

On quadrupole and octupole gravitational radiation in the ANK formalism

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Abstract Following the approach of Adamo–Newman–Kozameh (ANK) we derive the equations of motion for the center of mass and intrinsic angular momentum for isolated sources of gravitational waves in axially symmetric spacetimes. The original ANK formulation is generalized so that the angular momentum coincides with the Komar integral for a rotational Killing symmetry. This is done using the Winicour–Tamburino Linkages which yields the mass dipole-angular momentum tensor for the isolated sources. The ANK formalism then provides a complex worldline in a fiducial flat space to define the notions of center of mass and spin. The equations of motion are derived and then used to analyse a very simple astrophysical process where only quadrupole and octupole contributions are included. The results are then compared with those coming from the post newtonian approximation.

Keywords Center of mass · Spin · Gravitational radiation

1 Introduction

The Adamo–Newman–Kozameh (ANK) formalism defines the notions of center of mass and spin for isolated sources of gravitational radiation in asymptotically flat space times. The main idea of the formulation is to associate with a specific asymptotically shear free null congruence at null infinity a complex worldline in H-space where the real and complex part of the worldline are respectively the center of mass and the spin of the isolated source [1,2]. The formulation also provides equations of motion

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for those variables and gives kinematical relationships to other global variables like total linear and angular momentum. As a simplifying assumption, the theory only considers quadrupole radiation since essentially all definitions of angular momentum yield the same result for this case [2]. Nevertheless, octupole contributions also play an important role when creating templates of gravitational waveforms for the next generation of observers such as advanced LIGO [3], and thus they (as well as higher contributions) should be included in the definitions of center of mass and angular momentum.

To include quadrupole and octupole contributions from astrophysical processes we must generalize the ANK formulation to include general waveforms. The simplest scenario that could give nontrivial results and at the same time enable us to work with a robust definition of angular momentum is certainly the axially symmetric case. For spacetimes with this symmetry the Komar formula provides an adequate definition and, as expected, it is a constant of motion. However, the original definition used in the ANK formalism is not the Komar formula and thus, when including higher harmonic contributions, it gives a quantity that is not conserved for axially symmetric spacetimes. The idea then is to modify the ANK formulation to define center of mass and spin for axially symmetric spacetimes

Although it is very unlikely that an astrophysical process will be axisymmetric we want to stress that the main purpose of this work is to provide an adequate generalization of the ANK formulation for generic waveforms. At the same time we want to be able to give orders of magnitude between the emitted gravitational radiation and the dynamical evolution of the center of mass as defined in the ANK formulation. In particular, it is known that some processes result in a gravitational kick [4,5] that can send the center of mass outside the galaxy where the collision or radiation emission occurred. It is then interesting to first obtain a rough estimate of this recoil velocity and then compare with results in the literature.

In Sect. 2 we review the Newman Penrose Formalism for asymptotically flat spacetimes as our theoretical basis for subsequent results and write the Einstein field equations for the axisymmetric case. In particular we use the spin weighted Legendre Polynomials as our regular basis for expanding the relevant observables (see “Appendix A”).

In Sect. 3 we present the main results. We restrict ourselves to gravitational systems that only emit detectable gravitational radiation for a short period of time. For these systems it is reasonable to assume they are asymptotically stationary in the past and in the future.

We first review the ANK formulation presenting the main ideas and results applied to an axially symmetric, asymptotically flat spacetime [1]. As mentioned before, by assumption the original formulation only considers quadrupole waveforms and thus the equations reflect this ansatz. However, keeping the original definition of angular momentum to axially symmetric sources that emit radiation with higher harmonics gives non-physical results, namely, a direct calculation shows that this definition of angular momentum is not constant, as it should. We then review our earlier work [6] where the Komar integral associated with the rotational killing vector is used as the definition of angular momentum since it gives the correct conserved quantity. However, this earlier work had a conceptual flaw, to keep the ANK definition of center of mass

and use the Komar formula for the angular momentum. In special relativity the center of mass and angular momentum are the components of a special 2-form of the Poincaré algebra and any generalization to GR should carry on this geometrical feature. In this work we use the Winicour–Tamburino linkages [7] to define the mass dipole/angular momentum 2-form on the Lie algebra of the BMS group. We then define the center of mass and spin together and derive their corresponding equations of motions. As expected, they differ from either the original ANK or the revised versions.

In Sect. 4 we derive some predictions from the derived equations. We seek to predict the motion of the center of mass assuming different characters of the dominant gravitational radiation, i.e. at the quadrupole and octupole level. These types of radiation are dominant in certain types of head on collisions and for different orientations of the spins (facing in the same or opposite directions). We also compare our results for gravitational kicks with those from the post-Newtonian approximation. We determine from the knowledge of the gravitational radiation the change in velocity and position for the center of mass in both approaches.

Finally, in the conclusions we outline future areas of application and generalization of the present work.

2 Asymptotic flatness

The notion of asymptotically flatness is the adequate tool to analyze the gravitational and electromagnetic radiation coming from an arbitrary compact source. A spacetime can be thought of as asymptotically flat if the Weyl tensor vanishes as infinity is approached along the future-directed null geodesics of the spacetime. All the null geodesics end up at what is referred to as future null infinity, \mathcal{I}^+ , the future boundary of the spacetime [1, 8].

For those spacetimes we introduce a natural set of coordinates in the neighborhood of \mathcal{I}^+ called Bondi coordinates $(u_B, r, \zeta, \bar{\zeta})$. In this system, the Bondi time u_B labels a special family of null surfaces whose intersection with \mathcal{I}^+ are two spheres, r is the affine parameter along each null geodesic of the constant u_B surface and $\zeta = e^{i\varphi} \cot \frac{\theta}{2}$, is the complex stereographic angle that labels the null geodesics of the null surface.

Associated with Bondi coordinates is a null tetrad system based on these outgoing null hypersurfaces labeled by $(l_a, n_a, m_a, \bar{m}_a)$. The first tetrad vector l_a is defined as [8]

$$l_a = \nabla_a u_B. \quad (1)$$

Thus, $l^a = g^{ab} \nabla_b u_B$ is a null vector tangent to the geodesics of the surface. For the second tetrad vector we pick a null vector n^a normalized to l^a

$$n_a l^a = 1 \quad (2)$$

The tetrad is finally completed with the choice of a complex null vector m^a orthogonal to l^a and n^a

$$m_a \bar{m}^a = -1 \quad (3)$$

The spacetime metric is then [8]

$$g_{ab} = l_a n_b + n_a l_b - m_a \bar{m}_b - \bar{m}_a m_b \quad (4)$$

It is important to remember that at \mathcal{J}^+ , the vector n^a is tangent to \mathcal{J}^+ and the orbits of n^a are labelled by the coordinates $(\zeta, \bar{\zeta})$. There is a great deal of tetrad freedom at our disposal, the most important for us being the null rotations around n^a , performed in the neighborhood of \mathcal{J}^+ , that later play a major role. For an arbitrary complex function $L(u_B, \zeta, \bar{\zeta})$ on \mathcal{J}^+ , the null rotation about the vector n^a [8] is given by:

$$l^a \rightarrow l^{*a} = l^a - \frac{\bar{L}}{r} m^a - \frac{L}{r} \bar{m}^a + O(r^{-2}), \quad (5)$$

$$m^a \rightarrow m^{*a} = m^a - \frac{L}{r} n^a + O(r^{-2}) \quad (6)$$

$$n^a \rightarrow n^{*a} = n^a. \quad (7)$$

For each point $(u_B, \zeta, \bar{\zeta})$, the complex L gives the incidence angle of the null vector l^{*a} , as it reaches \mathcal{J}^+ .

Finally we introduce the concept of spin weight. A quantity η that transforms as $\eta \rightarrow e^{i s \lambda} \eta$ under a rotation $m^a \rightarrow e^{i \lambda} m^a$ is said to have a spin weight s . For any function $f(u, \zeta, \bar{\zeta})$, we define the differential operators \eth and $\bar{\eth}$ by [1]

$$\eth f = P^{1-s} \frac{\partial(P^s f)}{\partial \zeta}, \quad (8)$$

$$\bar{\eth} f = P^{1+s} \frac{\partial(P^{-s} f)}{\partial \bar{\zeta}}, \quad (9)$$

where f has a spin weight s and $P = (1 + \zeta \bar{\zeta})$ is the conformal factor of the unit sphere the metric [1].

2.1 The Newman–Penrose formalism

The Newman–Penrose (NP) formalism is the basic working tool for our analysis. We present here an outline of the formulation and leave reference [8] for details. We focus in the general form of the asymptotically flat solutions of Einstein–Maxwell equations in Bondi coordinates.

The NP version [8,9] of the vacuum Einstein equations uses the tetrad vectors $(l^a, n^a, m^a, \bar{m}^a)$ rather than the metric, as the basic variables. Using the tetrad one then writes the five complex tetrad components of the Weyl tensor as:

$$\psi_0 = -C_{abcd} l^a m^b l^c m^d; \quad \psi_1 = -C_{abcd} l^a n^b l^c m^d$$

$$\begin{aligned}\psi_2 &= -\frac{1}{2}(C_{abcd}l^a n^b l^c n^d - C_{abcd}l^a n^b m^c \bar{m}^d) \\ \psi_3 &= C_{abcd}l^a n^b n^c \bar{m}^d; \quad \psi_4 = -C_{abcd}n^a \bar{m}^b n^c \bar{m}^d\end{aligned}\quad (10)$$

The Peeling theorem of Sachs [10] tell us that the behavior of the Weyl scalar is given by:

$$\psi_0 = \psi_0^0 r^{-5} + O(r^{-6}) \quad (11)$$

$$\psi_1 = \psi_1^0 r^{-4} + O(r^{-5}) \quad (12)$$

$$\psi_2 = \psi_2^0 r^{-3} + O(r^{-4}) \quad (13)$$

$$\psi_3 = \psi_3^0 r^{-2} + O(r^{-3}) \quad (14)$$

$$\psi_4 = \psi_4^0 r^{-1} + O(r^{-2}) \quad (15)$$

Furthermore, at \mathcal{I}^+ we have:

$$\psi_4^0 = -\bar{\sigma}_0^{\cdot}, \quad (16)$$

$$\psi_3^0 = \bar{\partial}\bar{\sigma}_0, \quad (17)$$

$$\psi_2^0 - \bar{\psi}_2^0 = \bar{\partial}^2\sigma_0 - \bar{\partial}^2\bar{\sigma}_0 + \bar{\sigma}_0\sigma_0^{\cdot} - \sigma_0\bar{\sigma}_0^{\cdot}, \quad (18)$$

where the dot over the scalars means $\frac{\partial}{\partial u}$, and σ_0 is the asymptotic shear of the $u = \text{const.}$ cuts [8]. The complex scalar $\dot{\sigma}_0$ is called the Bondi news function and represents the gravitational radiation reaching null infinity.

A linear combination of Weyl scalars and tetrad connection components defines the mass aspect,

$$\Psi \equiv \psi_2^0 + \bar{\partial}^2\bar{\sigma}_0 + \sigma_0\bar{\sigma}_0^{\cdot}, \quad (19)$$

which satisfies the following reality condition:

$$\Psi = \bar{\Psi}. \quad (20)$$

Finally, from the asymptotic Bianchi Identities, we obtain the dynamical (or evolution) equations:

$$\dot{\psi}_2^0 = -\bar{\partial}\psi_3^0 + \sigma_0\psi_4^0, \quad (21)$$

$$\dot{\psi}_1^0 = -\bar{\partial}\psi_2^0 + 2\sigma_0\psi_3^0, \quad (22)$$

$$\dot{\psi}_0^0 = -\bar{\partial}\psi_1^0 + 3\sigma_0\psi_2^0, \quad (23)$$

Using the mass aspect, Ψ , with Eqs. (16) and (17), the first of the asymptotic Bianchi Identities can be rewritten as,

$$\dot{\Psi} = \dot{\sigma}_0\bar{\sigma}_0^{\cdot}. \quad (24)$$

One of the immediate physical interpretations arising from the asymptotically flat solutions was Bondi's [1] identifications, at \mathcal{I}^+ , of the underlying spacetime four-momentum (energy/momentum). Given the mass aspect, Eq. (19), and the spherical harmonic expansion

$$\Psi = \Psi^0 + \Psi^i Y_{1i}^0 + \Psi^{ij} Y_{2ij}^0 + \dots \quad (25)$$

one defines Bondi mass and three-momentum from the $l = 0$ and $l = 1$ harmonic coefficients of Ψ contributions;

$$M = -\frac{c^2}{2\sqrt{2}G} \Psi^0, \quad (26)$$

$$P^i = -\frac{c^3}{6G} \Psi^i. \quad (27)$$

It is also important to obtain relationships between the Weyl scalars when we make a null rotation. In particular ψ_1^0 transforms as

$$\psi_1^{*0} = (\psi_1^0 - 3L\psi_2^0 + 3L^2\psi_3^0 - L^3\psi_4^0). \quad (28)$$

This relation will be very useful later in this work.

2.2 Asymptotically shear-free null geodesic congruences

We present here a brief review of an extensive work on null geodesic congruences in asymptotically flat spacetimes [1]. The results that are needed for this work are given with only a sketch of a derivation but the reader can look for rigorous proofs in the main reference.

If an observer at \mathcal{I}^+ sees a null congruence that is asymptotically shear free, the null rays appear to be coming from a point in the interior. Since the shears associated with two null tetrads are related by

$$\sigma_0^* = \sigma_0 - \bar{\delta}L - LL', \quad (29)$$

where σ_0^* is the asymptotic shear of l^{*a} , that means there is a special choice of $L(u, \zeta, \bar{\zeta})$ such that $\sigma_0^* = 0$.

By requiring that $L(u, \zeta, \bar{\zeta})$ satisfies

$$\bar{\delta}L + LL' = \sigma_0(u, \zeta, \bar{\zeta}). \quad (30)$$

we have, at each point of \mathcal{I}^+ a null geodesic congruence that resembles a flat null cone from a point. This point of course does not belong to the spacetime, rather it is a point on a fiducial space (since the null rays are not coming from actual points).

The above remark can be made more precise by analyzing the freedom in the solution of Eq. (30). To do that we start with the solutions to the good cut equation, i.e.,

$$\eth^2 Z = \sigma_0(Z, \zeta, \bar{\zeta}). \quad (31)$$

It can be shown that the solution of the above equation depends on 4 complex numbers ξ^a , i.e. $u = Z(\xi^a, \zeta, \bar{\zeta})$ and the solution space is a 4-dim complex space with a lorentzian metric called H -space. Furthermore, if we select an arbitrary worldline $\xi^a(\tau)$, the function $Z(\xi^a(\tau), \zeta, \bar{\zeta})$, depends on a complex curve in H -space. Using $u = Z(\tau, \zeta, \bar{\zeta})$ together with its inverse $\tau = T(u, \zeta, \bar{\zeta})$ one can show [1] that the solution of Eq. (30) is given by

$$L(u, \zeta, \bar{\zeta}) = \eth_{(\tau)} Z|_{\tau=T(u, \zeta, \bar{\zeta})}.$$

In the above equation we first take the \eth operator keeping τ constant (the $\eth_{(\tau)}$ symbol), and then replace τ by $T(u, \zeta, \bar{\zeta})$. Thus, the freedom in the solution of Eq. (30) is given by arbitrary worldlines in H -Space and the virtual light-cones generated by the asymptotically shear-free null geodesic congruences emanate from a complex virtual world line in the associated H -space.

Using a gauge where $\xi^a = (\tau, \xi^i(\tau))$ for $i = 1, 2, 3$ and the slow motion approximation, one can obtain a set of useful equations,

$$Z(\tau, \zeta, \bar{\zeta}) = \frac{1}{\sqrt{2}}\tau - \frac{1}{2}\xi^i(\tau)Y_{1i}^0(\zeta, \bar{\zeta}) + \xi^{ij}(\tau)Y_{2ij}^0(\zeta, \bar{\zeta}) + \cdots \quad (32)$$

$$T(u, \zeta, \bar{\zeta}) = \sqrt{2}u + \frac{1}{\sqrt{2}}\xi^i(u)Y_{1i}^0(\zeta, \bar{\zeta}) - \sqrt{2}\xi^{ij}(u)Y_{2ij}^0(\zeta, \bar{\zeta}) + \cdots \quad (33)$$

$$L(u, \zeta, \bar{\zeta}) = \eth_{(\tau)} Z = \xi^i(T)Y_{1i}^1(\zeta, \bar{\zeta}) - 6\xi^{ij}(T)Y_{2ij}^1(\zeta, \bar{\zeta}) + \cdots \quad (34)$$

$$\sigma_0(u, \zeta, \bar{\zeta}) = \eth_{(\tau)}^2 Z = 24\xi^{ij}(T)Y_{2ij}^1(\zeta, \bar{\zeta}) + \cdots \quad (35)$$

Note that the last equation gives a direct relationship between the Bondi shear and Z . At the lowest order we find.

$$\xi^{ij}(u) = \sigma_0^{ij}(u).$$

We also assume that ξ^i is an analytical function of τ . We thus write

$$\xi^i(\tau) = \xi_R^i(\tau) + i\xi_I^i(\tau),$$

and, as we will see later, the real and imaginary components will be related to the center of mass and spin respectively.

Note that the above equation does not describe a worldline but rather a world sheet since $\tau = s + i\lambda$ is a complex parameter. A reality condition is imposed on Z and this fixes uniquely $\lambda = \lambda(s)$ and at the same time it yields a foliation of real cuts at null

infinity $u = Z_R(s, \zeta, \bar{\zeta})$, where the observers see the incoming asymptotically shear free null congruences. At the linearized level τ and the real cuts are given by

$$\tau = s + i \left(\frac{1}{\sqrt{2}} \xi_I^i(s) Y_{1i}^0 - \sqrt{2} \xi_I^{ij}(s) Y_{2ij}^0 \right), \quad (36)$$

$$Z_R = \frac{s}{\sqrt{2}} - \frac{1}{2} \xi_R^i(s) Y_{1i}^0 + \xi_R^{ij}(s) Y_{2ij}^0. \quad (37)$$

Thus, it is always possible to find a range of values for $\lambda(s)$ where the complex cuts intersect real null infinity with real Bondi coordinates. The restriction to real cuts however, must be performed after Eqs. (32) and (34) have been obtained. Since the \eth operator does not commute with the restriction, imposing the reality condition first will give wrong results.

2.3 Axially symmetric spacetimes

By assuming axial symmetry many problems in general relativity are greatly simplified and the resulting equations easier to derive. In spite of such a reduction on the degrees of freedom of the theory there are some physically relevant problems that can be analysed and, what could be more important, a well posed initial value problem can be set and a numerical evolution can be obtained.

In our case, working with an axially symmetric spacetime axial symmetry will allow us to give a robust definition of angular momentum for the gravitational source, which of course, must also have rotational symmetry.

Particularly, in this work we will use spin weighted Legendre polynomials $P_{lm}^s(\theta)$, instead of the tensorial spherical harmonics $Y_{lm}^s(\zeta, \bar{\zeta})$, previously used in many publications on this area. With this choice we can clearly identify the terms that depend on the angle θ and that are independent from ϕ in the sphere of null directions.

The definition of associated Legendre polynomials with spin weights $s = 0, 1, 2$; as well as the products between them are presented in “Appendix A”.

We assume that the gravitational radiation contains both quadrupole and octupole terms. On one hand this assumption gives nontrivial contributions to the equations of motion for the center of mass which are absent if one only assumes a quadrupole term. At the same time it simplifies the algebraic manipulations that would arise if all the higher harmonics are also present. We thus assume the radiation only contains Legendre polynomials with spin weight $s = 2$ for $l = 2$ and $l = 3$ (“Appendix A”). The Bondi shear can be written as

$$\sigma_0 = \sigma_2(u) P_2^2(\cos \theta) + \sigma_3(u) P_3^2(\cos \theta), \quad (38)$$

In the Newman–Penrose formalism the gravitational radiation is given by the scalar ψ_4^0 (10) since it decays as $\approx \frac{1}{r}$ (15) and it is related to the Bondi shear via $\psi_4^0 = -\ddot{\sigma}_0$.

A non trivial task is to relate the Bondi shear to the multipole moments of the source as one can see in the post newtonian approach [11, 12]. However, if one considers

linearized gravity, the multipole moments of the source are the multipole moments of the radiation.

Thus, for any spacetime σ_2 and σ_3 , to first order can be written as

$$\sigma_2 = \frac{2G}{\sqrt{2}c^4} (Q_+'' + iQ_-'') \quad (39)$$

$$\sigma_3 = \frac{3G}{\sqrt{2}c^5} (O_+''' + iO_-''') \quad (40)$$

where (\cdot) represent $\frac{\partial}{\partial u}$, and where Q and O are respectively the quadrupole and octupole moments of the source. The relations (39), (40) are written in the gauge TT (transverse traceless gauge) in terms of the corresponding multipolar expansion [11, 12].

The components Q_+ , O_+ represent the mass moments and Q_- , O_- represent the current moments and are responsible for the spin-orbit and spin-spin coupling effects in the postnewtonian approach [13, 14].

To contribute to a better understanding of the relationship between our work and the post newtonian (PN) approach (39) and (40) we give a very brief overview of the gravitational radiation formulae from the point of view of the PN approach in "Appendix C".

2.4 Rate of change of mass and linear momentum

Writing the mass aspect of the system as

$$\Psi = -\frac{2\sqrt{2}}{c^2} GMP_0^0(\cos\theta) - \frac{6G}{c^3} P^z P_1^0(\cos\theta) + \text{terms}(l \geq 2), \quad (41)$$

and inserting the above relation in the mass loss Eq. (24), together with (39), (40) and (41), yields the following equations

$$\dot{M} = -\frac{G}{5c^7} (Q_+''' Q_+''' + Q_-''' Q_-''') - \frac{9G}{28c^9} (O_+'''' O_+'''' + O_-'''' O_-'''), \quad (42)$$

$$\dot{P} = -\frac{3\sqrt{10}G}{35c^7} (Q_+''' O_+'''' + Q_-''' O_-'''). \quad (43)$$

We can see that the rate of change of mass (42) contains two contributions, one with quadrupole and another with octupole terms. On the other hand, for the rate of change of momentum (43) these contributions, quadrupole and octupole appear mixed so that if either Q''' or O'''' are null, then $\dot{P} = 0$, and then the Bondi three-momentum is constant. Note that if we only consider quadrupole contributions the Bondi 3-momentum is always conserved. It is thus important to keep higher order contributions to analyse the time evolution of P when a source is emitting gravitational radiation.

3 Centre of mass and angular momentum

3.1 Definition of the centre of mass and angular momentum in the ANK formalism

In this subsection we present the ANK [1] definitions of the center of mass and intrinsic angular momentum and its associated time evolutions. The basic strategy is to start with a suitable definition of dipole mass momentum and angular momentum associated with a given Bondi tetrad. Performing a null rotation and demanding that the mass dipole and angular momentum vanish on a suitable chosen null rotated basis one is able to select a worldline in H -space. This worldline is called center of mass and spin. However, as we will see below, these definitions fail to give physically meaningful quantities when generic gravitational radiation is taken into account.

Since we restrict ourselves to axisymmetric spacetimes we use the Legendre polynomials with spin weight listed in “Appendix A”.

We start with the leading order ψ_1^0 of the Weyl tensor at future null infinity

$$\lim_{r \rightarrow \infty} \psi_1^0 = - \lim_{r \rightarrow \infty} C_{abcd} l^a n^b l^c m^d r^4. \quad (44)$$

In the ANK approach, the mass dipole momentum D_{ANK} and angular momentum J_{ANK} are defined as the real and imaginary parts of $[\psi_1^0]_{l=1}$ i.e.,

$$(D + \frac{i}{c} J)_{ANK} \equiv - \frac{c^2}{6\sqrt{2}G} \left[\psi_1^0(u, \zeta, \bar{\zeta}) \right]_{l=1} \quad (45)$$

If we now perform an asymptotically shear free null rotation we have a new tetrad system $(l^{*a}, n^{*a}, m^{*a}, \bar{m}^{*a})$ with torsion and null shear, $\sigma_0^* = 0$. For the new system the corresponding variables are also defined as

$$(D^* + \frac{i}{c} J^*)_{ANK} \equiv - \frac{c^2}{6\sqrt{2}G} \left[\psi_1^{*0}(u, \zeta, \bar{\zeta}) \right]_{l=1}. \quad (46)$$

The relationships between the starred and un-starred variables directly come from

$$\psi_1^{*0} = \psi_1^0 - 3L\psi_2^0 + 3L^2\psi_3^0 - L^3\psi_4^0. \quad (47)$$

Note that for each complex worldline in the associated H -space we have a different starred tetrad.

We now demand that on a $\tau = \text{const.}$ slice the $l = 1$ harmonic in ψ_1^{*0} vanishes, i.e.,

$$\left[\psi_1^{*0}(Z(\tau, \zeta, \bar{\zeta}), \zeta, \bar{\zeta}) \right]_{l=1} = 0, \quad (48)$$

where we write Z instead of u to emphasize that only for a particular angle function L (which yields a unique complex worldline) we have a vanishing dipole moment.

Since the ANK approach uses the slow motion approximation we will only keep up to linear terms in L . Thus, the transformation equation (47) is reduced to

$$0 = \left[\psi_1^0 - 3L\psi_2^0 \right]_{l=1},$$

and the main equation from which the worldline is extracted is then written as

$$[\psi_1^0(Z, \zeta, \bar{\zeta}) - 3\bar{\partial}(Z)\psi_2^0(Z, \zeta, \bar{\zeta})]_{l=1} = 0, \quad (49)$$

We expand $Z(\tau, \zeta, \bar{\zeta})$ in the spin weighted Legendre polynomials P_l^s for $l = 0, 1, 2, \dots$, as

$$Z(\tau) = \tau P_0^0 + \xi(\tau) P_1^0 + \sigma_2(\tau) P_2^0 + \dots \quad (50)$$

with $\xi(\tau) = \xi_R(\tau) + i\xi_I(\tau)$ the z -component of the complex world line (the other components vanish in the axisymmetric case) and with $\sigma_2(\tau)$ the $l = 2$ component of the Bondi shear. The idea is to solve for $\xi(\tau)$ from the vanishing of $[\psi_1^{s0}]_{l=1}$. Assuming that initially $\xi(\tau_0)$ and $\frac{\partial \xi^i}{\partial \tau}$ are small, we write

$$Z = \tau + \delta\tau, \quad (51)$$

where $\delta\tau$ is small. This is particularly useful when keeping track of terms since we only keep up to order 2 in either the Bondi shear, $\xi(\tau)$ or bilinear terms of these two variables

Then, writing the expression (47) again, only up to the second order (i.e. third or higher order terms are discarded) and replacing in the Eq. (51) $Z(\tau) = \tau + \delta\tau$ we have

$$0 = \psi_1^0(\tau + \delta\tau, \zeta, \bar{\zeta}) - 3L\psi_2^0(\tau + \delta\tau, \zeta, \bar{\zeta}), \quad (52)$$

where the function $L = \bar{\partial}_\tau(Z) = \bar{\partial}_\tau(\delta\tau)$ is already order one and it is given by

$$L = \xi(\tau) P_1^1(\cos\theta) + 12\sigma_2(\tau) P_2^1(\cos\theta) + \frac{18\sqrt{10}}{5} \sigma_3(\tau) P_3^1(\cos\theta) + \dots \quad (53)$$

Expanding (52) to first order in $\delta\tau$ we have

$$0 = \psi_1^0(\tau, \zeta, \bar{\zeta}) + \delta\tau \dot{\psi}_1^0(\tau, \zeta, \bar{\zeta}) - 3\bar{\partial}(\delta\tau)\psi_2^0(\tau, \zeta, \bar{\zeta}).$$

In the above equation τ is a parameter independent of $(\zeta, \bar{\zeta})$ that plays the same role as the Bondi time at null infinity. Since the Weyl scalars involved in the calculation have the same harmonic expansion in terms of standard Bondi coordinates and their time evolution is given by the Bianchi identities at \mathcal{J}^+ one usually regards this parameter as the Bondi time u [1].

Using the Bianchi identity (22), and taking the $l = 1$ component of the above equation together with the definitions of mass dipole moment and angular momentum yields

$$D_{ANK} = M\dot{\xi}_R + \frac{32\sqrt{3}G}{5c^5} P Q_+'' + \frac{54\sqrt{5}G}{35c^7} (Q_+'' O_+''' + Q_\times'' O_\times'''), \quad (54)$$

$$J_{ANK} = cM\dot{\xi}_I + \frac{32\sqrt{3}G}{5c^4} P Q_\times'' + \frac{54\sqrt{5}G}{7c^6} (Q_+'' O_\times''' - Q_\times'' O_+'''). \quad (55)$$

The above equations give the definitions of center of mass ξ_R and intrinsic angular momentum $S \equiv cM\dot{\xi}_I$ in terms of asymptotic fields at null infinity.

Taking a time derivative on (54) and using the Bianchi identities gives

$$\begin{aligned} P = M(1 + \frac{32\sqrt{3}G}{5c^5} Q_+'' \dot{\xi}_R \\ + \frac{9\sqrt{5}G}{70c^7} [13(Q_+'' O_+''' + Q_\times'' O_\times''') + 11(Q_+'' O_\times''' + Q_\times'' O_+''')]). \end{aligned} \quad (56)$$

the relationship between the Bondi momentum and the velocity of the center of mass.

There is, however, a problem with the definition of angular momentum given in (45), since it is not consistent with the assumed axial symmetry of the space-time. There should be a conserved quantity associated with the rotational symmetry and this quantity should be the component of the angular momentum along the axis of symmetry. Computing \dot{J}_{ANK} yields

$$\dot{J}_{ANK} = \frac{54\sqrt{5}G}{7c^6} (Q_+'' O_\times''' + Q_\times'' O_+''' - Q_\times'' O_+''' - Q_+'' O_\times''') \neq 0, \quad (57)$$

i.e., when higher harmonics of the gravitational radiation are included, the above definition is not a conserved quantity. Note that if we set the octupole terms equal to zero (as it was done in the original ANK approach) J_{ANK} is constant. It is therefore clear that one must modify the ANK definition when a more general type of radiation is included

3.1.1 Angular momentum with the Komar integral

An initial attempt to modify the formulation was given in [6], where the definition of the mass dipole moment was taken as the real part of $(\psi_1^0)_{l=1}$, i.e., the ANK definition, but a new definition of angular momentum was introduced via the Komar integral.

Following Komar's approach to obtain conserved quantities from Killing symmetries we define the angular momentum as

$$J = \frac{1}{16\pi} \lim_{r \rightarrow \infty} \int \nabla^a K_{(\phi)}^b dS_{ab}, \quad (58)$$

where $K_{(\phi)}^a$ is a rotational vector that satisfies the Killing equation and the $r = \text{const.}$ 2-surface is taken on the $u = \text{const.}$ null surface. Integrating explicitly equation (58) (see “Appendix B”) we obtain

$$J = -\frac{c}{6\sqrt{2}G} \operatorname{Im} \left[\psi_1^0 - \sigma \bar{\sigma} \right]_{l=1}. \quad (59)$$

Using (59) together with the imaginary parts of (45) and $(\sigma \bar{\sigma})_{l=1}$, gives the following relationship

$$J = S + \frac{32\sqrt{3}G}{5c^4} P Q_{\times}'' - \frac{3(177)\sqrt{5}G}{70c^7} (Q_{\times}'' O_{+}''' - Q_{+}'' O_{\times}'''), \quad (60)$$

where the intrinsic angular momentum $S = cM\xi_I$ was used in the above equation.

Although this new definition of angular momentum given in [6] fixed one problem it created another one, i.e., the old definition of mass dipole momentum and the new definition of angular momentum are intrinsically different geometrical objects.

Below we use the so called Winicour–Tamburino linkages to give definitions with a well defined geometrical meaning.

3.2 Mass dipole/angular momentum using linkages

The Winicour–Tamburino linkage provides a mapping from two forms at null infinity to the Lie algebra of the BMS group [7]. We give a brief review of this formulation.

Given a Bondi null foliation $u = \text{const.}$ together with an affine parameter r and building a 2-surface $r = \text{const.}$ with surface element $l^{[a} n^{*b]} dS$ we define the linkage as [7].

$$L_{\chi} = -\frac{1}{16\pi} \lim_{r \rightarrow \infty} \int \left(\nabla^{[a} \chi^{b]} + \nabla_c \chi^c l^{[a} n^{*b]} \right) l_a n_b^* dS. \quad (61)$$

In the above equation the vector χ^b is an element of the BMS algebra \mathcal{S}^+ . Note that in the special case where χ^b is a Killing field the above equation yields the Komar formula and the integral is a conserved quantity. Note also that \hat{n}^{*b} is not one of the associated null vectors of the Bondi tetrad. Whereas n^b is parallel propagated along l^a , n^{*b} is orthogonal to the $u = \text{const.}$, $r = \text{const.}$ surface.

This integral gives a map from 2-forms at null infinity to the BMS algebra. If we demand that at \mathcal{S}^+ the vector χ^b is tangent to $u = \text{const.}$ surface one obtains the generators of the Lorentz subalgebra for axial symmetry, which in this case it is just a boost and a rotation along the axis of symmetry. Writing the integral in terms of the Newman Penrose coefficients for the two independent elements yields [15, 16]

$$D + \frac{i}{c} J = -\frac{c^2}{12\sqrt{2}G} \left[2\psi_1^0 - 2\sigma_0 \bar{\sigma}_0 - \bar{\sigma} (\sigma_0 \bar{\sigma}_0) \right]_{l=1}. \quad (62)$$

Restricting the above formula for the axially symmetric spacetimes and keeping up to octupole contributions yields,

$$D + \frac{i}{c}J = -\frac{c^2}{6\sqrt{2}G} \left[\psi_1^0 \right]_{l=1} + \frac{3c^2\sqrt{5}}{140G} (\sigma_2\sigma_3 - \sigma_3\sigma_2). \quad (63)$$

Note that the last term on the r.h.s. is pure imaginary. Therefore, as expected, the angular momentum differs from the original definition in the ANK formulation. The mass dipole moment, on the other hand, remains the same. However, they will have different formulae if higher order contributions are taken into account or when the axial symmetry is removed.

If we now define the equivalent scalars in the null rotated tetrad we obtain

$$D^* + \frac{i}{c}J^* = -\frac{c^2}{12\sqrt{2}G} \left[2\psi_1^{0*} - 2\sigma_0^*\bar{\sigma}_0^* - \bar{\sigma}(\sigma_0^*\sigma_0^*) \right]_{l=1} \quad (64)$$

$$= -\frac{c^2}{6\sqrt{2}G} \left[\psi_1^{0*} \right]_{l=1}. \quad (65)$$

Note that any definition of mass dipole/angular momentum that contains shear terms would give the same result in the null rotated basis since σ_0^* vanishes in that frame.

Imposing the condition

$$\left[D^* + \frac{i}{c}J^* \right]_{l=1} = 0, \quad (66)$$

in the null rotated basis, using Eq. (49) together with (62) yields the relationships

$$D + \frac{i}{c}J = M\xi(\tau) + \frac{16\sqrt{6}G}{5c}\sigma_2P - \frac{3c^2\sqrt{5}}{140G}(47\sigma_2\sigma_3 - 71\sigma_3\sigma_2). \quad (67)$$

From the above relations we derive the equations of motion for the center of mass and spin.

3.3 Equations of motion

3.3.1 Center of mass

We first take the real part of Eq. (67), giving

$$D = M\xi_R + \frac{32\sqrt{3}G}{5c^5}PQ_+'' + \frac{54\sqrt{5}G}{35c^7}(Q_+''O_+''' + Q_\times''O_\times'''). \quad (68)$$

Taking a time derivative on the above equation and using the Bianchi identities gives

$$P = M\left(1 + \frac{32\sqrt{3}G}{5c^5} Q_+^{\dots}\right) \dot{\xi}_R + \frac{9\sqrt{5}G}{70c^7} [13(Q_+^{\dots} O_+^{\dots} + Q_{\times}^{\dots} O_{\times}^{\dots}) + 11(Q_+^{\dots} O_+^{\dots} + Q_{\times}^{\dots} O_{\times}^{\dots})]. \quad (69)$$

The term $M(1 + \frac{32\sqrt{3}G}{5c^5} Q_+^{\dots})$ plays the role of an effective mass when quadrupole radiation is present and the second term on the r.h.s. of the equation is a radiation contribution to the linear momentum of the system that only appears when quadrupole and octupole terms are present.

Taking one more time derivative on (69) gives

$$\begin{aligned} M\left(1 + \frac{32\sqrt{3}G}{5c^5} Q_+^{\dots}\right) \ddot{\xi}_R = & -\frac{3\sqrt{10}G}{35c^7} (Q_+^{\dots} O_+^{\dots} + Q_{\times}^{\dots} O_{\times}^{\dots}) \\ & -\frac{9\sqrt{5}G}{70c^7} [13(Q_+^{\dots} O_+^{\dots} + Q_{\times}^{\dots} O_{\times}^{\dots}) \\ & + 11(Q_+^{\dots} O_+^{\dots} + Q_{\times}^{\dots} O_{\times}^{\dots})]. \end{aligned} \quad (70)$$

Equations (68) and (69), relate the radiation scalars at null infinity with the center of mass worldline and (70) gives the center of mass acceleration when gravitational radiation is present.

3.3.2 Angular momentum

Note that directly from Eq. (62) we get

$$J = -\frac{c}{6\sqrt{2}G} \text{Im}[\psi_1^0 - \sigma \bar{\sigma}]_{l=1}, \quad (71)$$

since the term $\bar{\sigma}(\sigma\sigma)$ is real, i.e., we recover Komar's expression (59).

The relationship with the spin is given by

$$J = S + \frac{32\sqrt{3}G}{5c^4} P Q_{\times}^{\dots} - \frac{3(177)\sqrt{5}G}{70c^7} (Q_{\times}^{\dots} O_+^{\dots} - Q_+^{\dots} O_{\times}^{\dots}). \quad (72)$$

Since J is a conserved quantity, the above relationship shows that when there is emission of gravitational radiation the spin could change if the initial and final quadrupole and octupole moments are different.

4 Some applications

As we mentioned in the introduction, the purpose of this work was to extend the ANK formulation when higher multipole moments are considered in the gravitational

radiation. However, the main goal of the formulation is to describe the motion of the center of mass and spin for astrophysical sources. From this point of view the assumption of axial symmetry virtually rules out any physical situation, and thus, the next step is to derive the formulae for spacetimes without symmetries. Nevertheless it is illustrative to obtain some orders of magnitude when this formalism is applied to head on collisions whose final stage is a single black hole. In this case a gravitational kick back could send the final black hole with a high receding velocity.

It is also important to compare this formalism with the postnewtonian approximation since they both should give similar results in the low speed limit.

4.1 Equations of motion for head on collisions

We assume the axis of symmetry is along the z direction and that the gravitational radiation contains quadrupole and octupole terms (see “Appendix B”). We also use the relationship between the radiative quadrupole and octupole moments and the corresponding source moments as in ref. [3]. Furthermore, we model the source moments using two points masses m_1 and m_2 with spin S_1 and S_2 (both spins are necessarily aligned with the axis of symmetry).

The equations for the linear and angular momentum up to second order are

$$P = M\dot{\xi}_R + \frac{9\sqrt{5}G}{70c^7} [13 (Q_+^{\dots} O_+^{\dots} + Q_{\times}^{\dots} O_{\times}^{\dots}) + 11 (Q_+^{\dots} O_+^{\dots} + Q_{\times}^{\dots} O_{\times}^{\dots})], \quad (73)$$

$$J = S + \frac{32\sqrt{3}G}{5c^4} P Q_{\times} - \frac{3(177)\sqrt{5}G}{70c^7} (Q_{\times}^{\dots} O_+^{\dots} - Q_+^{\dots} O_{\times}^{\dots}). \quad (74)$$

As mentioned before we assume for simplicity that initially the center of mass of the system is at the origin of coordinates. The angular momentum is a conserved quantity in physical axisymmetric spacetime, ie

$$\dot{J} = 0. \quad (75)$$

Newton’s equations for the center of mass read

$$\begin{aligned} Ma_R = & -\frac{3\sqrt{10}G}{35c^7} (Q_+^{\dots} O_+^{\dots} + Q_{\times}^{\dots} O_{\times}^{\dots}) \\ & - \frac{9\sqrt{5}G}{70c^7} [13 (Q_+^{\dots} O_+^{\dots} + Q_{\times}^{\dots} O_{\times}^{\dots}) + 11 (Q_+^{\dots} O_+^{\dots} + Q_{\times}^{\dots} O_{\times}^{\dots})]. \end{aligned} \quad (76)$$

The idea is to solve the equations for two different values of the mass parameters, ($m_1 \approx m_2$) or ($m_1 \neq m_2$), since the quadrupole and octupole moments values depend on these choices. As it is shown in “Appendix C”, the spins do not make a valuable contribution unless they are close to the extrem Kerr limit. Thus, in the equations presented below we neglect the contribution of the current multipole moments and we only consider the two choices of the mass parameters

4.1.1 Case 1: identical masses ($m_1 \approx m_2$)

In this case the gravitational radiation is dominated by the quadrupole moment of the source. We thus set $O_+^{\dots} = 0$; (see “Appendix C”).

The rate of change of the total mass (42) and linear momentum (43) are reduced to:

$$\dot{M} = -\frac{G}{5c^7} (Q_+^{\dots} Q_+^{\dots}), \quad (77)$$

$$\dot{P} = 0. \quad (78)$$

The above equation implies that

$$a_c = 0. \quad (79)$$

b) Angular Momentum: In the same way we can analyze the expression for the angular momentum (60) considering that in this case it is reduced to

$$J = cM\xi_I = S = \text{const}. \quad (80)$$

Both P and J are constants of motion when the masses are similar. The center of mass does not change position (or moves with constant velocity) and we can only observe mass loss in the system.

4.1.2 Case 2: different masses ($m_1 \neq m_2$)

If $m_1 \neq m_2$ the gravitational radiation also contains an octupole moment contribution since it is proportional to the mass difference (“Appendix C”).

The rate of change for the total mass and linear momentum are given by:

$$\dot{M} = -\frac{G}{5c^7} (Q_+^{\dots} Q_+^{\dots}) - \frac{9G}{28c^9} (O_+^{\dots} O_+^{\dots}), \quad (81)$$

$$\dot{P} = -\frac{3\sqrt{10}G}{35c^7} (Q_+^{\dots} O_+^{\dots}) \neq 0. \quad (82)$$

We observe that in this case the linear momentum is not conserved. Thus, the center of mass will accelerate and after all the radiation is emitted it will be moving with a final velocity.

To compute the instantaneous acceleration we write

$$Ma_c = -\frac{3\sqrt{10}G}{35c^7} (Q_+^{\dots} O_+^{\dots}) - \frac{9\sqrt{5}G}{70c^7} [13Q_+^{\dots} O_+^{\dots} + 11Q_+^{\dots} O_+^{\dots}]. \quad (83)$$

If we integrate Eq. (83) we obtain the change in the velocity of the center of mass of the system due to the emission of gravitational radiation, i.e.

$$\Delta v_c = -\frac{3\sqrt{5}G}{35c^7} \frac{1}{M} \int_{u_0}^{u_f} \left[\sqrt{2} (Q_+^{\dots} O_+^{\dots}) + \frac{3}{2} [13 Q_+^{\dots} O_+^{\dots} + 11 Q_+^{\dots} O_+^{\dots}] \right] du. \quad (84)$$

The minus sign in the speed difference shows that the center of mass recedes in the opposite direction to the emission of gravitational radiation. These results are consistent with estimates made by other approaches presented in previous publications [3, 5]. However, as we will see below, we found a difference when computing the change in the position of the center of mass.

Combining both equations and integrating once more, we have

$$\begin{aligned} \Delta \xi_R = & -\frac{3\sqrt{5}G}{35c^7} \frac{1}{M} \int_{u_i}^{u_f} \left[\frac{3}{2} [13 Q_+^{\dots} O_+^{\dots} + 11 Q_+^{\dots} O_+^{\dots}] \right. \\ & \left. + \sqrt{2} \int_{u_i}^{u_l} (Q_+^{\dots} O_+^{\dots}) du \right] du. \end{aligned} \quad (85)$$

This equation gives the change in the position of the center mass. As $\Delta P \neq 0$ the final value of the position ξ_R is different from the initial value.

Regarding the issue of angular momentum, the new expression reads exactly the same (when current multipole moments are neglected) as before,

$$J = S = \text{const.}, \quad (86)$$

and does not give any new insights in the evolution.

4.2 A comparison with the postnewtonian formalism

Although there are conceptual differences between our approach and the Postnewtonian (PN) approximation [3, 13, 14] it is worth asking if those differences can be measured for an astrophysical process whereby gravitational radiation is emitted.

For simplicity we assume the astrophysical system to be stationary before (after) and initial (final) time u_i (u_f).

While in the PN formalism the relationship between the velocity of the center of mass and linear momentum is given by

$$P = M V_{PN}. \quad (87)$$

In the ANK approach the analogous relation is given by (69)

$$P = M \dot{\xi}_R + \frac{9\sqrt{5}G}{70c^7} [13 (Q_+^{\dots} O_+^{\dots} + Q_-^{\dots} O_-^{\dots}) + 11 (Q_+^{\dots} O_+^{\dots} + Q_-^{\dots} O_-^{\dots})], \quad (88)$$

In spite of the clear difference between both equations the change in velocity in both approaches is the same since the initial and final velocities coincide when gravitational radiation is absent. Defining

$$\Delta V \equiv V_f - V_i$$

one has on both approaches

$$M \Delta V = \Delta P, \quad (89)$$

with

$$\Delta P = - \int_{u_0}^{u_f} (\dot{\sigma} \dot{\sigma}) l_z du. \quad (90)$$

We thus have

$$M \Delta V_R = M \Delta V_{PN} = \Delta P. \quad (91)$$

Although the initial and final velocity in both formalism are the same, the change in the position of the center of mass is different. To see this we first show that, directly from Eq. (87), in the PN formalism one has

$$M \Delta \xi_{PN} = \Delta D. \quad (92)$$

Using equation (68) we have,

$$M \Delta \xi_{PN} = M \Delta \xi_R + \Delta \left(\frac{32\sqrt{3}G}{5c^5} P Q_+'' + \frac{54\sqrt{5}G}{35c^7} (Q_+'' O_+''' + Q_\times'' O_\times''') \right). \quad (93)$$

where $\Delta \xi_R$ will be the change of position of center of mass and where $\Delta (Q_+'' O_+''' + Q_\times'' O_\times''') \neq 0$ for a generic situation when gravitational radiation is emitted. Thus, both approaches predict different positions for the final position of the center of mass even if initially one assumes that they start at the same place.

It is also worth asking whether this difference is significant for a given astrophysical process. As an example we can consider a head-to-head impact of two different masses m_1 and m_2 in a preferential direction z . The idea is to estimate the order of magnitude of the difference in the position of the center of mass between the two formalisms.

The predicted change is maximum for a particular relationship of the interacting masses and can be calculated using the equations for Q'' and O''' shown in "Appendix B". For this case the current octupole moment O_\times''' is of a lower order than other moments and therefore it is negligible here. The mass ratio that maximizes the change in position for the equation (93) is $m_1 \cong \frac{3}{4}M$ and $m_2 \cong \frac{1}{4}M$, where M is the total mass of the system. Performing an order of magnitude calculation using the expressions for

Q'' and O'' , the difference between the predicted values of the PN and our formulation is given by

$$\Delta\xi_R \cong \Delta\xi_{PN} + O\left(\frac{G}{c^2}M\right), \quad (94)$$

i.e., it is proportional to the Schwarzschild radius of the final black hole. A more careful calculation using a model for head on collision is needed to obtain the proportionality factor in the above equation.

5 Conclusions

In this work we have extended the original formulation of the ANK formalism from quadrupole to general radiation. For simplicity our derivation was done for axially symmetric spacetimes but in principle one can generalize the construction to spacetimes without symmetries.

This generalization was motivated for the fact that the Komar integral (58) provides a suitable definition of angular momentum for axisymmetric spaces times whereas the formula in the original formulation [1] does not yield a conserved quantity when considering arbitrary radiation.

However, since the mass dipole and angular momentum must be the components of a 2-form on the BMS algebra, a new definition of angular momentum necessarily induces a change in the notion of center of mass. The Winicour–Tamburino linkages [7] together with a null rotated bases were the mathematical tools involved in this construction.

We found that the mass loss (42) and momentum (43) equations change significantly when quadrupole and octupole contributions are taken into account. In particular one can show that the linear momentum is conserved if either the quadrupole or octupole term vanish. Although small, the quadrupole contribution is present in many astrophysical processes. One thus, concludes that in almost all situations the linear momentum will not be conserved and the center of mass will change its position as a result of the emitted radiation.

One can then compute the change in velocity and position of the center of mass from this gravitational kickback and compare the results with the postnewtonian formulation. we found that the change in the position is different in both formulations, a straightforward consequence of the difference in the definitions of these global variables. Although hardly any astrophysical process will preserve axial symmetry, it is illustrative to do some back of the envelope calculations to check orders of magnitude. In this sense a head on collision of two point masses moving in a single direction is a useful calculation to perform.

When the two masses are equal the octupole contribution is null and the radiation only contains a quadrupole contribution. Thus linear and angular momentum are conserved and there is no acceleration of the center of mass. It is worth mentioning that both the mass and current radiation quadrupole terms used in this calculation were

taken from [3], since the radiation and source quadrupole terms are the same at the lowest orders of approximation.

When the masses are different the linear momentum is not a conserved quantity and the center of mass is accelerated. Our calculation shows that the location of the center of mass, after the radiation is emitted, is different if computed in the postnewtonian or the ANK formalism and this difference is proportional to the total mass of the system. Care must be taken with this result since it is highly unlikely that an astrophysical collision or coalescence process will be axially symmetric.

It is interesting and important to generalize this work for generic spacetimes since it is very likely that the predicted velocities will be different in the generic case. The evolution equation for the linear momentum has a cross product term which vanishes in this symmetric situation but could give a non trivial contribution when symmetries are lifted. One could then compare the resulting equations with those that are used in the aLIGO waveform construction. In particular, the final outcome of some astrophysical processes, such as binary coalescence and the motion of the final black hole could be a testing ground to compare different approaches. One can also foresee that in collisions where the final black hole is ejected from the galactic plane at high speed any difference between the ANK and PN approaches could result in very different angular positions of the black hole in the celestial sphere.

Appendix A: Legendre polynomials with spin weight

The associated Legendre polynomials $P_l^s(x)$ and $P_l^{-s}(x)$ are solutions to the associated Legendre differential equation, where l is a positive integer and $s = 0 \dots l$. They are implemented in Mathematica as Legendre $P[l, m, x]$. For positive s , they can be written in terms of the unassociated polynomials by

$$P_l^s(x) = (-1)^s (1-x^2)^{s/2} \frac{d^s}{dx^s} P_l(x), \quad (95)$$

where $P_l(x)$ are the unassociated Legendre polynomials. To define the associated Legendre polynomials for negative s we follow the convention due to Abramowitz and Stegun [17],

$$P_l^{-s}(x) = (-1)^s P_l^s(x). \quad (96)$$

They are orthogonal over the interval $[-1; 1]$ with respect to l

$$\int_{-1}^1 P_l^s(x) P_l^s(x) dx = \frac{2}{2l+1} \frac{(l+s)!}{(-s)!} \delta_{ll}, \quad (97)$$

and orthogonal over $[-1; 1]$ with respect to s with the weighting function $(1-x^2)^{-1}$,

$$\int_{-1}^1 P_l^s(x) P_l^{s'}(x) \frac{dx}{(1-x^2)} = \frac{2}{2l+1} \frac{(l+s)!}{s(l-s)!} \delta_{ss'}. \quad (98)$$

The associated Legendre polynomials also obey the following recurrence relations

$$(l-s) P_l^s(x) = x(2l-1) P_{l-1}^s(x) - (l+s-1) P_{l-2}^s(x). \quad (99)$$

It is also very useful to consider the identities

$$P_l^l(x) = (-1)^l (2l-1)!! (1-x^2)^{1/2}, \quad (100)$$

$$P_{l+1}^l(x) = x(2l+1) P_l^l(x). \quad (101)$$

Letting $x = \cos \theta$ we obtain Legendre polynomials $P_l^s(\cos \theta)$. Below we list the $s = 0; 1; 2$ Polynomials together with some useful products.

Legendre polynomials with spin weight $s = 0, 1, 2$

The $s = 0$ (spin weight zero) associated Legendre polynomials are

$$P_0^0 = 1, P_1^0 = \cos \theta, P_2^0 = \frac{1}{2}(3 \cos^2 \theta - 1), \quad (102)$$

$$P_3^0 = \frac{1}{2}(5 \cos^3 \theta - 3 \cos \theta), P_4^0 = \frac{1}{8}(35 \cos^4 \theta - 30 \cos^2 \theta + 3), \quad (103)$$

$$P_5^0 = \frac{1}{8}(63 \cos^5 \theta - 70 \cos^3 \theta + 15 \cos \theta),$$

$$P_6^0 = \frac{1}{16}(231 \cos^6 \theta - 315 \cos^4 \theta + 105 \cos^2 \theta - 5), \quad (104)$$

The polynomials with spin weight s are obtained via the $\bar{\partial}$ derivative, defined as

$$\bar{\partial} F = -(\sin \theta)^s \frac{\partial}{\partial \theta} ((\sin \theta)^{-s} F), \quad \bar{\partial} F = -(\sin \theta)^{-s} \frac{\partial}{\partial \theta} ((\sin \theta)^s F). \quad (105)$$

Using the above formula we then define the $P_l^s(\cos \theta)$ as

$$P_l^s(\cos \theta) = \bar{\partial}^s P_l^0(\cos \theta), \quad 0 \leq s \leq l, \quad (106)$$

$$P_l^s(\cos \theta) = (-1)^s \bar{\partial}^{-s} P_l^0(\cos \theta), \quad -l \leq s \leq 0, \quad (107)$$

$$P_l^s(\cos \theta) = 0, \quad l < |s|. \quad (108)$$

and use the recurrence relationship $\sqrt{(l-s)(l+s+1)} P_l^{s+1}(\cos \theta) = \bar{\partial} P_l^s(\cos \theta)$, so that $\sqrt{(l)(l+1)} P_l^1(\cos \theta) = \bar{\partial} P_l^0(\cos \theta)$, where $\bar{\partial} P_l^0(\cos \theta) = -\frac{\partial}{\partial \theta} P_l^0(\cos \theta)$.

A few $s = 1$ Legendre polynomials are listed below

$$P_1^1 = -P_1^{-1} = \frac{1}{\sqrt{2}} \sin \theta, \quad P_2^1 = -P_2^{-1} = \frac{3}{\sqrt{6}} \sin \theta \cos \theta, \quad (109)$$

$$P_3^1 = -P_3^{-1} = \frac{1}{\sqrt{12}} \sin \theta \left(\frac{15}{2} \cos^2 \theta - \frac{3}{2} \right),$$

$$P_4^1 = -P_4^{-1} = \frac{1}{\sqrt{20}} \sin \theta \left(\frac{35}{2} \cos \theta^3 - \frac{15}{2} \cos \theta \right), \quad (110)$$

$$P_5^1 = -P_5^{-1} = \sin \theta \frac{5}{8\sqrt{30}} (63 \cos \theta^4 - 42 \cos \theta^2 + 3) \quad (111)$$

The polynomials with spin weight $s = 2$ can be calculated from the recurrence relation, i.e.,

$$\sqrt{(l-1)(l+2)} P_l^2(\cos \theta) = \partial P_l^1(\cos \theta). \quad (112)$$

We list a few polynomials,

$$P_2^2 = P_2^{-2} = \frac{3}{2\sqrt{6}} (\sin \theta)^2, \quad P_3^2 = P_3^{-2} = \frac{15}{\sqrt{120}} (\sin \theta)^2 \cos \theta, \quad (113)$$

Products with polynomials P_1^1, P_2^1, P_3^1

Using the above formulae together with their inverse relationships we can compute the following products with $s = 1$,

$$P_1^1 P_1^1 = \frac{1}{3} (P_0^0 - P_2^0), \quad P_1^1 P_2^1 = \sqrt{3} \left(\frac{4}{5} P_1^0 - \frac{1}{5} P_3^0 \right) \quad (114)$$

$$P_1^1 P_3^1 = \frac{\sqrt{6}}{7} (P_2^0 - P_4^0), \quad P_2^1 P_2^1 = \frac{1}{5} P_0^0 + \frac{1}{7} P_2^0 - \frac{12}{35} P_4^0 \quad (115)$$

$$P_2^1 P_3^1 = \frac{\sqrt{2}}{4} \left(-\frac{20}{21} P_5^0 + \frac{4}{15} P_3^0 - \frac{177}{70} P_1^0 \right),$$

$$P_3^1 P_3^1 = \frac{1}{12} \left(-\frac{300}{77} P_6^0 + \frac{158}{385} P_4^0 + \frac{1}{14} P_2^0 + \frac{443}{140} P_0^0 \right) \quad (116)$$

Products with polynomials $s = 2$ P_2^2 and P_3^2

We list below some products of polynomials that are useful for our work.

$$\begin{aligned} P_2^2 P_2^2 &= \frac{3}{35} P_4^0 - \frac{2}{7} P_2^0 + \frac{1}{5} P_0^0, \\ P_2^2 P_3^2 &= \sqrt{20} \left(\frac{1}{21} P_5^0 - \frac{1}{15} P_3^0 + \frac{3}{70} P_1^0 \right), \\ P_3^2 P_3^2 &= \frac{10}{77} P_6^0 - \frac{3}{11} P_4^0 + \frac{1}{7} P_2^0, \end{aligned} \quad (117)$$

Other useful products

Other products of polynomials that are also useful are

$$P_1^1 P_1^0 = \frac{\sqrt{3}}{3} P_2^1, \quad P_2^1 P_1^0 = \frac{\sqrt{6}}{15} \left(\sqrt{12} P_3^1 + \frac{3\sqrt{2}}{2} P_1^1 \right), \quad (118)$$

$$P_3^1 P_1^0 = \frac{1}{7\sqrt{12}} \left(3\sqrt{20} P_4^1 + 4\sqrt{6} P_2^1 \right), \quad P_1^1 P_2^0 = \frac{\sqrt{6}}{5} P_3^1 - \frac{1}{5} P_1^1, \quad (119)$$

$$P_1^1 P_3^0 = \frac{\sqrt{10}}{7} P_4^1 - \frac{\sqrt{3}}{7} P_2^1, \quad P_2^1 P_2^0 = \frac{3}{\sqrt{6}} \left(\frac{3\sqrt{20}}{35} P_4^1 + \frac{\sqrt{6}}{21} P_2^1 \right), \quad (120)$$

$$P_2^1 P_3^0 = \frac{3}{\sqrt{6}} \left(\frac{4\sqrt{30}}{63} P_5^1 - \frac{2\sqrt{12}}{315} P_3^1 - \frac{3\sqrt{2}}{35} P_1^1 \right),$$

$$P_3^1 P_2^0 = \frac{1}{\sqrt{12}} \left(\frac{2\sqrt{30}}{7} P_5^1 + \frac{9\sqrt{12}}{70} P_3^1 + \frac{18\sqrt{2}}{35} P_1^1 \right), \quad (121)$$

$$P_3^1 P_3^0 = \frac{1}{\sqrt{12}} \left(\frac{50\sqrt{42}}{231} P_6^1 - \frac{15 + 66\sqrt{20}}{77} P_4^1 + \left(\frac{4790}{1232} - \frac{15\sqrt{6}}{7} + \frac{3\sqrt{2}}{2} \right) P_2^1 \right), \quad (122)$$

$$P_2^2 P_2^1 = \frac{3}{4} \left(-\frac{2\sqrt{20}}{35} P_4^1 + \frac{4\sqrt{6}}{21} P_2^1 \right),$$

$$P_2^2 P_3^1 = \frac{\sqrt{2}}{8} \left(-\frac{4\sqrt{30}}{21} P_5^1 + \frac{61\sqrt{12}}{105} P_3^1 - \frac{12\sqrt{2}}{35} P_1^1 \right), \quad (123)$$

$$P_3^2 P_2^1 = \frac{3\sqrt{20}}{8} \left(-\frac{8\sqrt{30}}{315} P_5^1 + \frac{16\sqrt{12}}{315} P_3^1 + \frac{4\sqrt{2}}{35} P_1^1 \right), \quad (124)$$

$$P_3^2 P_3^1 = \frac{\sqrt{10}}{8} \left(-\frac{20\sqrt{42}}{231} P_6^1 + \frac{(180\sqrt{20} - 315)}{350} P_4^1 + \frac{44\sqrt{6} + 119}{56} P_2^1 \right), \quad (125)$$

Appendix B: Komar integral and angular momentum

The Komar integral associated with the rotation Killing field $K_{(\varphi)}^a$ yields a suitable definition of angular momentum for vacuum axially symmetric spacetimes,

Using Stokes theorem and the fact that $K_{(\varphi)}^a$ is a Killing field we have

$$\oint_{\partial\Sigma} \nabla^a K_{(\varphi)}^b dS_{ab} = 2 \int_{\Sigma} R_{ab} K_{(\varphi)}^b d\Sigma^a = 0, \quad (126)$$

where $\partial\Sigma$ is the boundary of the hypersurface Σ . Using Bondi coordinates on an asymptotically flat spacetime one can choose Σ as the $r = \text{const.}$ and $\partial\Sigma$ as two $u = \text{const.}$ 2-surfaces. It then follows from the above equations that

$$J^z = \frac{1}{16\pi} \lim_{r \rightarrow \infty} \oint_u \nabla^a K_{(\varphi)}^b dS_{ab} = \text{const.}, \quad (127)$$

i.e., the value of the integral does not depend on the Bondi time u . One can explicitly integrate this equation in the N-P formalism to obtain a formula at \mathcal{I}^+ in terms of the spin coefficients. We first write the Killing vector field $K_{(\varphi)}^b$ as a combination of the null tetrad vectors as

$$K_{(\varphi)}^b = \frac{\text{Im}(\sigma^0 \bar{\omega}^0)}{r} \cos \theta l^b - ir \cos \theta m^b + ir \cos \theta \bar{m}^b, \quad (128)$$

and the two-dimensional surface area can also be expressed as

$$dS_{ab} = -2n_{[a}l_{b]}r^2 \sin \theta d\theta d\varphi, \quad (129)$$

thus the Komar integral can be written as

$$J^z = -\frac{1}{16\pi} \lim_{r \rightarrow \infty} \int_0^\pi \int_0^{2\pi} \nabla^a K_{(\varphi)}^b (n_a l_b - l_a n_b) r^2 \sin \theta d\varphi d\theta. \quad (130)$$

Using Eq. (128), and writing this equation up to order $O(r^{-2})$ we get

$$\nabla^a K_{(\varphi)}^b (n_a l_b - l_a n_b) = \frac{\cos \theta}{r^2} \text{Im}(\psi_1^0 + \sigma^0 \bar{\omega}^0) \quad (131)$$

where $\bar{\omega}^0 = -\bar{\partial}\bar{\sigma}^0$ [8, 18]. Thus, the Komar integral can be written as

$$J^z = \frac{1}{8} \int_1^{-1} \text{Im}(\psi_1^0 - \sigma^0 \bar{\partial}\bar{\sigma}^0) \cos \theta d(\cos \theta). \quad (132)$$

where we have used the axial symmetry to integrate in the azimuth direction. Finally, inserting the appropriate constants back into the equation yields,

$$J^z = -\frac{c}{6\sqrt{2}G} \text{Im}(\psi_1^0 - \sigma^0 \bar{\partial}\bar{\sigma}^0)|_{l=1} \quad (133)$$

Appendix C: Gravitational radiation: gauge TT and postnewtonian methods

The equations of motion in the postnewtonian approximation are usually given in the Transverse Traceless (TT) gauge [11, 12]. To compare our results with those coming

from this approximation it is convenient to use the same gauge. We give a brief review below of the interaction of two point particles in the TT gauge.

Given a cartesian coordinate system, (x, y, z, t) with coordinate basis vectors will e_x, e_y, e_z , we introduce a standard spherical coordinate system (r, θ, ϕ) , where θ its the angle inclination from the axis z and ϕ its the phase angle.

If the waves propagate in the direction of the radial unit vector

$$e_r = e_x \sin \theta \cos \phi + e_y \sin \theta \sin \phi + e_z \cos \theta, \quad (134)$$

a natural set of vectors orthogonal basis of which to build the basic tensors TT is

$$e_\theta = e_x \cos \theta \cos \phi + e_y \cos \theta \sin \phi - e_z \sin \theta \quad (135)$$

$$e_\phi = -e_x \sin \phi + e_y \cos \phi \quad (136)$$

Our problem then is how to extract information from the radiation detected at null infinity in relation to the Weyl scalars, i.e., how to relate ψ_4 to the second time derivative h_{ij} , which can written as

$$h_{ij} = A_{ij} \frac{M}{r} + O(r^{-2}), \quad (137)$$

where A_{ij} does not depend on r and M is the total mass of the system.

It is convenient to separate h_{ij} in two polarizations (h_+ and h_\times) modes in the wave zone.

One can show that h_{ij} can be written as

$$h = \sum_{i,j} h_{ij} e_i \otimes e_j = h_+ (e_\theta \otimes e_\theta - e_\phi \otimes e_\phi) + h_\times (e_\theta \otimes e_\phi + e_\phi \otimes e_\theta), \quad (138)$$

Linearized gravity then tell us that two time derivatives of (138) is related to the Weyl tensor.

Using The Newman Penrose formalism one can show that the gravitational radiation is contained in a particular component of the Weyl tensor, ψ_4 ,

$$\psi_4 = C_{abcd} n^a m^b n^c m^d, \quad (139)$$

where n^a and m^a are two vectors of a null tetrad constructed from the orthonormal spherical basis

$$\begin{aligned} l^a &= \frac{1}{\sqrt{2}} (e_t^a + e_r^a), \\ n^a &= \frac{1}{\sqrt{2}} (e_t^a - e_r^a), \\ m^a &= \frac{1}{\sqrt{2}} (e_\theta^a + i e_\phi^a), \end{aligned}$$

$$m^a = \frac{1}{\sqrt{2}} \left(e_\theta^a - i e_\phi^a \right), \quad (140)$$

where e_t^a is the unit timelike vector and $e_r^a, e_\theta^a, e_\phi^a$ are usual orthonormal basis induced by the spherical coordinates.

One can then show that

$$\psi_4 = \ddot{h}_+ - i \ddot{h}_\times \quad (141)$$

At a sufficiently large distance from the source of the gravitational waves is write

$$\psi_4 = \psi_4^0 r^{-1} + \dots = -\ddot{\sigma}_B r^{-1} + \dots \quad (142)$$

where σ_B is the Bondi shear of the null congruence of the geodesics. We are assuming that σ_B (38) contains radiative quadrupole and octupole terms and that they are related with the two polarizations h_+ and h_\times .

The above equations show that $h_+ - i h_\times$ can be expressed in terms of the mass quadrupole and octupole moments ($Q_+^{\ddot{\cdot}}$ and $O_+^{\ddot{\cdot}}$) and current quadrupole and octupole moments ($Q_\times^{\ddot{\cdot}}$ and $O_\times^{\ddot{\cdot}}$), as it is written in the Eqs. (39, 40).

If one then write h^{ij} in terms of the mass and current source multiple moments up to post-Newtonian order 1/2 [3] we then have

$$h^{ij} = \frac{2}{r} \left\{ I^{(2)ij} + \frac{1}{3} I^{(3)ijk} N^k + \dots \right. \\ \left. + \epsilon^{kl(i} \left[\frac{4}{3} J^{(2)j)k} N^l + \frac{1}{2} J^{(3)j)km} N^l N^m + \dots \right] \right\}_{TT}$$

where $I^{ij\cdots}$ and $J^{ij\cdots}$ are respectively the mass and current multipole moments, D is the distance from the source to the observer, N^i is the unit vector from the center of mass of the source to the observation point (in the axisymmetric case $N^i = e_r^i$ because of the center of mass is located in the origin of coordinates), the number in parentheses (n) indicates the number of derivatives with respect to the delayed time and ϵ^{ijk} is the Levi-Civita symbol.

The source mass multipole moments can be written as

$$I^L(u) = \int (x^L)^{STF} \rho(\vec{x}, u) d^3x \\ - \frac{4(2l+1)}{(l+1)(2l+3)} \frac{d}{du} \int (x^{iL})^{STF} \rho^i(\vec{x}, u) d^3x \\ + \frac{1}{2(2l+3)} \frac{d^2}{du^2} \int |\vec{x}|^2 (x^L)^{STF} \rho(\vec{x}, u) d^3x \quad (143)$$

where L is a multiple index, i.e. indicates that $x^L \equiv x^{i_1} x^{i_2} x^{i_3} \dots x^{i_l}$, while STF (Symmetric Trace Free) indicates that this is a traceless symmetric tensor.

If we apply the above formulae to a binary system and in particular to the mass quadrupole moment one obtains [3]

$$\begin{aligned}
 I^{ij} = \sum_A \left\{ m_A \left[x_A^i x_A^j \right]^{STF} \left[1 + \frac{3}{2} v_A^2 - \sum_{B \neq A} \frac{M}{r_{AB}} \right] \right. \\
 - \frac{20}{21} \frac{d}{dt} \left[m_A \left[x_A^i x_A^j x_A^k \right]^{STF} v_A^k \right] \\
 + \frac{1}{14} \frac{d^2}{dt^2} \left[m_A x_A^2 \left[x_A^i x_A^j \right]^{STF} \right] + 4 \left[x_A^i \left(\vec{v}_A \times \vec{S}_A \right)^j \right]^{STF} \\
 \left. - \frac{4}{3} \frac{d}{dt} \left[x_A^i \left(\vec{x}_A \times \vec{S}_A \right)^j \right]^{STF} \right\} \quad (144)
 \end{aligned}$$

We now apply the above equation to the head on collision of two different point masses m_1 and m_2 along a z direction, keeping axisymmetric conditions.

As we deal with axisymmetric case the vectors are aligned then $\vec{v}_A \times \vec{S}_A = 0$ and $\vec{x}_A \times \vec{S}_A = 0$.

Using relative coordinates $x^i = x_2^i - x_1^i$; $v^i = v_2^i - v_1^i$, and post-Newtonian corrections [3], the mass quadrupole and octupole terms can be written as

$$I^{ij} = \mu \left(x^i x^j \right)^{STF} \left[1 + \frac{29}{42} (1 - 3\eta) v^2 - \frac{1}{7} (5 - 8\eta) \frac{m}{r} \right] + \dots \quad (145)$$

$$I^{ijk} = -\mu \frac{\delta m}{m} \left(x^i x^j x^k \right)^{STF} \left[1 + \frac{1}{6} (5 - 19\eta) v^2 - \frac{1}{6} (5 - 13\eta) \frac{m}{r} \right] + \dots \quad (146)$$

where $m = m_1 + m_2$; $\mu = \frac{m_1 m_2}{m}$ it is the reduced mass; $\delta m = m_1 - m_2$; $\eta = \frac{\mu}{m}$.

To exhibit the postnewtonian corrections, it is useful to identify the newtonian quadrupole (I_N^{ij}) and octupole (I_N^{ijk}) moments in the above equations, i.e.

$$I_{PN}^{ij} = I_N^{ij} \left[1 + \frac{29}{42} (1 - 3\eta) v^2 - \frac{1}{7} (5 - 8\eta) \frac{m}{r} + \dots \right], \quad (147)$$

$$I_{PN}^{ijk} = I_N^{ijk} \left[1 + \frac{1}{6} (5 - 19\eta) v^2 - \frac{1}{6} (5 - 13\eta) \frac{m}{r} + \dots \right], \quad (148)$$

where

$$I_N^{ij} = \mu \left[3x^i x^j - \delta^{ij} x^2 \right], \quad (149)$$

$$I_N^{ijk} = -\mu \frac{\delta m}{m} \left[15x^i x_j x^k - 3x^j \delta^{ik} x^2 - 3x^i \delta^{jk} x^2 - 3x^k \delta^{ij} x^2 \right]. \quad (150)$$

A direct calculation of the above components shows that the only nontrivial terms are given by

$$I_N^{zz} = 2\mu x^2, \quad I_N^{zz} = -6\mu \frac{\delta m}{m} x^3 \quad (151)$$

Although in many astrophysical situations the spin effects can be very small and negligible [13], in our situation they give the only nontrivial contribution to the current moments.

To lowest orders, the current multipole moments are given by

$$J^{iL} = \left\{ \epsilon^{iab} \int \sigma^b x^{aL} d^3x \right\}^{STF} \quad (152)$$

The current quadrupole moment J^{ij} has an orbital angular momentum as well as a spin contribution. Substituting $\sigma(x, t) = T^{00} + T^{ii}$; $\sigma(x, t) = T^{0i}$; $T^{0i} = \rho^* v^i$ and $T^{ij} = \rho^* v^i v^j + p \delta^{ij}$, the current quadrupole moment is given by

$$J^{ij} = \left\{ \sum_A \epsilon^{iab} \int_A \rho^* \left(\vec{x} \right) v^b x^a x^j d^3x \right\}^{STF}. \quad (153)$$

Using $\int_A \rho^* \left(\vec{x} \right) \vec{x}_A^a v_A^b d^3x = \frac{1}{2} \epsilon^{iab} S_A^i$ together with axial symmetry, the current quadrupole moment can be written as

$$J^{ij} = \frac{3}{2} \sum_A \left(x_A^i S_A^j \right)^{STF}, \quad (154)$$

and,

$$J^{ijk} = 2 \sum_A \left(x_A^i x_A^j S_A^k \right)^{STF}. \quad (155)$$

Writing the above equations written in relative coordinates and using the post-Newtonian correction for the quadrupole current moment of Wiseman [3], we obtain

$$J^{ij} = -\frac{3}{2} \eta \left(x^i \Delta^j \right)^{STF}, \quad (156)$$

and

$$J^{ijk} = 2\eta\mu (1 - 3\eta) \left(x^i x^j \xi^k \right)^{STF}, \quad (157)$$

where $\xi^z = S^z + \frac{\delta m}{m} \Delta^z$; $S^z = S_1^z + S_2^z$ and $\Delta^z = m \left(\frac{S_2^z}{m_2} - \frac{S_1^z}{m_1} \right)$.

The only nontrivial components are given by

$$J^{zz} = -\frac{3}{2}\eta x \Delta^z \quad J^{zzz} = 2\eta\mu (1 - 3\eta) x^2 \xi^z. \quad (158)$$

To compute the quadrupole and octupole current moments we take two time derivatives on the spin terms. As the spins of the bodies are aligned with the angular momentum which is constant in the axisymmetric case we have that $\dot{J} = \dot{S}^z = 0$. We can also assume that each individual spin does not change in magnitude. Thus the only allowed changes are its orientation, parallel or antiparallel, and they enter as parameters in the calculations.

Therefore, the quadrupole and octupole current moments depend on x and x^2 respectively but the parameter coefficients are much smaller than the mass counterparts [3]. Summarizing, in a typical head-on collision the mass quadrupole moment will be dominant in the emission of gravitational radiation, followed by the mass octupole moment and even a smaller contribution from the current quadrupole moment. Since the current octupole moment is at least an order of magnitude smaller than the above mentioned terms and for typical rotating objects is negligible [3]; we can safely assume in our equations that the octupole current moment is null.

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