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Weight multiplicity formulas for bivariate representations of classical Lie algebras

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A bivariate representation of a complex simple Lie algebra is an irreducible representation having highest weight a combination of the first two fundamental weights. For a complex classical Lie algebra, we establish an expression for the weight multiplicities of bivariate representations. *Published by AIP Publishing.* https://doi.org/10.1063/1.5043305

I. INTRODUCTION

This article concerns on giving weight multiplicity formulas, continuing the previous authors' article.[16] In that article, for a classical complex Lie algebra $\mathfrak{g}$, a closed explicit formula for the weight multiplicities of any representation of any p-fundamental string was determined. Such a representation is an irreducible representation of $\mathfrak{g}$ with highest weight $k\omega_1 + \omega_p$ for some non-negative integer k. Here, ω_j denotes the jth fundamental weight associated with the root system of $\mathfrak{g}$.

The primary goal of the present article is to find an expression for the weight multiplicity of every *bivariate representation* of a classical complex Lie algebra $\mathfrak{g}$. A bivariate representation is an irreducible representation with highest weight $a\omega_1 + b\omega_2$ for some non-negative integers a and b (cf. Ref. 17). See Sec. 1 in Ref. 16 for references of classical and recent previous results on this problem.

In Sec. II, we introduce the standard notation used to describe the root system associated with a classical complex Lie algebra $\mathfrak{g}$. In particular, for $\mathfrak{g}$ of type B_n, C_n, or D_n and $\mathfrak{h}$ a fixed Cartan subalgebra of $\mathfrak{g}$, $\{\varepsilon_1, \ldots, \varepsilon_n\}$ denotes the basis of $\mathfrak{h}^*$ satisfying that the set of simple roots are $\{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n\}$ for type B_n, $\{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n\}$ for type C_n, and $\{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\}$ for type D_n. According to this notation, bivariate representations have highest weight of the form $k\varepsilon_1 + l\varepsilon_2$ for integers $k \geq l \geq 0$.

The obtained weight multiplicity formulas for types B_n, C_n, and D_n are in Theorems III.1, III.2, and III.3, respectively. The expressions involve a sum over partitions of the integer numbers $\leq l$, so they may not be considered "closed explicit formulas" like in Ref. 16. An immediate and curious consequence of the formulas is the next result.

Theorem I.1. *Let $\mathfrak{g}$ be a classical complex Lie algebra of type B_n, C_n, or D_n. Let $k \geq l \geq 0$ integers and $\mu = \sum_{i=1}^n a_i\varepsilon_i$ with $a_i \in \mathbb{Z}$ for all i. The multiplicity of μ in the irreducible representation $\pi_{k\varepsilon_1+l\varepsilon_2}$ of $\mathfrak{g}$ with highest weight $k\varepsilon_1 + l\varepsilon_2$, denoted by $m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(\mu)$, depends only on*

$$\|\mu\|_1 := \sum_{i=1}^n |a_i| \quad \textit{and} \quad Z_t(\mu) := \#\{i : 1 \leq i \leq n,\ |a_i| = t\} \quad \textit{for all } 0 \leq t \leq l-1. \tag{1}$$

In other words, if μ and μ' satisfy $\|\mu\|_1 = \|\mu'\|_1$ and $Z_t(\mu) = Z_t(\mu')$ for all $0 \leq t \leq l-1$, then $m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(\mu) = m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(\mu')$.

This theorem is analogous to Corollary 1.1 in Ref. 16 (see also Lemma 3.3 in Ref. 15), which states that the multiplicity of a weight μ in representations in p-fundamental strings depends only

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on $\|\mu\|_1$ and $Z_0(\mu)$. Such representations have highest weights of the form $k\omega_1 + \omega_p$ for $k \geq 0$ and $1 \leq p \leq n-1$ for type B_n, $1 \leq p \leq n$ for type C_n, and $1 \leq p \leq n-2$ for type D_n.

Q4 In the best authors' knowledge, the weight multiplicity formulas in Theorems III.1–III.3 are not in the literature. Nevertheless, Maddox[17] obtained a multiplicity formula for bivariate representations when $\mathfrak{g}$ is of type C_n. However, her expression differs significantly from ours. In particular, Theorem I.1 does not follow immediately from her formula. See Remark III.14 for more details.

We compare from a computational point of view, the multiplicity formulas obtained in Theorems III.1–III.3 with Freudenthal's famous formula (see Subsection III C). We used the open-source mathematical software `Sage`[19] to do the calculations. It was evidenced in the computational results shown in Table I that the bivariate algorithm based on Theorems III.1–III.3 is faster than

TABLE I. Computational comparison between the bivariate algorithm and Freudenthal's formula. Each column shows, for the
Q5 corresponding algorithm and type X, the required time for returning the set of weights with multiplicities of the representation $\pi_{k\varepsilon_1+l\varepsilon_2}$ of $\mathfrak{g}$ of type X_n according to the row. The column $\mathrm{D}_n(\star)$ refers to the version of the bivariate algorithm returning only the dominant weights.

			Time bivariate				Time Freudenthal		
n	k	l	B_n	C_n	D_n	$\mathrm{D}_n(\star)$	B_n	C_n	D_n
2	5	3	0.15	0.07			0.32	0.13	
3	5	3	0.26	0.14	0.15	0.13	3.88	1.87	1.27
4	5	3	0.46	0.35	0.28	0.19	32.58	14.90	12.05
5	5	3	0.99	0.67	0.62	0.23	187.43	94.08	79.82
6	5	3	2.82	1.89	1.78	0.24	876.17	527.69	451.43
7	5	3	6.94	5.22	4.74	0.27	3436.25	1898.23	1961.54
8	5	3	17.77	14.11	12.59	0.36			
9	5	3	43.23	35.47	32.11	0.51			
10	5	3	97.55	87.67	78.66	0.84			
2	10	3	0.29	0.13			1.92	0.48	
3	10	3	0.82	0.49	0.45	0.37	23.68	10.23	7.85
4	10	3	2.09	1.16	1.22	0.58	291.63	130.93	108.61
5	10	3	8.30	5.06	4.84	0.80	2630.09	1193.45	1028.42
6	10	3	38.42	24.86	23.90	1.16			
7	10	3	183.73	146.06	126.82	1.96			
2	50	3	3.47	1.79			78.13	28.54	
3	50	3	36.40	17.62	17.50	13.50	5146.69	2108.21	1578.78
4	50	3	472.14	325.76	267.25	58.00			
2	6	6	2.20	1.32			0.50	0.19	
3	6	6	9.28	5.35	5.23	5.21	11.90	4.74	3.53
4	6	6	19.29	11.19	11.44	11.09	157.23	67.93	54.79
5	6	6	30.81	18.05	18.04	15.78	1443.41	663.86	553.57
6	6	6	53.76	32.65	32.64	19.50			
2	10	6	3.58	2.04			1.66	0.62	
3	10	6	18.14	9.86	9.91	9.77	43.50	18.34	13.59
4	10	6	44.69	25.20	24.96	23.98	695.71	298.50	243.70
5	10	6	87.71	49.55	49.36	38.89	8114.00	3571.44	2966.84
6	10	6	235.76	158.26	133.51	52.88			
2	20	6	8.77	4.61			8.41	3.16	
3	20	6	63.71	33.25	33.16	32.61	312.83	129.92	98.02
4	20	6	216.46	115.58	117.99	109.08	7486.63	3199.29	2620.10
5	20	6	654.96	390.98	393.92	220.11			
2	15	9	40.04	22.33			4.93	1.81	
3	15	9	390.59	209.87	209.12	208.37	191.24	78.85	59.11
4	15	9	1594.63	865.10	853.83	851.50	4710.03	1908.34	1642.50
5	15	9	3794.15	2112.98	2051.99	1962.57	71389.97	32013.22	28179.33
2	50	9	231.18	116.93			96.16	35.20	
3	50	9	4800.55	2423.05	2553.71	2492.15	7851.47	3117.85	2346.84

the Freudenthal algorithm for most of the small values of k and l. Moreover, for any choice of k and l, the same conclusion would hold for n big enough. It is probably a more significant the fact that Theorems III.1–III.3 return in a speedy way the multiplicity of a single weight. The situation is very different with Freudenthal's formula since it is defined recursively, and moreover, it has to calculate the multiplicities of many intermediate weights in case the original weight is far away from the highest weight. Many more related remarks are made in Subsection III C.

We have already mentioned that the expressions for the weight multiplicities in Theorems III.1–III.3 and IV.2 are not closed explicit formulas since they involve a sum over partitions. However, in some particular cases, one can write down the corresponding partitions obtaining a (long) closed expression. For instance, the statements for the multiplicity of the weight $\mu = 0$ for types B_n, C_n, and D_n are included in Subsection III D. Furthermore, a formula for $\pi_{k\varepsilon_1+2\varepsilon_2}$ in type Q6 D_n is also part of Subsection III D.

Although weight multiplicity formulas are interesting in themselves, the authors were motivated by their application in spectral geometry (see Sec. 7 in Ref. 16). In Remark III.17, we mention possible applications for the weight multiplicity formulas obtained in this article for the determination of the spectra of some natural differential operators on spaces covered by compact symmetric spaces with Abelian fundamental groups.

The weight multiplicity formula for $\mathfrak{g}$ of type A_n is determined in Sec. IV. Furthermore, the corresponding expression for the case $l = 2$ is stated in Corollary IV.3. This case, $\mathfrak{g}$ of type A_n, is much simpler than the previous ones. The obtained expressions are probably already present in the extensive literature on this area.

The article is organized as follows. Section II introduces the necessary notation to read the statements of the primary results. Section III, which considers classical Lie algebras of type B_n, C_n, and D_n, is divided in five subsections. Subsection III A states the weight multiplic- Q7 ity formulas which are proven in Subsection III B. The computational comparison is made in Subsection III C. Subsection III D shows closed explicit formulas in particular cases. Section III ends with some remarks. The case when $\mathfrak{g}$ is of type A_n is considered in Section IV.

II. NOTATION

Throughout this section, $\mathfrak{g}$ denotes a classical complex Lie algebra of type B_n, C_n, and D_n, namely, $\mathfrak{so}(2n+1,\mathbb{C})$, $\mathfrak{sp}(n,\mathbb{C})$, and $\mathfrak{so}(2n,\mathbb{C})$, respectively. We assume $n \geq 2$ for types B_n and C_n, and $n \geq 3$ for D_n. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Let $\{\varepsilon_1, \ldots, \varepsilon_n\}$ be the standard basis of $\mathfrak{h}^*$. Then, the sets of positive roots $\Sigma^+(\mathfrak{g},\mathfrak{h})$ and the space of integral weights $P(\mathfrak{g})$ are, respectively, given by

$$\begin{aligned}
&\{\varepsilon_i \pm \varepsilon_j : i<j\} \cup \{\varepsilon_i\} \text{ and } \left\{ \textstyle\sum_i a_i\varepsilon_i : a_i \in \mathbb{Z}\,\forall i, \text{ or } a_i - \frac{1}{2} \in \mathbb{Z}\,\forall i \right\} && \text{for } \mathfrak{g} \text{ of type } B_n,\\
&\{\varepsilon_i \pm \varepsilon_j : i<j\} \cup \{2\varepsilon_i\} \text{ and } \left\{ \textstyle\sum_i a_i\varepsilon_i : a_i \in \mathbb{Z}\,\forall i \right\} && \text{for } \mathfrak{g} \text{ of type } C_n,\\
&\{\varepsilon_i \pm \varepsilon_j : i<j\} \text{ and } \left\{ \textstyle\sum_i a_i\varepsilon_i : a_i \in \mathbb{Z}\,\forall i, \text{ or } a_i - \frac{1}{2} \in \mathbb{Z}\,\forall i \right\} && \text{for } \mathfrak{g} \text{ of type } D_n.
\end{aligned}$$

Furthermore, $\sum_i a_i\varepsilon_i \in P(\mathfrak{g})$ is dominant if and only if $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ for types B_n and C_n and $a_1 \geq \cdots \geq a_{n-1} \geq |a_n|$ for type D_n.

By the highest weight theorem, irreducible representations of $\mathfrak{g}$ are in correspondence with integral dominant weights. We denote by $P^{++}(\mathfrak{g})$ the set of integral dominant weights and by π_λ the irreducible representation of $\mathfrak{g}$ with highest weight $\lambda \in P^{++}(\mathfrak{g})$.

The first two fundamental weights are $\omega_1 = \varepsilon_1$ and $\omega_2 = \varepsilon_1 + \varepsilon_2$. Hence, any non-negative integer combination of them is of the form $k\varepsilon_1 + l\varepsilon_2$ for some integers $k \geq l \geq 0$.

The following notation is essential to read the weight multiplicity formulas in Sec. III. We will Q8 write $\mathbb{Z}^n$ for the set of elements $\sum_i a_i\varepsilon_i \in P(\mathfrak{g})$ such that $a_i \in \mathbb{Z}$ for all i. For a weight $\mu = \sum_{i=1}^n a_i\varepsilon_i \in \mathbb{Z}^n$

and t a non-negative integer, define

$$\|\mu\|_1 = \sum_{i=1}^{n} |a_i|, \qquad Z_t(\mu) = \#\{i : 1 \le i \le n,\ |a_i| = t\}. \tag{2}$$

Given $N \geq 0$, let $\mathcal{Q}_n(N)$ be the set of all partitions of N with length $\leq n$, that is,

$$\mathcal{Q}_n(N) = \left\{\mathbf{q} = (q_1, q_2, \dots, q_n) \in \mathbb{Z}^n : q_1 \geq q_2 \geq \dots \geq q_n \geq 0,\ \textstyle\sum_{i=1}^{n} q_i = N\right\}. \tag{3}$$

Furthermore, for $\mathbf{q} \in \mathcal{Q}_n(N)$ and $1 \leq j \leq N$, we set

$$s_j^{\mathbf{q}} := \#\{i : 1 \le i \le n,\ q_i = j\}, \tag{4}$$

$$\mathcal{B}^{\mathbf{q}} := \{\left(\beta_1^1, \beta_1^2, \beta_2^2, \beta_1^3, \beta_2^3, \beta_3^3, \dots, \beta_N^N\right) : \beta_t^j \geq 0,\ \textstyle\sum_{t=1}^{j} \beta_t^j \leq s_j^{\mathbf{q}}\}, \tag{5}$$

$$\mathcal{A}_\beta^{\mathbf{q}} := \{(\alpha_1^1, \alpha_1^2, \alpha_2^2, \alpha_1^3, \alpha_2^3, \alpha_3^3, \dots, \alpha_N^N) : 0 \leq \alpha_t^j \leq \beta_t^j\} \qquad \text{for any } \beta \in \mathcal{B}^{\mathbf{q}}. \tag{6}$$

Throughout the article, we use the convention $\binom{b}{a} = 0$ if $a < 0$ or $b < a$.

III. TYPES B$_n$, C$_n$, AND D$_n$

In this section, we consider $\mathfrak{g}$ a classical complex Lie algebra of types B$_n$, C$_n$, and D$_n$. We assume that $n \geq 2$ for types B$_n$ and C$_n$ and $n \geq 3$ for type D$_n$.

A. Main results

We now state the three theorems which establish the weight multiplicity formulas for types
Q9 B$_n$, C$_n$, and D$_n$, respectively. The notation required was introduced in Sec. II. The formulas consider weights in $\mathbb{Z}^n$, since the multiplicity in $\pi_{k\varepsilon_1 + l\varepsilon_2}$ of any weight in $P(\mathfrak{g}) \setminus \mathbb{Z}^n$ vanishes (see Remark III.4).

Theorem III.1 (Type B$_n$). *Let $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$ for some $n \geq 2$. Let $k \geq l \geq 0$ integers and $\mu \in \mathbb{Z}^n$. If $r(\mu) := (k + l - \|\mu\|_1)/2$ is negative, then $m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(\mu) = 0$, and otherwise*

$$\begin{aligned} m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(\mu) = {} & B_n(l, r(\mu), Z_0(\mu), \dots, Z_{l-1}(\mu)) \\ & - B_n(l-1, r(\mu), Z_0(\mu), \dots, Z_{l-1}(\mu)) \\ & - B_n(l-1, r(\mu)-1, Z_0(\mu), \dots, Z_{l-1}(\mu)) \\ & + B_n(l-2, r(\mu)-1, Z_0(\mu), \dots, Z_{l-1}(\mu)), \end{aligned}$$

where

$$\begin{aligned} B_n(l, r, Z_0, \dots, Z_{l-1}) = {} & \sum_{0 \leq N \leq l} \sum_{\mathbf{q} \in \mathcal{Q}_n(N)} \sum_{\beta \in \mathcal{B}^{\mathbf{q}}} \sum_{\alpha \in \mathcal{A}_\beta^{\mathbf{q}}} \binom{\lfloor (l-N)/2 \rfloor + n - 1}{n-1} \\ & \binom{\lfloor r - (l+N)/2 \rfloor + \sum_{j=1}^{N} \sum_{i=1}^{j} (j+1-i)\alpha_i^j + n - 1}{n-1} \\ & \prod_{j=1}^{N} \Bigg(2^{s_j^{\mathbf{q}} - \sum_{i=1}^{j} \beta_i^j} \binom{\beta_1^j}{\alpha_1^j} \binom{n - \sum_{t=0}^{j-1} Z_t - \sum_{r=j+1}^{N} \sum_{s=1}^{r-j+1} \beta_s^r}{\beta_1^j} \\ & \binom{Z_0 - \sum_{h=j+1}^{N} (s_h^{\mathbf{q}} - \sum_{s=1}^{h} \beta_s^h)}{s_j^{\mathbf{q}} - \sum_{t=1}^{j} \beta_t^j} \prod_{i=2}^{j} \binom{Z_{j-i+1} - \sum_{t=1}^{N-j} \beta_{i+1}^{j+t}}{\beta_i^j} \binom{\beta_i^j}{\alpha_i^j} \Bigg). \end{aligned}$$

Theorem III.2 (Type C_n). *Let $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$ for some $n \geq 2$. Let $k \geq l \geq 0$ integers and $\mu \in \mathbb{Z}^n$. If $r(\mu) := (k + l - \|\mu\|_1)/2$ is not a non-negative integer, then $m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(\mu) = 0$, and otherwise*

$$\begin{aligned} m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(\mu) = {} & C_n(l, r(\mu), Z_0(\mu), \dots, Z_{l-1}(\mu)) \\ & - C_n(l-1, r(\mu), Z_0(\mu), \dots, Z_{l-1}(\mu)) \\ & - C_n(l-1, r(\mu)-1, Z_0(\mu), \dots, Z_{l-1}(\mu)) \\ & + C_n(l-2, r(\mu)-1, Z_0(\mu), \dots, Z_{l-1}(\mu)), \end{aligned}$$

where

$$\begin{aligned} C_n(l, r, Z_0, \dots, Z_{l-1}) = {} & \sum_{\substack{0 \leq N \leq l, \\ N \equiv l \pmod 2}} \sum_{\mathbf{q} \in \mathcal{Q}_n(N)} \sum_{\beta \in \mathcal{B}^{\mathbf{q}}} \sum_{\alpha \in \mathcal{A}^{\mathbf{q}}_{\beta}} \binom{(l-N)/2 + n - 1}{n-1} \\ & \binom{r - (l+N)/2 + \sum_{j=1}^{N} \sum_{i=1}^{j} (j+1-i)\alpha_i^j + n - 1}{n-1} \\ & \prod_{j=1}^{N} \Bigg(2^{s_j^{\mathbf{q}} - \sum_{i=1}^{j} \beta_i^j} \binom{\beta_1^j}{\alpha_1^j} \binom{n - \sum_{t=0}^{j-1} Z_t - \sum_{r=j+1}^{N} \sum_{s=1}^{r-j+1} \beta_s^r}{\beta_1^j} \\ & \binom{Z_0 - \sum_{h=j+1}^{N} (s_h^{\mathbf{q}} - \sum_{s=1}^{h} \beta_s^h)}{s_j^{\mathbf{q}} - \sum_{t=1}^{j} \beta_t^j} \prod_{i=2}^{j} \binom{Z_{j-i+1} - \sum_{t=1}^{N-j} \beta_{i+1}^{j+t}}{\beta_i^j} \binom{\beta_i^j}{\alpha_i^j} \Bigg). \end{aligned}$$

Theorem III.3 (Type D_n). *Let $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ for some $n \geq 3$. Let $k \geq l \geq 0$ integers and $\mu \in \mathbb{Z}^n$. If $r(\mu) := (k + l - \|\mu\|_1)/2$ is not a non-negative integer, then $m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(\mu) = 0$, and otherwise*

$$\begin{aligned} m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(\mu) = {} & D_n(l, r(\mu), Z_0(\mu), \dots, Z_{l-1}(\mu)) \\ & - D_n(l-1, r(\mu), Z_0(\mu), \dots, Z_{l-1}(\mu)) \\ & - D_n(l-1, r(\mu)-1, Z_0(\mu), \dots, Z_{l-1}(\mu)) \\ & + D_n(l-2, r(\mu)-1, Z_0(\mu), \dots, Z_{l-1}(\mu)), \end{aligned}$$

where

$$\begin{aligned} D_n(l, r, Z_0, \dots, Z_{l-1}) = {} & \sum_{\substack{0 \leq N \leq l, \\ N \equiv l \pmod 2}} \sum_{\mathbf{q} \in \mathcal{Q}_n(N)} \sum_{\beta \in \mathcal{B}^{\mathbf{q}}} \sum_{\alpha \in \mathcal{A}^{\mathbf{q}}_{\beta}} \binom{(l-N)/2 + n - 2}{n-2} \\ & \binom{r - (l+N)/2 + \sum_{j=1}^{N} \sum_{i=1}^{j} (j+1-i)\alpha_i^j + n - 2}{n-2} \\ & \prod_{j=1}^{N} \Bigg(2^{s_j^{\mathbf{q}} - \sum_{i=1}^{j} \beta_i^j} \binom{\beta_1^j}{\alpha_1^j} \binom{n - \sum_{t=0}^{j-1} Z_t - \sum_{r=j+1}^{N} \sum_{s=1}^{r-j+1} \beta_s^r}{\beta_1^j} \\ & \binom{Z_0 - \sum_{h=j+1}^{N} (s_h^{\mathbf{q}} - \sum_{s=1}^{h} \beta_s^h)}{s_j^{\mathbf{q}} - \sum_{t=1}^{j} \beta_t^j} \prod_{i=2}^{j} \binom{Z_{j-i+1} - \sum_{t=1}^{N-j} \beta_{i+1}^{j+t}}{\beta_i^j} \binom{\beta_i^j}{\alpha_i^j} \Bigg). \end{aligned}$$

Remark III.4. For all types considered, it turns out that $m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(\mu) = 0$ for all $\mu \in P(\mathfrak{g}) \smallsetminus \mathbb{Z}^n$. Indeed, the subset $\mathbb{Z}^n$ coincides with the set of G-integral weights, where G is the only compact linear group G whose Lie algebra is a compact real form in $\mathfrak{g}$. We have that G is isomorphic to $\mathrm{SO}(2n+1)$, $\mathrm{Sp}(n)$, and $\mathrm{SO}(2n)$ for types B_n, C_n, and D_n, respectively. Since $k\varepsilon_1 + l\varepsilon_2 \in \mathbb{Z}^n$, the representation $\pi_{k\varepsilon_1+l\varepsilon_2}$ descends to a representation of G, and consequently, their weights are in $\mathbb{Z}^n$ (see Lemma 5.106 in Ref. 11).

B. Proofs

This subsection contains a unified proof of Theorems III.1–III.3. We first need two lemmas. The first one gives well-known closed explicit formulas for the weight multiplicities of representations having highest weights of the form $k\varepsilon_1$ for k a non-negative integer. A proof can be found in Lemmas 3.2, 4.3, and 5.3 in Ref. 16. The second lemma is the first step to prove the theorems.

Lemma III.5. Let $\mathfrak{g}$ be a complex Lie algebra of type B_n, C_n, or D_n for some $n \geq 2$. Let $k \geq 0$ integer and $\mu \in \mathbb{Z}^n$. Then

$$m_{\pi_{k\varepsilon_1}}(\mu) = \binom{\lfloor r(\mu)\rfloor + n-1}{n-1} \quad \textit{where } r(\mu) = \tfrac{k-\|\mu\|_1}{2}, \qquad \textit{for } \mathfrak{g} \textit{ of type } \mathrm{B}_n, \tag{7}$$

$$m_{\pi_{k\varepsilon_1}}(\mu) = \begin{cases} \binom{r(\mu)+n-1}{n-1} & \textit{if } r(\mu) := \frac{k-\|\mu\|_1}{2} \in \mathbb{N}_0, \\ 0 & \textit{otherwise}, \end{cases} \qquad \textit{for } \mathfrak{g} \textit{ of type } \mathrm{C}_n, \tag{8}$$

$$m_{\pi_{k\varepsilon_1}}(\mu) = \begin{cases} \binom{r(\mu)+n-2}{n-2} & \textit{if } r(\mu) := \frac{k-\|\mu\|_1}{2} \in \mathbb{N}_0, \\ 0 & \textit{otherwise}, \end{cases} \qquad \textit{for } \mathfrak{g} \textit{ of type } \mathrm{D}_n. \tag{9}$$

Lemma III.6. Let $\mathfrak{g}$ be a classical Lie algebra of type B_n, C_n, or D_n. For integers $k \geq l \geq 0$, write $\tau_{k,l} = \pi_{k\varepsilon_1} \otimes \pi_{l\varepsilon_1}$. Then, in the virtual ring of representations of $\mathfrak{g}$, we have that

$$\pi_{k\varepsilon_1 + l\varepsilon_2} \simeq \tau_{k,l} - \tau_{k+1,l-1} - \tau_{k-1,l-1} + \tau_{k,l-2}.$$

Proof. We have the fusion rule (see, for instance, page 510, Example 2 in Ref. 12)

$$\tau_{k,l} = \pi_{k\varepsilon_1} \otimes \pi_{l\varepsilon_1} \simeq \bigoplus_{j=0}^{l} \bigoplus_{i=0}^{j} \pi_{(k+l-j-i)\varepsilon_1 + (j-i)\varepsilon_2}.$$

As an immediate consequence, we obtain

$$\tau_{k,l} - \tau_{k+1,l-1} = \sum_{j=0}^{l}\sum_{i=0}^{j} \pi_{(k+l-j-i)\varepsilon_1+(j-i)\varepsilon_2} - \sum_{j=0}^{l-1}\sum_{i=0}^{j} \pi_{(k+l-j-i)\varepsilon_1+(j-i)\varepsilon_2} = \sum_{i=0}^{l} \pi_{(k-i)\varepsilon_1+(l-i)\varepsilon_2},$$

$$\begin{aligned}\tau_{k-1,l-1} - \tau_{k,l-2} &= \sum_{j=0}^{l-1}\sum_{i=0}^{j} \pi_{(k+l-j-i-2)\varepsilon_1+(j-i)\varepsilon_2} - \sum_{j=0}^{l-2}\sum_{i=0}^{j} \pi_{(k+l-j-i-2)\varepsilon_1+(j-i)\varepsilon_2} \\ &= \sum_{i=0}^{l-1} \pi_{(k+1-i)\varepsilon_1+(l-1-i)\varepsilon_2} = \sum_{i=1}^{l} \pi_{(k-i)\varepsilon_1+(l-i)\varepsilon_2}.\end{aligned}$$

Subtracting the previous identities, we obtain the desired formula. □

Proofs of Theorems III.1–III.3. Without loss of generality, we can assume that $\mu \in \mathbb{Z}^n$ is dominant since the Weyl group preserves weight multiplicities. Recall that $\tau_{k,l} = \pi_{k\varepsilon_1} \otimes \pi_{l\varepsilon_1}$ for $k \geq l \geq 0$ integers. Since

$$m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(\mu) = m_{\tau_{k,l}}(\mu) - m_{\tau_{k+1,l-1}}(\mu) - m_{\tau_{k-1,l-1}}(\mu) + m_{\tau_{k,l-2}}(\mu) \tag{10}$$

by Lemma III.6, we are left with the task of showing that

$$m_{\tau_{k,l}}(\mu) = \begin{cases} B_n(l, r(\mu), Z_0(\mu), \ldots, Z_{l-1}(\mu)) & \text{for } \mathfrak{g} \text{ of type } \mathrm{B}_n, \\ C_n(l, r(\mu), Z_0(\mu), \ldots, Z_{l-1}(\mu)) & \text{for } \mathfrak{g} \text{ of type } \mathrm{C}_n, \\ D_n(l, r(\mu), Z_0(\mu), \ldots, Z_{l-1}(\mu)) & \text{for } \mathfrak{g} \text{ of type } \mathrm{D}_n. \end{cases} \tag{11}$$

It is well known that (see Exercise V.14 in Ref. 11)

$$m_{\tau_{k,l}}(\mu) = \sum_{\eta} m_{\pi_{k\varepsilon_1}}(\mu - \eta)\, m_{\pi_{l\varepsilon_1}}(\eta), \tag{12}$$

where the sum is restricted to $\mathcal{P}(\pi_{l\varepsilon_1})$, the set of weights of $\pi_{l\varepsilon_1}$. By Lemma III.5, the weights of $\pi_{l\varepsilon_1}$ are those η such that $l - \|\eta\|_1 \in \mathbb{N}_0$ for type B_n and $l - \|\eta\|_1 \in 2\mathbb{N}_0$ for types C_n and D_n. In order to calculate $\|\mu - \eta\|_1$ and to determine $m_{k\varepsilon_1}(\mu - \eta)$, we make a convenient partition of $\mathcal{P}(\pi_{l\varepsilon_1})$.

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We write $Z_t = Z_t(\mu)$ for all $0 \le t \le l-1$. For $0 \le N \le l$ integer, $\mathbf{q} = (q_1, \ldots, q_n) \in \mathcal{Q}_n(N)$, $\beta = (\beta_1^1, \beta_1^2, \ldots, \beta_N^N) \in \mathcal{B}^{\mathbf{q}}$, and $\alpha = (\alpha_1^1, \alpha_1^2, \ldots, \alpha_N^N) \in \mathcal{A}_\beta^{\mathbf{q}}$ (see Sec. II for notation), we set

$$\mathcal{P}^{\mathbf{q}}_{\beta,\alpha} = \left\{ \sum_{j=1}^{N} \sum_{k=0}^{s_j^{\mathbf{q}}} b_{j,k}\varepsilon_{i_k^j} : \begin{array}{l} i_k^j \neq i_{s'}^{j'} \text{ if } (j,k)\neq(j',k');\ b_{j,k}=\pm j \text{ for all } k; \\ i_1^j < \cdots < i^j_{\beta_1^j} \le n - \sum_{t=0}^{j-1} Z_t < i^j_{\beta_1^j+1} < \cdots < i^j_{\beta_1^j+\beta_2^j} \\ \le n - \sum_{t=0}^{j-2} Z_t < \cdots \le n - Z_0 < i^j_{\sum_{i=1}^{j} \beta_i^j +1} < \cdots < i^j_{s_j^{\mathbf{q}}}, \\ \text{for all } 1\le j\le N, \\ \#\{k : \sum_{t=1}^{h-1}\beta_t^j + 1 \le k \le \sum_{t=1}^{h} \beta_t^j,\ b_{j,k}=j\} = \alpha_h^j \end{array}\right\}. \tag{13}$$

We now list some properties shared by all the elements in $\mathcal{P}^{\mathbf{q}}_{\beta,\alpha}$. Let $\eta = \sum_{i=1}^n c_i\varepsilon_i \in \mathcal{P}^{\mathbf{q}}_{\beta,\alpha}$. The multiset (i.e., a set where an element can be repeated) given by the elements $|c_1|, \ldots, |c_n|$ coincides with the multiset of elements $q_1, \ldots, q_n$; thus, $\|\eta\|_1 = N$. For a fixed $1 \le j \le N$, the number of entries equal to $\pm j$ is $s_j^{\mathbf{q}}$ located as follows: we divide the integral interval $[1, n]$ in $(j+1)$-blocks as the identity $n = (n - \sum_{t=0}^{j-1} Z_t) + Z_{j-1} + Z_{j-2} + \cdots + Z_1 + Z_0$ suggests; that is, the first block has the first $(n - \sum_{r=0}^{j-1} Z_r)$ integers, the second block has the next Z_{j-1} elements, the third one has the next Z_{j-2} elements, and so on. For each $1 \le t \le j$, there are β_t^j entries in the tth block equal to $\pm j$—α_t^j of them are positive. In the last block, there are $s_j^{\mathbf{q}} - \sum_{t=1}^{j-1} \beta_t^j$ entries equal to $\pm j$.

As a consequence of the previous paragraph, we have partitioned the set of weights of $\pi_{l\varepsilon_1}$ as

$$\mathcal{P}(\pi_{l\varepsilon_1}) = \bigcup_{N} \bigcup_{\mathbf{q}\in\mathcal{Q}_n(N)} \bigcup_{\beta\in\mathcal{B}^{\mathbf{q}}} \bigcup_{\alpha\in\mathcal{A}_\beta^{\mathbf{q}}} \mathcal{P}^{\mathbf{q}}_{\beta,\alpha}, \tag{14}$$

where the first union is over $N \in \mathbb{N}_0$ satisfying $l - N \in \mathbb{N}_0$ for type B_n and $l - N \in 2\mathbb{N}_0$ for types C_n and D_n. All the unions are disjoint.

Fix an integer $0 \le N \le l$, $\mathbf{q} \in \mathcal{Q}_n(N)$, $\beta \in \mathcal{B}^{\mathbf{q}}$, $\alpha \in \mathcal{A}_\beta^{\mathbf{q}}$, and $\eta \in \mathcal{P}^{\mathbf{q}}_{\beta,\alpha}$. One may check that

$$\begin{aligned} \|\mu - \eta\|_1 &= k + l - 2r + \sum_{j=1}^{N}\sum_{i=1}^{j} (j(\beta_i^j - \alpha_i^j) + (2i - j - 2)\alpha_i^j) + \sum_{j=1}^{N} j(s_j^{\mathbf{q}} - \sum_{i=1}^{j}\beta_i^j) \\ &= k + l - 2r + \sum_{j=1}^{N}\sum_{i=1}^{j} 2(i - j - 1)\alpha_i^j + j s_j^{\mathbf{q}} \\ &= k + l + N - 2\Big(r + \sum_{j=1}^{N}\sum_{i=1}^{j} (j + 1 - i)\alpha_i^j\Big) \\ &= k - 2\Big(r + \sum_{j=1}^{N}\sum_{i=1}^{j} (j + 1 - i)\alpha_i^j - (l + N)/2\Big). \end{aligned}$$

Since $m_{\pi_{l\varepsilon_1}}(\eta)$ and $m_{\pi_{k\varepsilon_1}}(\mu - \eta)$ are given in Lemma III.5 in terms of $l - \|\eta\|_1$ and $k - \|\mu - \eta\|_1$, respectively, it follows that $m_{\pi_{l\varepsilon_1}}(\eta)$ and $m_{\pi_{k\varepsilon_1}}(\mu - \eta)$ are constant, independent of the choice of $\eta \in \mathcal{P}^{\mathbf{q}}_{\beta,\alpha}$.

From the above fact, the partition (14), and the formula (12), we conclude that

$$m_{\tau_{k,l}}(\mu) = \sum_{N} \sum_{\mathbf{q}\in\mathcal{Q}_n(N)} \sum_{\beta\in\mathcal{B}^{\mathbf{q}}} \sum_{\alpha\in\mathcal{A}_\beta^{\mathbf{q}}} m_{\pi_{k\varepsilon_1}}(\mu - \eta^{\mathbf{q}}_{\beta,\alpha})\, m_{\pi_{l\varepsilon_1}}(\eta^{\mathbf{q}}_{\beta,\alpha})\, \#\mathcal{P}^{\mathbf{q}}_{\beta,\alpha}, \tag{15}$$

where $\eta^{\mathbf{q}}_{\beta,\alpha}$ is any element in $\mathcal{P}^{\mathbf{q}}_{\beta,\alpha}$, and the first sum is over $N \in \mathbb{N}_0$ satisfying $l - N \in \mathbb{N}_0$ for type B_n and $l - N \in 2\mathbb{N}_0$ for types C_n and D_n.

By tedious but straightforward combinatorial arguments, we have

$$
\begin{aligned}
\#\mathcal{P}^{\mathbf{q}}_{\beta,\alpha} &= 2^{\sum_{j=1}^{N}(s_j^{\mathbf{q}}-\sum_{i=1}^{j}\beta_i^j)}\binom{n-\sum_{j=0}^{N-1}Z_j}{\beta_1^N}\binom{Z_{N-1}}{\beta_2^N}\cdots\binom{Z_1}{\beta_N^N}\binom{Z_0}{s_N^{\mathbf{q}}-\sum_{j=1}^{N}\beta_j^N}\\
&\quad \binom{n-\sum_{j=0}^{N-2}Z_j-\beta_1^N-\beta_2^N}{\beta_1^{N-1}}\binom{Z_{N-2}-\beta_3^N}{\beta_2^{N-1}}\cdots\binom{Z_1-\beta_N^N}{\beta_{N-1}^{N-1}}\binom{Z_0-(s_N^{\mathbf{q}}-\sum_{j=1}^{N}\beta_j^N)}{s_{N-1}^{\mathbf{q}}-\sum_{j=1}^{N-1}\beta_j^{N-1}}\\
&\quad \cdots\binom{n-Z_0-\sum_{j=2}^{N}\sum_{i=1}^{j}\beta_i^j}{\beta_1^1}\binom{Z_0-\sum_{j=2}^{N}(s_j^{\mathbf{q}}-\sum_{i=1}^{j}\beta_i^j)}{s_1^{\mathbf{q}}-\beta_1^1}\binom{\beta_1^1}{\alpha_1^1}\binom{\beta_1^2}{\alpha_1^2}\cdots\binom{\beta_N^N}{\alpha_N^N}\\
&= \prod_{j=1}^{N}\Bigg(2^{s_j^{\mathbf{q}}-\sum_{i=1}^{j}\beta_i^j}\binom{n-\sum_{t=0}^{j-1}Z_t-\sum_{r=j+1}^{N}\sum_{s=1}^{r-j+1}\beta_s^r}{\beta_1^j}\\
&\quad \binom{Z_0-\sum_{r=j+1}^{N}(s_r^{\mathbf{q}}-\sum_{s=1}^{r}\beta_s^r)}{s_j^{\mathbf{q}}-\sum_{t=1}^{j}\beta_t^j}\binom{\beta_1^j}{\alpha_1^j}\prod_{i=2}^{j}\binom{Z_{j-i+1}-\sum_{t=1}^{N-j}\beta_{i+1}^{j+t}}{\beta_i^j}\binom{\beta_i^j}{\alpha_i^j}\Bigg).
\end{aligned}
$$

Replacing in (15) the values of $m_{\pi_{k\varepsilon_1}}(\mu-\eta^{\mathbf{q}}_{\beta,\alpha})$ and $m_{\pi_{l\varepsilon_1}}(\eta^{\mathbf{q}}_{\beta,\alpha})$ given by Lemma III.5 and $\#\mathcal{P}^{\mathbf{q}}_{\beta,\alpha}$ by the above expression, we obtain the desired weight multiplicity formula for $\tau_{k,l}$. According to (10), the Proofs of Theorems III.1–III.3 are complete. □

C. Computational comparison

We now include a non-serious computational comparison between the weight multiplicity formulas in Theorems III.1–III.3 and Freudenthal's formula (see, for instance, Sec. 22.3 in Ref. 10). We use the open-source mathematical software `Sage`[19] and its algebraic combinatorics features developed by the `Sage-Combinat` community,[20] which has implemented Freudenthal's formulas. The source code containing the bivariate algorithm can be found in the public project[18] available in `CoCalc`. (To see the corresponding hyperlink go to the electronic version of this article.)

The word "non-serious" in the previous paragraph has been added for several reasons that we now explain. The formulas proved above have been implemented in `Sage` by the first named author, who lacks computer programming skills. Thus, their implementations are done poorly and inefficiently. On the contrary, the `Sage-Combinat` community programmed Freudenthal's formula in `Sage` in a very efficient way. Furthermore, the calculations have been made using an old version of `Sage`[19] and a slow computer.

The implementation of Freudenthal's formula in `Sage`, called *Freudenthal algorithm* in the sequel, returns all the weights with their corresponding multiplicities. We suspect that this tactic is due to a matter of efficiency since Freudenthal's formula is defined recursively. On the other hand, Theorems III.1–III.3 compute the multiplicity of a single weight. Thus, in order to make a fair comparison between them, the bivariate algorithm will also determine the set of weights of $\pi_{k\varepsilon_1+l\varepsilon_2}$. To this end, we first find a subset of $\mathbb{Z}^n$ containing the set of weights of $\pi_{k\varepsilon_1+l\varepsilon_2}$, namely, $\{\mu\in\mathbb{Z}^n : \|\mu\|_1 \le k+l\}$. Here is a summary of the algorithm.

Algorithm III.7 (Bivariate algorithm).

INPUT: $\mathfrak{g}$ a classical complex Lie algebra of type B_n or C_n with $n \geq 2$, or D_n with $n \geq 3$, and $k \geq l$ non-negative integers.

OUTPUT: the sequence of pairs $[\mu, m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(\mu)]$, where μ runs over every weight of the representation $\pi_{k\varepsilon_1+l\varepsilon_2}$ of $\mathfrak{g}$ and $m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(\mu)$ is its multiplicity.

1. Initialize S as an empty list.
2. Determine the set P of vectors $\mu=(a_1,\dots,a_n)\in\mathbb{Z}^n$ such that $\|\mu\|_1 \leq k+l$ and $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$.
3. Run over all elements μ in P.
4. Compute $m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(\mu)$ by Theorems III.1–III.3.

5. In case $m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(\mu) > 0$, determine the orbit of μ by the group $W_n \simeq \mathrm{Sym}(n) \times \{\pm 1\}^n$, which acts by permutations and multiplication by ± 1 on its entries.
6. For each ν in the above orbit, add in S the entry $[\nu, m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(\mu)]$.
7. Return S.

Remark III.8. Notice that the set of dominant weights for $\pi_{k\varepsilon_1+l\varepsilon_2}$ is included in P introduced in (ii) when $\mathfrak{g}$ is of types B_n and C_n. Although this fact is not true for $\mathfrak{g}$ of type D_n, each remaining element has the form $\bar{\mu} := (a_1, \ldots, a_{n-1}, -a_n)$ for some $\mu = (a_1, \ldots, a_n)$ in P with $a_n > 0$, and it satisfies $m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(\bar{\mu}) = m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(\mu)$ since $n \geq 3$. Consequently, step (v) obtains all the weights of $\pi_{k\varepsilon_1+l\varepsilon_2}$ when $\mathfrak{g}$ is of type B_n and C_n for $n \geq 2$ and D_n for $n \geq 3$. Likewise, the group W_n introduced in (v) coincides with the Weyl group when $\mathfrak{g}$ is of type B_n or C_n. For $\mathfrak{g}$ of type D_n and $n \geq 3$, the Weyl group is isomorphic to $\mathrm{Sym}(n) \times \{\pm 1\}^{n-1}$; thus, it is strictly included in W_n. This fact is consistent with the previous comment on the set of dominant weights that is not contained in P. Q11 Q12 Q13

Table I displays the times (in seconds) required by both the bivariate and Freudenthal algorithms for different choices of n, k, and l. Let us introduce the notation $B(\mathrm{X}_n, k, l)$ for the time required by our implementation in `Sage` of the bivariate algorithm for $\mathfrak{g}$ of type X_n (=B_n, C_n, or D_n) and the irreducible representation of $\mathfrak{g}$ having highest weight $k\varepsilon_1 + l\varepsilon_2$. Similarly, write $F(\mathrm{X}_n, k, l)$ for the corresponding required time for the implementation in `Sage` of Freudenthal algorithm. This abuse of notation (the numbers are periods of time not uniquely determined) will be advantageous to express the numerical conclusions.

We now indicate some conclusions evidenced by the numerical experiments. It is clear that $B(\mathrm{X}_n, k, l)$ is much smaller than $F(\mathrm{X}_n, k, l)$ for coherent small values of n, k, and l. Furthermore, the function $n \mapsto B(\mathrm{X}_n, k, l)/F(\mathrm{X}_n, k, l)$ seems to be increasing for any fixed choice of X, k, and l. Moreover, for n big enough, one would have $B(\mathrm{X}_n, k, l) < F(\mathrm{X}_n, k, l)$.

On the one hand, we see that $F(\mathrm{D}_n, k, l) < F(\mathrm{C}_n, k, l) < F(\mathrm{B}_n, k, l)$ and the gaps among them increase when n grows. The reason is that Freudenthal's formula depends heavily on the root system associated with $\mathfrak{g}$, which is simpler for type D_n and more complicated for type B_n. On the other hand, $B(\mathrm{C}_n, k, l)$ and $B(\mathrm{D}_n, k, l)$ look similar and $B(\mathrm{B}_n, k, l)$ larger. In this case, the reason is the number of weights. Roughly speaking, the set of weights of $\pi_{k\varepsilon_1+l\varepsilon_2}$ is almost equal to $\{\mu \in \mathbb{Z}^n : \|\mu\|_1 \leq k + l, \|\mu\|_1 \equiv k + l \pmod 2\}$ for types C_n and D_n and to $\{\mu \in \mathbb{Z}^n : \|\mu\|_1 \leq k + l\}$ for type B_n. In fact, this is a consequence of $\|\alpha\|_1 = 2$ for every root α in types C_n and D_n and $\|\alpha\|_1 \in \{1, 2\}$ for every root α in type B_n. Summing up, the bivariate algorithm is not sensible to the number of roots in the corresponding root system, but it is sensible to the one-norm of the roots.

Throughout this paragraph fix a type X_n. The times required by both algorithms depend on $k + l$. In fact, the set of weights of $\pi_{k\varepsilon_1+l\varepsilon_2}$ does not vary considerably among the different choices of k and l with $k + l$ fixed. Likewise, Freudenthal's formula is slightly faster when l grows since the size of the set of weights decreases. However, the bivariate algorithm strongly depends on l. Indeed, as this algorithm involves partitions of all non-negative integers less than or equal to l, its speed reduces when l increases. In conclusion, fixing the value $m = k + l$, the function $l \mapsto B(\mathrm{X}_n, m - l, l)/F(\mathrm{X}_n, m - l, l)$ attains its minimum when l is as large as possible, that is, when $l = k$ or $l = k - 1$ according to the parity of $k + l$. This situation is exemplified in Table II.

The authors believe that the weight multiplicity formulas in Theorems III.1–III.3 could be implemented on new versions of `Sage`. Bivariate representations are a non-trivial class of irreducible

TABLE II. Comparison among representations of $\mathfrak{g}$ of type D_4 with $k + l = 14$ fixed.

l	0	1	2	3	4	5	6	7
$B(\mathrm{D}_4, 14 - l, l)$	1.03	1.04	1.09	1.69	3.28	7.45	17.50	39.99
$F(\mathrm{D}_4, 14 - l, l)$	152.84	152.77	152.60	151.50	146.38	137.41	124.40	106.22

representations, which frequently appear on users' calculations. Not only the time required by the bivariate algorithm for n large enough is reduced, but there is also a great advantage in the possibility of calculating the multiplicity of a single weight in a very short period of time. For example, when $\mathfrak{g}$ is of type D_5, $k = 20$, and $l = 6$, its implemented program in `Sage` takes only between 0.40 and 0.65 s for each single weight μ. Furthermore, the efficiency of the algorithm improves significantly when it returns only the multiplicities of dominant weights [i.e., step (v) is omitted in Algorithm III.7], which is in general what users really need. This can be appreciated in the fourth column of Table I, denoted by $D_n(\star)$. There, we list the times required by this simplified version of the bivariate algorithm for $\mathfrak{g}$ of type D_n. Of course, the fact that the bivariate algorithm works only for particular simple complex Lie algebras and bivariate representations is a big disadvantage.

D. Closed explicit weight formulas in particular cases

The weight multiplicity formulas obtained in Theorems III.1–III.3 are not closed expressions because they involve a sum over partitions of non-negative integers. However, in some particular cases (e.g., small values of l, particular choices of μ), it is possible to write out the partitions, and therefore, the formulas become closed expressions. For example, if $l = 0$, then the formulas reduce to the closed explicit expressions in Lemma III.5.

When $l = 1$, only sums over the set of partitions of 0 or 1 are involved. These sets have exactly one element, so the sums disappear. For example, when $\mathfrak{g}$ is of type D_n, we get

$$\begin{aligned} m_{\pi_{k\varepsilon_1+\varepsilon_2}}(\mu) &= D_n(1, r(\mu), Z_0(\mu)) - D_n(0, r(\mu)) - D_n(0, r(\mu)-1) \\ &= \sum_{\beta_1^1=0}^{1} \sum_{\alpha_1^1=0}^{1-\beta_1^1} \binom{r-1+\alpha_1^1+n-2}{n-2} 2^{1-\beta_1^1} \binom{\beta_1^1}{\alpha_1^1} \binom{n-Z_0(\mu)}{\beta_1^1} \binom{Z_0(\mu)}{1-\beta_1^1} \\ &\quad - \binom{r(\mu)+n-2}{n-2} - \binom{r(\mu)-1+n-2}{n-2} \end{aligned} \tag{16}$$

for every $\mu \in \mathbb{Z}^n$ satisfying that $r(\mu) = (k+1-\|\mu\|_1)/2$ is a non-negative integer. Notice that this formula is a particular case of Theorem 4.1 in Ref. 16.

Similarly, when $l = 2$, there are only sums over the set of partitions of N for $N = 0, 1, 2$. Since $2 = 2$ and $2 = 1 + 1$ are the only partitions of 2, the corresponding sum splits into two. We now state the multiplicity formula for $l = 2$ and type D_n. We pick type D_n for citing purposes.

Corollary III.9. Let $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ *for* $n \geq 3$, *let* $k \geq 2$ *integer, and let* $\mu \in \mathbb{Z}^n$. *If* $r(\mu) := (k+2-\|\mu\|_1)/2$ *is a non-negative integer, then*

$$\begin{aligned} m_{\pi_{k\varepsilon_1+2\varepsilon_2}}(\mu) &= \binom{r(\mu)+n-4}{n-2}\left(2Z_0(\mu)(n-1) + \binom{n-Z_0(\mu)}{2}\right) \\ &\quad + \binom{r(\mu)+n-3}{n-2}\left(2Z_0(\mu)(n-Z_0(\mu)) + Z_1(\mu) - n + 2\binom{n-Z_0(\mu)}{2}\right) \\ &\quad + \binom{r(\mu)+n-2}{n-2}\left(\binom{n-Z_0(\mu)}{2} - Z_1(\mu)\right), \end{aligned}$$

and $m_{\pi_{k\varepsilon_1+2\varepsilon_2}}(\mu) = 0$ *otherwise.*

Furthermore, we can obtain a closed explicit multiplicity formula for the weight $\mu = 0$ in the representation $\pi_{k\varepsilon_1+l\varepsilon_2}$ of $\mathfrak{g}$. We next state the formulas for types B_n, C_n, and D_n, but we prove it only for the case D_n, since types B_n and C_n follow in a similar way.

Corollary III.10 (Type D_n). *Let $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ for some $n \geq 3$ and let $k \geq l \geq 0$ integers. We have that $m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(0) = 0$ if $k + l$ is odd. Moreover, if $k + l$ is even, then*

$$m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(0) = 2 \sum_{0\leq N\leq l} (-1)^{N+l} R(n,k,l,N) \binom{\lfloor (l-N)/2\rfloor + n-2}{n-2} \binom{\lfloor (k-N+1)/2\rfloor + n-2}{n-2} \sum_{t=0}^{n} \binom{n}{t}\binom{N-t+n-1}{n-1},$$

where

$$R(n,k,l,N) = \begin{cases} \dfrac{l-N+n-2}{l-N+2n-4} & \text{if } N \equiv l \pmod 2, \\ \dfrac{k+1-N+n-2}{k+1-N+2n-4} & \text{if } N \equiv l+1 \pmod 2. \end{cases}$$

Proof. The asserted formula can be obtained by Theorem III.3. However, we will prove it in a simplified way, by following the Proof of Theorem III.3. The reason is that the partition in (14) of the set of weights of $\pi_{l\varepsilon_1}$ is (unnecessarily) too fine for $\mu = 0$. By (10), we have that

$$m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(0) = m_{\tau_{k,l}}(0) - m_{\tau_{k+1,l-1}}(0) - m_{\tau_{k-1,l-1}}(0) + m_{\tau_{k,l-2}}(0). \tag{17}$$

As before, for arbitrary $k \geq l \geq 0$ integers, it holds

$$m_{\tau_{k,l}}(0) = \sum_{\eta} m_{\pi_{k\varepsilon_1}}(-\eta)\, m_{\pi_{l\varepsilon_1}}(\eta),$$

where the sum is restricted to the weights of $\pi_{l\varepsilon_1}$. From Lemma III.5, we see that η is a weight of $\pi_{l\varepsilon_1}$ if and only if $l - \|\eta\|_1 \in 2\mathbb{N}_0$. For such a weight η, $m_{\pi_{k\varepsilon_1}}(-\eta) = 0$ unless $2\mathbb{N}_0 \ni k - \|-\eta\|_1 = (k-l) + (l - \|\eta\|_1)$, equivalently $k - l \in 2\mathbb{N}_0$. We conclude that $m_{\tau_{k,l}}(0) = 0$ if $k + l$ is odd. Moreover, $m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(0) = 0$ if $k + l$ is odd by (17).

We now proceed to compute $m_{\tau_{k,l}}(0)$ for arbitrary $k \geq l \geq 0$ integers satisfying that $k + l$ is even. Fix $N \in \mathbb{N}_0$ such that $l - N \in 2\mathbb{N}_0$. For each $\eta \in \mathbb{Z}^n$ with $\|\eta\|_1 = N$, we know that $m_{\pi_{k\varepsilon_1}}(-\eta)$ and $m_{\pi_{l\varepsilon_1}}(\eta)$ are constant, independent of the choice of η. Hence,

$$m_{\tau_{k,l}}(0) = \sum_{\substack{0\leq N\leq l,\\ N\equiv l \pmod 2}} \binom{(l-N)/2+n-2}{n-2}\binom{(k-N)/2+n-2}{n-2} \#\{\eta\in\mathbb{Z}^n : \|\eta\|_1 = N\}. \tag{18}$$

It is well known (see, for instance, Sec. 2.5 in Ref. 4) that $\#\{\eta\in\mathbb{Z}^n : \|\eta\|_1 = N\} = \sum_{t=0}^{n}\binom{n}{t}\binom{N-t+n-1}{n-1}$. Thus, by replacing (18) in (17), one obtains the desired formula. □

Corollary III.11 (Type C_n). *Let $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$ for some $n \geq 2$ and let $k \geq l \geq 0$ integers. We have that $m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(0) = 0$ if $k + l$ is odd. Moreover, if $k + l$ is even, then*

$$m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(0) = 2 \sum_{0\leq N\leq l} (-1)^{N+l} R(n+1,k,l,N) \binom{\lfloor (l-N)/2\rfloor + n-1}{n-1} \binom{\lfloor (k-N+1)/2\rfloor + n-1}{n-1} \sum_{t=0}^{n} \binom{n}{t}\binom{N-t+n-1}{n-1},$$

where $R(n, k, l, N)$ is as in Corollary III.10.

Corollary III.12 (Type B_n). *Let $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$ for some $n \geq 2$ and let $k \geq l \geq 0$ integers. Then*

$$m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(0) = \sum_{0\leq N\leq l} (-1)^{N+l} S(n,k,l,N) \binom{\lfloor (l-N)/2\rfloor + n-1}{n-1} \binom{\lfloor (k+1-N)/2\rfloor + n-1}{n-1} \sum_{t=0}^{n} \binom{n}{t}\binom{N-t+n-1}{n-1},$$

where

$$S(n,k,l,N)=\begin{cases} 1-\dfrac{\lfloor (l-N)/2\rfloor \lfloor (k+1-N)/2\rfloor}{(\lfloor (l-N)/2\rfloor+n-1)(\lfloor (k+1-N)/2\rfloor+n-1)} & \textit{if } k+l \textit{ is even},\\ \dfrac{\lfloor (k+1-N)/2\rfloor}{\lfloor (k+1-N)/2\rfloor+n-1}-\dfrac{\lfloor (l-N)/2\rfloor}{\lfloor (l-N)/2\rfloor+n-1} & \textit{if } k+l \textit{ is odd}. \end{cases}$$

E. Remarks

We end this section with a few remarks.

Remark III.13 .The weight multiplicity formula for type D_n in Theorem III.3 also holds when $n = 2$ with $\pi_{k\varepsilon_1+l\varepsilon_2}$ replaced by $\pi_{k\varepsilon_1+l\varepsilon_2} \oplus \pi_{k\varepsilon_1-l\varepsilon_2}$. It is important to note that $\mathfrak{so}(4,\mathbb{C})$ (type D_2) is not simple. Indeed, $\mathfrak{so}(4,\mathbb{C}) \simeq \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ or $D_2 = A_1 \oplus A_1$. Hence, a weight multiplicity formula for the representations $\pi_{k\varepsilon_1\pm l\varepsilon_2}$ with $k \ge l \ge 0$ of $\mathfrak{so}(4,\mathbb{C})$ can be obtained using this fact.

Remark III.14. Maddox[17] determined a weight multiplicity formula for any bivariate representation for $\mathfrak{g}$ of type C_n. Her expression (Theorem 4.3 in Ref. 17) looks more elegant than the one in Theorem III.2. However, it includes a sum over ordered partitions of $r(\mu)$ of length n and another sum over the subsets of a set of $2n$ elements. In conclusion, her shorter formula hides in the mentioned sums the involved terms appearing in the expression given in Theorem III.2. Furthermore, the neat dependence condition in Theorem I.1 does not follow immediately from Theorem 4.3 in Ref. 17.

We now compare Maddox's method with ours. Both employ the expression in Lemma III.6 for an irreducible representation as a sum of tensor products in the virtual ring of representations. The significant difference arises in the calculation of the weight multiplicity in a tensor product. Roughly speaking, the Proofs of Theorems III.1–III.3 use the identity (12) and then a convenient partition of the set of weights of the small component in the tensor product. On the other hand, Maddox makes use of $\tau_{k,l} = \pi_{k\varepsilon_1} \otimes \pi_{l\varepsilon_1} \simeq \mathrm{Sym}^k(\mathbb{C}^{2n}) \otimes \mathrm{Sym}^l(\mathbb{C}^{2n})$ for $\mathfrak{g}$ of type C_n and counts the weight vectors in terms of a function which has a combinatorial expression.

Remark III.15. There are in the literature several algorithms to compute weight multiplicities. The one based on Freudenthal's formula is the most classical and is still used for several computer programs (e.g., `Sage`[19]). Nowadays, there exist faster algorithms. A possible time comparison with any of them would require an implementation on `Sage`, which would be unfair because of the poor computer programming skills of the authors.

Q14 Among the mentioned faster algorithms, it is the distinguished one by Baldoni and Vergne[3] (see Refs. 1, 2, and 8 for related results), which is based on symbolic computations of Kostant partition functions. See also Refs. 6, 7, and 21 for recent different approaches.

Remark III.16. This interesting remark about the behavior of $m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(\mu)$ as a function on k and l was pointed out by the referee. For simplicity, we take $\mu = 0$, we fix l a non-negative integer and we consider $\mathfrak{g}$ a classical Lie algebra of type D_n for some $n \ge 3$, although the general case is very similar. Corollary III.10 implies that $k \mapsto m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(0)$ is a quasi-polynomial in the variable $k \ge l$ whose degree does not depend on l. In fact, its degree coincides with the degree of the polynomial $k \mapsto m_{\pi_{2k\varepsilon_1}}(0) = \binom{k+n-2}{n-2}$ (i.e., when $l = 0$), which is equal to $n-2$.

An interesting problem, also suggested by the referee, is to understand the behavior of the function $l \mapsto m_{\pi_{(l+h)\varepsilon_1+l\varepsilon_2}}(0)$, for some h fixed. This does not seem to be computable from Corollary III.10.

Remark III.17. In Sec. 7 of Ref. 16, there is a detailed account of some applications of weight multiplicity formulas in spectral geometry (see Refs. 5, 13–5). These expressions for the weight multiplicities are used to determine explicitly the spectra of certain natural differential operators on a manifold (or a good orbifold) of the form $\Gamma\backslash G/K$, where G is a semisimple compact Lie group, K is a closed subgroup of G, and Γ is a finite subgroup of the maximal torus T of G.

We next specify some cases where the formulas obtained in this article could be applied. When $G = \mathrm{Sp}(n)$ and $K = \mathrm{Sp}(n-1) \times \mathrm{Sp}(1)$, the spherical representations associated with the Gelfand pair (G, K) (i.e., the set of irreducible representations of G containing non-zero vectors fixed by K) have highest weight of the form $k(\varepsilon_1 + \varepsilon_2)$ for $k \geq 0$. Consequently, Theorem III.2 may be applied to determine the spectrum of the Laplace–Beltrami operator acting on functions on spaces covered by the n-dimensional quaternionic projective space $\mathrm{Sp}(n)/\mathrm{Sp}(n-1) \times \mathrm{Sp}(1)$ with the Abelian fundamental group.

When $G = \mathrm{SO}(m)$ and $K = \mathrm{SO}(m-2) \times \mathrm{SO}(2)$, the corresponding spherical representations for (G, K) have highest weight of the form $k\varepsilon_1 + l\varepsilon_2$ for $k \geq l \geq 0$. Thus, according to m is odd or even, Theorem III.1 or III.3 could be applied to the same purpose as above, for spaces covered by the 2-Grassmannian space G/K with the Abelian fundamental group.

In a slightly different way, we now consider $n \geq 3$, $G = \mathrm{SO}(2n)$, $K = \mathrm{SO}(2n-1)$, and more general natural differential operators. An irreducible representation τ of K induces a natural G-homogeneous complex vector bundle E_τ on G/K. There is an associated natural differential operator Δ_τ acting on smooth sections of E_τ, which induces the differential operator $\Delta_{\tau,\Gamma}$ acting on smooth sections of $\Gamma\backslash E_\tau$, that is, Γ-invariant smooth sections of E_τ. We now fix $\tau = \tau_{b\varepsilon_1}$, the irreducible representation of K with highest weight $b\varepsilon_1$. The corresponding $\tau_{b\varepsilon_1}$-spherical representations of $(G, K, \tau_{b\varepsilon_1})$ (i.e., the set of $\pi \in \widehat{G}$ such that $\mathrm{Hom}_K(\tau_{b\varepsilon_1}, \pi|_K) \neq 0$) is equal to $\{\pi_{k\varepsilon_1 + l\varepsilon_2} : k \geq b \geq l \geq 0\}$. Consequently, Theorem III.3 might be used to determine the spectrum of $\Delta_{\tau_{b\varepsilon_1},\Gamma}$ for Γ a finite subgroup of the maximal torus of G. An analogous process can be done in the case $G = \mathrm{SO}(2n-1)$ and $K = \mathrm{SO}(2n-2)$.

IV. TYPE A_n

Consider $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$ the Cartan subalgebra

$$\mathfrak{h} = \{\mathrm{diag}(\theta_1, \ldots, \theta_{n+1}) : \theta_i \in \mathbb{C}\ \forall\, i,\ \sum_{i=1}^{n+1} \theta_i = 0\}.$$

Set $\varepsilon_i(\mathrm{diag}(\theta_1, \ldots, \theta_{n+1})) = \theta_i$ for each $1 \leq i \leq n+1$. We will use the conventions of Lecture 15 in Ref. 9; that is, we correspondingly write

$$\mathfrak{h}^* = \bigoplus_{i=1}^{n+1} \mathbb{C}\varepsilon_i / \langle \sum_{i=1}^{n+1} \varepsilon_i = 0 \rangle,$$

and we write ε_i for its image in $\mathfrak{h}^*$. Consequently, the set of positive roots is given by $\Sigma^+(\mathfrak{g}, \mathfrak{h}) := \{\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq n+1\}$, and the weight lattice is $P(\mathfrak{g}) := \bigoplus_{i=1}^{n+1} \mathbb{Z}\varepsilon_i / \langle \sum_{i=1}^{n+1} \varepsilon_i = 0 \rangle$. Two weights $\mu = \sum_{i=1}^{n+1} b_i \varepsilon_i$ and $\nu = \sum_{i=1}^{n+1} c_i \varepsilon_i$ in $P(\mathfrak{g})$ coincide if and only if $b_i - c_i$ is constant, independent of i.

A weight $\lambda = \sum_{i=1}^{n+1} a_i \varepsilon_i$ in $P(\mathfrak{g})$ is dominant if and only if $a_1 \geq a_2 \geq \cdots \geq a_{n+1}$. By the highest weight theorem, the irreducible representations of $\mathfrak{g}$ are in correspondence with dominant weights. We denote by π_λ the irreducible representation with highest weight λ, which will be always written as $\lambda = \sum_{i=1}^{n+1} a_i \varepsilon_i$ with $a_{n+1} = 0$. Thus, the irreducible representations of G are in correspondence with elements in the set

$$P^{++}(\mathfrak{g}) := \left\{ \sum_{i=1}^{n} a_i \varepsilon_i : a_i \in \mathbb{Z}\ \forall\, i,\ a_1 \geq a_2 \geq \cdots \geq a_n \geq 0 \right\}.$$

The fundamental weights are given by $\omega_p = \varepsilon_1 + \cdots + \varepsilon_p$ for each $1 \leq p \leq n$. Thus, any integer combination between ω_1 and ω_2 has the form $k\varepsilon_1 + l\varepsilon_2$ for some integers $k \geq l \geq 0$.

For $\lambda = \sum_{i=1}^{n} a_i \varepsilon_i \in P^{++}(\mathfrak{g})$, any weight μ of π_λ (i.e., the multiplicity of μ in π_λ is non-zero) can be written as $\mu = \sum_{i=1}^{n+1} b_i \varepsilon_i$ for some $b_1, \ldots, b_{n+1} \in \mathbb{N}_0$ satisfying $\sum_{i=1}^{n+1} b_i = \sum_{i=1}^{n} a_i$. Indeed, every weight in π_λ is a difference between λ and a sum of positive roots.

Let λ and μ be as in the previous paragraph. It is well known that [see, for instance, (A.19) in Ref. 9] the multiplicity of μ in π_λ is given by the *Kostka number* $K_{\lambda,\mu}$: the number of *semistandard* Q15 *tableaux* on the Young diagram associated with λ (i.e., a diagram with a_i boxes in the ith row, with the rows of boxes lined up on the left) of type μ. More precisely, $K_{\lambda,\mu}$ is the number of ways one can fill the boxes of the Young diagram associated with λ with b_1 1's, b_2 2's, up to b_{n+1} $(n+1)$'s, in such a way that the entries in each row are non-decreasing, and those in each column are strictly increasing.

The next lemma will be needed in the proof of the main result of this section.

Lemma IV.1. Let $\mathfrak{g}$ be a classical Lie algebra of type A_n. For integers $k \geq l \geq 0$, write $\tau_{k,l} = \pi_{k\varepsilon_1} \otimes \pi_{l\varepsilon_1}$. Then, in the virtual ring of representations, we have that

$$\pi_{k\varepsilon_1+l\varepsilon_2} \simeq \tau_{k,l} - \tau_{k+1,l-1}.$$

Proof. The well-known fusion rule (see, for instance, Proposition 15.25 in Ref. 9)

$$\tau_{k,l} = \pi_{k\varepsilon_1} \otimes \pi_{l\varepsilon_1} \simeq \bigoplus_{p=0}^{l} \pi_{(k+p)\varepsilon_1+(l-p)\varepsilon_2}$$

implies

$$\tau_{k,l} - \tau_{k+1,l-1} = \sum_{p=0}^{l} \pi_{(k+p)\varepsilon_1+(l-p)\varepsilon_2} - \sum_{p=1}^{l} \pi_{(k+p)\varepsilon_1+(l-p)\varepsilon_2} = \pi_{k\varepsilon_1+l\varepsilon_2},$$

and the lemma follows. □

Q16 We now want to calculate the weight multiplicities of the representation with highest weight a non-negative integer combination of the first two fundamental weights. The following multiplicity formula is probably already known, but it is included here for completeness.

Theorem IV.2 (Type A_n). *Let $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{C})$ for some $n \geq 2$ and let $k \geq l \geq 0$ integers. Let $\mu = \sum_{i=1}^{n+1} a_i\varepsilon_i \in P(\mathfrak{g})$ with $a_i \in \mathbb{N}_0$ for all i and $\sum_{i=1}^{n+1} a_i = k + l$. If $a_i \leq k$ for all i, then*

$$m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(\mu) = \sum_{\mathbf{q}\in\mathcal{Q}_{n+1}(l)} \prod_{j=1}^{l} \binom{n+1-\sum_{t=0}^{j-1} Z_t(\mu) - \sum_{i=j+1}^{l} s_i^{\mathbf{q}}}{s_j^{\mathbf{q}}} - \sum_{\mathbf{q}'\in\mathcal{Q}_{n+1}(l-1)} \prod_{j=1}^{l-1} \binom{n+1-\sum_{t=0}^{j-1} Z_t(\mu) - \sum_{i=j+1}^{l-1} s_i^{\mathbf{q}'}}{s_j^{\mathbf{q}'}},$$

and $m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(\mu) = 0$ otherwise, where $Z_t(\mu) = \#\{i\colon 1 \leq i \leq n+1,\ a_i = t\}$,

$$\mathcal{Q}_{n+1}(N) = \{\mathbf{q} = (q_1, q_2, \ldots, q_{n+1}) \in \mathbb{Z}^{n+1} : q_1 \geq q_2 \geq \cdots \geq q_{n+1} \geq 0,\ \sum_{i=1}^{n+1} q_i = N\},$$

and $s_j^{\mathbf{q}} := \#\{i : 1 \leq i \leq n+1,\ q_i = j\}$ for $\mathbf{q} \in \mathcal{Q}_{n+1}(N)$ and $1 \leq j \leq N$.

Proof. Since $\tau_{k,l} = \pi_{k\varepsilon_1} \otimes \pi_{l\varepsilon_1}$, we have that

$$m_{\tau_{k,l}}(\mu) = \sum_{\eta} m_{\pi_{k\varepsilon_1}}(\mu - \eta)\, m_{\pi_{l\varepsilon_1}}(\eta), \tag{19}$$

where the sum is restricted to the weights of $\pi_{l\varepsilon_1}$. For h any positive integer, the Young diagram associated with $\pi_{h\varepsilon_1}$ has only one row, of length h. Thus, the set of weights of $\pi_{h\varepsilon_1}$ is given by elements of the form $\nu = \sum_{i=1}^{n+1} c_i\varepsilon_i$ with $c_1, \ldots, c_{n+1} \in \mathbb{N}_0$ and $\sum_{i=1}^{n+1} c_i = h$, and all of them have multiplicity 1. Consequently, $m_{\tau_{k,l}}(\mu)$ is equal to the number of weights η of $\pi_{l\varepsilon_1}$ satisfying that $\mu - \eta$ is a weight of $\pi_{k\varepsilon_1}$.

Let $\mathbf{q} \in \mathcal{Q}_{n+1}(l)$. We want to count the number of weights $\eta = \sum_{i=1}^{n+1} b_i\varepsilon_i$ contributing to (19) (i.e., η is a weight of $\pi_{l\varepsilon_1}$ and $\mu - \eta$ is a weight of $\pi_{k\varepsilon_1}$) satisfying that $s_j^{\mathbf{q}}$ entries of η are equal to j for each $1 \le j \le l$. Clearly, $\mu - \eta$ is a weight of $\pi_{k\varepsilon_1}$ if and only if $a_i - b_i \ge 0$ for all $1 \le i \le n+1$. Since for each $1 \le j \le l$ there are $n+1-\sum_{t=0}^{j-1} Z_t(\mu)$ a_i's greater than $j-1$, then the required number is

$$\binom{n+1-\sum_{t=0}^{l-1} Z_t(\mu)}{s_l^{\mathbf{q}}}\binom{n+1-\sum_{t=0}^{l-2} Z_t(\mu)-s_l^{\mathbf{q}}}{s_{l-1}^{\mathbf{q}}}\cdots\binom{n+1-Z_0(\mu)-\sum_{j=2}^{l} s_j^{\mathbf{q}}}{s_1^{\mathbf{q}}}. \tag{20}$$

We have shown that $m_{\tau_{k,l}}(\mu)$ is equal to the sum over $\mathbf{q} \in \mathcal{Q}_{n+1}(l)$ of (20). The theorem now follows by Lemma IV.1. □

We now state the closed explicit formulas for the particular cases $l = 0$, 1, and 2. When $l = 0$, since $\mathcal{Q}_{n+1}(0) = \{(0,\dots,0)\}$, Theorem IV.2 immediately implies that every weight as in the hypotheses (i.e., $\mu = \sum_{i=1}^{n+1} b_i\varepsilon_i$ with $b_i \in \mathbb{N}_0$ for all i and $\sum_{i=1}^{n+1} b_i = k$) has multiplicity one. This fact is very well known because the Young diagram associated with $\pi_{k\varepsilon_1}$ has only one row, and consequently, the number of semistandard tableaux on this diagram of type μ is one.

We now assume $l = 1$. Let μ be again as in the hypotheses of Theorem IV.2. The number of partitions of 1 is obviously one, i.e., $\mathcal{Q}_{n+1}(1) = \{(1,0,\dots,0)\}$; thus, $m_{\pi_{k\varepsilon_1+\varepsilon_2}}(\mu) = \binom{n+1-\ell_0(\mu)}{1} - 1 = n - \ell_0(\mu)$, where $\ell_0(\mu)$ is the number of zeros coordinates of μ. It is not difficult to check that the number of semistandard tableaux of type μ is $n - \ell_0(\mu)$. Q17

We conclude the article stating the multiplicity formula for the irreducible representation of $\mathfrak{sl}(n+1,\mathbb{C})$ with highest weight $k\varepsilon_1 + 2\varepsilon_2$. Similar to the above, the proof follows immediately from Theorem IV.2, since it reduces to consider the only two partitions of 2. The reader may try to obtain this formula by counting semistandard tableaux of type μ and convince his/herself that the difficulty will increase for a higher l.

Corollary IV.3. Let $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{C})$ for some $n \ge 2$ and let $k \ge 2$ integer. Let $\mu = \sum_{i=1}^{n+1} a_i\varepsilon_i \in P(\mathfrak{g})$ with $a_i \in \mathbb{N}_0$ for all i and $\sum_{i=1}^{n+1} a_i = k+2$. If $a_i \le k$ for all i, then

$$m_{\pi_{k\varepsilon_1+2\varepsilon_2}}(\mu) = \binom{n+1-Z_0(\mu)}{2} - Z_1(\mu)$$

and $m_{\pi_{k\varepsilon_1+2\varepsilon_2}}(\mu) = 0$ otherwise, where $Z_t(\mu) = \#\{i\colon 1 \le i \le n+1,\ a_i = t\}$.

We end this article with an observation pointed out by the referee, in the same spirit of Remark III.16.

Remark IV.4. We consider the "weight zero" in $\pi_{k\varepsilon_1+l\varepsilon_2}$, which in our convention is given by $0_{k+l} := \sum_{i=1}^{n+1} \frac{k+l}{n+1}\varepsilon_i$. Clearly, $m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(0_{k+l}) = 0$ unless $n+1$ divides $k+l$. Theorem IV.2 does not give an explicit expression for $m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(0_{k+l})$ like in Corollaries III.10–III.12. However, for $l \ge 0$ fixed, it implies that $m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(0_{k+l})$ does not depend on k, for k sufficiently large satisfying that $n+1$ divides $k+l$. Moreover, the function $k \mapsto m_{\pi_{k\varepsilon_1+l\varepsilon_2}}(0_{k+l})$ is constant for every $k \in ln + (n+1)\mathbb{N}_0$. Indeed, for such k, we have that $\frac{k+l}{n+1} \ge l$, thus $\ell_t(0_{k+l}) = 0$ for every $0 \le t \le l-1$.

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