

TWO-WEIGHT NORM INEQUALITIES FOR THE LOCAL MAXIMAL FUNCTION

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ABSTRACT. For a local maximal function defined on a certain family of cubes lying “well inside” of Ω , a proper open subset of \mathbb{R}^n , we characterize the couple of weights (u, v) for which it is bounded from $L^p(v)$ on $L^q(u)$.

1. INTRODUCTION

Let Ω be a proper open and non empty subset of \mathbb{R}^n . Let $Q = Q(x, l)$ be a cube with sides parallel to the axes. Here x and l denotes its center and half its side length respectively. For $0 < \beta < 1$ we consider the family of cubes well-inside of Ω defined by

$$\mathcal{F}_\beta = \{Q(x, l) : x \in \Omega, l < \beta d(x, \Omega^c)\},$$

where, as in all of this work, d denotes the d_∞ metric. Related to this family we have the following local maximal function on Ω :

$$(1.1) \quad M_\beta f(x) = \sup_{x \in Q \in \mathcal{F}_\beta} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

for every $f \in L^1_{\text{loc}}(\Omega)$ and every $x \in \Omega$.

In 2014 E. Harboure and the two last authors ([3]) considered this operator in the more general setting of a metric spaces X instead of \mathbb{R}^n with the Lebesgue measure replaced by a Borel measure μ defined only on Ω and doubling on the balls of \mathcal{F}_β (i.e.: $\mu(B(x, 2r)) \leq c\mu(B(x, r))$, whenever $B(x, 2r) \in \mathcal{F}_\beta$). The main result of [3] was a characterization of the weights w such that M_β is bounded from $L^p(\Omega, w\mu)$ to $L^p(\Omega, w\mu)$, $1 < p < \infty$, that is there exists a constant C such that

$$\int_\Omega |M_\beta f|^p w d\mu \leq C \int_\Omega |f|^p w d\mu,$$

for every function $f \in L^p(\Omega, w d\mu)$. The classes of weights related to this boundedness are a local version of the well known A_p -Muckenhoupt classes, associated to the Hardy-Littlewood maximal operator ([5]), more precisely non negative functions $w \in L^1_{\text{loc}}(\Omega, w d\mu)$ such that

$$\left(\frac{1}{\mu(B)} \int_B w d\mu \right) \left(\frac{1}{\mu(B)} \int_B w^{-\frac{1}{p-1}} d\mu \right)^{p-1} \leq C_\beta,$$

for every ball B in \mathcal{F}_β .

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After seeing this result, it is natural to ask ourselves about the problem for a couple of weights (v, w) . In connection with it, we should recall that the situation in the case $\Omega = \mathbb{R}^n$ do not have an easy answer. In fact, E. Sawyer ([7]) proved that the necessary and sufficient condition is

$$\int_Q |M(v^{-\frac{1}{p-1}} \chi_Q)|^p w \, dx \leq C \int_Q v^{-\frac{1}{p-1}} \, dx,$$

for every cube $Q \subset \mathbb{R}^n$. The problem becomes a little worse if we want to consider the boundedness from L^p to L^q with $1 < p \leq q < \infty$. In this case, Sawyer again, but this time as a particular case of his solution of the problem for fractional maximal ([8]), showed that the condition turns out to be

$$(1.2) \quad \left(\int_Q |M(v^{-\frac{1}{p-1}} \chi_Q)|^q w \, dx \right)^{\frac{1}{q}} \leq C \left(\int_Q v^{-\frac{1}{p-1}} \, dx \right)^{\frac{1}{p}}.$$

Our setting is even a bit more complicated since the family \mathcal{F}_β does not include all the balls needed to consider Ω as a metric space itself. At this point, if we restrict the problem to the case $p = q$, a simple application of a result due to B. Jawerth (Theorem 3.1, p. 382 [4]) allows us to get

Theorem 1.3. *Given $1 < p < \infty$, $0 < \beta < 1$. Let (u, v) be a pair of weights. Then assuming that $\sigma = v^{1-p'}$ is a weight, the following statement are equivalent:*

$$(1.4) \quad M_\beta : L^p(v) \rightarrow L^p(u),$$

if and only if there is a constant c such that

$$(1.5) \quad \int_F M_\beta(\sigma \chi_F)^p u \leq C \int_F \sigma < \infty,$$

for all finite unions F of cubes in \mathcal{F}_β , $F = \cup_{finite} Q_i$, $Q_i \in \mathcal{F}_\beta$; provided

$$(1.6) \quad M_{\beta, \sigma} : L^p(\sigma) \rightarrow L^p(\sigma).$$

where

$$M_{\beta, \sigma} f(x) = \sup_{x \in Q \in \mathcal{F}_\beta} \frac{1}{\sigma(Q)} \int_Q |f(y)| \sigma(y) \, dy.$$

Leaving aside that we are not getting an answer to the whole problem, the hypothesis on (u, v) have two drawbacks. In the first place, integrals over finite unions of cubes must be calculated instead of only integrals over cubes like in (1.2). In the second place the conditions involve the operator itself, which looks worse. The first disadvantage can be overcome by assuming an extra hypothesis on the weight v : a doubling condition related to $v^{-\frac{1}{p-1}}$ over balls of \mathcal{F}_β . We say that a weight u satisfies a doubling condition on cubes of \mathcal{F}_β , denoted by $u \in D_\beta$, whenever there exists a constant $C = C(\beta)$ such that

$$u(2Q) \leq C_\beta u(Q) < \infty,$$

for every cube in \mathcal{F}_β such that $2Q \in \mathcal{F}_\beta$, where $2Q$ means the concentric cube with side length two times the side length of Q , and $u(Q) = \int_Q u \, dx$.

By assuming a \mathcal{D}_β condition on $v^{-\frac{1}{p-1}}$, an application of results in [3] shows that our context fulfill the hypothesis about the boundedness of $M_{\beta, \sigma}$ in Theorem 1.3. But taking into account the additional geometric information we get about the sets on which the maximal is defined (notice that the Theorem of Jawerth is related to

general basis of open sets in \mathbb{R}^n), a better result can be obtained. Indeed we can prove the following Theorem.

Theorem 1.7. *Given $1 < p \leq q < \infty$, $0 < \beta < 1$. Let (u, v) be a pair of weights such that $\sigma = v^{1-p'} \in D_\beta$, then*

$$(1.8) \quad \left(\int_{\Omega} (M_\beta f)^q u \right)^{1/q} \leq C \left(\int_{\Omega} |f|^p v \right)^{1/p},$$

for every function $f \in L^p(v)$ if and only if

$$(1.9) \quad \left(\int_Q M_\beta(\sigma \chi_Q)^q u \right)^{1/q} \leq C \left(\int_Q \sigma \right)^{1/p} < \infty,$$

for every cube $Q \in \mathcal{F}_\beta$.

Note that the hypothesis (1.9) looks like (1.2). However the appearance of the operator, the second problem we have mentioned, makes it difficult to check the condition. In the case $\Omega = \mathbb{R}^n$ C. Pérez (Theorem 1.1, [6]) gave a solution by adding an A_∞ -condition on $v^{-\frac{1}{p-1}}$. We recall that a weight u belongs to the A_∞ class of Muckenhoupt if there are positive constants c and δ such that

$$(1.10) \quad \frac{u(E)}{u(Q)} \leq c \left(\frac{|E|}{|Q|} \right)^\delta,$$

for every measurable set $E \subset Q$ and every cube Q . With this extra assumption, the necessary and sufficient condition for the boundedness of the maximal is the existence of a constant C such that

$$(1.11) \quad \frac{u(Q)^{\frac{p}{q}} \left(v^{-\frac{1}{p-1}}(Q) \right)^{p-1}}{|Q|^p} \leq C,$$

for every cube Q ; which sometimes is referred to as $A_{p,q}$ condition. It is important to note that we cannot apply the solution given by C. Pérez because, as it was said before, our setting is not even a metric space. However, it served as a source of inspiration for our second result. In order to formulate it we introduce a couple of definitions.

Definition 1.12. *Given $0 < \beta < 1$, we say that a weight u belongs to A_∞^β if it there are positive constants c and δ such that (1.10) holds for every $Q \in \mathcal{F}_\beta$.*

Definition 1.13. *Let $1 < p \leq q < \infty$ and $0 < \beta < 1$. We say that the weights u and v lies in the class $A_{p,q}^\beta$ if and only if*

$$(1.14) \quad \frac{u(Q)^{p/q}}{|Q|} \left(\frac{\sigma(Q)}{|Q|} \right)^{p-1} \leq C,$$

for every cube $Q \in \mathcal{F}_\beta$, where $\sigma = v^{-\frac{1}{p-1}}$. In this cases we write $(u, v) \in A_{p,q}^\beta$.

Now we are in position to enunciate our second theorem where the reference to the operator in the hypothesis on the weights is completely avoided.

Theorem 1.15. *Let p, q, β and the weights u and v as in the Theorem above. In addition if $u \in D_\beta$ and $\sigma = v^{-1/(p-1)}$ belongs to A_∞^β , then*

$$(1.16) \quad M_\beta : L^p(v) \rightarrow L^q(u);$$

if and only if

$$(1.17) \quad (u, v) \in A_{p,q}^\beta .$$

We note that, under the hypothesis of Theorem 1.15, the classes $A_{p,q}^\beta$ coincides for different values of β . So, as is the one-weight case, we can refer to those weights as local weights (see Lemma 4.1 in section 4.)

As an important tool to prove the theorem above we consider the centered local maximal function on Ω , namely M_β^c given by

$$(1.18) \quad M_\beta^c f(x) = \sup_{\substack{Q=Q(x,l) \\ Q \in \mathcal{F}_\beta}} \frac{1}{|Q|} \int_Q |f(y)| dy ,$$

for every $f \in L_{\text{loc}}^1(\Omega)$ and every $x \in \Omega$. For this operator we show that the following theorem holds. We enunciate it here because it is important itself.

Theorem 1.19. *Let $1 < p \leq q < \infty$, $0 < \beta < 1$ and let u and v be two weights such that $\sigma = v^{-1/(p-1)}$ belongs to A_∞^β . Then*

$$(1.20) \quad M_\beta^c : L^p(v) \rightarrow L^q(u);$$

if and only if

$$(1.21) \quad (u, v) \in A_{p,q}^\beta .$$

Remark 1.22. Although the statements of our theorems are in terms of the maximal operator we want to remark that minor modifications in the proofs lead us to corresponding results for a fractional maximal function defined over \mathcal{F}_β .

The structure of the paper is as follows. Section 2 contains some useful geometrical lemmas. The proofs of Theorem 1.7 is in section 3. Finally the proofs of Theorems 1.15 and 1.19 are in section 4.

2. TECHNICAL LEMMAS

In this section we present a covering theorem and several covering results necessary for the proof of results below. We will write the following well-known theorem adapted to the context in our work and without proof.

Theorem 2.1 (Besicovitch Covering Theorem). *Let $E \in \mathbb{R}^n$. For each $x \in E$, let Q_x be a cube centered at x . Assume that E is bounded or that $\sup_{x \in E} l_{Q_x} < 1$. Then, there exists a countable set $E_0 \subset E$ and a constant $C(n) \in \mathbb{N}$ such that*

$$(2.2) \quad E \subset \bigcup_{x \in E_0} Q_x ;$$

$$(2.3) \quad \sum_{x \in E_0} \chi_{Q_x} \leq C(n) .$$

Now, we need to explain the notion of “cloud” of a given cube. That is, given $0 < \beta < 1$ and a cube $Q \in \mathcal{F}_\beta$, we shall denote the set

$$(2.4) \quad \mathcal{N}_\beta(Q) = \bigcup_{\substack{R \cap Q \neq \emptyset \\ R \in \mathcal{F}_\beta}} R,$$

and we say that these are the “cloud” of Q . This idea was introduced in [3] and the proof of the following lemmas can be found there in the context of the metric spaces.

Lemma 2.5. *Let $Q = Q(x, l) \in \mathcal{F}_\beta$ such that $10Q \notin \mathcal{F}_\beta$. We consider $k_0 \in \mathbb{Z}$ such that $2^{k_0-1} \leq d(x, \Omega^c) < 2^{k_0}$. Then there exists natural numbers h_1, h_2 independent of Q such that*

$$2^{k_0-h_1-1} \leq d(y, \Omega^c) < 2^{k_0+h_2}, \quad \text{for every } y \in \mathcal{N}_\beta(Q).$$

Proof. The proof is a consequence of the claim 1 and 2 contained in the proof of the lemma 2.3 in [3]. \square

Now, denoting by \mathcal{D} the usual family of dyadic cubes belonging to \mathcal{F}_β we have the following lemma.

Lemma 2.6. *Let Ω be an open proper subset of \mathbb{R}^n . Given $0 < \beta < 1$, for each $t \in \mathbb{N}$ such that $2^{-t} \leq \beta/5$, there exists a covering \mathcal{W}_t of Ω by dyadic cubes belonging to \mathcal{F}_β and satisfying the following properties*

i) If $R = R(x_R, l_R) \in \mathcal{W}_t$, then $10R \in \mathcal{F}_\beta$ and

$$2^{-t-3} d(x_R, \Omega^c) \leq l_R \leq 2^{-t-1} d(x_R, \Omega^c).$$

ii) There is a number M , only depending on β and t , such that for any cube $Q_0 = Q(x_0, l_0) \in \mathcal{F}_\beta$ with $10Q_0 \notin \mathcal{F}_\beta$, the cardinal of the set

$$\mathcal{W}_t(Q_0) = \{R \in \mathcal{W}_t : R \cap \mathcal{N}_\beta(Q_0) \neq \emptyset\},$$

is at most M . We will call the union of this cubes as

$$\mathcal{W}_{t, Q_0} = \bigcup_{R \in \mathcal{W}_t(Q_0)} R.$$

Proof. We will follow the ideas of Lemma 2.3 in [3]. So, we only show how we take the covering \mathcal{W}_t . For $k \in \mathbb{Z}$, we consider the bands defined by

$$\Omega_k = \{x \in \Omega : 2^{k-1} \leq d(x, \Omega^c) < 2^k\}.$$

If Ω_k is non empty, let us consider the collection G_k of all usual dyadic cubes $Q_j = Q(x_j, l_j)$ such that

$$l_j = 2^{k-t-2}, \quad Q_j \cap \Omega_k \neq \emptyset,$$

where t is given as in the hypothesis. It is clear that $\Omega_k \subset G_k$. Moreover, taking $y \in Q_j$ and $z \in Q_j \cap \Omega_k$ we get

$$d(y, \Omega^c) \leq d(z, \Omega^c) + d(y, z) \leq 2^k + 2l_j = 2^k + 2^{k-t-1} < 2^{k+1},$$

and

$$(2.7) \quad d(y, \Omega^c) \geq d(z, \Omega^c) - d(y, z) > 2^{k-1} - l_j = 2^{k-1} - 2^{k-t-2} > 2^{k-2},$$

so the inclusion

$$(2.8) \quad Q_j \subset \Omega_{k-1} \cup \Omega_k \cup \Omega_{k+1},$$

holds. However there are not cubes intersecting three bands simultaneously. In fact, suppose that there exists $z, w \in Q_j$ such that $z \in \Omega_{k-1}$ and $w \in \Omega_{k+1}$. Then

$$2^{k-1} = 2^k - 2^{k-1} \leq d(w, \Omega^c) - d(z, \Omega^c) \leq d(w, z) \leq 2l_j = 2^{k-t-1}.$$

This implies that t is less than or equal to 0 which is a contradiction. In conclusion, we can say that, for a fixed k there exists in G_k three classes of cubes

$$Q_j \cap \Omega_{k-1} \neq \emptyset, \quad \text{ó} \quad Q_j \subset \Omega_k, \quad \text{ó} \quad Q_j \cap \Omega_{k+1} \neq \emptyset.$$

Next, for each k we define the new collection E_k as follows: if either $Q_j \subset \Omega_k$ or $Q_j \cap \Omega_{k+1} \neq \emptyset$ we put the cube Q_j in E_k . If $Q_j \cap \Omega_{k-1} \neq \emptyset$ we consider the 2^n dyadic sub-cubes and put them in E_{k-1} . So, we note that E_k contains some cubes from E_{k+1} that have been subdivided into 2^n sub-cubes. Thus, the collections E_k are pairwise disjoint and for each $Q_j(x_j, l_j) \in E_k$ we have that $l_j = 2^{k-t-2}$ and

$$2^{k-1} < d(x_j, \Omega^c) \leq 2^{k+1}.$$

Now, we are able to define a disjoint collection of dyadic cubes by

$$(2.9) \quad \mathcal{W}_t = \bigcup_k E_k.$$

This is the family of cubes that we will consider. Then, the properties of the lemma follows by analogous arguments of [3]. \square

Lemma 2.10. *Let $0 < \beta < 1$, $\Omega \subset \mathbb{R}^n$ and μ be a measure doubling on \mathcal{F}_β . We consider $t \in \mathbb{Z}$ such that $2^{-t} \leq \beta/20$ and the covering \mathcal{W}_t of the Lemma above. Then, for any cube Q such that $10Q \notin \mathcal{F}_\beta$ there exists a constant K depending only on β and the constant of the doubling property of μ such that*

$$\mu(\mathcal{W}_{t,Q}) \leq K \nu(Q),$$

where $\mathcal{W}_{t,Q}$ is as in Lemma above.

Proof. The proof follows the same lines as in Remark 3.2 in [3] in the general setting of metric spaces. \square

Remark 2.11. Since $\mathcal{N}_\beta(Q) \subset \mathcal{W}_{t,Q}$ for every cube in \mathcal{F}_β , by the Lemma above we can deduce that

$$(2.12) \quad \mu(\mathcal{N}_\beta(Q)) \leq C \mu(Q).$$

We observe by the construction (2.9) that for each cube $Q_j(x_j, l_j) \in E_k \subset \mathcal{W}_t$ we get

$$\frac{1}{2} 2^{-t-2} \leq \frac{l_j}{d(x_j, \Omega^c)} < 2 2^{-t-2}.$$

In general, we will say that a collection of cubes $\{Q_i\}$ is of Whitney's type if there exists constants $0 < c_1 < c_2 < 1$ such that

$$c_1 < l_{Q_i}/d(x_{Q_i}, \Omega^c) < c_2.$$

Lemma 2.13. *Let $\{Q_i\}$ be a pairwise disjoint collection of Whitney's type cubes. Then their clouds have bounded overlapping. More precisely, there exists a natural number $M > 0$ such that*

$$\sum_i \chi_{\mathcal{N}_\beta(Q_i)}(x) \leq M,$$

for every x in Ω .

Proof. Let $\{Q_i\}$ be a such collection, x_i and l_i their centers and length sides respectively. Take again the bands Ω_k as in Lemma 2.6. We consider $x \in \Omega_k$ and assume that

$$x \in \bigcap_{i \in F} \mathcal{N}_\beta(Q_i),$$

for some family of index F . Let us prove that there is a constant M such that the cardinal of this family is controlled by M for every point $x \in \Omega$. By lemma 2.5, if the center $x_i \in \Omega_{k_i}$, we can say that

$$\mathcal{N}_\beta(Q_i) \subset \bigcup_{j=k_i-h_1}^{k_i+h_2} \Omega_j.$$

Thus, the range of j is independent of Q_i and equal to $h = h_2 + h_1$. Now, since $k_i - h_1 \leq k \leq k_i + h_2$ for every $i \in F$ it is easy to see that

$$(2.14) \quad \bigcup_{i \in F} \mathcal{N}_\beta(Q_i) \subset \bigcup_{j=k-h}^{k+h} \Omega_j,$$

that is, the range of values that may be the union of the clouds is $2h$.

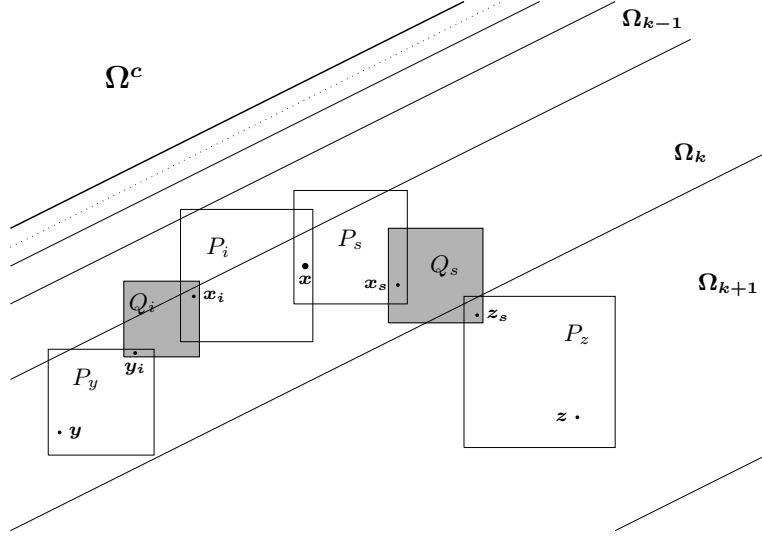
Now, suppose that there exists $y, z \in \mathcal{N}_\beta(Q_i) \cup \mathcal{N}_\beta(Q_s)$ with $i, s \in F$ and $l_i \leq l_s$. Let P_y, P_z, P_i and P_s be cubes such that

$$y \in P_y, \quad P_y \cap Q_i \neq \emptyset, \quad z \in P_z, \quad P_z \cap Q_s \neq \emptyset,$$

$$Q_i \cap P_i \neq \emptyset, \quad Q_s \cap P_s \neq \emptyset \quad \text{and} \quad x \in P_i \cap P_s.$$

Now, we take as in the figure the points

$$y_i \in P_y \cap Q_i, \quad z_s \in P_z \cap Q_s, \quad x_i \in Q_i \cap P_i \quad \text{and} \quad x_s \in Q_s \cap P_s.$$



Then, since all the cubes belongs to \mathcal{F}_β and considering (2.14) we have the following estimation

$$\begin{aligned} d(y, z) &\leq d(y, y_i) + d(y_i, x_i) + d(x_i, x) + d(x, x_s) + d(x_s, z_s) + d(z_s, z) \\ &\leq 6\beta 2^{k+h}. \end{aligned}$$

On the other hand

$$l_i > c_1 d(x_i, \Omega^c) > c_1 2^{k-h-1}.$$

Thus, there exists at most

$$\frac{6\beta 2^{k+h}}{l_i} \leq \frac{6\beta}{c_1} 2^{k+h-k+h+1} = C_\beta,$$

disjoint cubes of the family $\{Q_i\}$. This fact and (2.14) say that the family F is finite and then there exists a fixed natural number M , depending only on β such that

$$\sum_i \chi_{\mathcal{N}_\beta(Q_i)}(x) \leq M,$$

as we wanted to prove. \square

Lemma 2.15. *Let f be a non-negative, locally integrable function and μ be a doubling measure on \mathbb{R}^n . Suppose that for some $h > 0$ and some cube $Q = Q(x_Q, l_Q) \in \mathcal{F}_\beta$*

$$\frac{1}{|Q|} \int_Q f > h.$$

(i) *If $10Q \in \mathcal{F}_\beta$ then there exists a dyadic cube $P = P(x_P, l_P)$ such that $Q \subset 5P \in \mathcal{F}_\beta$ and a positive constant c_1 , independent of Q , such that*

$$(2.16) \quad \frac{1}{|P|} \int_P f > c_1 h.$$

(ii) If $10Q \notin \mathcal{F}_\beta$ then there exists a dyadic cube $R = R(x_R, l_R)$ such that $Q \subset \mathcal{W}_{t,R}$ and a positive constant c_2 , independent of Q , such that

$$(2.17) \quad \frac{1}{|R|} \int_R f > c_2 h,$$

where \mathcal{W}_t is as in the Lemma 2.6.

Proof. Let $Q = Q(x_Q, l_Q)$ be a such cube of the hypothesis. For (i) we consider $k \in \mathbb{Z}$ such that $2^{k-1} < l_Q \leq 2^k$. Considering dyadic cubes with side length equal to 2^{k-1} , there exists a finite collection of dyadic cubes P_1, \dots, P_N , with $1 \leq N \leq 3^n$, which intersect the interior of Q . Calling P any of these and taking $z \in Q \cap P$, we have

$$d(x_Q, x_P) \leq d(x_Q, z) + d(z, x_P) \leq \frac{1}{2} l_Q + \frac{1}{2} l_P \leq 2^{k-1} + 2^{k-2} = \frac{3}{2} l_P.$$

Now, if $w \in Q$ we get

$$d(w, x_P) \leq d(w, x_Q) + d(x_Q, x_P) \leq \frac{1}{2} l_Q + \frac{3}{2} l_P \leq \frac{5}{2} l_P,$$

which implies that $Q \subset 5P$. Moreover, for each $z \in 5P$

$$d(z, x_Q) \leq d(z, x_P) + d(x_Q, x_P) \leq \frac{5}{2} l_P + \frac{3}{2} l_P = 4l_P.$$

Thus, we can deduce that $Q \subset 5P \subset 8Q$. Now, a simpler estimation show that $5P \in \mathcal{F}_\beta$ whenever $10Q$ do it. In fact

$$l_P < l_Q \leq \frac{\beta}{10} d(x_Q, \Omega^c) \leq \frac{\beta}{10} (d(x_Q, x_P) + d(x_P, \Omega^c)) = \frac{\beta}{5} l_P + \frac{\beta}{10} d(x_P, \Omega^c),$$

then, recalling that $0 < \beta < 1$ we get

$$\frac{1}{2} l_P < (1 - \frac{\beta}{5}) l_P < \frac{\beta}{10} d(x_P, \Omega^c),$$

this implies that $5l_P < \beta d(x_P, \Omega^c)$ as required. Furthermore, for at least one of these dyadic cubes, which we denote by P ,

$$\int_P f > \frac{h|Q|}{3^n},$$

since otherwise we get a contradiction. In fact

$$\int_Q f \leq \sum_{t=1}^N \int_P f \leq \frac{N h |Q|}{3^n} \leq h |Q|.$$

Now, since $5P \subset 8Q$, the Lebesgue measure say that inequality (2.16) follows with $c_1 = 5^n/24^n$.

In order to prove (ii), by the Lemma 2.6ii) the cardinal of $\mathcal{W}_{t,Q}$ is finite and independent of Q , and its cubes are comparable size with Q , the same argument can be applied to take one of them, namely R such that (2.17) holds. \square

3. PROOF OF THE RESULTS

Proof of the Theorem 1.7. Assume that (1.8) holds. In particular, it is for $f = \sigma\chi_Q$, $Q \in \mathcal{F}_\beta$. Then

$$\left(\int_Q M_\beta(\sigma\chi_Q)^q u \right)^{1/q} \leq \left(\int_\Omega (\sigma\chi_Q)^p v \right)^{1/p} = \left(\int_Q \sigma \right)^{1/p} < \infty.$$

To show that (1.9) implies (1.8), fix a non negative function $f \in L^p(\Omega, v)$. By a standard argument, we may assume without loss of generality that f is bounded and has compact support. Now, for each $k \in \mathbb{Z}$, we consider the sets

$$A_k = \{x \in \Omega : 2^k < M_\beta f(x) \leq 2^{k+1}\}.$$

Considering a collection $\{Q_x^k\}_{x \in A_k}$ of cubes such that

$$\frac{1}{|Q_x^k|} \int_{Q_x^k} |f| > 2^k,$$

we define

$$\mathcal{Q}_1 = \{Q_x^k : 10Q_x^k \in \mathcal{F}_\beta\} \quad \text{and} \quad \mathcal{Q}_2 = \{Q_x^k : 10Q_x^k \notin \mathcal{F}_\beta\}.$$

For the cubes in \mathcal{Q}_1 by (i) of Lemma 2.15 there exists a dyadic cube P_x^k such that $Q_x^k \subset 5P_x^k$, $5P_x^k \in \mathcal{F}_\beta$ and

$$\frac{1}{|P_x^k|} \int_{P_x^k} f > c 2^k.$$

On the other hand, for the cubes in \mathcal{Q}_2 we take t such that $2^{-t} \leq \beta/20$ and consider the covering \mathcal{W}_t of the Lemma 2.6. Now, we can apply (ii) of Lemma 2.15 to have a dyadic cube R_x^k such that its cloud contain the original cube Q_x^k and

$$\frac{1}{|R_x^k|} \int_{R_x^k} f > c 2^k.$$

Since the P_x^k 's and R_x^k 's are dyadic and bounded in size (since f has compact support) we can obtain a maximal disjoint sub-collection $\{P_j^k\}$ such that for each x , either $Q_x^k \subset 5P_j^k$ or $Q_x^k \subset \mathcal{W}_{t, P_j^k}$ for some j .

We define $\tilde{P}_j^k = 5P_j^k$, if P_j^k was chosen from a cube in \mathcal{Q}_1 and $\tilde{P}_j^k = \mathcal{W}_{t, P_j^k}$ if P_j^k was chosen from a cube in \mathcal{Q}_2 . It is clear that $A_k \subset \cup_j \tilde{P}_j^k$. Now we define the following sets:

$$E_1^k = \tilde{P}_1^k \cap A_k, \quad E_2^k = (\tilde{P}_2^k \setminus \tilde{P}_1^k) \cap A_k, \quad \dots, \quad E_j^k = \left(\tilde{P}_j^k \setminus \bigcup_{i=1}^{j-1} \tilde{P}_i^k \right) \cap A_k, \quad \dots$$

Thus $A_k = \cup_j E_j^k$ and since the A_k 's are disjoint, the sets E_j^k 's are pairwise disjoint for all j and k .

In order to prove (1.8) we proceed as follows

$$\begin{aligned} \int_\Omega (M_\beta f)^q u &= \sum_k \int_{A_k} (M_\beta f)^q u \\ &= \sum_{j,k} \int_{E_j^k} (M_\beta f)^q u \\ &\leq C \sum_{j,k} u(E_j^k) 2^{kq} \end{aligned}$$

$$\leq C \sum_{j,k} u(E_j^k) \left(\frac{1}{|P_j^k|} \int_{P_j^k} f \right)^q.$$

Now, multiplying and dividing by $\left(\int_{\tilde{P}_j^k} \sigma \right)^q$ we have

$$\begin{aligned} \int_{\Omega} (M_{\beta} f)^q u &\leq C \sum_{j,k} u(E_j^k) \left(\frac{1}{|P_j^k|} \int_{\tilde{P}_j^k} \sigma \right)^q \left(\frac{\int_{P_j^k} (f/\sigma) \sigma}{\int_{\tilde{P}_j^k} \sigma} \right)^q \\ &= C \int_X T(f/\sigma)^q d\omega, \end{aligned}$$

where $X = \mathbb{N} \times \mathbb{Z}$, the discrete measure ω on X is given by

$$\omega(j, k) = u(E_j^k) \left(\frac{1}{|P_j^k|} \int_{\tilde{P}_j^k} \sigma \right)^q,$$

and for a non-negative, measurable function g , the operator T is defined by

$$(3.1) \quad Tg(j, k) = \frac{\int_{P_j^k} g \sigma}{\int_{\tilde{P}_j^k} \sigma}.$$

By interpolation's theory it is sufficient to show that T is weak-type $(1, q/p)$ for getting (1.8). For this, fix g bounded and with compact support. Then for $\lambda > 0$ we consider

$$B_{\lambda} = \{(j, k) \in X : Tg(j, k) > \lambda\}.$$

By the definition of \tilde{P}_j^k we have

$$\begin{aligned} B_{\lambda}^1 &= \{(j, k) \in X : Tg(j, k) > \lambda, \tilde{P}_j^k = 5P_j^k\}; \\ B_{\lambda}^2 &= \{(j, k) \in X : Tg(j, k) > \lambda, \tilde{P}_j^k = \mathcal{W}_{t, P_j^k}\}. \end{aligned}$$

Then, we can estimate

$$\omega(B_{\lambda}) = \sum_{(j,k) \in B_{\lambda}} u(E_j^k) \left(\frac{1}{|P_j^k|} \int_{\tilde{P}_j^k} \sigma \right)^q = \sum_{(j,k) \in B_{\lambda}^1} + \sum_{(j,k) \in B_{\lambda}^2} = I + II.$$

Remembering that $5P_j^k \in \mathcal{F}_{\beta}$ and since $E_j^k \subset 5P_j^k$, it is not difficult to see that

$$I \leq \sum_{(j,k) \in B_{\lambda}^1} \int_{E_j^k} M_{\beta}(\sigma \chi_{5P_j^k})^q u.$$

Let now $\{P_i\}$ be the maximal disjoint sub-collection of $\{P_j^k : (j, k) \in B_{\lambda}^1\}$. Then, since the E_j^k are pairwise disjoint and the hypothesis (1.9) we have

$$\begin{aligned} I &\leq \sum_i \sum_{P_j^k \subset P_i} \int_{E_j^k} M_{\beta}(\sigma \chi_{5P_j^k})^q u \\ &\leq \sum_i \int_{5P_i} M_{\beta}(\sigma \chi_{5P_i})^q u \\ &\leq C \sum_i \left(\int_{5P_i} \sigma \right)^{q/p}. \end{aligned}$$

Finally, by the definition of B_λ^1 , the cubes P_i 's are disjoint and $q/p \geq 1$

$$(3.2) \quad \begin{aligned} I &\leq C \sum_i \left(\frac{1}{\lambda} \int_{P_i} g \sigma \right)^{q/p} \\ &\leq C \left(\frac{1}{\lambda} \int_{\Omega} g \sigma \right)^{q/p}. \end{aligned}$$

The estimation above follows similar lines of the proof of Theorem 1.1 in [1].

Now, we need to estimate II . For this, let $(j, k) \in B_\lambda^2$ and we write

$$\mathcal{W}_{t, P_j^k} = \bigcup_{m=1}^{t_j^k} P_{j,m}^k,$$

where $P_{j,m}^k \in \mathcal{W}_t(P_j^k)$ are disjoint. By Lemma 2.6 *ii*) we know that $t_j^k \leq M$ where M is independent of the cubes. Then, considering t_j^k disjoint sets defined by

$$E_{j,m}^k = (P_{j,m}^k \setminus \bigcup_{i=1}^{j-1} \tilde{P}_i^k) \cap A_k.$$

So, we get

$$E_j^k = \bigcup_{m=1}^{t_j^k} E_{j,m}^k,$$

where the sets $E_{j,m}^k$ are disjoint in j , m and k . Then

$$\begin{aligned} II &= \sum_{(j,k) \in B_\lambda^2} u(E_j^k) \left(\frac{1}{|P_j^k|} \int_{\mathcal{W}_{t, P_j^k}} \sigma \right)^q \\ &= \sum_{(j,k) \in B_\lambda^2} \sum_{m=1}^{t_j^k} \int_{E_{j,m}^k} \left(\frac{1}{|P_j^k|} \sum_{l=1}^{t_j^k} \int_{P_{j,l}^k} \sigma \right)^q u. \end{aligned}$$

Now, we consider for each $P_{j,l}^k$ a finite chain joining $P_{j,l}^k$ with $P_{j,m}^k$, that is, a finite subset of $\mathcal{W}_t(P_j^k)$, say R_1, \dots, R_n which are all different, with $R_1 = P_{j,l}^k$ and $R_n = P_{j,m}^k$ and for R_i and R_{i+1} neither $R_i \subset R_{i+1}$ or $R_{i+1} \subset R_i$.

Moreover, part *i*) and *ii*) of the Lemma 2.6 say that $P_{j,m}^k \in \mathcal{F}_\beta$ and $n \leq M$. Thus, since σ is doubling on \mathcal{F}_β we can deduce that $\sigma(P_{j,l}^k) \leq C \sigma(P_{j,m}^k)$. Then, by the Lemma 2.10 again we have

$$\begin{aligned} II &\leq C \sum_{(j,k) \in B_\lambda^2} \sum_{m=1}^{t_j^k} \int_{E_{j,m}^k} \left(\frac{1}{|\mathcal{W}_{t, P_j^k}|} \sum_{l=1}^{t_j^k} \int_{P_{j,m}^k} \sigma \right)^q u \\ &\leq C \sum_{(j,k) \in B_\lambda^2} \sum_{m=1}^{t_j^k} \int_{E_{j,m}^k} \left(\frac{1}{|P_{j,m}^k|} \int_{P_{j,m}^k} \sigma \right)^q u \\ &\leq C \sum_{(j,k) \in B_\lambda^2} \sum_{m=1}^{t_j^k} \int_{E_{j,m}^k} M_\beta (\sigma \chi_{P_{j,m}^k})^q u, \end{aligned}$$

where the last inequality holds because $E_{j,m}^k \subset P_{j,m}^k$. Let $\{P_i\}$ be a maximal disjoint sub-collection of $\{P_{j,m}^k\}$ with $1 \leq m \leq t_j^k$ and $(j,k) \in B_\lambda^2$. Then, since the $E_{j,m}^k$'s are pairwise disjoint, we have that

$$\begin{aligned} II &\leq C \sum_i \sum_{\substack{(j,k) \in B_\lambda^2 \\ P_{j,m}^k \subset P_i}} \int_{E_{j,m}^k} M_\beta(\sigma \chi_{P_{j,m}^k})^q u \\ &\leq C \sum_i \int_{P_i} M_\beta(\sigma \chi_{P_i})^q u. \end{aligned}$$

Now, by inequality (1.9) and the fact that $q \geq p$ we get

$$\begin{aligned} II &\leq C \sum_i \left(\int_{P_i} \sigma \right)^{q/p} \\ &\leq C \left(\sum_i \int_{P_i} \sigma \right)^{q/p}. \end{aligned}$$

Finally, since the operator T is defined on the cubes P_j^k we need to take again a maximal disjoint sub-collection of the family $\{P_j^k\}$ with $(j,k) \in B_\lambda^2$. Let $\{P_s\}$ be such sub-collection. Thus, since the P_i 's are disjoint and $P_i \in \mathcal{W}_{t,P_j^k} \subset \mathcal{W}_{t,P_s}$ for some $(j,k) \in B_\lambda^2$ and some s , by the definition of the operator T we can estimate

$$\begin{aligned} II &\leq C \left(\sum_s \sum_{i: P_i \subset \mathcal{W}_{t,P_s}} \int_{P_i} \sigma \right)^{q/p} \\ &\leq C \left(\sum_s \int_{\mathcal{W}_{t,P_s}} \sigma \right)^{q/p} \\ &\leq C \left(\frac{1}{\lambda} \sum_s \int_{P_s} g \sigma \right)^{q/p} \\ &\leq C \left(\frac{1}{\lambda} \int_\Omega g \sigma \right)^{q/p}, \end{aligned}$$

as we wanted to prove. Then the proof of the Theorem is complete. \square

4. MORE MANAGEABLE CONDITIONS ON CUBES

Now we concentrate in the classes $A_{p,q}^\beta$. Since $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$, whenever $\alpha \leq \beta$ we observe that $A_{p,q}^\beta \subset A_{p,q}^\alpha$. Moreover, if $A_{p,q}$ consist in all weights for what (1.11) holds for every cube $Q \in \mathbb{R}^n$, it is clear that $A_{p,q} \subset A_{p,q}^\beta$. This inclusion is proper. In fact taking $u(x) = \|x\|^\alpha$ and $v(x) = \|x\|^\gamma$, with $\gamma = (\alpha+n)\frac{p}{q} - n$, it is not difficult to see that $(u,v) \in A_{p,q}$ whenever $-n < \alpha < n(q-1)$. However, if $\Omega = \mathbb{R}^n - \{0\}$ and $0 < \beta < 1$, we can check that $(u,v) \in A_{p,q}^\beta$ for every power $\alpha \in \mathbb{R}$.

However, in the next Lemma, we show that, under certain conditions on the weights the classes $A_{p,q}^\beta$ really are independent of β .

Lemma 4.1. *Let $0 < \alpha < \beta < 1$. Let u and v be weights such that $u, \sigma \in D_\alpha$. Then*

$$A_{p,q}^\beta \equiv A_{p,q}^\alpha.$$

Proof. Let $0 < \alpha < \beta < 1$. By the previous observation, the Lemma is proved if we show the inclusion $A_{p,q}^\alpha \subset A_{p,q}^\beta$. For this, given a cube $Q \in \mathcal{F}_\beta \setminus \mathcal{F}_\alpha$ we consider the cube $\tilde{Q} = \frac{\alpha}{\beta}Q$. Taking $k \in \mathbb{Z}$ such that $2^{k-1} < \frac{\beta}{\alpha} \leq 2^k$, since $\tilde{Q} \in \mathcal{F}_\alpha$, by the doubling condition on u and σ we get

$$u(Q)^{p/q} \sigma(Q)^{p-1} \leq u(2^k \tilde{Q})^{p/q} \sigma(2^k \tilde{Q})^{p-1} \leq C u(\tilde{Q})^{p/q} \sigma(\tilde{Q})^{p-1} \leq C |Q|,$$

which proves the lemma. \square

Lemma 4.2. *Let $\sigma \in A_\infty^\beta$. Then σ satisfies a Reverse Hölder inequality, i.e.*

$$(4.3) \quad \left(\frac{1}{|Q|} \int_Q \sigma^{1+\epsilon} \right)^{1/(1+\epsilon)} \leq C \frac{1}{|Q|} \int_Q \sigma,$$

for every cube $Q \in \mathcal{F}_\beta$.

Proof. We only need to observe that for every cube $Q = Q(x, l) \in \mathcal{F}_\beta$ and any cube $\tilde{Q} = \tilde{Q}(x', l') \subset Q$ such that $l = 2l'$ it follow that $\tilde{Q} \in \mathcal{F}_\beta$. In fact, since $d(x, x') \leq l'$ and $\beta < 1$

$$l = 2l' \leq \beta d(x, \Omega^c) \leq \beta d(x, x') + \beta d(x', \Omega^c) \leq \beta l' + \beta d(x', \Omega^c),$$

implies

$$l' \leq \frac{\beta}{2-\beta} d(x', \Omega^c) \leq \beta d(x', \Omega^c).$$

Then, the proof follows a similar way as in [2]. \square

Lemma 4.4. *Let $0 < \beta < 1$ and we consider $(u, v) \in A_{p,q}^\beta$. If $\sigma = v^{-1/(p-1)} \in A_\infty^\beta$ then there exists $\tilde{p} < p$ and $\tilde{q} < q$ such that $(u, v) \in A_{\tilde{p}, \tilde{q}}^\beta$.*

Proof. Since $\sigma \in A_\infty^\beta$, by the Lemma 4.2 it follows that σ satisfies a Reverse Hölder inequality as in (4.3). Thus, from the hypothesis on the weights we get

$$(4.5) \quad \frac{u(Q)^{p/q}}{|Q|} \left(\frac{1}{|Q|} \int_Q \sigma^{1+\epsilon} \right)^{(p-1)/(1+\epsilon)} \leq C.$$

Now, taking $\delta > 0$ such that $(p-1)/(1+\epsilon) = p - \delta - 1$ the Lemma is proved defining $\tilde{p} = p - \delta$ and $\tilde{q} = \frac{p-\delta}{p} q < q$. In fact,

$$\sigma^{1+\epsilon} = v^{-(1+\epsilon)/(p-1)} = v^{-1/(\tilde{p}-1)},$$

and noting that $p/q = \tilde{p}/\tilde{q}$ defining $\tilde{\sigma} = v^{-1/(\tilde{p}-1)}$ by (4.5) we have $(u, v) \in A_{\tilde{p}, \tilde{q}}^\beta$. \square

In the next Lemma we show the relation between the local and the centered local maximal function.

Lemma 4.6. *Let $0 < \alpha < 1/4$. There exists $0 < \gamma < 1$ such that*

$$(4.7) \quad M_\alpha f(x) \leq 2^n M_\gamma^c f(x),$$

for every locally integrable function f and every x in Ω .

Proof. Let f be a locally integrable function. For $0 < \alpha < 1/4$ and $x \in \Omega$ we consider cubes Q, \tilde{Q} such that $x \in Q \in \mathcal{F}_\alpha$ and \tilde{Q} is centered at x with $l_{\tilde{Q}} = 2l_Q$. If we show that $\tilde{Q} \in \mathcal{F}_\gamma$, for some $0 < \gamma < 1$ then the Lemma will be proved. In fact,

$$\frac{1}{|Q|} \int_Q |f| \leq \frac{2^n}{|\tilde{Q}|} \int_{\tilde{Q}} |f| \leq 2^n M_\gamma^c f(x),$$

then, taking the supremum over all cubes in \mathcal{F}_α containing x we get (4.7). So, we observe that

$$l_{\tilde{Q}} \leq 2\alpha d(x_Q, \Omega^c) \leq 2\alpha d(x_Q, x) + 2\alpha d(x, \Omega^c) \leq \alpha l_{\tilde{Q}} + 2\alpha d(x, \Omega^c),$$

thus

$$l_{\tilde{Q}} \leq \frac{2\alpha}{1-\alpha} d(x, \Omega^c).$$

Then, it is clear that $\tilde{Q} \in \mathcal{F}_\gamma$ with $\gamma = \frac{2\alpha}{1-\alpha} < 1$ since the choose of α . \square

Proof of the Theorem 1.19. That (1.20) implies (1.21) is trivial using the test function $\sigma\chi_Q$ for each cube $Q \in \mathcal{F}_\beta$ and the definition of M_β^c .

On the other hand, since it is clear that $\|M_\beta^c\|_\infty \leq \|f\|_\infty$, if we prove that M_β^c is of weak type (\tilde{p}, \tilde{q}) for some number $\tilde{p} < p$ and $\tilde{q} < q$, by applying the Marcinkiewicz interpolation theorem we will get the result.

In order to do this, let $U_\lambda = \{x \in \Omega : M_\beta^c f(x) > \lambda\}$ and let

$$\{Q_x\}_{x \in U_\lambda} = \left\{ Q_x : Q_x \in \mathcal{F}_\beta, \text{ centered at } x \text{ and } \frac{1}{|Q_x|} \int_{Q_x} |f| > \lambda \right\},$$

a covering for U_λ . Then, by the Theorem 2.1 we can select a countable subfamily of cubes $\{Q_j\}$ which still cover U_λ and such that $\sum_j \chi_{Q_j}(x) \leq C(n)$.

Then, considering \tilde{p} and \tilde{q} provided by the Lemma 4.4 and taking into account the property of the cubes in the covering we can write

$$\begin{aligned} u(U_\lambda) &\leq u(\cup_j Q_j) \\ &\leq \sum_j \frac{u(Q_j)}{|Q_j|^{\tilde{q}}} |Q_j|^{\tilde{q}} \\ &\leq \frac{C}{\lambda^{\tilde{q}}} \sum_j \frac{u(Q_j)}{|Q_j|^{\tilde{q}}} \left(\int_{Q_j} |f| \right)^{\tilde{q}} \\ &= \frac{C}{\lambda^{\tilde{q}}} \sum_j \frac{u(Q_j)}{|Q_j|^{\tilde{q}}} \left(\int_{Q_j} |f| v^{1/\tilde{p}} v^{-1/\tilde{p}} \right)^{\tilde{q}}. \end{aligned}$$

Hölder inequality with $\tilde{p} > 1$ and Lemma 4.4 allows us to get

$$\begin{aligned} u(U_\lambda) &\leq \frac{C}{\lambda^{\tilde{q}}} \sum_j \frac{u(Q_j)}{|Q_j|^{\tilde{q}}} \left(\int_{Q_j} |f|^{\tilde{p}} v \right)^{\tilde{q}/\tilde{p}} \left(\int_{Q_j} v^{-1/(\tilde{p}-1)} \right)^{(\tilde{p}-1)\tilde{q}/\tilde{p}} \\ &= \frac{C}{\lambda^{\tilde{q}}} \sum_j \left\{ \frac{u(Q_j)^{\tilde{p}/\tilde{q}}}{|Q_j|^{\tilde{p}/\tilde{q}}} \left(\frac{\tilde{\sigma}(Q_j)}{|Q_j|} \right)^{\tilde{p}-1} \right\}^{\tilde{q}/\tilde{p}} \left(\int_{Q_j} |f|^{\tilde{p}} v \right)^{\tilde{q}/\tilde{p}} \end{aligned}$$

$$\begin{aligned} &\leq \frac{C}{\lambda^{\bar{q}}} \left(\sum_j \int_{Q_j} |f|^{\bar{p}} v \right)^{\bar{q}/\bar{p}} \\ &\leq \frac{C}{\lambda^{\bar{q}}} \left(\int_{\Omega} |f|^{\bar{p}} v \right)^{\bar{q}/\bar{p}}, \end{aligned}$$

and the proof of the Theorem is complete. \square

Now, we introduce the following maximal function. For each $0 < \beta < 1$ we get

$$(4.8) \quad M_{(\beta/4, \beta]} f(x) = \sup_{x \in Q \in \mathcal{F}_{\beta} \setminus \mathcal{F}_{\beta/4}} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

Proposition 4.9. *Let $0 < \beta < 1$ and let $1 < p \leq q < \infty$. Given two weights u and v such that $(u, v) \in \mathbf{A}_{p, q}^{\beta}$ and $\sigma \in \mathcal{D}_{\beta}$, we have*

$$(4.10) \quad M_{(\beta/4, \beta]} : L^p(v) \rightarrow L^q(u).$$

Proof. For each $x \in \Omega$ we choose a cube Q_x such that $x \in Q_x \in \mathcal{F}_{\beta} \setminus \mathcal{F}_{\beta/4}$ and

$$M_{(\beta/4, \beta]} f(x) \leq \frac{2}{|Q_x|} \int_{Q_x} |f|.$$

Now, let t be such that $2^{-t} \leq \beta/20$ and we consider the covering \mathcal{W}_t of Ω provided by the Lemma 2.6. For simplicity we write

$$\mathcal{W}_t = \bigcup_j Q_j = \bigcup_j Q_j(x_j, l_j),$$

where the cubes Q_j are disjoint. Since $x \in Q_j$ for some j , denoting x_Q the center of Q_x , we get

$$\begin{aligned} d(x_Q, \Omega^c) &\leq d(x_j, \Omega^c) + d(x_Q, x) + d(x_j, x) \\ &\leq d(x_j, \Omega^c) + \beta d(x_Q, \Omega^c) + \beta d(x_j, \Omega^c). \end{aligned}$$

This implies that their centers holds

$$d(x_j, \Omega^c) \leq \frac{1 + \beta}{1 - \beta} d(x_Q, \Omega^c).$$

Since $10Q_j \in \mathcal{F}_{\beta}$ by part *i*) of the Lemma 2.6, the inequality above and the fact $Q_x \notin \mathcal{F}_{\beta/4}$ we have that

$$l_j < \frac{\beta}{10} d(x_j, \Omega^c) \leq \frac{\beta}{10} \frac{1 + \beta}{1 - \beta} d(x_Q, \Omega^c) \leq \frac{\beta}{10} \frac{1 + \beta}{1 - \beta} \frac{4}{\beta} l_Q = \frac{2}{5} \frac{1 + \beta}{1 - \beta} l_Q = c_{\beta} l_Q.$$

Thus, $|Q_j| \leq C |Q_x|$. Now, it is clear that $x \in \mathcal{N}_{\beta}(Q_j)$ since $x \in Q_x \cap Q_j$. Then, by Hölder inequality we can proceed as follows

$$\begin{aligned} \int_{\Omega} (M_{(\beta/4, \beta]} f)^q u &\leq C \sum_j \int_{Q_j} \frac{1}{|Q_x|^q} \left(\int_{Q_x} |f(y)| dy \right)^q u \\ &\leq C \sum_j \frac{u(Q_j)}{|Q_j|^q} \left(\int_{\mathcal{N}_{\beta}(Q_j)} |f| \right)^q \\ &\leq C \sum_j \frac{u(Q_j)}{|Q_j|^q} \sigma(\mathcal{N}_{\beta}(Q_j))^{q(p-1)/p} \left(\int_{\mathcal{N}_{\beta}(Q_j)} |f|^p v \right)^{q/p}. \end{aligned}$$

Since $\sigma \in \mathcal{D}_\beta$, by Remark 2.11 we get $\sigma(\mathcal{N}_\beta(Q_j)) \leq C \sigma(Q_j)$. In addition, we use the fact that $(u, v) \in A_{p,q}^\beta$, $p \leq q$ and by the Lemma 2.13 we can conclude that

$$\begin{aligned} \int_{\Omega} (M_{(\beta/4,\beta]} f)^q u &\leq C \sum_j \left(\frac{u(Q_j)^{p/q}}{|Q_j|^p} \sigma(Q_j)^{p-1} \right)^{q/p} \left(\int_{\mathcal{N}_\beta(Q_j)} |f|^p v \right)^{q/p} \\ &\leq C \left(\sum_j \int_{\mathcal{N}_\beta(Q_j)} |f|^p v \right)^{q/p} \\ &\leq C \left(\int_{\Omega} |f|^p v \right)^{q/p}, \end{aligned}$$

which gives (4.10). \square

Now, we are going to apply the results mentioned above to prove the analogous result for the local maximal.

Proof of the Theorem 1.15. Clearly (1.16) implies (1.14). Conversely let $0 < \beta < 1$. By Lemma 4.6 with $\alpha = \beta/4$ there exists $0 < \gamma < 1$ such that the inequality

$$(4.11) \quad \begin{aligned} M_\beta f(x) &\leq M_{\beta/4} f(x) + M_{(\beta/4,\beta]} f(x) \\ &\leq M_\gamma^c f(x) + M_{(\beta/4,\beta]} f(x), \end{aligned}$$

for every $x \in \Omega$. Then, by Theorem 1.19 and Proposition 4.9 we have

$$\int_{\Omega} M_\beta f^q u \leq C \left(\int_{\Omega} M_\gamma^c f^q u + \int_{\Omega} M_{(\beta/4,\beta]} f^q u \right) \leq C \left(\int_{\Omega} f^p v \right)^{q/p},$$

and the proof of the Theorem is complete. \square

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