{x}^* \in O^n$ and $\overline{1}$, since we may write:

$$\overline{x} = (1-\lambda)\overline{x}^* + \lambda\overline{1}, \tag{2.1}$$

$$\lambda = \min\{x_1,\dots,x_n\}, \qquad x_i^* = \frac{x_i - \lambda}{1-\lambda}, \quad 1 \leq i \leq n. \tag{2.2}$$

From this we get that $f(\overline{x}) = (1-\lambda)f(\overline{x}^*) + \lambda$. In addition, $f(\overline{1}) = 1$ also holds. Let $R(O^n)$ be the restriction to O^n of all functions in $R([0,1]^n)$. Clearly $\mathbf{R}(O^n)$ is a Łukasiewicz implication algebra isomorphic to $\mathbf{R}([0,1]^n)$.

Lemma 2.4. *If $f \in M([0,1]^n)$ and $\ell \in \{1,\dots,n\}$, then there exists an implicative term $u(x_1,\dots,x_n)$ such that $u^{\mathbf{[0,1]}}(\overline{x}) = f(\overline{x}) \vee x_\ell$ for every $\overline{x} \in O^n$.*

Proof. Let f be any McNaughton function on $[0,1]^n$. Since the MV-operations are recoverable from $\to$ and 0 (recall that $x \oplus y = (x \to 0) \to y$, $\neg x = x \to 0$), there exists an implicative term $t(x_1,\dots,x_n,x_{n+1})$ such that $f(x_1,\dots,x_n) = t^{\mathbf{[0,1]}}(x_1,\dots,x_n,0)$. Consider the $\{\wedge,\to,1\}$-term $t\left(x_1,\dots,x_n,\bigwedge_i x_i\right)$. Note that in any MV-algebra the following equations hold:

- $x \to (y \wedge z) \approx (x \to y) \wedge (x \to z)$,
- $(x \wedge y) \to z \approx (x \to z) \vee (y \to z)$,
- $x \vee y \approx (x \to y) \to y$.

These equations imply that there exist implicative terms $t_1,\dots,t_k$ such that $t\left(x_1,\dots,x_n,\bigwedge_i x_i\right) \approx \bigwedge_j t_j(x_1,\dots,x_n)$. Now consider the implicative term

$$u(\overline{x}) = \left(\bigvee_j (t_j(\overline{x}) \to x_\ell)\right) \to x_\ell.$$

Then, for every $\overline{x} \in O^n$, we have

$$\begin{aligned} u^{[\mathbf{0},\mathbf{1}]}(\overline{x}) &= \left(\bigwedge_j t_j^{[\mathbf{0},\mathbf{1}]}(\overline{x})\right) \vee x_\ell \\ &= t^{[\mathbf{0},\mathbf{1}]}\left(\overline{x}, \bigwedge_i x_i\right) \vee x_\ell \\ &= t^{[\mathbf{0},\mathbf{1}]}(\overline{x}, 0) \vee x_\ell \\ &= f(\overline{x}) \vee x_\ell. \end{aligned}$$

□

The sufficiency in Rose's Theorem is now an easy consequence of the last lemma. Indeed, if a McNaughton function f on $[0,1]^n$ satisfies conditions (i) and (ii), then $u(\overline{x}) = f(\overline{x}) \vee x_\ell = f(\overline{x})$ for every $\overline{x} \in O^n$ and, by condition (i), $u(\overline{x}) = f(\overline{x})$ for every $\overline{x} \in [0,1]^n$, thus proving that $f \in R([0,1]^n)$.

Thus far, in light of Rose's Theorem, we have that $\mathbf{Free}_{\mathbb{L}}(n) \cong \mathbf{R}([0,1]^n) \cong \mathbf{R}(O^n)$. We now turn to study the relation between Rose's Theorem and Díaz Varela's Theorem from [7]. The latter showed that $\mathbf{Free}_{\mathbb{L}}(n)$ is isomorphic to the restriction to O^n of *all* McNaughton functions that satisfy condition (ii). We define the subset $M^+([0,1]^n) \subseteq M([0,1]^n)$ as the one consisting of McNaughton functions that satisfy condition (ii). The subset $M^+(S) \subseteq M(S)$ may be defined accordingly for any $S \subseteq [0,1]^n$. Since $M^+(S)$ is an increasing subset in $\mathbf{M}(S)$, it is the universe of a Łukasiewicz implication algebra $\mathbf{M}^+(S) = \langle M^+(S), \to, 1\rangle$. Thus the result in [7] becomes the following.

Theorem 2.5 ([7])**.** $\mathbf{Free}_{\mathbb{L}}(n) \cong \mathbf{M}^+(O^n)$.

Note that, by definition, the members of $R(O^n)$ are the restrictions to O^n of *some* McNaughton functions that satisfy condition (ii) (the ones that also satisfy condition (i)). In fact, by Theorem 2.5, $R(O^n)$ must coincide with the restrictions to O^n of *all* McNaughton functions that satisfy condition (ii). We give a direct proof of this fact in the following proposition.

Proposition 2.6. $R(O^n) = M^+(O^n)$.

Proof. The forward inclusion is trivial. Now let $f \in M([0,1]^n)$ be a McNaughton function such that $f(\overline{x}) \geq x_\ell$ for every $\overline{x} \in [0,1]^n$ and some fixed $\ell \in \{1,\dots,n\}$. By Lemma 2.4, there exists an implicative term u such that $u^{[\mathbf{0},\mathbf{1}]}(\overline{x}) = f(\overline{x}) \vee x_\ell = f(\overline{x})$ for every $\overline{x} \in O^n$. Thus, since clearly $u^{[\mathbf{0},\mathbf{1}]} \in R([0,1]^n)$, it follows that $f \restriction O^n \in R(O^n)$. □

Remark 2.7. There is a more direct way to prove the last proposition which we want to delineate here. Let $f \in M([0,1]^n)$ be a McNaughton function such that $f(\overline{x}) \geq x_\ell$ for every $\overline{x} \in [0,1]^n$ and some fixed $\ell \in \{1,\dots,n\}$. Define a function $g\colon [0,1]^n \to [0,1]$ in the following way: $g(\overline{1}) = 1$, and if $\overline{x} \neq \overline{1}$, we put $g(\overline{x}) = (1-\lambda) f(\overline{x}^*) + \lambda$, with λ and $\overline{x}^*$ as in (2.2). It is clear that

$f \restriction O^n = g \restriction O^n$. Thus, to show that $f \restriction O^n \in R(O^n)$, we only need to show that $g \in R([0,1]^n)$.

First note that it is intuitively clear and quite straightforward to prove that g is continuous. Moreover, g satisfies condition (i) in Rose's Theorem by the way it was defined and, since $g(\overline{x}) = (1-\lambda)f(\overline{x}^*) + \lambda \geq (1-\lambda)x^*_\ell + \lambda = x_\ell$, g also satisfies condition (ii). It remains to show that g is a McNaughton function on $[0,1]^n$.

Let $p_1, \dots, p_k$ be the linear pieces of the McNaughton function f. Fix $\overline{x} \in [0,1]^n$, $\overline{x} \neq \overline{1}$, and let $\lambda = \min\{x_1, \dots, x_n\} = x_s$. Then $x^*_i = \dfrac{x_i - x_s}{1 - x_s}$ and $g(\overline{x}) = (1-x_s)f(\overline{x}^*)+x_s$. There exists $r \in \{1, \dots, k\}$ such that $f(\overline{x}^*) = p_r(\overline{x}^*)$. Assume $p_r(\overline{y}) = a_0 + a_1y_1 + \cdots + a_ny_n$, $a_0, a_1, \dots, a_n \in \mathbb{Z}$. Then

$$g(\overline{x}) = (1-x_s)(a_0 + \sum_i a_ix^*_i) + x_s = a_0 + \sum_i a_ix_i + (1 - a_0 - \sum_i a_i)x_s.$$

Thus, $p_{r,s}(\overline{y}) = a_0 + \sum_i a_iy_i + (1 - a_0 - \sum_i a_i)y_s$ is a linear polynomial with integer coefficients such that $g(\overline{x}) = p_{r,s}(\overline{x})$. We have thus shown that there is a finite family of linear polynomials with integer coefficients, namely $\{p_{r,s} : 1 \leq r \leq k, 1 \leq s \leq n\}$, such that for every $\overline{x} \in [0,1]^n$, there is a polynomial $p_{r,s}$ such that $g(\overline{x}) = p_{r,s}(\overline{x})$. This proves that g is a McNaughton function.

Example 2.8. To illustrate the last proposition and the subsequent remark, consider the McNaughton function $f\colon [0,1]^2 \to [0,1]$ given by $f(x,y) = x \oplus x$ which clearly belongs to $M^+([0,1]^2)$ but not to $R([0,1]^2)$. We write f explicitly using linear polynomials with integer coefficients on O^2 and obtain

$$f(x,0) = \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases} \qquad f(0,y) = 0.$$

Following the construction in the remark, we get

$$g(x,y) = \begin{cases} x & \text{if } (x,y) \in A_1, \\ 2x - y & \text{if } (x,y) \in A_2, \\ 1 & \text{if } (x,y) \in A_3, \end{cases}$$

where the regions A_1, A_2 and A_3 are depicted in Figure 1. This McNaughton function satisfies conditions (i) and (ii) in Rose's Theorem and, therefore, is the interpretation on $[0,1]$ of an implicative term. We may find the corresponding implicative term from the original MV-term $f(x,y) = x \oplus x$ applying the procedure described in the proof of Lemma 2.4. We first write $f(x,y) = x \oplus x = \neg x \to x = (x \to 0) \to x$. Then we replace 0 by $x \wedge y$, producing $\widetilde{f}(x,y) = (x \to (x \wedge y)) \to x$ and then we rewrite this expression as a meet of implicative terms. In this case, $\widetilde{f}(x,y) = ((x \to x) \wedge (x \to y)) \to x = (x \to y) \to x$ is already the desired implicative term.

To end this section on free algebras, we summarize the different characterizations of free Łukasiewicz implication algebras in the following theorem.

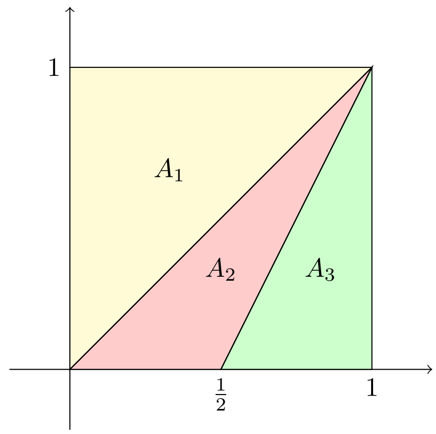


FIGURE 1. Decomposition of $[0,1]^2$ in Example 2.8

Theorem 2.9. $\mathbf{Free}_{\mathbb{L}}(n) \cong \mathbf{R}([0,1]^n) \cong \mathbf{R}(O^n) = \mathbf{M}^+(O^n)$.

3. Projective and finitely presented Łukasiewicz implication algebras

A study of projectivity of MV-algebras may be found in [2]. In this article we study projective algebras in the variety $\mathbb{L}$ as well as their relation to projective MV-algebras. Remember that we are omitting the adverb *weakly* everywhere.

We start with a universal characterization of finitely-generated projective algebras as certain subalgebras of free algebras.

Theorem 3.1. *Let* $\mathbb{V}$ *be a variety. An* n*-generated algebra* $\mathbf{A}$ *is projective in* $\mathbb{V}$ *if and only if* $\mathbf{A}$ *is isomorphic to a subalgebra of* $\mathbf{Free}_{\mathbb{V}}(n)$ *generated by* n *elements* $\{t_1, \ldots, t_n\}$ *such that* $t_i(t_1, \ldots, t_n) = t_i$ *for each* $i \in \{1, \ldots, n\}$.

Proof. Let $\mathbf{A}$ be a projective algebra in $\mathbb{V}$ generated by elements $a_1, \ldots, a_n \in A$. Let $x_1, \ldots, x_n$ be the free generators of $\mathbf{Free}_{\mathbb{V}}(n)$. Thus there exists a surjective homomorphism $\pi\colon \mathbf{Free}_{\mathbb{V}}(n) \to \mathbf{A}$ such that $\pi(x_i) = a_i$. Hence, there exists an embedding $\iota\colon \mathbf{A} \to \mathbf{Free}_{\mathbb{V}}(n)$ such that $\pi \circ \iota = id_A$ (identity map on A). It follows that $\mathbf{A} \cong \iota(\mathbf{A})$ and $\iota(\mathbf{A})$ is a subalgebra of $\mathbf{Free}_{\mathbb{V}}(n)$ generated by $\{t_i := \iota(a_i) : 1 \leq i \leq n\}$. Now observe that $\iota(\pi(x_i)) = \iota(a_i) = t_i$ and also $t_i(t_1, \ldots, t_n) = t_i(\iota(\pi(x_1)), \ldots, \iota(\pi(x_n))) = \iota(\pi(t_i(x_1, \ldots, x_n)) = \iota(\pi(t_i)) = \iota(\pi(\iota(\pi(x_i)))) = \iota(\pi(x_i)) = t_i$, since $\pi \circ \iota = id_A$.

Conversely, let $\mathbf{S}$ be the subalgebra of $\mathbf{Free}_{\mathbb{V}}(n)$ generated by $\{t_1, \ldots, t_n\}$ with the condition stated in the theorem. Let $\pi\colon \mathbf{Free}_{\mathbb{V}}(n) \to \mathbf{S}$ be the surjective homomorphism defined by $\pi(x_i) = t_i$. Then $\pi(t_i) = \pi(t_i(x_1, \ldots, x_n)) = t_i(\pi(x_1), \ldots, \pi(x_n)) = t_i(t_1, \ldots, t_n) = t_i$. Moreover, for any term t, we have that $\pi(t(t_1, \ldots, t_n)) = t(\pi(t_1), \ldots, \pi(t_n)) = t(t_1, \ldots, t_n)$. This shows that π is a retraction of $\mathbf{Free}_{\mathbb{V}}(n)$ onto $\mathbf{S}$ proving that $\mathbf{S}$ is projective. ∎

This theorem, together with the representation of free Łukasiewicz implication algebras as algebras of McNaughton functions, allows us to give a characterization of finitely generated projective Łukasiewicz implication algebras as algebras of McNaughton functions on certain polyhedra in the real cube. We recall some definitions from [2]. A *$\mathbb{Z}$-retraction* of $[0,1]^n$ is a map $\eta\colon [0,1]^n \to [0,1]^n$ such that $\eta = (f_1,\dots,f_n)$, where $f_1,\dots,f_n \in M([0,1]^n)$, and $\eta\circ\eta = \eta$. We say that $P \subseteq [0,1]^n$ is a *$\mathbb{Z}$-retract* of $[0,1]^n$ if $P = \eta([0,1]^n)$ for some $\mathbb{Z}$-retraction of $[0,1]^n$. We now define the corresponding notions suitable for the case we are interested in. A $\mathbb{Z}$-retraction $\eta = (f_1,\dots,f_n)$ is an *implicative $\mathbb{Z}$-retraction* if $f_1,\dots,f_n \in R([0,1]^n)$. Analogously, $P \subseteq [0,1]^n$ is an *implicative $\mathbb{Z}$-retract* of $[0,1]^n$ if $P = \eta([0,1]^n)$ for some implicative $\mathbb{Z}$-retraction of $[0,1]^n$.

Theorem 3.2. *A Łukasiewicz implication algebra $\mathbf{A}$ is an n-generated projective algebra if and only if $\mathbf{A} \cong \mathbf{R}(P)$ for some implicative $\mathbb{Z}$-retract of $[0,1]^n$.*

Proof. By Theorem 3.1, it is enough to show that, given $f_1,\dots,f_n \in R([0,1]^n)$ such that $f_i(f_1,\dots,f_n) = f_i$, $i = 1,\dots,n$, the subalgebra $\mathbf{A}$ of $\mathbf{R}([0,1]^n)$ generated by $f_1,\dots,f_n$ is isomorphic to $\mathbf{R}(P)$, where $P = \eta([0,1]^n)$, $\eta = (f_1,\dots,f_n)$. To prove this, consider the following surjective homomorphisms:

- $\psi_1\colon \mathbf{R}([0,1]^n) \to \mathbf{A}$ given by $\psi_1(\pi_i) = f_i$, where $\pi_1,\dots,\pi_n$ are the projection maps (free generators),
- $\psi_2\colon \mathbf{R}([0,1]^n) \to \mathbf{R}(P)$ given by $\psi_2(f) = f \restriction P$, the restriction of f to P.

Now observe that $\psi_1(f) = 1$ iff $f(f_1,\dots,f_n) = 1$ iff $f \restriction P = 1$ iff $\psi_2(f) = 1$. Thus, $\ker\psi_1 = \ker\psi_2$, so $\mathbf{A} \cong \mathbf{R}(P)$. □

In order to give a simpler characterization of n-generated projective algebras, we need to study the structure of implicative $\mathbb{Z}$-retracts of $[0,1]^n$. We will find that their structure is quite simple and, as a by-product, we will show that finitely generated projective algebras coincide with finitely presented ones (recall that an algebra is *finitely presented* if it is isomorphic to a quotient of a finitely generated free algebra via a finitely generated congruence). Before pursuing this task, we prove that the known characterization of finitely presented MV-algebras extends straightforwardly to Łukasiewicz implication algebras. Recall that, given a subset $S \subseteq M([0,1]^n)$, the *one-set* of S is the set $\{\overline{x} \in [0,1]^n : f(\overline{x}) = 1 \text{ for every } f \in S\}$.

Theorem 3.3. *A Łukasiewicz implication algebra $\mathbf{A}$ is finitely presented if and only if $\mathbf{A} \cong \mathbf{R}(P)$ for some one-set P of a finite subset of $R([0,1]^n)$.*

Proof. It enough to show that, given a finite subset $\{f_1,\dots,f_k\} \subseteq R([0,1]^n)$, the quotient algebra $\mathbf{R}([0,1]^n)/F$, where F is the implicative filter generated by $\{f_1,\dots,f_k\}$, is isomorphic to the algebra $\mathbf{R}(P)$, where P is the one-set of $\{f_1,\dots,f_k\}$.

Indeed, consider the homomorphism $\psi\colon \mathbf{R}([0,1]^n) \to \mathbf{R}(P)$ given by $\psi(f) = f \restriction P$. Note that $\ker\psi = \{f \in R([0,1]^n) : f(\overline{x}) = 1 \text{ for every } \overline{x} \in P\}$. Now observe that the definition of filters in MV-algebras is equivalent to the definition of implicative filters. Thus, if F denotes the filter generated by $\{f_1, \dots, f_k\}$ in $\mathbf{R}([0,1]^n)$ and G denotes the filter generated by the same functions in the MV-algebra $\mathbf{M}([0,1]^n)$, then $F = G \cap R([0,1]^n)$. For MV-algebras, it is known that $G = \{f \in M([0,1]^n) : f(\overline{x}) = 1 \text{ for every } \overline{x} \in P\}$. Hence, $F = \{f \in R([0,1]^n) : f(\overline{x}) = 1 \text{ for every } \overline{x} \in P\} = \ker\psi$. This concludes the proof. □

In order to describe the structure of implicative $\mathbb{Z}$-retracts of $[0,1]^n$, we recall some more definitions and results from [2]. A *rational polyhedron* is the pointset union of finitely many simplexes with rational vertices. A set $X \subseteq [0,1]^n$ is *star-shaped* if there exists an element $p \in X$ (called a *pole* of X) such that, for every $y \in X$, the linear segment $[p, y]$ is contained in X. We also denote by $\operatorname{conv}(X)$ the convex hull generated by a set X and by $[X, \overline{1}]$ the set of convex combinations of points in $X \subseteq [0,1]^n$ and the vertex $\overline{1} = (1,1,\dots,1) \in [0,1]^n$, that is, $[X, \overline{1}] = \{(1-\lambda)\overline{x} + \lambda\overline{1} : \overline{x} \in X, \lambda \in [0,1]\}$.

Lemma 3.4 ([2])**.**

(*a*) *If* $\eta\colon [0,1]^n \to [0,1]^n$ *is given by* $\eta = (f_1, \dots, f_n)$ *where* $f_1, \dots, f_n \in M([0,1]^n)$*, then* $\eta([0,1]^n)$ *is a rational polyhedron. In particular, every (implicative)* $\mathbb{Z}$*-retract of* $[0,1]^n$ *is a rational polyhedron.*

(*b*) *If* $P \subseteq [0,1]^n$ *is a star-shaped rational polyhedron with a pole* $\overline{p}$ *such that* $p_i \in \{0,1\}$ *for each* $i = 1, \dots, n$*, then* P *is a* $\mathbb{Z}$*-retract of* $[0,1]^n$*.*

Recall that given $x \in \mathbb{Q}^n$, the least common denominator of the coordinates of x is called the *denominator of* x and is denoted by $\operatorname{den}(x)$. The *homogeneous coordinates* of x are then defined as the integer vector $\operatorname{den}(x)(x,1) \in \mathbb{Z}^{n+1}$. Moreover, recall that a rational simplex (a simplex whose vertices have rational coordinates) is *unimodular* if the set of homogeneous coordinates of its vertices may be extended to a basis of the free abelian group $\mathbb{Z}^{n+1}$.

Lemma 3.5. *If* S *is a unimodular rational simplex contained in* O^n*, then* $S' = \operatorname{conv}(S \cup \{\overline{1}\}) = [S, \overline{1}]$ *is a unimodular rational simplex contained in* $[0,1]^n$*.*

Proof. Since S is convex and $S \subseteq O^n$, there exists $j \in \{1, \dots, n\}$ such that S is contained in the hyperplane $x_j = 0$. Without loss of generality we may assume $j = 1$. Let $\overline{v}_1, \dots, \overline{v}_k$ be the vertices of S. Then $\overline{v}_1, \dots, \overline{v}_k, \overline{1}$ are affine independent and thus $S' = \operatorname{conv}(S \cup \{\overline{1}\}) = \operatorname{conv}(\{\overline{v}_1, \dots, \overline{v}_k, \overline{1}\})$ is a rational simplex.

Now let $\widetilde{v}_1, \dots, \widetilde{v}_k$ be the corresponding homogeneous coordinates of the vertices of S. Since S is unimodular, $\{\widetilde{v}_1, \dots, \widetilde{v}_k\}$ may be extended to a basis of the free abelian group $\mathbb{Z}^{n+1}$, or, equivalently, if M denotes the $k \times (n+1)$ matrix whose rows are the vectors $\widetilde{v}_1, \dots, \widetilde{v}_k$, the greatest common divisor of the k-minors of M is 1. Now observe that the vertices of S' are $\overline{1}, \overline{v}_1, \dots, \overline{v}_k$, whose homogeneous coordinates are $\widetilde{1}, \widetilde{v}_1, \ldots, \widetilde{v}_k$, where $\widetilde{1} = (1, 1, \ldots, 1) \in \mathbb{Z}^{n+1}$. Let M' be the $(k+1) \times (n+1)$ matrix whose rows are the vectors $\widetilde{1}, \widetilde{v}_1, \ldots, \widetilde{v}_k$. The matrix M' has the following shape: $M' = \left[\begin{array}{c|c} 1 & \overline{1} \\ \hline 0 & A \end{array}\right]$. Observe that the $(k+1)$-minors of M' that include the first column coincide with the k-minors of M that do not include the first column. Also note that the k-minors of M that include the first column are zero. Then the greatest common divisor of all $(k+1)$-minors of M' must be 1. Thus $\{\widetilde{1}, \widetilde{v}_1, \ldots, \widetilde{v}_k\}$ may be extended to a basis of $\mathbb{Z}^{n+1}$ and S' is unimodular. □

The following lemma characterizes the implicative $\mathbb{Z}$-retracts of $[0,1]^n$. Note that the vertex $\overline{1} = (1, 1, \ldots, 1)$ belongs to any implicative $\mathbb{Z}$-retract of $[0,1]^n$ since any $f \in R([0,1]^n)$ satisfies $f(\overline{1}) = 1$. Note also that the endomorphisms φ_λ from Proposition 2.2 extend naturally to endomorphisms $\varphi_\lambda \colon [0,1]^n \to [0,1]^n$ given by $\varphi_\lambda(\overline{x}) = (\varphi_\lambda(x_1), \ldots, \varphi_\lambda(x_n)) = (1-\lambda)\overline{x} + \lambda\overline{1}$.

Lemma 3.6. *For $P \subseteq [0,1]^n$, the following are equivalent:*

(i) *P is an implicative $\mathbb{Z}$-retract of $[0,1]^n$.*
(ii) *P is the one-set of a finite subset of $R([0,1]^n)$.*
(iii) *$P = \{\overline{1}\}$ or $P = [Q, \overline{1}]$ for some rational polyhedron $Q \subseteq O^n$.*

Proof. (i) $\Rightarrow$ (ii) Assume $P = \eta([0,1]^n)$ for some implicative $\mathbb{Z}$-retraction $\eta \colon [0,1]^n \to [0,1]^n$. Put $\eta = (f_1, \ldots, f_n)$ for some $f_1, \ldots, f_n \in R([0,1]^n)$. Since $\eta \circ \eta = \eta$, $P = \{\overline{x} \in [0,1]^n : \eta(\overline{x}) = \overline{x}\}$. Hence $P = \{\overline{x} \in [0,1]^n : f_i(\overline{x}) = x_i, 1 \leq i \leq n\}$. Defining $g_i(\overline{x}) = f_i(\overline{x}) \to x_i$ and $h_i(\overline{x}) = x_i \to f_i(\overline{x})$, it follows that P is the one-set of the finite set $\{g_1, \ldots, g_n, h_1, \ldots, h_n\}$.

(ii) $\Rightarrow$ (iii) Let P be the one-set of $\{f_1, \ldots, f_k\} \subseteq R([0,1]^n)$. Given $\overline{x} \in P$ and $1 \leq i \leq k$, we have that $f_i(\varphi_\lambda(\overline{x})) = \varphi_\lambda(f_i(\overline{x})) = \varphi_\lambda(1) = 1$. This shows that the line segment $[\overline{x}, \overline{1}]$ is contained in P.

Now let $\overline{x} \in P$, $\overline{x} \neq \overline{1}$. As in the proof of Proposition 2.6, note that we may write $\overline{x} = (1-\lambda)\overline{x}^* + \lambda\overline{1}$ for some $\overline{x}^* \in O^n$, setting $\lambda = \min\{x_1, \ldots, x_n\}$, $x_i^* = \dfrac{x_i - \lambda}{1-\lambda}$. Thus, for $1 \leq i \leq k$, $1 = f_i(\overline{x}) = (1-\lambda)f_i(\overline{x}^*) + \lambda$, so $f_i(\overline{x}^*) = 1$. This shows that $\overline{x}^* \in P$.

Consequently, if $P \neq \{\overline{1}\}$, $P = [Q, \overline{1}]$ for the rational polyhedron $Q = P \cap O^n$.

(iii) $\Rightarrow$ (i) The proof follows that of Theorem 1.4 in [2]. Let Q be any rational polyhedron contained in O^n and put $P = [Q, \overline{1}]$. P is a rational polyhedron and is star-shaped with a pole in $\overline{1}$. We want to show that P is an implicative $\mathbb{Z}$-retract of $[0,1]^n$. By Lemma 3.4 we know that there is a $\mathbb{Z}$-retraction η of $[0,1]^n$ such that $\eta([0,1]^n) = P$. However, we need to look closer at the way in which η is constructed in the proof of this lemma in order to check that η may be taken to be an *implicative* $\mathbb{Z}$-retraction.

Following the proof in [2], let Δ be a unimodular triangulation of O^n such that $Q = \bigcup\{S \in \Delta : S \subseteq Q\}$. By Lemma 3.5, for every $S \in \Delta$, $S' = [S, \overline{1}]$ is a unimodular simplex contained in $[0,1]^n$. Moreover, $\Delta' = \{[S, \overline{1}] : S \in \Delta\}$ is a unimodular triangulation of $[0,1]^n$ such that $P = \bigcup\{[S,\overline{1}] : S \in \Delta, S \subseteq Q\}$. For each simplex $S' \in \Delta'$, let $\mathrm{ext}(S')$ be the set of vertices of S'. Then there exists a unique affine linear map $\eta_{S'}\colon S' \to [0,1]^n$ with integer coefficients determined by

$$\eta_{S'}(\overline{v}) = \begin{cases} \overline{v} & \text{if } \overline{v} \in \mathrm{ext}(S') \cap P, \\ \overline{1} & \text{if } \overline{v} \in \mathrm{ext}(S') \setminus P. \end{cases}$$

Let $\eta = \bigcup\{\eta_{S'} : S' \in \Delta'\}$. The fact that η is a $\mathbb{Z}$-retraction of $[0,1]^n$ whose range is P is shown in detail in [2]. Here we will show that η is, in this particular case, an implicative $\mathbb{Z}$-retraction.

Let $S' = [S,\overline{1}] \in \Delta'$, $\mathrm{ext}(S') \setminus P = \{\overline{v}_1,\dots,\overline{v}_m\}$ and $\mathrm{ext}(S') \cap P = \{\overline{w}_1,\dots,\overline{w}_k\}$. Then, if $\overline{x} = \sum_i \alpha_i \overline{v}_i + \sum_j \beta_j \overline{w}_j$ is a convex combination of the vertices of S', then, since $\eta_{S'}$ is affine linear,

$$\eta_{S'}(\overline{x}) = \sum_i \alpha_i \eta_{S'}(\overline{v}_i) + \sum_j \beta_j \eta_{S'}(\overline{w}_j) = \sum_i \alpha_i \overline{1} + \sum_j \beta_j \overline{w}_j.$$

Thus

$$\eta_{S'}(\overline{x}) \geq \sum_i \alpha_i \overline{v}_i + \sum_j \beta_j \overline{w}_j = \overline{x},$$

where $\leq$ stands for the product partial ordering on $[0,1]^n$. This shows that $\eta(\overline{x}) \geq \overline{x}$ for every $\overline{x} \in [0,1]^n$. Hence, if $\eta = (f_1,\dots,f_n)$, $f_i(\overline{x}) \geq x_i$ for every $\overline{x} \in [0,1]^n$, $1 \leq i \leq n$. This way condition (ii) in Rose's Theorem is met.

It remains to check condition (i) in Rose's Theorem. Using the same notation as in the previous paragraph, note that $\varphi_\lambda(\overline{x}) \in S'$ for every $\lambda \in [0,1]$. Thus

$$\begin{aligned} \eta(\varphi_\lambda(\overline{x})) = \eta_{S'}(\varphi_\lambda(\overline{x})) &= \eta_{S'}((1-\lambda)\overline{x} + \lambda\overline{1}) = (1-\lambda)\eta_{S'}(\overline{x}) + \lambda\eta_{S'}(\overline{1}) \\ &= (1-\lambda)\eta_{S'}(\overline{x}) + \lambda\overline{1} = \varphi_\lambda(\eta_{S'}(\overline{x})) = \varphi_\lambda(\eta(\overline{x})). \end{aligned}$$

This shows that $\varphi_\lambda(f_i(\overline{x})) = f_i(\varphi_\lambda(\overline{x}))$ for every $\overline{x} \in [0,1]^n$, $1 \leq i \leq n$. Thus we have shown that the components of η belong to $R([0,1]^n)$. □

Using the last lemma we can simplify the representation of $\mathbf{R}(P)$.

Lemma 3.7. $\mathbf{R}(P) \cong \mathbf{R}(P \cap O^n) = \mathbf{M}^+(P \cap O^n)$ *for any implicative* $\mathbb{Z}$*-retract* $P \neq \{\overline{1}\}$ *in* $[0,1]^n$.

Proof. The equality $\mathbf{R}(P \cap O^n) = \mathbf{M}^+(P \cap O^n)$ follows directly from Proposition 2.6. To show that $\mathbf{R}(P) \cong \mathbf{R}(P \cap O^n)$, it is enough to show that the restriction homomorphism $\psi\colon \mathbf{R}(P) \to \mathbf{R}(P \cap O^n)$ is a bijection. Indeed, surjectivity is straightforward. In addition, from Lemma 3.6, we know that $P = [P \cap O^n, \overline{1}]$. Thus condition (i) from Rose's Theorem guarantees that any $f \in R(P)$ is completely determined by its values on $P \cap O^n$. This shows that ψ is injective. □

Putting together Theorem 3.2, Theorem 3.3, Lemma 3.6, and Lemma 3.7, we get the following summary theorem. (For the sake of this theorem, consider $\mathbf{M}^+(\emptyset) = \mathbf{R}(\emptyset) =$ trivial algebra.)

Theorem 3.8. *The following are equivalent:*

(*i*) $\mathbf{A}$ *is an n-generated projective Łukasiewicz implication algebra,*
(*ii*) $\mathbf{A}$ *is an n-generated finitely presented Łukasiewicz implication algebra.*
(*iii*) $\mathbf{A} \cong \mathbf{R}(P)$ *for some implicative $\mathbb{Z}$-retract P of $[0,1]^n$.*
(*iv*) $\mathbf{A} \cong \mathbf{M}^+(Q) = \mathbf{R}(Q)$ *for some rational polyhedron $Q \subseteq O^n$.*

Remark 3.9. An interesting consequence of this theorem is the fact that all finite Łukasiewicz implication algebras are projective. In particular, since free algebras in the proper subvarieties of $\mathbb{L}$ are finite, they are projective and, thus, retractions of free algebras in $\mathbb{L}$. We show in the Appendix an explicit retraction from $\mathbf{Free}_{\mathbb{L}}(n)$ onto $\mathbf{Free}_{\mathbb{L}_k}(n)$.

Moreover, since the subvarieties of $\mathbb{L}$ are all generated by their finite members and all finite algebras are projective, it is easy to show that every subquasivariety of $\mathbb{L}$ is a variety. This had already been shown in [3]. Note also that the logical counterpart of this result is the fact that the implicative fragment of Łukasiewicz logic is hereditarily structurally complete. The structural completeness of this logic had already been proven in [5].

Finally, to close this section we will show that the implicative reducts of all projective MV-algebras are projective in $\mathbb{L}$. This may be proved, in the finitely generated case, as a consequence of our characterization given in Theorem 3.8 and the corresponding theorem of Cabrer and Munidici in [2]. However, we give here a very simple direct proof that encompasses also the non-finitely generated case.

Lemma 3.10. *For any set of free generators X, $\mathbf{Free}_{\mathbb{MV}}(X)^{\rightarrow}$ is projective in $\mathbb{L}$.*

Proof. Consider a set $Y = X \cup \{y\}$ such that $y \notin X$ and let $\mathbf{A}$ be the subalgebra of $\mathbf{Free}_{\mathbb{L}}(Y)$ with universe $[y)$ (increasing set generated by y). We can define a homomorphism of MV-algebras $\iota\colon \mathbf{Free}_{\mathbb{MV}}(X) \to \mathbf{A}_y$ by putting $\iota(x) = x \vee y$ for every $x \in X$. Note that $\iota\colon \mathbf{Free}_{\mathbb{MV}}(X)^{\rightarrow} \to \mathbf{Free}_{\mathbb{L}}(Y)$ is a homomorphism of Łukasiewicz implication algebras. Now we define a homomorphism $\pi\colon \mathbf{Free}_{\mathbb{L}}(Y) \to \mathbf{Free}_{\mathbb{MV}}(X)^{\rightarrow}$ by setting $\pi(x) = x$ for every $x \in X$ and $\pi(y) = 0$. It immediately follows that $\pi \circ \iota = id$. Thus $\mathbf{Free}_{\mathbb{MV}}(X)^{\rightarrow}$ is a retract of a free algebra in $\mathbb{L}$, so it is projective in $\mathbb{L}$. □

Proposition 3.11. *If $\mathbf{A}$ is a projective MV-algebra, then $\mathbf{A}^{\rightarrow}$ is projective in $\mathbb{L}$.*

Proof. Let $\mathbf{A}$ be a projective MV-algebra. Then $\mathbf{A}$ is a retract of $\mathbf{Free}_{\mathbb{MV}}(X)$ for some set of free generators X, that is, there are homomorphisms $\iota\colon \mathbf{A} \to \mathbf{Free}_{\mathbb{MV}}(X)$ and $\pi\colon \mathbf{Free}_{\mathbb{MV}}(X) \to \mathbf{A}$ such that $\pi \circ \iota = id$. Both maps are homomorphisms between the corresponding $\{\rightarrow, 1\}$-reducts. Hence $\mathbf{A}^{\rightarrow}$ is a retract of the projective algebra $\mathbf{Free}_{\mathbb{MV}}(X)^{\rightarrow}$, so it is also projective. □

It is interesting to note that the converse of the last proposition is not true. If $\mathbf{A} \in \mathbb{L}$ has a least element 0, it may be the case that $\mathbf{A}$ is projective in

$\mathbb{L}$ but $\mathbf{A}_0$ is not projective in $\mathbb{MV}$. For example, finite chains with three or more elements are not MV-projective, but we have already seen that they are projective as Łukasiewicz implication algebras.

4. Projective chains

In this section we characterize totally ordered projective Łukasiewicz implication algebras. In fact, we prove that the only projective chains are finite chains. It is interesting to note that, for the finitely generated case, this is a simple corollary of Theorem 3.8. Indeed, if $\mathbf{A}$ is a nontrivial n-generated projective algebra in $\mathbb{L}$, then $\mathbf{A} \cong \mathbf{R}(Q)$ for some rational polyhedron $Q \subseteq O^n$. If Q has more than one element, then it is easy to see that $\mathbf{R}(Q)$ is not totally ordered. On the other hand, if $Q = \{\overline{a}\}$ consists of one (rational) point in O^n, then $\mathbf{R}(Q)$ is a finite chain. In what follows, we prove the same result in the general case.

We start by showing that the infinite chain $\mathbf{L}_\omega$ is not projective.

Lemma 4.1. *In any MV-algebra, for any $n \in \mathbb{N}$, if $a^n \to b = a$, then $b = a^{n+1}$.*

Proof. We use induction on n. If $a \to b = a$, then $a^2 = a * (a \to b) = a \wedge b = b$. This shows the case $n = 1$. Now assume $a^{n+1} \to b = a$. Then $a^n \to (a \to b) = a$, so, by the induction hypothesis, $a \to b = a^{n+1}$. Thus $a^{n+2} = a * (a \to b) = a \wedge b = b$. □

Corollary 4.2. *Given an MV-algebra $\mathbf{A}$, every subalgebra of $\mathbf{A}^{\to}$ isomorphic to $\mathbf{L}_\omega$ has universe $\{1, a, a^2, a^3, \dots\}$ for some $a \in A$, $a \neq 1$.*

Proof. Suppose $S = \{a_0 = 1, a_1, a_2, \dots\}$ is the universe of a subalgebra of $\mathbf{A}^{\to}$ isomorphic to $\mathbf{L}_\omega$, that is, we may assume that $a_n \to a_{n+m} = a_m$ for every $n, m \in \mathbb{N} \cup \{0\}$. We show by induction on n that $a_n = a_1^n$ for every $n \geq 1$. The case $n = 1$ is obvious. Assuming $a_n = a_1^n$, note that $a_1^n \to a_{n+1} = a_n \to a_{n+1} = a_1$. Thus, by the previous lemma, $a_{n+1} = a_1^{n+1}$. □

Lemma 4.3. *In the MV-algebra $[\mathbf{0}, \mathbf{1}]$, for each $n \in \mathbb{N}$, $a \to a^{n+1} = a^n$ if and only if $a \geq 1 - \frac{1}{n+1}$.*

Proof. Let $a \in [0, 1]$. Note that $a \to a^{n+1} = na - n + 1$ if $a \geq 1 - \frac{1}{n+1}$ and $a \to a^{n+1} = 1 - a$ otherwise. Also $a^n = na - n + 1$ if $a \geq 1 - \frac{1}{n}$ and $a^n = 0$ otherwise. The lemma is now straightforward. □

Theorem 4.4. *The $\{\to, 1\}$-reduct of any free MV-algebra has no subalgebra isomorphic to $\mathbf{L}_\omega$.*

Proof. Assume $\mathbf{S} \leq \mathbf{Free}_{\mathbb{MV}}(X)^{\to}$ and $\mathbf{S} \cong \mathbf{L}_\omega$. By Corollary 4.2, $S = \{1, t, t^2, t^3, \dots\}$ for some $t \in Free_{\mathbb{MV}}(X)$. Since t depends on a finite number of variables, we may assume, without loss of generality, that $X = \{x_1, \dots, x_m\}$. Note that $t(\overline{x}) \to t(\overline{x})^{n+1} = t(\overline{x})^n$ for every $n \in \mathbb{N}$. By the previous lemma, $t^{[\mathbf{0},\mathbf{1}]}(\overline{a}) = 1$ for every $\overline{a} \in [0, 1]^m$. Thus $t(\overline{x}) = 1$, which is a contradiction. □

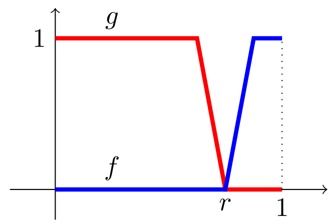


FIGURE 2. Proof of Lemma 4.6

Corollary 4.5. $\mathbf{L}_\omega$ *is not projective.*

Proof. Since $\mathbf{Free}_{\mathbb{L}}(X) \leq \mathbf{Free}_{\mathbb{MV}}(X)^{\to}$, if $\mathbf{Free}_{\mathbb{L}}(X)$ had a subalgebra isomorphic to $\mathbf{L}_\omega$, $\mathbf{Free}_{\mathbb{MV}}(X)^{\to}$ would too, which is impossible. Thus $\mathbf{L}_\omega$ cannot be a retract of a free algebra in $\mathbb{L}$, that is, it is not projective. □

We show now a stronger result than Theorem 4.4: no infinite chain may be a subalgebra of the $\{\to, 1\}$-reduct of a free MV-algebra. Thus, the only totally ordered projective Łukasiewicz implication algebras are the finite chains. We will need several lemmas. We write $[0,1]_{\mathbb{Q}} := [0,1] \cap \mathbb{Q}$ for short.

Lemma 4.6. *Given* $r \in [0,1]_{\mathbb{Q}}$*, there exist MV-terms* $\alpha(x), \beta(x)$ *such that for every* $a \in [0,1]$*:*

- $\alpha^{\mathbf{[0,1]}}(a) \leq \beta^{\mathbf{[0,1]}}(a) \iff a \leq r$,
- $\alpha^{\mathbf{[0,1]}}(a) \geq \beta^{\mathbf{[0,1]}}(a) \iff a \geq r$.

Proof. If $r = 1$, we may consider $\alpha(x) = x$ and $\beta(x) = 1$. Also, if $r = 0$, we let $\alpha(x) = x$ and $\beta(x) = 0$. Assume $0 < r < 1$ and write $r = \frac{m}{n}$ with $\gcd(m,n) = 1$. Let $k \in \mathbb{Z}$ large enough so that $0 < \frac{m}{n} - \frac{1}{nk} < \frac{m}{n} + \frac{1}{nk} < 1$ and define the following McNaughton functions on $[0,1]$:

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{m}{n}, \\ nkx - mk & \text{if } \frac{m}{n} \leq x \leq \frac{m}{n} + \frac{1}{nk}, \\ 1 & \text{if } \frac{m}{n} + \frac{1}{nk} \leq x \leq 1, \end{cases}$$

$$g(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \frac{m}{n} - \frac{1}{nk}, \\ -nkx + mk & \text{if } \frac{m}{n} - \frac{1}{nk} \leq x \leq \frac{m}{n}, \\ 0 & \text{if } \frac{m}{n} \leq x \leq 1. \end{cases}$$

Clearly the MV-terms corresponding to f and g have the desired property, see Figure 2. □

Recall from Proposition 2.2 that $\mathbf{[0,1]}^{\to} \cong [\boldsymbol{b}, \mathbf{1}]^{\to}$ for any $b \in [0,1)$. We also showed that $\varphi_b \colon [0,1] \to [b,1]$ given by $\varphi(x) = (1-b)x + b$ is an implicative isomorphism. Note also that, since $[\boldsymbol{b}, \mathbf{1}]^{\to}$ has least element b, the interval $[b,1]$ has a natural structure of MV-algebra, which we denote by $[\boldsymbol{b}, \mathbf{1}]_b^{\to}$.

Lemma 4.7. *Given $r \in [0,1]_{\mathbb{Q}}$, there are implicative terms $\alpha(x,y)$, $\beta(x,y)$ such that for $0 \leq b \leq a \leq 1$:*

- $\alpha^{\mathbf{[0,1]}}(a,b) \leq \beta^{\mathbf{[0,1]}}(a,b) \iff b \leq a \leq b + r(1-b)$,
- $\alpha^{\mathbf{[0,1]}}(a,b) \geq \beta^{\mathbf{[0,1]}}(a,b) \iff b + r(1-b) \leq a \leq 1$.

Proof. Let $\alpha'(x)$ and $\beta'(x)$ be the MV-terms from Lemma 4.6 corresponding to r. Let $\alpha(x,y)$ and $\beta(x,y)$ be implicative terms such that $\mathbb{MV} \models \alpha(x,0) = \alpha'(x)$ and $\mathbb{MV} \models \beta(x,0) = \beta'(x)$. (These terms may be produced by writing $\alpha'(x)$ and $\beta'(x)$ in the language $\{\to, 0\}$ and then replacing every 0 by the variable y.) Let $\mathbf{B} = [\boldsymbol{b}, \mathbf{1}]_b^{\to}$, which is isomorphic to $\mathbf{[0,1]}$ via the isomorphism $\varphi_b(x)$. Let $a^* \in [0,1]$ be such that $\varphi_b(a^*) = a$. Then

$$\begin{aligned}
\alpha^{\mathbf{[0,1]}}(a,b) \leq \beta^{\mathbf{[0,1]}}(a,b) &\iff \alpha^{[\boldsymbol{b},\mathbf{1}]^{\to}}(a,b) \leq \beta^{[\boldsymbol{b},\mathbf{1}]^{\to}}(a,b) \\
&\iff \alpha^{\mathbf{B}}(a,b) \leq \beta^{\mathbf{B}}(a,b) \\
&\iff \alpha^{\mathbf{B}}(\varphi_b(a^*), \varphi_b(0)) \leq \beta^{\mathbf{B}}(\varphi_b(a^*), \varphi_b(0)) \\
&\iff \varphi_b(\alpha^{\mathbf{[0,1]}}(a^*, 0)) \leq \varphi_b(\beta^{\mathbf{[0,1]}}(a^*, 0)) \\
&\iff \alpha^{\mathbf{[0,1]}}(a^*, 0) \leq \beta^{\mathbf{[0,1]}}(a^*, 0) \\
&\iff \overline{\alpha}^{\mathbf{[0,1]}}(a^*) \leq \overline{\beta}^{\mathbf{[0,1]}}(a^*) \\
&\iff a^* \leq r \\
&\iff \varphi_b(a^*) \leq \varphi_b(r) \\
&\iff a \leq b + r(1-b).
\end{aligned}$$

The other equivalence is completely analogous. □

Recall that if κ is a (possibly infinite) cardinal, a McNaughton function over the κ-cube is a function $f\colon [0,1]^\kappa \to [0,1]$ which depends on finitely many variables $x_{i_1}, \ldots, x_{i_n}$ and such that $f(x_{i_1}, \ldots, x_{i_n})$ is a McNaughton function over the n-cube. We denote the algebra of McNaughton functions on the κ-cube by $\mathbf{M}([0,1]^\kappa)$.

Lemma 4.8. *Let $\mathbf{A}$ be a totally ordered subalgebra of $\mathbf{M}([0,1]^\kappa)^{\to}$. Given $s, t \in A$ such that $1 \neq t \leq s$, there is $r \in [0,1]_{\mathbb{Q}}$ such that $s = t + r(1-t)$.*

Proof. Consider the following subsets of $[0,1]_{\mathbb{Q}}$: $I = \{r \in [0,1]_{\mathbb{Q}} : t + r(1-t) \leq s\}$ and $J = \{r \in [0,1]_{\mathbb{Q}} : s \leq t + r(1-t)\}$. Both sets are nonempty and every element in I is less than or equal to every element in J.

We claim that $I \cup J = [0,1]_{\mathbb{Q}}$. Indeed, fix $r \in [0,1]_{\mathbb{Q}}$ and let $\alpha(x,y)$ and $\beta(x,y)$ be the corresponding implicative terms from the previous lemma. Then $\alpha(s,t), \beta(s,t) \in A$, so either $\alpha(s,t) \leq \beta(s,t)$ or otherwise $\alpha(s,t) \geq \beta(s,t)$. Hence, the previous lemma asserts that either $r \in I$ or $r \in J$.

Put $r = \sup I = \inf J$. We will show that $s = t + r(1-t)$. By way of contradiction, assume that $s(\overline{a}) \neq t(\overline{a}) + r(1 - t(\overline{a}))$ for some $\overline{a} \in [0,1]^\kappa$. Suppose first that $s(\overline{a}) > t(\overline{a}) + r(1 - t(\overline{a}))$. Then $r < \dfrac{s(\overline{a}) - t(\overline{a})}{1 - t(\overline{a})}$ (note that

$t(\overline{a}) < 1)$. Choose $r' \in \mathbb{Q}$ such that $r < r' < \dfrac{s(\overline{a}) - t(\overline{a})}{1 - t(\overline{a})}$. Then $t(\overline{a}) + r(1 - t(\overline{a})) < t(\overline{a}) + r'(1 - t(\overline{a})) < s(\overline{a})$. Hence $r' \notin J$, so $r' \in I$, which contradicts the fact that $r < r'$. The case $s(\overline{a}) < t(\overline{a}) + r(1 - t(\overline{a}))$ is analogous.

We have proved that $s = t + r(1 - t)$ for some $r \in [0, 1]$. Now observe that, since $t \neq 1$, there is $\overline{a} \in [0, 1]^{\kappa}_{\mathbb{Q}}$ such that $t(\overline{a}) \neq 1$ (recall that t has a finite number of linear pieces that depend on finitely many variables). Then $r = \dfrac{s(\overline{a}) - t(\overline{a})}{1 - t(\overline{a})} \in \mathbb{Q}$. □

Theorem 4.9. *Every totally ordered subalgebra of* $\mathbf{M}([0, 1]^{\kappa})^{\rightarrow}$ *is finite.*

Proof. Let $\mathbf{A}$ be a totally ordered subalgebra of $\mathbf{M}([0, 1]^{\kappa})^{\rightarrow}$. Fix $t \in A$. We claim that $A_t = \{s \in A : s \geq t\}$ is a finite chain. Indeed, if $t = 1$ there is nothing to show. Suppose $t \neq 1$ and t depends on variables $x_{i_1}, \ldots, x_{i_n}$. There is $\overline{a} \in [0, 1]^{\kappa}_{\mathbb{Q}}$ such that $t(\overline{a}) \neq 1$. We can assume that $a_i = 0$ for every $i \notin \{i_1, \ldots, i_n\}$. Let $\{b_1, \ldots, b_k\}$ be the set of rational numbers in $[0, 1]$ whose denominator is a divisor of $\mathrm{den}(a_{i_1}, \ldots, a_{i_n})$. It follows that, for every $s \in M([0, 1]^{\kappa})$, $s(\overline{a}) \in \{b_1, \ldots, b_k\}$. If $s \in A$ and $t \leq s$, by the previous lemma, there is $r \in [0, 1]_{\mathbb{Q}}$ such that $s = t + r(1 - t)$. In particular, $s(\overline{a}) = t(\overline{a}) + r(1 - t(\overline{a})) \in \{b_1, \ldots, b_k\}$. Hence $r \in \{\frac{b_i - t(\overline{a})}{1 - t(\overline{a})} : 1 \leq i \leq k\}$, that is, there is only a finite number of possible values for r. Consequently, A_t is a finite chain.

We have thus proved that $\mathbf{A}$ is a totally ordered Łukasiewicz implication algebra such that for every $t \in A$ the segment $[t, 1]$ in $\mathbf{A}$ is a finite chain. There are only two possibilities: either $\mathbf{A}$ itself is a finite chain, or $\mathbf{A} \cong \mathbf{L}_{\omega}$. However, Theorem 4.4 rules out the second option. □

Corollary 4.10. *If* $\mathbf{A}$ *is a totally ordered Łukasiewicz implication algebra, then* $\mathbf{A}$ *is projective if and only if* $\mathbf{A}$ *is finite.*

Appendix A. Appendix: An explicit retraction from $\mathbf{Free}_{\mathbb{L}}(n)$ onto $\mathbf{Free}_{\mathbb{L}_k}(n)$

As stated in Remark 3.9, every finite algebra in $\mathbb{L}$ is projective and, consequently, a retract of a free algebra. Now, given a finite algebra $\mathbf{A}$, in order to find an explicit retraction from the free algebra to $\mathbf{A}$, we would have to find a presentation of $\mathbf{A}$, the correspondent implicative $\mathbb{Z}$-retract and its implicative $\mathbb{Z}$-retraction. Another way to find this retraction that involves a different kind of calculations would be the following. First write $\mathbf{A}$ as a homomorphic image of some $\mathbf{Free}_{\mathbb{L}_k}(n)$, the free algebra in some proper subvariety $\mathbf{L}_k$. Applying Proposition 2.6 in [3] this surjective homomorphism is a retraction and the corresponding embedding is easly constructible. Thus it only remains to find an explicit retraction from $\mathbf{Free}_{\mathbb{L}}(n)$ onto $\mathbf{Free}_{\mathbb{L}_k}(n)$. We will provide this in this appendix.

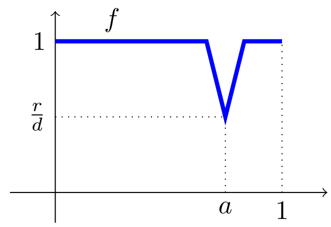


Figure 3. Proof of Lemma A.3

Let $O^n_k = \{\overline{x} \in O^n \cap \mathbb{Q}^n : \mathrm{den}(\overline{x}) \leq k\}$ (recall the definition of $\mathrm{den}(\overline{x})$ given before Lemma 3.5). Díaz Varela proved in [7] that $\mathbf{Free}_{\mathbb{L}_k}(n) \cong \mathbf{M}^+(O^n_k)$, based on the fact that $\mathbb{L}_k$ is the $\{\to, 1\}$-subreduct of the subvariety of MV-algebras $\mathbb{MV}_k = V(\mathbf{S}_1, \mathbf{S}_2, \dots, \mathbf{S}_k)$ and the characterization of free algebras in MV-subvarieties given by Panti in [9], which states that $\mathbf{Free}_{\mathbb{MV}_k}(n) \cong \mathbf{M}(\{\overline{x} \in [0,1]^n_{\mathbb{Q}} : \mathrm{den}(\overline{x}) \leq k\})$. This characterization of free algebras in $\mathbb{L}_k$ may be also derived as we did in Section 2 for the free algebras in $\mathbb{L}$.

We will prove in a constructive way that $\mathbf{M}^+(O^n_k)$ is a retract of $\mathbf{M}^+(O^n)$. We need some lemmas concerning properties of the MV-algebra $[\mathbf{0}, \mathbf{1}]$ and McNaughton functions.

Lemma A.1. *In the MV-algebra $[\mathbf{0}, \mathbf{1}]$, for $a, b \in [0,1]$:*

(1) *$b^m \geq a$ if and only if $a = 0$ or $b \geq 1 - \frac{1-a}{m}$;*
(2) *if $\frac{k-1}{k} \leq a < 1$, then $\langle \{1, a, a^2, \dots, a^k\}, \to, 1\rangle$ is a subalgebra of $[\mathbf{0}, \mathbf{1}]^{\to}$ isomorphic to $\mathbf{L}_k$.*

Proof. (1) follows easily from the fact that $b^m = \max\{0, mb - m + 1\}$. To prove (2), note that using Lemma 4.3 several times it follows that $a^r \to a^m = a^{m-r}$ for $0 \leq r \leq m \leq k$. □

Item (2) in the last lemma has the following interesting corollary.

Corollary A.2. *If a nonconstant McNaughton function $f\colon [0,1]^n \to [0,1]$ satisfies the condition $f(x) \geq \frac{k-1}{k}$ for every $x \in [0,1]^n$, then the algebra $\langle \{1, f, f^2, \dots, f^k\}, \to, 1\rangle$ is a subalgebra of $\mathbf{M}([0,1]^n)^{\to}$ isomorphic to $\mathbf{L}_k$.*

Lemma A.3. *Let $a \in [0,1]_{\mathbb{Q}}$, $a = \frac{m}{d}$, $d = \mathrm{den}(a)$ and $r \in \mathbb{Z}$, $0 \leq r \leq d$. Given $\varepsilon > 0$, there is a McNaughton function $f\colon [0,1] \to [0,1]$ such that $f(a) = \frac{r}{d}$, $f(x) \geq \frac{r}{d}$ for every $x \in [0,1]$, and $f(x) = 1$ whenever $|x - a| \geq \varepsilon$.*

Proof. Similarly as in the proof of Lemma 4.6, it is easy to find a v-shaped McNaughton function with the desired properties with its vertex at $(a, \frac{r}{d})$ as shown in Figure 3. □

We need some elementary lemmas concerning the decomposition of fractions in $[0,1]$.

Lemma A.4. *Let $a, b_1, b_2 \in \mathbb{N}$ such that $\gcd(b_1, b_2) = 1$.*

(*a*) *There are unique* $q, r_1, r_2 \in \mathbb{Z}$, $0 \le r_i < b_i$, *such that* $\frac{a}{b_1 b_2} = q + \frac{r_1}{b_1} + \frac{r_2}{b_2}$.
(*b*) *If* $a < b_1 b_2$, *then* $q = 0$ *or* $q = -1$, *so* $\frac{a}{b_1 b_2} = \frac{r_1}{b_1} \oplus \frac{r_2}{b_2}$ *or* $\frac{a}{b_1 b_2} = \frac{r_1}{b_1} * \frac{r_2}{b_2}$.
(*c*) *If* $1 - \frac{1}{b_1} - \frac{1}{b_2} < \frac{a}{b_1 b_2} < 1$, *then* $q = 0$ *and* $\frac{a}{b_1 b_2} = \frac{r_1}{b_1} \oplus \frac{r_2}{b_2}$.

Proof. Part (a) is elementary. To prove (b), note that if $a < b_1 b_2$, then $0 < q + \frac{r_1}{b_1} + \frac{r_2}{b_2} < 1$, so $-\frac{r_1}{b_1} - \frac{r_2}{b_2} < q < 1 - \frac{r_1}{b_1} - \frac{r_2}{b_2}$. This proves that $-2 < q < 1$. Finally, for (c), just observe that if $q = -1$, then $q + \frac{r_1}{b_1} + \frac{r_2}{b_2} \le -1 + \frac{b_1 - 1}{b_1} + \frac{b_2 - 1}{b_2} = 1 - \frac{1}{b_1} - \frac{1}{b_2}$. □

Corollary A.5. *Let* $a, b_1, \dots, b_n \in \mathbb{N}$, $a < b = b_1 \dots b_n$, $\gcd(b_i, b_j) = 1$ *for* $i \ne j$.

(*a*) *There are unique* $q, r_1, \dots, r_n \in \mathbb{Z}$, $0 \le r_i < b_i$, *such that* $\frac{a}{b} = q + \frac{r_1}{b_1} + \dots + \frac{r_n}{b_n}$.
(*b*) *There is a term* $t(x_1, \dots, x_n) = x_1 \circ x_2 \circ \dots \circ x_n$ *(associating to the left), where each* $\circ$ *stands for* $\oplus$ *or* $*$, *such that* $\frac{a}{b} = t\left(\frac{r_1}{b_1}, \dots, \frac{r_n}{b_n}\right)$.
(*c*) *Given* $i \in \{1, \dots, n\}$, *there is a term* $t(x_1, \dots, x_{n-1}) = x_1 \circ \dots \circ x_{n-1}$ *(associating to the left), where each* $\circ$ *stands for* $\oplus$ *or* $*$, *such that* $\frac{b-1}{b} = \frac{r_i}{b_i} \oplus t\left(\frac{r_1}{b_1}, \dots, \frac{r_{i-1}}{b_{i-1}}, \frac{r_{i+1}}{b_{i+1}}, \dots, \frac{r_n}{b_n}\right)$.

Proof. Parts (a) and (b) come from applying the last lemma several times, and uniqueness in (a) is an easy exercise. To show part (c), we use item (c) in the previous lemma for the fraction $\frac{b-1}{b} = \frac{b-1}{b_i b^*}$ where $b^* = b_1 \dots b_{i-1} b_{i+1} \dots b_n$, to obtain a decomposition $\frac{b-1}{b} = \frac{s_i}{b_i} \oplus \frac{s}{b^*}$. By (a) and (b), there is a term t in the language $\{\oplus, *\}$ such that $\frac{b-1}{b} = \frac{s_i}{b_i} \oplus t\left(\frac{s_1}{b_1}, \dots, \frac{s_{i-1}}{b_{i-1}}, \frac{s_{i+1}}{b_{i+1}}, \dots, \frac{s_n}{b_n}\right)$, where $0 \le s_j < b_j$, $1 \le j \le n$. Finally, by the way in which the decomposition is generated and the fact that in each step the Łukasiewicz sum or product gives a result different from 0 or 1, it is clear that $\frac{s_i}{b_i} \oplus t\left(\frac{s_1}{b_1}, \dots, \frac{s_{i-1}}{b_{i-1}}, \frac{s_{i+1}}{b_{i+1}}, \dots, \frac{s_n}{b_n}\right) = \frac{s_1}{b_1} + \dots + \frac{s_n}{b_n} - p$, where p is the number of occurrences of the Łukasiewicz product in the term t. Using now the uniqueness part of item (a), it follows that $s_i = r_i$ for $1 \le i \le n$. □

Given $x = (x_1, \dots, x_n) \in [0,1]^n$, we denote $\|x\| = \max\{|x_1|, \dots, |x_n|\}$.

Lemma A.6. *Let* $a \in [0,1]^n_{\mathbb{Q}}$, $d = \mathrm{den}(a)$. *Given* $\varepsilon > 0$, *there is a McNaughton function* $f\colon [0,1]^n \to [0,1]$ *such that* $f(a) = \frac{d-1}{d}$, $f(x) \ge \frac{d-1}{d}$ *for every* $x \in [0,1]^n$, *and* $f(x) = 1$ *whenever* $\|x - a\| \ge \varepsilon$. *Moreover,* $f(x)^m \ge x_i$ *for every* $x \in [0,1]^n$ *if* $\left(\frac{d-1}{d}\right)^m \ge a_i$, $0 \le m \le d$.

Proof. Put $a = (a_1, a_2, \dots, a_n)$, $a_i = \frac{m_i}{d}$, $0 \le m_i \le d$, $1 \le i \le n$. Note that, if $d_i = \mathrm{den}(a_i)$, then $d = \mathrm{den}(a) = \mathrm{lcm}(d_1, \dots, d_n)$. An elementary exercise proves that there are $b_1, \dots, b_n \in \mathbb{N}$ such that $d = b_1 \dots b_n$, $\gcd(b_i, b_j) = 1$ for $i \ne j$ and b_i divides d_i for $1 \le i \le n$. Now, by Corollary A.5, there are $r_1, \dots, r_n \in \mathbb{Z}$, $0 \le r_i < b_i$, and terms $t_1, \dots, t_n$ in the language $\{\oplus, *\}$ such that

$$\frac{d-1}{d} = \frac{d-1}{b_1 \dots b_n} = \frac{r_i}{b_i} \oplus t_i\left(\frac{r_1}{b_1}, \dots, \frac{r_{i-1}}{b_{i-1}}, \frac{r_{i+1}}{b_{i+1}}, \dots, \frac{r_n}{b_n}\right).$$

Since b_i divides d_i, we may write $\frac{r_i}{b_i} = \frac{s_i}{d_i}$ for suitable $s_i \in \mathbb{Z}$, $0 \leq s_i < d_i$, $1 \leq i \leq n$. Using Lemma A.3, for each $1 \leq i \leq n$ there is a McNaughton function $f_i\colon [0,1] \to [0,1]$ such that:

- $f_i(a_i) = \frac{s_i}{d_i} = \frac{r_i}{b_i}$,
- $f_i(x) \geq \frac{s_i}{d_i} = \frac{r_i}{b_i}$ for every $x \in [0,1]$,
- $f_i(x) = 1$ whenever $|x - a_i| \geq \varepsilon$.

Using the same lemma once again, for each $1 \leq i \leq n$, there is a McNaughton function $g_i\colon [0,1] \to [0,1]$ such that:

- $g_i(a_i) = 0$,
- $g_i(x) = 1$ whenever $|x - a_i| \geq \varepsilon$.

We now define the following McNaughton function on $[0,1]^n$:

$$f(x_1, x_2, \ldots, x_n) = \bigvee_{i=1}^{n} (f_i(x_i) \oplus t_i(f_1(x_1), \ldots, f_{i-1}(x_{i-1}), f_{i+1}(x_{i+1}), \ldots, f_n(x_n))) \vee \bigvee_{i=1}^{n} g_i(x_i).$$

We now check that f has the desired properties.

- $f(a) = \frac{d-1}{d}$. Straightforward.
- $f(x) \geq \frac{d-1}{d}$ for each $x \in [0,1]^n$.

  Indeed, given $x = (x_1, \ldots, x_n) \in [0,1]^n$,

$$f(x) \geq f_1(x_1) \oplus t_1(f_2(x_2), \ldots, f_n(x_n)) \geq \frac{r_1}{b_1} \oplus t_1\left(\frac{r_2}{b_2}, \ldots, \frac{r_n}{b_n}\right) = \frac{d-1}{d}$$

  since $\oplus$ and $*$ are increasing in each argument.
- $f(x)^m \geq x_i$ for every $x \in [0,1]^n$ if $\left(\frac{d-1}{d}\right)^m \geq a_i$.

  Assume $\left(\frac{d-1}{d}\right)^m \geq a_i$ for some index i, $0 \leq m \leq d$. Then, by Lemma A.1, $\frac{d-1}{d} \geq 1 - \frac{1-a_i}{m}$, so $m \leq d(1-a_i) = d - m_i$. Observe that:

$$\begin{aligned} f(x) &\geq f_i(x_i) \oplus t_i(f_1(x_1), \ldots, f_{i-1}(x_{i-1}), f_{i+1}(x_{i+1}), \ldots, f_n(x_n)) \\ &\geq f_i(x_i) \oplus t_i\left(\frac{r_1}{b_1}, \ldots, \frac{r_{i-1}}{b_{i-1}}, \frac{r_{i+1}}{r_{i+1}}, \ldots, \frac{r_n}{b_n}\right) \\ &= f_i(x_i) \oplus \left(\frac{d-1}{d} - \frac{r_i}{b_i}\right) \\ &= \left(f_i(x_i) + \left(\frac{d-1}{d} - \frac{r_i}{b_i}\right)\right) \wedge 1 \\ &\geq 1 - \frac{1-x_i}{d-m_i}. \end{aligned}$$

  The last inequality follows from the fact that

$$f_i(x_i) \geq 1 - \frac{1-x_i}{d-m_i} - \left(\frac{d-1}{d} - \frac{r_i}{b_i}\right)$$

for every $x_i \in [0,1]$ such that $f_i(x_i) \neq 1$ (recall that $f_i(x_i)$ was defined in Lemma A.3 as a v-shaped McNaughton function and the slope of the right hand side in the last equality is $\frac{1}{d-m_i}$).

By Lemma A.1 it follows that $f(x)^{d-m_i} \geq x_i$ for every $x \in [0,1]^n$. Therefore, since $m \leq d - m_i$, we have that $f(x)^m \geq f(x)^{d-m_i} \geq x_i$.

- $f(x) = 1$ whenever $\|x - a\| \geq \varepsilon$.

  Indeed, if $\|x-a\| \geq \varepsilon$, there is some index j such that $|x_j - a_j| \geq \varepsilon$, so $f(x) \geq g_j(x_j) = 1$.

□

We are now ready to make the retraction explicit.

Theorem A.7. $\mathbf{M}^+(O_k^n)$ *is a retract of* $\mathbf{M}^+(O^n)$.

Proof. It is clear that $\pi\colon \mathbf{M}^+(O^n) \to \mathbf{M}^+(O_k^n)$, defined by restriction to O_k^n of the functions in $M^+(O^n)$, is a homomorphism. We look for a homomorphism $\iota\colon \mathbf{M}^+(O_k^n) \to \mathbf{M}^+(O^n)$ such that $\pi \circ \iota = id_{M^+(O_k^n)}$.

Since O_k^n is finite, there is some $\varepsilon > 0$ so that for every pair $a_1, a_2 \in O_k^n$ the sets $\{x \in [0,1]^n : \|x - a_1\| < \varepsilon\}$ and $\{x \in [0,1]^n : \|x - a_2\| < \varepsilon\}$ are disjoint. For this value of ε, we apply the previous lemma for each $a \in O_k^n$ and produce the corresponding McNaughton functions f_a. From now on we write d_a instead of $\mathrm{den}(a)$. Clearly, since $f_a(a) = \frac{d_a-1}{d_a}$, $f_a(a)^m = \frac{d_a-m}{d_a}$ for $0 \leq m \leq d_a$. Moreover, note that since $f_a(x) \geq \frac{d_a-1}{d_a}$, by Lemma A.2, $\langle\{1, f_a, f_a^2, \ldots, f_a^{d_a}\}, \to, 1\rangle$ is a subuniverse of $\mathbf{M}([0,1]^n)^{\to}$ isomorphic to $\mathbf{L}_{d_a}$.

Given $g \in M^+(O_k^n)$, put $g(a) = \frac{d_a - m_a}{d_a}$, $0 \leq m_a \leq d_a$, for each $a \in O_k^n$. We define

$$\iota(g)(x) = \bigwedge_{a \in O_k^n} f_a(x)^{m_a}.$$

We claim that $\iota(g) \in M^+(O^n)$. Indeed, since $g \in M^+(O_k^n)$ there is an index $\ell \in \{1, \ldots, n\}$ such that $g(x) \geq x_\ell$ for every $x \in O_k^n$. Therefore, given $a \in O_k^n$, $g(a) = \frac{d_a-m_a}{d_a} = \left(\frac{d_a-1}{d_a}\right)^{m_a} \geq a_\ell$. Hence, we know that $f_a(x)^{m_a} \geq x_\ell$ for every $x \in [0,1]^n$. Since this holds for every $a \in O_k^n$, it follows that $\iota(g)(x) \geq x_\ell$ for every $x \in O^n$, so $\iota(g) \in M^+(O^n)$.

We have just defined a map $\iota\colon M^+(O_k^n) \to M^+(O^n)$. We now show that ι is a homomorphism of Łukasiewicz implication algebras. Indeed, given $g, g' \in M^+(O_k^n)$, assume that $g(a) = \frac{d_a-m_a}{d_a}$ and $g'(a) = \frac{d_a-m'_a}{d_a}$ for every $a \in O_k^n$. Then

$$\begin{aligned}\iota(g)(x) \to \iota(g')(x) &= \bigwedge_{a \in O_k^n} f_a(x)^{m_a} \to \bigwedge_{b \in O_k^n} f_b(x)^{m'_b} \\ &= \bigwedge_{b \in O_k^n} \bigvee_{a \in O_k^n} \left(f_a(x)^{m_a} \to f_b(x)^{m'_b}\right)\end{aligned}$$

Now note that, if $a \neq b$, $f_a(x) = 1$ for every x such that $f_b(x) \neq 1$. Then

$$\iota(g)(x) \to \iota(g')(x) = \bigwedge_{b \in O_k^n} \left(f_b(x)^{m_b} \to f_b(x)^{m'_b} \right).$$

Finally, since $\langle \{1, f_b, f_b^2, \ldots, f_b^{d_b}\}, \to, 1\rangle$ is a subalgebra of $\mathbf{M}^+([0,1]^n)$ isomorphic to $\mathbf{L}_{d_b}$ and $\frac{d_b - m_b}{d_b} \to \frac{d_b - m'_b}{d_b} = \frac{d_b - \max\{m'_b - m_b, 0\}}{d_b}$, we conclude that

$$\iota(g)(x) \to \iota(g')(x) = \bigwedge_{b \in O_k^n} f_b(x)^{\max\{m'_b - m_b, 0\}} = \iota(g \to g')(x).$$

We have thus a homomorphism $\iota \colon \mathbf{M}^+(O_k^n) \to \mathbf{M}^+(O^n)$ that clearly satisfies the condition $\pi \circ \iota = id_{M^+(O_k^n)}$. Consequently, $\mathbf{M}^+(O_k^n)$ is a retract of $\mathbf{M}^+(O^n)$. $\square$

Acknowledgment. The author would like to thank the anonymous referee for suggesting an improvement to the original version of Theorem 3.8.

References

[1] Berman, J., Blok, W.J.: Free Łukasiewicz and hoop residuation algebras. Studia Logica **77**, 153–180 (2004)
[2] Cabrer, L., Mundici, D.: Projective MV-algebras and rational polyhedra. Algebra Universalis **62**, 63–74 (2009)
[3] Campercholi, M., Castaño, D., Díaz Varela, J.P.: Quasivarieties and congruence permutability of Łukasiewicz implication algebras. Studia Logica **98**, 267–283, (2011)
[4] Cignoli, R.L., D'Ottaviano, I.M.L., Mundici, D.: Algebraic foundations of many-valued reasoning. Kluwer Academic Publishers, Dordrecht (2000)
[5] Cintula, P., Metcalfe, G.: Structural completeness in fuzzy logics. Notre Dame J. Form. Log. **50**, 153–182 (2009)
[6] Díaz Varela, J.P., Torrens, A.: Decomposability of free Łukasiewicz implication algebras. Arch. Math. Logic **45**, 1011–1020 (2006)
[7] Díaz Varela, J.P.: Free Łukasiewicz implication algebras. Arch. Math. Logic **47**, 25–33 (2008)
[8] Komori, Y.: Super-Łukasiewicz implicational logics. Nagoya Math. J. **72**, 127–133 (1978)
[9] Panti, G.: Varieties of MV-algebras. J. Appl. Non-Classical Logics **9**, 141–157 (1999)
[10] Rose, A.: Extensions de quelques thormes de McNaughton. C. R. Acad. Sci. Paris Sér. I Math. **292**, 979–981 (1981) (French)

Diego N. Castaño
Departamento de Matemática (Universidad Nacional del Sur), INMABB (UNS-CONICET), Bahía Blanca (8000), Argentina
e-mail: diego.castano@uns.edu.ar