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Derivatives of Horn hypergeometric functions with respect to their parameters

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The derivatives of eight Horn hypergeometric functions [four Appell F_1, F_2, F_3, and F_4, and four (degenerate) confluent Φ_1, Φ_2, Ψ_1, and Ξ_1] with respect to their parameters are studied. The first derivatives are expressed, systematically, as triple infinite summations or, alternatively, as single summations of two-variable Kampé de Fériet functions. Taking advantage of previously established expressions for the derivative of the confluent or Gaussian hypergeometric functions, the generalization to the nth derivative of Horn's functions with respect to their parameters is rather straightforward in most cases; the results are expressed in terms of $n+2$ infinite summations. Following a similar procedure, mixed derivatives are also treated. An illustration of the usefulness of the derivatives of F_1, with respect to the first and third parameters, is given with the study of autoionization of atoms occurring as part of a post-collisional process. Their evaluation setting the Coulomb charge to zero provides the coefficients of a Born-like expansion of the interaction. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4994059]

I. INTRODUCTION

Two-variable hypergeometric series have been studied extensively from their mathematical point of view.[1–4] Horn essentially identified 34 distinct convergent series (see, e.g., p. 224-226 of Ref. 3 or Sec. 1.3 of Ref. 1). Amongst them, we have selected eight Horn series that occur more frequently in a wide variety of problems in theoretical physics, applied mathematics, chemistry, statistics and engineering sciences (see, e.g., Refs. 2–16). They are the four Appell hypergeometric functions (F_1, F_2, F_3, and F_4) and four (degenerate) confluent hypergeometric ones (Φ_1, Φ_2, Ψ_1, and Ξ_1). The purpose of this paper is to provide ready to use compact expressions for the derivatives of such functions with respect to their parameters.

Amongst the properties of two-variable (say z_1 and z_2) hypergeometric functions, one finds compact expressions for the derivatives to nth order with respect to z_1 and/or z_2. In some applications, on top of the variables, the quantity of interest may be one of the parameters, say α. The dependence on such a parameter is therefore related to the study of Horn series as a function of α, rather than their variables z_1 and z_2. Similarly, for one-variable hypergeometric series $_pF_q$, the dependence on a parameter may be of special interest; this is the case for applications in fields as varied as physics,[17–24] engineering,[25–28] neurosciences,[29] biochemistry,[30] statistics,[31,32] and even in mathematical finance.[33] One important tool is then provided by the derivatives of such functions with respect to their parameters since they allow, for example, to write a Taylor expansion around a given value α_0. The first derivative, in particular, is the required quantity if one focuses on the linear dependence on a given parameter.

The derivative of hypergeometric series with respect to a parameter implies expressing the derivative of a Pochhammer symbol (or the reciprocal Pochhammer symbol) with respect to its argument. In this *direct method*, appears the digamma function Ψ or polygamma functions if higher derivatives are at stake; although recurrence relations may then be effectively used to reformulate the sought after results, the approach is cumbersome as discussed in the Introduction of Ref. 34. Following this direct route, derivatives with respect to parameters of Appell hypergeometric functions have been derived by Sahai and Verma.[35] With a different scope, useful expressions and properties of derivatives of the Pochhammer symbol were provided recently by Greynat *et al.*[7,24] who then applied the results for the so-called ε-expansions of some Appell and Kampé de Fériet hypergeometric functions (the analytical method seems particularly well adapted to the studied cases in which ε appears linearly in several parameters at the same time). Such ε-expansion is expressible also in terms of multiple polylogarithms (see Refs. 23 and 36–38 and the references therein). Without going into details of each, other analytical and/or numerical approaches exist. For certain classes of functions, small parameter expansions can be achieved by algebraic tools based on nested sums and can be formulated as algorithms suitable for an implementation on a computer.[38,39] Another numerical tool is provided by the HYPERDIRE project which is devoted to the creation of a set of Mathematica-based programs for the differential reduction of generalized hypergeometric functions without the construction of ε-expansions.[16,37,40]

As we already stated, the Horn series appear in many branches of science. In high energy physics, in particular, it was shown that any one-loop Feynman diagram has a representation in terms of Horn-type hypergeometric functions of few variables (see, e.g., Refs. 5, 6, and 15). In the framework of dimensional regularization, it is necessary to construct the derivative of such functions with respect to the parameters. It is therefore in this field, in particular, that analytical properties have been obtained and different algorithms have been developed and computer packages published (see, e.g., Refs. 7, 16, 23, 24, and 36–41 and the references therein). Some techniques are specific to a class of multivariate hypergeometric functions, or to integer, half-integer, or rational parameters, others have limitations in their application. We emphasize that the present contribution does not aim to review and compare different algorithms. Furthermore, it does not aim to be exhaustive with respect to the 34 Horn two-variable series.

In previous investigations, we studied the derivatives of any order n, with respect to their parameters, of one-variable (z) hypergeometric series: the confluent hypergeometric function ${}_1F_1$,[34] the Gaussian hypergeometric function ${}_2F_1$,[42] and the general hypergeometric function ${}_pF_q$.[43] This was made possible using the second-order linear differential equation that they satisfy, together with Babister's solution[44] to closely related non-homogeneous differential equations. The nth derivatives could finally be expressed in a systematic way in terms of Kampé de Fériet functions[2] in $n+1$ equal variables z. In contrast to the direct method, we found this *differential equation method* simpler to implement and generalize. Besides, according to the rather wide range of applications it has encountered (see, e.g., Refs. 17–22 and 25–33), our formulation seems to be sufficiently accessible to non-mathematician end users.

The differential equation method employed for the one-variable hypergeometric functions can also be applied, in principle, for some two-variable Horn functions (like the Appell F_1) that can be related to a high order ordinary differential equation.[45] However, since this association is not generally applicable to all the functions of Horn's list, we propose here a different strategy. Essentially, we express Horn functions as single series of Gaussian or confluent hypergeometric functions and then exploit, when possible, our previous findings.[34,42] This provides us with a systematic way of writing the nth derivatives of Horn functions with respect to the parameters; they are given, in most cases, as a single sum of generalized multivariable Kampé de Fériet functions, noted as ${}_2\Theta_1^{(n)}$ and $\Theta^{(n)}$, whose definition and properties were presented in Refs. 42 and 34, respectively.

We consider derivatives to any order n, but we shall provide details of the procedure mainly for all $n = 1$ cases; higher order derivatives, as well as mixed derivatives, are obtained in a similar fashion since our approach is quite systematic. Also, results for only eight Horn functions will be detailed. Further two-variable Horn series, as well as triple hypergeometric functions, can be considered in a very similar way.

To illustrate the usefulness of some of the found expressions, we have considered a physical application: the autoionization of atoms[46] when the decay occurs as part of a post-collisional process. After an ion-molecule double electron capture, say $He^{++} + H_2 \rightarrow He^{**} + H_2^{++}$, the doubly excited He^{**} autoionizes, i.e., $He^{**} \rightarrow He^{+} + e^{-}$. We propose to use the Φ_2 model[47,48] to describe the interaction between the autoionized electron and the two nuclei of the doubly ionized molecule. Such a model leads to interference fringes on the electron spectra.[49] As we shall explain, the nuclear charges of the molecule play the role of physical parameters; a Taylor expansion around them corresponds to a Born series approach. Since these nuclear charges appear as mathematical parameters in an Appell F_1 function, derivatives with respect to them are needed to build up the corresponding series and allow for a physical interpretation of the involved Coulomb interactions. Note that for sufficiently high relative electron velocities, a first-Born treatment is generally enough, and thus only the first derivative is needed.

In Sec. II, we describe the method and provide the expressions for the first derivatives with respect to the parameters. The generalization to order n, mixed derivatives and some further properties are given in Sec. III. A physical application is presented in Sec. IV. Section V provides a summary of our results.

II. FIRST DERIVATIVE OF EIGHT HORN HYPERGEOMETRIC FUNCTIONS WITH RESPECT TO THE PARAMETERS

We assume hereafter that all variables and parameters of the Horn functions are complex numbers and that the parameters appearing in the denominator of all series are neither zero nor negative integers. Also, unless otherwise indicated, in all summations, the integers run from 0 to ∞. We start by recalling some results of Refs. 34 and 42 which we shall need below.

Consider first the Gaussian hypergeometric function

$$ {}_2F_1\,(a,b,c;z) = \sum_n \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{1}$$

where the Pochhammer symbol $(\gamma)_n = \Gamma(\gamma+n)/\Gamma(\gamma)$ is defined in terms of the Gamma function;[3] it is assumed that $|z| < 1$. The derivatives with respect to the parameters a or c of the function ${}_2F_1\,(a,b,c;z)$ can be written as[42]

$$ \frac{d}{da}\,{}_2F_1(a,b,c;z) = \frac{z}{a}\,\frac{a\,b}{c}\,{}_2\Theta_1^{(1)}\left(\begin{matrix} 1,1|a,a+1,b+1 \\ a+1|2,c+1 \end{matrix}\middle|\,;z,z\right), \tag{2a}$$

$$ \frac{d}{dc}\,{}_2F_1(a,b,c;z) = -\frac{z}{c}\,\frac{a\,b}{c}\,{}_2\Theta_1^{(1)}\left(\begin{matrix} 1,1|c,a+1,b+1 \\ c+1|2,c+1 \end{matrix}\middle|\,;z,z\right), \tag{2b}$$

where ${}_2\Theta_1^{(1)}$ stands for a two-variable Kampé de Fériet function[2] defined as[42]

$$ {}_2\Theta_1^{(1)}\left(\begin{matrix} a_1,a_2|b_1,b_2,b_3 \\ c_1|d_1,d_2 \end{matrix}\middle|\,;z_1,z_2\right) = \sum_{m,n} \frac{(a_1)_m (a_2)_n (b_1)_m}{(c_1)_m}\,\frac{(b_2)_{m+n}(b_3)_{m+n}}{(d_1)_{m+n}(d_2)_{m+n}}\,\frac{z_1^m}{m!}\,\frac{z_2^n}{n!}. \tag{3}$$

As a and b play a similar role, the derivatives with respect to b may be obtained by interchanging a and b in (2a).

For the confluent hypergeometric function

$$ {}_1F_1\,(a,b;z) = \sum_n \frac{(a)_n}{(b)_n}\frac{z^n}{n!}, \tag{4}$$

the derivatives with respect to a and b read[34]

$$ \frac{d}{da}\,{}_1F_1(a,b;z) = \frac{z}{b}\Theta^{(1)}\left(\begin{matrix} 1,1|\,a,a+1 \\ a+1|\,2,b+1 \end{matrix}\middle|\,;z,z\right) \tag{5a}$$

$$ \frac{d}{db}\,{}_1F_1(a,b;z) = -\,\frac{z}{b}\,\frac{a}{b}\,\Theta^{(1)}\left(\begin{matrix} 1,1|\,b,a+1 \\ b+1|\,2,b+1 \end{matrix}\middle|\,;z,z\right), \tag{5b}$$

where $\Theta^{(1)}$ stands for a two-variable Kampé de Fériet function[2] defined as[34]

$$ \Theta^{(1)}\left(\begin{matrix} a_1,a_2|\,b_1,b_2 \\ c_1|\,d_1,d_2 \end{matrix}\middle|\,;z_1,z_2\right) = \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} \frac{(a_1)_{m_1}(a_2)_{m_2}(b_1)_{m_1}(b_2)_{m_1+m_2}}{(c_1)_{m_1}(d_1)_{m_1+m_2}(d_2)_{m_1+m_2}}\,\frac{z_1^{m_1} z_2^{m_2}}{m_1!\,m_2!}. \tag{6}$$

We can now proceed with the eight Horn series F_1, F_2, F_3, F_4, Φ_1, Φ_2, Ψ_1, and Ξ_1, whose definitions can be found, e.g., in Ref. 1.

A. Function F_2

For presentation convenience, we shall start with the Appell F_2 function which is defined by the two-variable series

$$F_2(a,b_1,b_2,c_1,c_2;z_1,z_2) = \sum_{m,n} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c_1)_m(c_2)_n} \frac{z_1^m}{m!} \frac{z_2^n}{n!}, \quad |z_1| + |z_2| < 1. \tag{7}$$

One may also express the F_2 function as a series about $z_1 = 0$ for fixed z_2

$$F_2(a,b_1,b_2,c_1,c_2;z_1,z_2) = \sum_k \frac{(a)_k(b_1)_k}{(c_1)_k} \, {}_2F_1(a+k,b_2,c_2;z_2) \frac{z_1^k}{k!}. \tag{8}$$

Using the derivatives of the Gaussian hypergeometric function with respect to its second (respectively, third) parameter [i.e., relations (2a), respectively, (2b)], we find

$$\frac{d}{db_2} F_2(a,b_1,b_2,c_1,c_2;z_1,z_2)$$

$$= z_2 \frac{a}{c_2} \sum_k \frac{(a+1)_k(b_1)_k}{(c_1)_k} \, {}_2\Theta_1^{(1)} \left(\begin{matrix} 1,1|b_2,b_2+1,a+k+1 \\ b_2+1|2,c_2+1 \end{matrix} \middle| ;z_2,z_2 \right) \frac{z_1^k}{k!} \tag{9a}$$

$$= z_2 \frac{a}{c_2} \sum_{k,m,n} (a+1)_{k+m+n} \frac{(b_2+1)_{m+n}}{(2)_{m+n}(c_2+1)_{m+n}} \frac{(b_1)_k}{(c_1)_k} \frac{(1)_m(b_2)_m}{(b_2+1)_m} (1)_n \frac{z_1^k}{k!} \frac{z_2^m}{m!} \frac{z_2^n}{n!}, \tag{9b}$$

$$\frac{d}{dc_2} F_2(a,b_1,b_2,c_1,c_2;z_1,z_2)$$

$$= -z_2 \frac{a\,b_2}{c_2^2} \sum_k \frac{(a+1)_k(b_1)_k}{(c_1)_k} \, {}_2\Theta_1^{(1)} \left(\begin{matrix} 1,1|c_2,a+k+1,b_2+1 \\ c_2+1|2,c_2+1 \end{matrix} \middle| ;z_2,z_2 \right) \frac{z_1^k}{k!}, \tag{9c}$$

$$= -z_2 \frac{a\,b_2}{c_2^2} \sum_{k,m,n} (a+1)_{k+m+n} \frac{(b_2+1)_{m+n}}{(2)_{m+n}(c_2+1)_{m+n}} \frac{(b_1)_k}{(c_1)_k} \frac{(1)_m(c_2)_m}{(c_2+1)_m} (1)_n \frac{z_1^k}{k!} \frac{z_2^m}{m!} \frac{z_2^n}{n!}. \tag{9d}$$

In each case, the second equality is obtained by using the identity

$$\frac{1}{(a+k)} = \frac{1}{a} \frac{(a)_k}{(a+1)_k}. \tag{10}$$

Thus the derivative of the Appell function is expressed either as an infinite series of functions ${}_2\Theta_1^{(1)}$ or, equivalently, as a triple infinite summation.

Making use of the symmetry relation $F_2(a, b_1, b_2, c_1, c_2; z_1, z_2) = F_2(a, b_2, b_1, c_2, c_1; z_2, z_1)$, we have similar expressions for the derivatives with respect to b_1 and c_1 where in the above one interchanges (z_1, b_1, c_1) with (z_2, b_2, c_2).

Finally, we consider the derivative with respect to the parameter a which appears in the numerator of series (7) with combined index $m+n$ or, alternatively, in a more cumbersome manner in expansion (8). For this case, we use a different approach, based on the derivative of the Pochhammer symbol

$$\frac{d(a)_{n+m}}{da} = (a)_{n+m}[\Psi(a+n+m) - \Psi(a)] = (a)_{n+m} \sum_{k=0}^{m+n-1} \frac{1}{a+k}, \tag{11}$$

the second equality coming from a recurrence relation of the digamma function Ψ (Eq. (6.3.6) of Ref. 50). Note that for $n = m = 0$, this derivative is obviously zero. It is convenient to split the sum into two,

$$\frac{d(a)_{n+m}}{da} = (a)_{n+m} \left[\sum_{k=0}^{m-1} \frac{1}{a+k} + \sum_{k=0}^{n-1} \frac{1}{a+m+k} \right]. \tag{12}$$

The derivative of $F_2(a, b_1, b_2, c_1, c_2; z_1, z_2)$ with respect to a can therefore be written as

$$\frac{d}{da} F_2(a,b_1,b_2,c_1,c_2;z_1,z_2)$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a)_{n+m}\frac{(b_1)_m(b_2)_n}{(c_1)_m(c_2)_n}\frac{z_1^m}{m!}\frac{z_2^n}{n!}\left[\sum_{k=0}^{m-1}\frac{1}{a+k}+\sum_{k=0}^{n-1}\frac{1}{a+m+k}\right] \tag{13a}$$
$$= \sum_{n=0}^{\infty}\frac{(b_2)_n}{(c_2)_n}\frac{z_2^n}{n!}\sum_{m=0}^{\infty}\frac{(b_1)_{m+1}}{(c_1)_{m+1}}\frac{z_1^{m+1}}{(m+1)!}(a)_{n+m+1}\sum_{k=0}^{m}\frac{1}{a}\frac{(a)_k}{(a+1)_k}$$
$$+ \sum_{m=0}^{\infty}\frac{(b_1)_m}{(c_1)_m}\frac{z_1^m}{m!}\sum_{n=0}^{\infty}\frac{(b_2)_{n+1}}{(c_2)_{n+1}}\frac{z_2^{n+1}}{(n+1)!}(a)_{n+m+1}\sum_{k=0}^{n}\frac{1}{a}\frac{(a)_{m+k}}{(a+1)_{m+k}}, \tag{13b}$$

where for the second equality, we shifted the index m (respectively, n), and we made use of relation (10). Using then the rearrangement series technique (see, for example, Chap. 2 of Ref. 51)

$$\sum_{p=0}^{\infty}\sum_{k=0}^{p}B(k,p) = \sum_{p=0}^{\infty}\sum_{k=0}^{\infty}B(k,p+k), \tag{14}$$

we obtain two separate triple infinite summations

$$\frac{d}{da} F_2(a,b_1,b_2,c_1,c_2;z_1,z_2)$$
$$= z_1\frac{b_1}{c_1}\sum_{k,m,n=0}^{\infty}(a+1)_{n+m+k}\frac{(b_1+1)_{m+k}}{(c_1+1)_{m+k}(2)_{m+k}}\frac{(1)_k(a)_k}{(a+1)_k}(1)_m\frac{(b_2)_n}{(c_2)_n}\frac{z_1^k}{k!}\frac{z_1^m}{m!}\frac{z_2^n}{n!} \tag{15}$$
$$+ z_2\frac{b_2}{c_2}\sum_{k,m,n=0}^{\infty}(a+1)_{n+m+k}\frac{(b_2+1)_{n+k}}{(c_2+1)_{n+k}(2)_{n+k}}\frac{(a)_{m+k}}{(a+1)_{m+k}}(1)_k\frac{(b_1)_m}{(c_1)_m}(1)_n\frac{z_2^k}{k!}\frac{z_2^n}{n!}\frac{z_1^m}{m!}.$$

Each of these triple summations can also be expressed as single series of ${}_2\Theta_1^{(1)}$ functions.

B. Function F_1

We now turn to the F_1 function which is defined as

$$F_1(a,b_1,b_2,c;z_1,z_2) = \sum_{m,n}\frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}}\frac{z_1^m}{m!}\frac{z_2^n}{n!}, \qquad |z_1|<1, |z_2|<1. \tag{16}$$

As a series around the $z_1 = 0$ point, for fixed z_2, one has

$$F_1(a,b_1,b_2,c;z_1,z_2) = \sum_k\frac{(a)_k(b_1)_k}{(c)_k}\,{}_2F_1(a+k,b_2,c+k;z_2)\frac{z_1^k}{k!}. \tag{17}$$

Using expression (2a), we immediately get

$$\frac{d}{db_2} F_1(a,b_1,b_2,c;z_1,z_2)$$
$$= z_2\frac{a}{c}\sum_k\frac{(a+1)_k(b_1)_k}{(c+1)_k}\,{}_2\Theta_1^{(1)}\left(\begin{matrix}1,1|b_2,b_2+1,a+k+1\\ b_2+1|2,c+k+1\end{matrix}\middle|;z_2,z_2\right)\frac{z_1^k}{k!} \tag{18a}$$
$$= z_2\frac{a}{c}\sum_{k,m,n}\frac{(a+1)_{k+m+n}}{(c+1)_{k+m+n}}\frac{(b_2+1)_{m+n}}{(2)_{m+n}}(b_1)_k\frac{(1)_m(b_2)_m}{(b_2+1)_m}(1)_n\frac{z_1^k}{k!}\frac{z_2^m}{m!}\frac{z_2^n}{n!}, \tag{18b}$$

and a similar expression, for the derivative with respect to b_1 by interchanging (z_1, b_1) with (z_2, b_2), since $F_1(a, b_1, b_2, c; z_1, z_2) = F_1(a, b_2, b_1, c; z_2, z_1)$.

For the derivative with respect to parameters a and c, we first use the identity

$$F_2(a,b_1,b_2,c,a;z_1,z_2) = (1-z_2)^{-b_2}\, F_1\left(b_1, a-b_2, b_2, c; z_1, \frac{z_1}{1-z_2}\right) \tag{19}$$

from which

$$F_1(a,b_1,b_2,c;z_1,z_2) = \left(\frac{z_1}{z_2}\right)^{b_2} F_2\left(b_1+b_2, a, b_2, c, b_1+b_2; z_1, 1-\frac{z_1}{z_2}\right). \tag{20}$$

Then, applying the expressions found previously for F_2, i.e., Eqs. (9a)–(9d), one easily finds

$$\begin{aligned}
&\frac{d}{da} F_1(a,b_1,b_2,c;z_1,z_2)\\
&= \left(\frac{z_1}{z_2}\right)^{b_2} \frac{d}{da} F_2\left(b_1+b_2, a, b_2, c, b_1+b_2; z_1, 1-\frac{z_1}{z_2}\right) && \text{(21a)}\\
&= \left(\frac{z_1}{z_2}\right)^{b_2} z_1\, \frac{b_1+b_2}{c} \sum_k \frac{(b_1+b_2+1)_k (b_2)_k}{(b_1+b_2)_k} \frac{1}{k!} \left(1-\frac{z_1}{z_2}\right)^k\\
&\times\, {}_2\Theta_1^{(1)}\left(\begin{matrix}1,1|a,a+1,b_1+b_2+1+k\\ a+1|2,c+1\end{matrix}\middle|; z_1, z_1\right) && \text{(21b)}\\
&= \left(\frac{z_1}{z_2}\right)^{b_2} z_1\, \frac{b_1+b_2}{c} \sum_{k,m,n} (b_1+b_2+1)_{k+m+n} \frac{(a+1)_{m+n}}{(2)_{m+n}(c+1)_{m+n}} \frac{(b_2)_k}{(b_1+b_2)_k}\\
&\times \frac{(1)_m (a)_m}{(a+1)_m} (1)_n \frac{1}{k!} \left(1-\frac{z_1}{z_2}\right)^k \frac{z_1^m}{m!} \frac{z_1^n}{n!}, && \text{(21c)}
\end{aligned}$$

$$\begin{aligned}
&\frac{d}{dc} F_1(a,b_1,b_2,c;z_1,z_2)\\
&= \left(\frac{z_1}{z_2}\right)^{b_2} \frac{d}{dc} F_2\left(b_1+b_2, a, b_2, c, b_1+b_2; z_1, 1-\frac{z_1}{z_2}\right) && \text{(22a)}\\
&= -\left(\frac{z_1}{z_2}\right)^{b_2} z_1\, \frac{(b_1+b_2)\,a}{c^2} \sum_k \frac{(b_1+b_2+1)_k (b_2)_k}{(b_1+b_2)_k} \frac{1}{k!} \left(1-\frac{z_1}{z_2}\right)^k\\
&\times\, {}_2\Theta_1^{(1)}\left(\begin{matrix}1,1|c,b_1+b_2+1+k,a+1\\ c+1|2,c+1\end{matrix}\middle|; z_1, z_1\right) && \text{(22b)}\\
&= -\left(\frac{z_1}{z_2}\right)^{b_2} z_1\, \frac{(b_1+b_2)\,a}{c^2} \sum_{k,m,n} (b_1+b_2+1)_{k+m+n} \frac{(a+1)_{m+n}}{(2)_{m+n}(c+1)_{m+n}} \frac{(b_2)_k}{(b_1+b_2)_k}\\
&\times \frac{(1)_m (c)_m}{(c+1)_m} (1)_n \frac{1}{k!} \left(1-\frac{z_1}{z_2}\right)^k \frac{z_1^m}{m!} \frac{z_1^n}{n!}. && \text{(22c)}
\end{aligned}$$

C. Function F_3

Next, we consider the F_3 function which is defined as

$$F_3(a_1,a_2,b_1,b_2,c;z_1,z_2) = \sum_{m,n} \frac{(a_1)_m (a_2)_n (b_1)_m (b_2)_n}{(c)_{m+n}} \frac{z_1^m}{m!} \frac{z_2^n}{n!}, \quad |z_1|<1, |z_2|<1. \tag{23}$$

Using the series around the $z_1 = 0$ point, for fixed z_2,

$$F_3(a_1,a_2,b_1,b_2,c;z_1,z_2) = \sum_k \frac{(a_1)_k (b_1)_k}{(c)_k}\, {}_2F_1(a_2,b_2,c+k;z_2) \frac{z_1^k}{k!}, \tag{24}$$

and following the same procedure [i.e., using result (2a)], we obtain

$$\frac{d}{da_2}F_3(a_1,a_2,b_1,b_2,c;z_1,z_2)$$
$$= z_2\,\frac{b_2}{c}\sum_k \frac{(a_1)_k(b_1)_k}{(c+1)_k}\,\frac{z_1^k}{k!}\,{}_2\Theta_1^{(1)}\left(\begin{array}{c}1,1|a_2,a_2+1,b_2+1\\ a_2+1|2,c+k+1\end{array}\middle|;z_2,z_2\right) \tag{25a}$$
$$= z_2\,\frac{b_2}{c}\sum_{k,m,n}\frac{1}{(c+1)_{k+m+n}}\,\frac{(a_2+1)_{m+n}(b_2+1)_{m+n}}{(2)_{m+n}}\,(a_1)_k(b_1)_k\,\frac{(1)_m(a_2)_m}{(a_2+1)_m}\,(1)_n\,\frac{z_1^k}{k!}\,\frac{z_2^m}{m!}\,\frac{z_2^n}{n!}. \tag{25b}$$

Since a_2 and b_2 play a similar role in the definition of F_3, the derivative with respect to b_2 is the same as the above by simply interchanging a_2 with b_2. Moreover, since $F_3(a_1, a_2, b_1, b_2, c; z_1, z_2) = F_3(a_2, a_1, b_2, b_1, c; z_2, z_1)$, the derivative with respect to a_1 (and similarly to b_1) is given by expressions (25a) or (25b), by interchanging (z_1, a_1, b_1) with (z_2, a_2, b_2).

For the derivative with respect to c, the calculation is different, as c appears with an index $m+n$ in (23). In this case, we have

$$\frac{d}{dc}\frac{1}{(c)_{n+m}} = -\frac{1}{[(c)_{n+m}]^2}\frac{d(c)_{n+m}}{dc} = -\frac{1}{(c)_{n+m}}\left[\sum_{k=0}^{m-1}\frac{1}{c+k}+\sum_{k=0}^{n-1}\frac{1}{c+m+k}\right], \tag{26}$$

the second equality coming from (12). Proceeding similarly to the derivative of F_2 with respect to a, one arrives to

$$\frac{d}{dc}F_3(a_1,a_2,b_1,b_2,c;z_1,z_2)$$
$$= -z_1\frac{a_1b_1}{c^2}\sum_{m,n,k}\frac{(1)_m(a_2)_n(b_2)_n(1)_k(c)_k}{(c+1)_k}\,\frac{(a_1+1)_{m+k}(b_1+1)_{m+k}}{(2)_{m+k}}\,\frac{1}{(c+1)_{m+n+k}}$$
$$\times\ \frac{z_1^m}{m!}\,\frac{z_1^k}{k!}\,\frac{z_2^n}{n!}$$
$$-\ z_2\frac{a_2b_2}{c^2}\sum_{m,n,k}(a_1)_m(b_1)_m(1)_n(1)_k\,\frac{(c)_{m+k}}{(c+1)_{m+k}}\,\frac{(a_2+1)_{n+k}(b_2+1)_{n+k}}{(2)_{n+k}}\,\frac{1}{(c+1)_{m+n+k}}$$
$$\times\ \frac{z_1^m}{m!}\,\frac{z_2^k}{k!}\,\frac{z_2^n}{n!}, \tag{27}$$

which may be written, equivalently, as two single summations of ${}_2\Theta_1^{(1)}$ functions.

D. Function F_4

Finally, the Appell function F_4 is defined as

$$F_4(a,b,c_1,c_2;z_1,z_2) = \sum_{m,n}\frac{(a)_{m+n}(b)_{m+n}}{(c_1)_m(c_2)_n}\,\frac{z_1^m}{m!}\,\frac{z_2^n}{n!},\qquad |\sqrt{z_1}|+|\sqrt{z_2}|<1. \tag{28}$$

Alternatively, as a series around the $z_1 = 0$ point, for fixed z_2, one has

$$F_4(a,b,c_1,c_2;z_1,z_2) = \sum_k\frac{(a)_k(b)_k}{(c_1)_k}\,{}_2F_1(a+k,b+k,c_2;z_2)\,\frac{z_1^k}{k!}. \tag{29}$$

Applying result (2b), one immediately finds

$$\frac{d}{dc_2}F_4(a,b,c_1,c_2;z_1,z_2)$$
$$= -z_2\,\frac{a\,b}{c_2^2}\sum_k\frac{(a+1)_k(b+1)_k}{(c_1)_k}\,\frac{z_1^k}{k!}\,{}_2\Theta_1^{(1)}\left(\begin{array}{c}1,1|c_2,a+k+1,b+k+1\\ c_2+1|2,c_2+1\end{array}\middle|;z_2,z_2\right) \tag{30a}$$
$$= -z_2\,\frac{a\,b}{c_2^2}\sum_{k,m,n}(a+1)_{k+m+n}(b+1)_{k+m+n}\,\frac{1}{(2)_{m+n}(c_2+1)_{m+n}}\,\frac{1}{(c_1)_k}$$
$$\times\ \frac{(1)_m(c_2)_m}{(c_2+1)_m}\,(1)_n\,\frac{z_1^k}{k!}\,\frac{z_2^m}{m!}\,\frac{z_2^n}{n!} \tag{30b}$$

and a similar expression for the derivative with respect to c_1 can be obtained by interchanging (z_1, c_1) with (z_2, c_2), since $F_4(a, b, c_1, c_2; z_1, z_2) = F_4(a, b, c_2, c_1; z_2, z_1)$.

For the derivative with respect to a (and similarly with respect to b, by permutation), the calculation is longer, as a appears with an index $m + n$ in (28). We can proceed similarly to the derivative with respect to a of function F_2, and we end up again with two triple infinite summations.

E. Function Φ_1

The confluent hypergeometric series Φ_1 has the following alternative representations

$$\Phi_1(a,b,c;z_1,z_2) = \sum_{m,n} \frac{(a)_{m+n}(b)_m}{(c)_{m+n}} \frac{z_1^m}{m!} \frac{z_2^n}{n!}, \qquad |z_1| < 1 \tag{31a}$$

$$= \sum_{n} \frac{(a)_n}{(c)_n} \frac{z_2^n}{n!}\, {}_2F_1(a+n,b,c+n;z_1). \tag{31b}$$

The derivative with respect to the b parameter is given by the direct application of formula (2a) and reads

$$\frac{d}{db}\Phi_1(a,b,c;z_1,z_2)$$

$$= z_1 \frac{a}{c} \sum_{n} \frac{(a+1)_n}{(c+1)_n} \frac{z_2^n}{n!} {}_2\Theta_1^{(1)}\left(\begin{array}{c} 1,1|b,b+1,a+n+1 \\ b+1|2,c+n+1 \end{array} \middle| ;z_1,z_1 \right) \tag{32a}$$

$$= z_1 \frac{a}{c} \sum_{k,m,n} \frac{(a+1)_{k+m+n}}{(c+1)_{k+m+n}} \frac{(b+1)_{m+k}}{(2)_{m+k}} \frac{(1)_m(b)_m}{(b+1)_m} (1)_k \frac{z_1^k}{k!} \frac{z_1^m}{m!} \frac{z_2^n}{n!}. \tag{32b}$$

The case of the derivative with respect to a is similar to that of the derivative with respect to the first parameter of the Appell F_2 function, while the situation with parameter c is similar to the one found with the fifth parameter of the Appell F_3 function. Thus they can be calculated, proceeding as described in those two cases, and the results are

$$\frac{d}{da}\Phi_1(a,b,c;z_1,z_2)$$

$$\begin{aligned} &= z_1 \frac{b}{c} \sum_{m,n,k} \frac{(1)_m(1)_k(a)_k}{(a+1)_k} \frac{(b+1)_{m+k}}{(2)_{m+k}} \frac{(a+1)_{m+n+k}}{(c+1)_{m+n+k}} \frac{z_1^m}{m!} \frac{z_1^k}{k!} \frac{z_2^n}{n!} \\ &+ z_2 \frac{1}{c} \sum_{m,n,k} (b)_m(1)_n(1)_k \frac{(a)_{m+k}}{(a+1)_{m+k}} \frac{1}{(2)_{n+k}} \frac{(a+1)_{m+n+k}}{(c+1)_{m+n+k}} \frac{z_1^m}{m!} \frac{z_2^k}{k!} \frac{z_2^n}{n!}, \end{aligned} \tag{33}$$

$$\frac{d}{dc}\Phi_1(a,b,c;z_1,z_2)$$

$$\begin{aligned} &= -z_1 \frac{ab}{c^2} \sum_{m,n,k} \frac{(1)_m(1)_k(c)_k}{(c+1)_k} \frac{(b+1)_{m+k}}{(2)_{m+k}} \frac{(a+1)_{m+n+k}}{(c+1)_{m+n+k}} \frac{z_1^m}{m!} \frac{z_1^k}{k!} \frac{z_2^n}{n!} \\ &- z_2 \frac{a}{c^2} \sum_{m,n,k} (b)_m(1)_n(1)_k \frac{(c)_{m+k}}{(c+1)_{m+k}} \frac{1}{(2)_{n+k}} \frac{(a+1)_{m+n+k}}{(c+1)_{m+n+k}} \frac{z_1^m}{m!} \frac{z_2^k}{k!} \frac{z_2^n}{n!}, \end{aligned} \tag{34}$$

each triple summation being equivalent to a single summation of ${}_2\Theta_1^{(1)}$ functions.

F. Function Φ_2

The confluent hypergeometric series Φ_2 has the following alternative representations

$$\Phi_2(a,b,c;z_1,z_2) = \sum_{m,n} \frac{(a)_m(b)_n}{(c)_{m+n}} \frac{z_1^m}{m!} \frac{z_2^n}{n!} \tag{35a}$$

$$= \sum_{m} \frac{(a)_m}{(c)_m} \frac{z_1^m}{m!}\, {}_1F_1(b,c+m;z_2). \tag{35b}$$

The derivative with respect to the b parameter is given by the direct application of formula (5b) and reads

$$\begin{aligned}
&\frac{d}{db}\Phi_2(a,b,c;z_1,z_2)\\
&= z_2\frac{1}{c}\sum_n \frac{(a)_m}{(c+1)_m}\frac{z_1^m}{m!}\,\Theta^{(1)}\left(\begin{array}{c}1,1|b,b+1\\ b+1|2,c+m+1\end{array}\middle|;z_2,z_2\right) &(36a)\\
&= z_2\frac{1}{c}\sum_{k,m,n}\frac{1}{(c+1)_{k+m+n}}\frac{(b+1)_{n+k}}{(2)_{n+k}}(a)_m\frac{(1)_n(b)_n}{(b+1)_n}(1)_k\frac{z_1^m}{m!}\frac{z_2^k}{k!}\frac{z_2^n}{n!}. &(36b)
\end{aligned}$$

As a consequence of the symmetry $\Phi_2(a,b,c;z_1,z_2)=\Phi_2(b,a,c;z_2,z_1)$, the derivative of Φ_2 with respect to a can be obtained by simply interchanging (z_1, a) with (z_2, b) in the previous formula.

For the derivative with respect to c, we repeat the procedure used to calculate $\frac{d}{dc}F_3$ to obtain

$$\begin{aligned}
&\frac{d}{dc}\Phi_2(a,b,c;z_1,z_2)\\
&=-z_1\frac{a}{c^2}\sum_{m,n,k}\frac{(1)_m(b)_n(1)_k(c)_k}{(c+1)_k}\frac{(a+1)_{m+k}}{(2)_{m+k}}\frac{1}{(c+1)_{m+n+k}}\frac{z_1^m}{m!}\frac{z_1^k}{k!}\frac{z_2^n}{n!}\\
&\quad-z_2\frac{b}{c^2}\sum_{m,n,k}(a)_m(1)_n(1)_k\frac{(c)_{m+k}}{(c+1)_{m+k}}\frac{(b+1)_{n+k}}{(2)_{n+k}}\frac{1}{(c+1)_{m+n+k}}\frac{z_1^m}{m!}\frac{z_2^k}{k!}\frac{z_2^n}{n!}. &(37)
\end{aligned}$$

G. Function Ψ_1

For the confluent hypergeometric series Ψ_1, one has the equivalent series expressions

$$\begin{aligned}
\Psi_1(a,b,c_1,c_2;z_1,z_2) &= \sum_{m,n}\frac{(a)_{m+n}(b)_m}{(c_1)_m(c_2)_n}\frac{z_1^m}{m!}\frac{z_2^n}{n!}, \qquad |z_1|<1 &(38a)\\
&= \sum_m \frac{(a)_m(b)_m}{(c_1)_m}\frac{z_1^m}{m!}\,{}_1F_1(a+m,c_2;z_2) &(38b)\\
&= \sum_n \frac{(a)_n}{(c_2)_n}\frac{z_2^n}{n!}\,{}_2F_1(a+n,b,c_1;z_1). &(38c)
\end{aligned}$$

For the derivative with respect to parameter c_2, we can use the series representation in terms of confluent hypergeometric functions (38b) and apply directly formula (5b) to get

$$\frac{d}{dc_2}\Psi_1(a,b,c_1,c_2;z_1,z_2)=-z_2\frac{a}{c_2^2}\sum_m\frac{(a+1)_m(b)_m}{(c_1)_m}\frac{z_1^m}{m!}\,\Theta^{(1)}\left(\begin{array}{c}1,1|c_2,a+m+1\\ c_2+1|2,c_2+1\end{array}\middle|;z_2,z_2\right). \qquad (39)$$

For derivatives with respect to b (respectively, c_1), it is more convenient to start from series (38c); by applying directly (2a) [respectively, (2b)], we get

$$\frac{d}{db}\Psi_1(a,b,c_1,c_2;z_1,z_2)=z_1\frac{a}{c_1}\sum_m\frac{(a+1)_m}{(c_2)_m}\frac{z_2^m}{m!}\,{}_2\Theta_1^{(1)}\left(\begin{array}{c}1,1|b,b+1,a+m+1\\ b+1|2,c_1+1\end{array}\middle|;z_1,z_1\right), \qquad (40)$$

$$\frac{d}{dc_1}\Psi_1(a,b,c_1,c_2;z_1,z_2)=-z_1\frac{ab}{c_1^2}\sum_m\frac{(a+1)_m}{(c_2)_m}\frac{z_2^m}{m!}\,{}_2\Theta_1^{(1)}\left(\begin{array}{c}1,1|c_1,a+m+1,b+1\\ c_1+1|2,c_1+1\end{array}\middle|;z_1,z_1\right). \qquad (41)$$

The derivative with respect to a can be calculated similarly to the derivative with respect to the first parameter of the F_2 function. The result reads

$$\begin{aligned}
&\frac{d}{da}\Psi_1(a,b,c_1,c_2;z_1,z_2)\\
&= z_1\frac{b}{c_1}\sum_{m,n,k}\frac{(1)_m(1)_k(a)_k}{(c_2)_n(a+1)_k}\frac{(b+1)_{m+k}}{(2)_{m+k}(c_1+1)_{m+k}}(a+1)_{m+n+k}\frac{z_1^m}{m!}\frac{z_1^k}{k!}\frac{z_2^n}{n!}\\
&+ z_2\frac{1}{c_2}\sum_{m,n,k}\frac{(b)_m(1)_n(1)_k}{(c_1)_m}\frac{(a)_{m+k}}{(a+1)_{m+k}}\frac{1}{(2)_{n+k}(c_2+1)_{n+k}}(a+1)_{m+n+k}\frac{z_1^m}{m!}\frac{z_2^k}{k!}\frac{z_2^n}{n!}.
\end{aligned}\tag{42}$$

H. Function Ξ_1

For the confluent hypergeometric series Ξ_1, one has the equivalent series expressions

$$\Xi_1(a_1,a_2,b,c;z_1,z_2) = \sum_{m,n}\frac{(a_1)_m(a_2)_n(b)_m}{(c)_{m+n}}\frac{z_1^m}{m!}\frac{z_2^n}{n!}, \qquad |z_1|<1 \tag{43a}$$

$$= \sum_{m}\frac{(a_1)_m(b)_m}{(c)_m}\frac{z_1^m}{m!}\,{}_1F_1(a_2,c+m;z_2) \tag{43b}$$

$$= \sum_{n}\frac{(a_2)_n}{(c)_n}\frac{z_2^n}{n!}\,{}_2F_1(a_1,b,c+n;z_1). \tag{43c}$$

The derivative with respect to a_1 is, once more, easily deduced from formula (2a) and reads

$$\frac{d}{da_1}\Xi_1(a_1,a_2,b,c;z_1,z_2) = z_1\frac{b}{c}\sum_{n}\frac{(a_2)_n}{(c+1)_n}\frac{z_2^n}{n!}{}_2\Theta_1^{(1)}\left(\begin{matrix}1,1|a_1,a_1+1,b+1\\ a_1+1|2,c+n+1\end{matrix}\middle|;z_1,z_1\right). \tag{44}$$

The derivative with respect to b is obtained by interchanging a_1 with b in the last formula as a consequence of the symmetry that the Gaussian hypergeometric functions satisfy. On the other hand, by applying (5a), for the parameter a_2, one finds

$$\frac{d}{da_2}\Xi_1(a_1,a_2,b,c;z_1,z_2) = z_2\frac{1}{c}\sum_{m}\frac{(a_1)_m(b)_m}{(c+1)_m}\frac{z_1^m}{m!}\,\Theta^{(1)}\left(\begin{matrix}1,1|a_2,a_2+1\\ a_2+1|2,c+m+1\end{matrix}\middle|;z_2,z_2\right). \tag{45}$$

For the derivative with respect to c, we proceed similarly to $\frac{dF_3}{dc}$ to obtain

$$\begin{aligned}
&\frac{d}{dc}\Xi_1(a_1,a_2,b,c;z_1,z_2)\\
&= -z_1\frac{a_1b}{c^2}\sum_{m,n,k}\frac{(1)_m(a_2)_n(1)_k(c)_k}{(c+1)_k}\frac{(a_1+1)_{m+k}(b+1)_{m+k}}{(2)_{m+k}}\frac{1}{(c+1)_{m+n+k}}\frac{z_1^m}{m!}\frac{z_1^k}{k!}\frac{z_2^n}{n!}\\
&- z_2\frac{a_2}{c^2}\sum_{m,n,k}(a_1)_m(b)_m(1)_n(1)_k\frac{(c)_{m+k}}{(c+1)_{m+k}}\frac{(a_2+1)_{n+k}}{(2)_{n+k}}\frac{1}{(c+1)_{m+n+k}}\frac{z_1^m}{m!}\frac{z_2^k}{k!}\frac{z_2^n}{n!}.
\end{aligned}\tag{46}$$

III. *n*TH DERIVATIVES, MIXED DERIVATIVES, AND OTHER PROPERTIES

A. *n*th derivatives

Similarly to the case of the first derivatives of the Gaussian hypergeometric function ${}_2F_1$ for which we introduced a two-variable ${}_2\Theta_1^{(1)}$ function, for the nth derivatives, it is convenient to introduce a hypergeometric function in $n+1$ variables[42]

$$
\begin{aligned}
&{}_2\Theta_1^{(n)}\left(\begin{array}{c} a_1, a_2, \ldots, a_{n+1} | \, b_1, b_2, \ldots, b_{n+2} \\ c_1, \ldots, c_n | \, d_1, d_2 \end{array} \middle| ; z_1, \ldots, z_{n+1}\right) \\
&= \sum_{m_1=0}^{\infty} \cdots \sum_{m_{n+1}=0}^{\infty} (a_1)_{m_1} (a_2)_{m_2} \ldots (a_{n+1})_{m_{n+1}} \frac{(b_1)_{m_1} (b_2)_{m_1+m_2} \ldots (b_{n+1})_{m_1+m_2+\cdots+m_{n+1}}}{(c_1)_{m_1} (c_2)_{m_1+m_2} \ldots (c_n)_{m_1+m_2+\cdots+m_n}} \\
&\times \frac{(b_{n+2})_{m_1+m_2+\cdots+m_{n+1}}}{(d_1)_{m_1+m_2+\cdots+m_{n+1}} (d_2)_{m_1+m_2+\cdots+m_{n+1}}} \frac{z_1^{m_1} z_2^{m_2} \ldots z_{n+1}^{m_{n+1}}}{m_1! m_2! \ldots m_{n+1}!},
\end{aligned} \tag{47}
$$

which follow some recurrence relations and possess alternative series representations[42] which may be of use in certain cases. In terms of these functions (which are also Kampé de Fériet functions[42]), the nth derivatives of the Gaussian hypergeometric function with respect to the parameters read

$$
\frac{d^n}{da^n} {}_2F_1(a, b, c; z) = \frac{(b)_n}{(c)_n} z^n \, {}_2\Theta_1^{(n)}\left(\begin{array}{c} 1, 1, \ldots, 1 | \, a, a+1, \ldots, a+n, b+n \\ a+1, \ldots, a+n | \, n+1, c+n \end{array} \middle| ; z, \ldots, z\right), \tag{48a}
$$

$$
\frac{d^n}{dc^n} {}_2F_1(a, b, c; z) = (-1)^n \frac{n!}{c^n} \frac{ab}{c} z \, {}_2\Theta_1^{(n)}\left(\begin{array}{c} 1, 1, \ldots, 1 | \, c, c, \ldots, c, a+1, b+1 \\ c+1, \ldots, c+1 | \, 2, c+1 \end{array} \middle| ; z, \ldots, z\right). \tag{48b}
$$

Obviously, the nth derivative with respect to the b parameter can be obtained from (48a) simply by interchanging a and b. Note that in these nth derivatives, the same variable z appears $n + 1$ times.

Applying the same procedure as described in Sec. II, the nth derivatives of the Appell functions with respect to their parameters are given by $n + 2$ infinite summations. In most cases, they can be obtained straightforwardly and expressed as a single sum of these ${}_2\Theta_1^{(n)}$ functions [for the derivatives of F_2 (respectively, F_3 or F_4) with respect to a (respectively, c or a), the generalization of the results to nth order is not as compact]. For example, from (17), one immediately finds

$$
\begin{aligned}
&\frac{d^n}{db_2^n} F_1(a, b_1, b_2, c; z_1, z_2) \\
&= z_2^n \frac{(a)_n}{(c)_n} \sum_k \frac{(a+n)_k (b_1)_k}{(c+n)_k} \frac{z_1^k}{k!} \\
&\times {}_2\Theta_1^{(n)}\left(\begin{array}{c} 1, 1, \ldots, 1 | \, b_2, b_2+1, \ldots, b_2+n, a+k+n \\ b_2+1, \ldots, b_2+n | \, n+1, c+k+n \end{array} \middle| ; z_2, \ldots, z_2\right).
\end{aligned} \tag{49}
$$

For the b_1 parameter, a simple symmetry relation can be invoked. To obtain the nth derivative of F_1 with respect to a or c, it is convenient to use relation (20) and the series representation (8). Thus we have

$$
\begin{aligned}
&\frac{d^n}{da^n} F_1(a, b_1, b_2, c; z_1, z_2) \\
&= \left(\frac{z_1}{z_2}\right)^{b_2} z_1^n \frac{1}{(c)_n} \sum_k \frac{(b_2)_k (b_1+b_2)_{n+k}}{(b_1+b_2)_k \, k!} \left(1 - \frac{z_1}{z_2}\right)^k \\
&\times {}_2\Theta_1^{(n)}\left(\begin{array}{c} 1, 1, \ldots, 1 | \, a, a+1, \ldots, a+n, b_1+b_2+k+n \\ a+1, \ldots, a+n | \, n+1, c+n \end{array} \middle| ; z_1, \ldots, z_1\right),
\end{aligned} \tag{50}
$$

$$
\begin{aligned}
&\frac{d^n}{dc^n} F_1(a, b_1, b_2, c; z_1, z_2) \\
&= \left(\frac{z_1}{z_2}\right)^{b_2} z_1 (-1)^n n! \frac{a(b_1+b_2)}{c^{n+1}} \sum_p \frac{(b_2)_p (b_1+b_2+1)_p}{(b_1+b_2)_p \, p!} \left(1 - \frac{z_1}{z_2}\right)^p \\
&\times {}_2\Theta_1^{(n)}\left(\begin{array}{c} 1, 1, \ldots, 1 | \, c, c, \ldots, c, b_1+b_2+p+1, a+1 \\ c+1, \ldots, c+1 | \, 2, c+1 \end{array} \middle| ; z_1, \ldots, z_1\right).
\end{aligned} \tag{51}
$$

For confluent Horn functions, the generalization depends on which parameter one is considering. When series expressions in terms of Gaussian hypergeometric functions appear, one proceeds similarly to the Appell situations; in most cases, the result is expressed as a single sum of ${}_2\Theta_1^{(n)}$ functions.

On the other hand, when expressions in terms of confluent hypergeometric functions appear, one needs the nth derivatives[34]

$$\frac{d^n}{da^n}\,{}_1F_1(a,b;z) = \frac{z^n}{(b)_n}\,\Theta^{(n)}\left(\left.\begin{matrix} 1,1,\ldots,1|\,a,a+1,..,a+n \\ a+1,a+2,\ldots,a+n|\,n+1,b+n\end{matrix}\right|;z,z,..,z\right), \tag{52a}$$

$$\frac{d^n}{db^n}\,{}_1F_1(a,b;z) = (-1)^n\frac{n!}{b^n}\frac{a}{b}z\,\Theta^{(n)}\left(\left.\begin{matrix} 1,1,\ldots,1|\,b,b,\ldots,b,a+1 \\ b+1,b+1,\ldots,b+1|\,2,b+1\end{matrix}\right|;z,z,\ldots,z\right), \tag{52b}$$

where the $\Theta^{(n)}$ function is defined by

$$\begin{aligned}&\Theta^{(n)}\left(\left.\begin{matrix} a_1,a_2,\ldots,a_{n+1}|\,b_1,..,b_{n+1} \\ c_1,\ldots,c_n|\,d_1,d_2\end{matrix}\right|;z_1,z_2,..,z_{n+1}\right) \\ &= \sum_{m_1=0}^{\infty}\cdots\sum_{m_{n+1}=0}^{\infty}\frac{z_1^{m_1}z_2^{m_2}\cdots z_{n+1}^{m_{n+1}}}{m_1!m_2!\cdots m_{n+1}!} \\ &\quad\times\frac{(a_1)_{m_1}(a_2)_{m_2}\ldots(a_{n+1})_{m_{n+1}}(b_1)_{m_1}(b_2)_{m_1+m_2}\ldots(b_{n+1})_{m_1+m_2+\cdots+m_{n+1}}}{(c_1)_{m_1}(c_2)_{m_1+m_2}\ldots(c_n)_{m_1+m_2+\cdots+m_n}(d_1)_{m_1+m_2+\cdots+m_{n+1}}(d_2)_{m_1+m_2+\cdots+m_{n+1}}}.\end{aligned} \tag{53}$$

(Similarly to the functions ${}_2\Theta_1^{(n)}$, $\Theta^{(n)}$ are Kampé de Fériet functions with recurrence relations and alternative series representations.[34]) For example, starting from series (35b), the nth derivative with respect to b of the confluent Horn function $\Phi_2(a,b,c;z_1,z_2)$ reads

$$\begin{aligned}\frac{d^n}{db^n}\Phi_2(a,b,c;z_1,z_2) &= \sum_m\frac{(a)_m}{(c)_m}\frac{z_1^m}{m!}\frac{z_2^n}{(c+m)_n} \\ &\quad\times\Theta^{(n)}\left(\left.\begin{matrix} 1,1,\ldots,1|\,b,b+1,..,b+n \\ b+1,b+2,\ldots,b+n|\,n+1,c+m+n\end{matrix}\right|;z_2,z_2,..,z_2\right),\end{aligned} \tag{54}$$

where the same variable z_2 appears $n+1$ times in the $\Theta^{(n)}$ function.

For length purposes, we do not provide all nth derivatives explicitly. However, it is clear that expressing Horn functions as single series of Gaussian or confluent hypergeometric functions, the results, (48a) and (48b) or (52a) and (52b), can be applied straightforwardly.

B. Other representations and properties

Most of the derivatives of the eight Horn functions considered in Sec. II have been expressed in terms of the functions ${}_2\Theta_1^{(1)}$ or $\Theta^{(1)}$, with two equal variables $z_1 = z_2$. Such functions have a number of properties such as series and integral representations, and contiguous relations.[34,42] A deeper study on functions $\Theta^{(1)}$ and ${}_2\Theta_1^{(1)}$ and their generalizations (including the system of differential equations they satisfy, the holonomic rank and the locus of singularities) should be the subject of future investigations.

As it will be useful in the subsection on mixed derivatives, we provide here alternative series representations of the ${}_2\Theta_1^{(1)}$ function,[42]

$$\begin{aligned}&{}_2\Theta_1^{(1)}\left(\left.\begin{matrix} a_1,a_2|\,b_1,b_2,b_3 \\ c_1|\,d_1,d_2\end{matrix}\right|;z_1,z_2\right) \\ &= \sum_{m_1=0}^{\infty}\frac{(a_1)_{m_1}(b_1)_{m_1}(b_2)_{m_1}(b_3)_{m_1}}{(c_1)_{m_1}(d_1)_{m_1}(d_2)_{m_1}}\frac{z_1^{m_1}}{m_1!}\,{}_3F_2\left(a_2,b_2+m_1,b_3+m_1,d_1+m_1,d_2+m_1;z_2\right) \\ &= \sum_{m_2=0}^{\infty}\frac{(a_2)_{m_2}(b_2)_{m_2}(b_3)_{m_2}}{(d_1)_{m_2}(d_2)_{m_2}}\frac{z_2^{m_2}}{m_2!}\,{}_4F_3\left(a_1,b_1,b_2+m_2,b_3+m_2,c_1,d_1+m_2,d_2+m_2;z_1\right).\end{aligned}$$

Thus the derivatives of Horn functions with respect to the parameters can be written in alternative forms which may result to be more practical. For example, for the F_1 function, we have

$$\frac{d}{db_1} F_1(a,b_1,b_2,c;z_1,z_2) = z_1 \frac{a}{c} \sum_{k,m} \frac{(a+1)_{k+m}}{(c+1)_{k+m}} (b_2)_k \frac{(1)_m (b_1)_m}{(2)_m} \frac{z_2^k}{k!} \frac{z_1^m}{m!} \tag{55a}$$

$$\times\, {}_3F_2(1, b_1+1+m, a+1+k+m, 2+m, c+1+k+m; z_1)$$

$$= z_1 \frac{a}{c} \sum_{k,m} \frac{(a+1)_{k+m}}{(c+1)_{k+m}} (b_2)_k \frac{(1)_m (b_1+1)_m}{(2)_m} \frac{z_2^k}{k!} \frac{z_1^m}{m!} \tag{55b}$$

$$\times\, {}_4F_3(1, b_1, b_1+1+m, a+1+k+m, b_1+1, 2+m, c+1+k+m; z_1).$$

It is also possible to express such derivatives in terms of Gaussian hypergeometric functions

$$\frac{d}{db_1} F_1(a,b_1,b_2,c;z_1,z_2) = z_1 \frac{a}{c} \sum_{m,n} \frac{(a+1)_{m+n} (b_1+1)_{m+n}}{(c+1)_{m+n} (2)_{m+n}} \frac{(b_1)_m}{(b_1+1)_m} z_1^{m+n}$$

$$\times\, {}_2F_1(b_2, a+1+m+n, c+1+m+n; z_2), \tag{56}$$

a result that shall be used in Subsection III C.

One more remark. In some subcases, one may easily recover previously published results. As an example, consider the derivative with respect to a of the F_1 function in the case $z_1 = z_2$; by inspection of result (21b), only the $k = 0$ term survives in the summation and the derivative is given as a single ${}_2\Theta_1^{(1)}$ function. At the same time, the Appell function F_1 is known to reduce to a Gaussian hypergeometric function, $F_1(a,\ b_1,\ b_2,\ c;\ z_1,\ z_1) = {}_2F_1(a,\ b_1 + b_2,\ c;\ z_1)$, so that the derivative $\frac{d}{da} F_1(a,b_1,b_2,c;z_1,z_1) = \frac{d}{da}\, {}_2F_1(a,b_1+b_2,c;z_1)$ is also directly provided by (2a) as published in Ref. 42. The results obviously coincide.

C. Mixed derivatives

Let us now provide some mixed derivatives, that is to say derivatives with respect to two or more parameters at the same time. To illustrate the procedure, we start from the alternative series representations (56) for the derivative $\frac{d}{db_1} F_1$. From it, together with relation (2a), one can easily deduce a result for the mixed derivative $\frac{\partial^2}{\partial b_1 \partial b_2} F_1$, in terms of a double infinite summation of ${}_2\Theta_1^{(1)}$ functions

$$\frac{\partial^2}{\partial b_1 \partial b_2} F_1(a,b_1,b_2,c;z_1,z_2) = z_1 z_2 \frac{a(a+1)}{c(c+1)} \sum_{m,n} \frac{(a+2)_{m+n} (b_1+1)_{m+n}}{(c+2)_{m+n} (2)_{m+n}} \frac{(b_1)_m}{(b_1+1)_m} z_1^{m+n}$$

$$\times\, {}_2\Theta_1^{(1)} \left(\begin{matrix} 1, 1 | b_2, b_2+1, a+2+m+n \\ b_2+1 | 2, c+2+m+n \end{matrix} \middle| ; z_2, z_2 \right), \tag{57}$$

or, alternatively, a quadruple infinite summation.

Let us now go further and consider the nth derivative (49) with respect to b_2. Making use of the series representation (47), algebraic manipulations allow one to perform the summation over k and obtain

$$\frac{d^n}{db_2^n} F_1(a,b_1,b_2,c;z_1,z_2)$$

$$= z_2^n \frac{(a)_n}{(c)_n} \sum_{\ell_1} \cdots \sum_{\ell_{n+1}} (1)_{\ell_1} \ldots (1)_{\ell_{n+1}} \frac{(a+n)_{\ell_1+\ell_2+\cdots+\ell_{n+1}}}{(n+1)_{\ell_1+\ell_2+\cdots+\ell_{n+1}} (c+n)_{\ell_1+\ell_2+\cdots+\ell_{n+1}}}$$

$$\times \frac{(b_2)_{\ell_1} (b_2+1)_{\ell_1+\ell_2} \ldots (b_2+n)_{\ell_1+\ell_2+\ldots+\ell_{n+1}}}{(b_2+1)_{\ell_1} (b_2+2)_{\ell_1+\ell_2} \ldots (b_2+n)_{\ell_1+\ell_2+\ldots+\ell_n}} \frac{z_2^{\ell_1}}{\ell_1!} \cdots \frac{z_2^{\ell_{n+1}}}{\ell_{n+1}!}$$

$$\times {}_2F_1 \left(b_1, a+n+\ell_1+\ell_2+\ldots+\ell_{n+1}, c+n+\ell_1+\ell_2+\ldots+\ell_{n+1}; z_1\right). \tag{58}$$

Since the b_1 parameter appears now only in ${}_2F_1$, we may apply directly formula (48a) and obtain

$$\frac{\partial^{m+n}}{\partial b_1^m\,\partial b_2^n}F_1(a,b_1,b_2,c;z_1,z_2)$$
$$= z_1^m\, z_2^n\, \frac{(a)_{m+n}}{(c)_{m+n}} \sum_{\ell_1}\cdots\sum_{\ell_{n+1}} (1)_{\ell_1}\ldots(1)_{\ell_{n+1}} \frac{(a+m+n)_{\ell_1+\ell_2+\cdots+\ell_{n+1}}}{(n+1)_{\ell_1+\ell_2+\cdots+\ell_{n+1}}(c+m+n)_{\ell_1+\ell_2+\ldots+\ell_{n+1}}}$$
$$\times \frac{(b_2)_{\ell_1}(b_2+1)_{\ell_1+\ell_2}\ldots(b_2+n)_{\ell_1+\ell_2+\cdots+\ell_{n+1}}}{(b_2+1)_{\ell_1}(b_2+2)_{\ell_1+\ell_2}\ldots(b_2+n)_{\ell_1+\ell_2+\ldots+\ell_n}}\, \frac{z_2^{\ell_1}}{\ell_1!}\cdots\frac{z_2^{\ell_{n+1}}}{\ell_{n+1}!}$$
$$\times\, {}_2\Theta_1^{(m)}\left(\begin{matrix}1,\ldots,1|\,b_1,b_1+1,\ldots,b_1+m,a+m+n+\ell_1+\ell_2+\cdots+\ell_{n+1}\\ b_1+1,\ldots,b_1+m|\,m+1,c+m+n+\ell_1+\ell_2+\cdots+\ell_{n+1}\end{matrix}\right|;z_2,\ldots,z_2\Bigg). \quad (59)$$

The same procedure can be applied to easily find compact results for mixed derivatives of any order n and m with respect to (b_1, b_2), (c_1, c_2), (b_1, c_2), and (b_2, c_1) for function F_2; to (a_1, a_2),(a_1, b_2), (a_2, b_1), and (b_1, b_2) for function F_3; and to (c_1, c_2) for function F_4.

The same thing can be done, for example, with the function Φ_2 and its mixed derivatives with respect to a and b. From (54), using the series representation (53) and performing the sum over m, we find

$$\frac{d^n}{db^n}\Phi_2(a,b,c;z_1,z_2)$$
$$= z_2^n\,\frac{1}{(c)_n}\sum_{\ell_1}\cdots\sum_{\ell_{n+1}}\frac{(1)_{\ell_1}\ldots(1)_{\ell_{n+1}}}{(n+1)_{\ell_1+\ell_2+\cdots+\ell_{n+1}}(c+n)_{\ell_1+\ell_2+\cdots+\ell_{n+1}}}$$
$$\times \frac{(b)_{\ell_1}(b+1)_{\ell_1+\ell_2}\ldots(b+n)_{\ell_1+\ell_2+\cdots+\ell_{n+1}}}{(b+1)_{\ell_1}(b+2)_{\ell_1+\ell_2}\ldots(b+n)_{\ell_1+\ell_2+\cdots+\ell_n}}\,\frac{z_2^{\ell_1}}{\ell_1!}\cdots\frac{z_2^{\ell_{n+1}}}{\ell_{n+1}!}$$
$$\times\, {}_1F_1\left(a,c+n+\ell_1+\ell_2+\cdots+\ell_{n+1};z_1\right). \quad (60)$$

Thus using (52a), we obtain

$$\frac{\partial^{m+n}}{\partial a^m\,\partial b^n}\Phi_2(a,b,c;z_1,z_2)$$
$$= z_1^m\, z_2^n\,\frac{1}{(c)_{m+n}}\sum_{\ell_1}\cdots\sum_{\ell_{n+1}}\frac{(1)_{\ell_1}\ldots(1)_{\ell_{n+1}}}{(n+1)_{\ell_1+\ell_2+\cdots+\ell_{n+1}}(c+m+n)_{\ell_1+\ell_2+\cdots+\ell_{n+1}}}$$
$$\times \frac{(b)_{\ell_1}(b+1)_{\ell_1+\ell_2}\ldots(b+n)_{\ell_1+\ell_2+\ldots+\ell_{n+1}}}{(b+1)_{\ell_1}(b+2)_{\ell_1+\ell_2}\cdots(b+n)_{\ell_1+\ell_2+\cdots+\ell_n}}\,\frac{z_2^{\ell_1}}{\ell_1!}\cdots\frac{z_2^{\ell_{n+1}}}{\ell_{n+1}!}$$
$$\times\, \Theta^{(m)}\left(\begin{matrix}1,\ldots,1|\,a,a+1,\ldots,a+m\\ a+1,\ldots,a+m|\,m+1,c+m+n+\ell_1+\ell_2+\cdots+\ell_{n+1}\end{matrix}\right|;z_1,\ldots,z_1\Bigg). \quad (61)$$

The same procedure can be applied to easily find results for mixed derivatives of any order with respect to (c_1, c_2) and (b, c_2) for function Ψ_1, and to (a_1, a_2) and (a_2, b) for function Ξ_1.

D. Extensions

We have studied in detail the derivatives of 8 of the 34 two-variables Horn functions (listed, e.g., in Sec. 1.3 of Ref. 1). For many of them, the study of the derivative with respect to parameters can be performed following the procedures indicated in Sec. II. The starting point consists in expressing, when possible, such functions as a sum of one variable hypergeometric functions ${}_pF_q$, and then apply the expressions presented in Refs. 34, 42, and 43 for the derivatives with respect to their parameters.

Without being exhaustive, for example, for the derivative of function H_4 with respect to the second (β), third (γ), or fourth (δ) parameters, we could easily find expressions in terms of a single sum of ${}_2\Theta_1^{(1)}$ functions. For the function G_1, which presents Pochhammer symbols with index $n - m$, the situation is slightly trickier. We could find, however, a closed form for the derivative with respect to the second (β) or third (β') parameters in terms of two sums of ${}_3\Theta_2^{(1)}$ functions; the same can be said for function G_2 with respect to the third (β) or fourth (β') parameters. Other functions, such as G_3 or H_6, present Pochhammer symbols with more intricate summation indices such as $2m - n$. These difficult cases will be investigated and presented in a future work.

IV. APPLICATION

Among the many possible applications of the results presented above, we would like to briefly discuss one that is related to collision problems, namely, autoionization of atoms[46] when the decay occurs as part of a post-collisional process; incidentally, the considered application is also closely connected to the theory of many-variable hypergeometric functions.

Let us consider the situation where a heavy charged ion, e.g., He^{2+}, collides with a simple molecule, say H_2. As a result of the collision, both molecular electrons are captured by the projectile to form a doubly excited state He^{**}. After a given time, an atomic transition occurs whereby one of the electrons is emitted to the continuum while the other decays to the ground state of He^+; this autoionization process is schematized by $He^{**} \to He^+ + e^-$. The emitted electron interacts with three charged particles (the two molecular nuclei and the projectile, now singly ionized, He^+). It moves away from the system with velocity $\mathbf{v}$ relative to the target, and velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ relative to each of the H_2^{++} nuclei, which we shall label 1 (charge Z_1) and 2 (charge Z_2). The two Sommerfeld parameters defined as $\eta_1 = Z_1/v_1$ and $\eta_2 = Z_2/v_2$ measure the strength of the Coulomb interaction.

The probability for this autoionization process to occur is an extension of the transition amplitude corresponding to ion-atom considered in Refs. 52–54,

$$A = -i \int_0^\infty \langle \Psi_f \mid V \mid \Psi_i \rangle t^{i\eta_1 + i\eta_2} e^{i\mathcal{E}t} dt. \tag{62}$$

Here $\mathcal{E} = E - E_0 + i\frac{\Gamma}{2}$, where $E_0 = \varepsilon_i - \varepsilon_f + \Delta$, ε_i and ε_f are the initial and final energies of the atom and Γ is the life-time of the autoionizing state (Δ is a correction of the energy due to the interaction between the atom and the molecule); E is the final energy of the system and includes part of the energy exchanged by the projectile with the molecule. The matrix element $\langle \Psi_f \mid V \mid \Psi_i \rangle$ gives the transition amplitude related to the decay:[49] Ψ_i is the doubly excited state He^{**}, and V is the interaction potential. In the final state, one of the electrons is bound in the ground state He^+, and the other moves in the presence of both the atomic nucleus and the two centers of the ionized molecule. This continuum state can be represented as follows:

$$\Psi_f^- \simeq \Psi^-(\mathbf{r})\, \tilde{D}^-(\mathbf{r}, \mathbf{v}_1 t, \mathbf{v}_2 t), \tag{63}$$

where the continuum wave function $\Psi^-(\mathbf{r})$ of the emitted electron (here taken with incoming behaviour) is distorted by the presence of the molecular nuclei; mathematically, this is expressed by a distortion factor $\tilde{D}^-(\mathbf{r}, \mathbf{v}_1 t, \mathbf{v}_2 t)$ where the classical approximation $\mathbf{r}_i = \mathbf{v}_i t$ ($i = 1, 2$) is considered. Since the integrand is strongly confined to the vicinity of the target nuclei (localization of the ground state), the matrix element can be simplified by means of a peaking approximation, namely,

$$\langle \Psi_f \mid V \mid \Psi_i \rangle \simeq \langle \Psi^- \mid V \mid \Psi_i \rangle D^-(\mathbf{v}_1 t, \mathbf{v}_2 t). \tag{64}$$

The resulting transition amplitude A reads

$$A = -iA_0 \int_0^\infty D^-(\mathbf{v}_1 t, \mathbf{v}_2 t) t^{i\eta_1 + i\eta_2} e^{i\mathcal{E}t} dt, \tag{65}$$

where $A_0 = \langle \Psi^- \mid V \mid \Psi_i \rangle$.

Different approaches for the distortion factor $D^-(\mathbf{v}_1 t,\ \mathbf{v}_2 t)$ can be considered. One of them is the simple, uncorrelated, product of two Coulomb distortion factors.[52] Martínez and co-workers[49] used the Φ_2 model introduced in the context of ion-atom collisions[47,48] to include correlation. In the process we are considering here, the distortion factor is built using the Φ_2 model; it can be understood as a sum of products of Coulomb functions, corresponding to the Sommerfeld parameters $\eta' = Z_1/v'$ and $\eta'' = Z_2/v''$ where v' and v'' are the relative electron-ion velocities with respect to each nuclei.

It reads

$$D^{-}(\mathbf{v}_1 t, \mathbf{v}_2 t) = \Gamma\left(1 + i\eta' + i\eta''\right) e^{\frac{\pi}{2}(\eta'+\eta'')} \sum_{m=0}^{\infty} \frac{(-i\eta')_m(-i\eta'')_m}{m!(m)_m(1)_{2m}}\, d_m^{-}(\mathbf{v}_1 t)\, d_m^{-}(\mathbf{v}_2 t), \tag{66}$$

where

$$d_m^{-}(\mathbf{v}_1 t) = [-ix_1]^m \; {}_1F_1\left[-i\eta' + m, 1 + 2m; -ix_1\right], \tag{67a}$$

$$d_m^{-}(\mathbf{v}_2 t) = [-ix_2]^m \; {}_1F_1\left[-i\eta'' + m, 1 + 2m; -ix_2\right], \tag{67b}$$

where x_1 and x_2 depend on a combination of the velocities v', v'', v_1, v_2 and t. Omitting further intermediate steps, one finds that the amplitude A can be expressed in terms of a F_1 function

$$A(Z_1, Z_2) = \mathcal{A}(Z_1, Z_2) F_1(a, b_1, b_2, c; z_1, z_2), \tag{68}$$

where

$$a = 1 + i\eta_1 + i\eta_2, \tag{69a}$$

$$b_1 = i\eta', \tag{69b}$$

$$b_2 = i\eta'', \tag{69c}$$

$$c = 1, \tag{69d}$$

and the purely imaginary variables z_1 and z_2 depend on all velocities involved. The coefficient $\mathcal{A}(Z_1, Z_2)$ is given by

$$\mathcal{A}(Z_1, Z_2) = -iA_0 \Gamma(a)\Gamma(1 + b_1 + b_2) e^{\pi(\eta'+\eta'')/2} [-i\mathcal{E}]^{-a}. \tag{70}$$

Contrary to a simple product of Coulomb functions, the Φ_2 approach correlates the interaction of the electron with both centers in a non-separable way. The autoionization probability is then representing the emission of the electron with the full interaction with the charged centers. It is then interesting to analyze the effect of the interaction with, separately, each of them. The quantity $A(Z_1, Z_2 = 0)$ provides the full interaction of the electron with Z_1; indeed, in this case, the Φ_2 distortion factor reduces to a single Coulomb distortion. Similarly, $A(Z_1 = 0, Z_2)$ includes the full interaction but with the other center. Including the correlation to first order, we have

$$\begin{aligned} A(Z_1, Z_2) \simeq{}& A(Z_1 = 0, Z_2) + A(Z_1, Z_2 = 0) \\ &+ Z_1 \left.\frac{dA(Z_1, Z_2)}{dZ_1}\right|_{Z_1=0} + Z_2 \left.\frac{dA(Z_1, Z_2)}{dZ_2}\right|_{Z_2=0} + \cdots . \end{aligned} \tag{71}$$

In a first Born-like expansion for the autoionization process, the first derivative $\left.\frac{dA}{dZ_2}\right|_{Z_2=0}$ stands for the full interaction with the center Z_1 after interacting once with the other (Z_2). It is given by

$$\left.\frac{dA}{dZ_2}\right|_{Z_2=0} = \left.\frac{d\mathcal{A}(Z_1, Z_2)}{dZ_2}\right|_{Z_2=0} {}_2F_1(1 + i\eta_1, i\eta', 1; z_1) + \mathcal{A}(Z_1, 0) \left.\frac{dF_1(a, b_1, b_2, c; z_1, z_2)}{dZ_2}\right|_{Z_2=0}, \tag{72}$$

where, in the first term, the Appell function reduces to ${}_2F_1$. Of course, a similar expression results when considering the full interaction with Z_2 after interacting once with Z_1. The derivative of the Appell function with respect to Z_2 is further complicated because it involves the derivative with respect to both a and b_2

$$\left.\frac{dF_1(a, b_1, b_2, c; z_1, z_2)}{dZ_2}\right|_{Z_2=0} = \frac{i}{v_2} \left.\frac{dF_1(a, b_1, b_2, c; z_1, z_2)}{da}\right|_{Z_2=0} + \frac{i}{v''} \left.\frac{dF_1(a, b_1, b_2, c; z_1, z_2)}{db_2}\right|_{Z_2=0}. \tag{73}$$

However, using the results presented above [specifically, Eqs. (21b) and (18b)], a closed form expression in terms of multivariable hypergeometric functions results

$$\begin{aligned} \frac{d}{dZ_2} F_1(a, b_1, b_2, c; z_1, z_2)\bigg|_{Z_2=0} ={}& -\frac{z_1 \eta'}{v_2}\, {}_2\Theta_1^{(1)}\left(\begin{array}{c} 1, 1 |\, \tilde{a}, \tilde{a}+1, b_1+1 \\ \tilde{a}+1 |\, 2, 2 \end{array} \middle| ; z_1, z_1 \right) \\ &+ i\frac{\tilde{a} z_2}{v''}\, {}_2\Theta_1^{(1)}\left(\begin{array}{c} 1, b_1 |\, 1, \tilde{a}+1, - \\ 2 |\, 2, - \end{array} \middle| ; z_2, z_1 \right), \end{aligned} \tag{74}$$

where $\tilde{a} = 1 + i\eta_1$.

The result given in Eq. (71) differs substantially from the one that would come from the use a distortion factor $D^{-}(\mathbf{v}_1 t, \mathbf{v}_2 t)$ given by a simple product of two Coulomb functions. With the present Φ_2 approach, the first order allows for an interference between the electron scattering by each of the molecular nuclei individually. Indeed, when calculating the square modulus of the amplitude, an interference arises from the cross product involving the first derivatives of Eq. (73), both of which can be easily obtained with the formulas provided in Sec. II. Higher orders in powers of Z_1 and Z_2 can be included in expansion (71), summing one by one the interactions between the electron and the charged centers of the residual molecule. Such a construction, or the aforementioned interference, is not possible with a distortion factor represented by a simple product of Coulomb factors. The physics associated with the included Coulomb interaction is contained in the coefficients of the Taylor expansion (71). The latter are given by derivatives of the Appell function F_1, that is to say—as shown in the present contribution—in terms of the ${}_2\Theta_1^{(n)}$ functions.

V. SUMMARY

We have studied the derivatives to any order n, with respect to their parameters, of eight Horn hypergeometric functions. They can be written as $n+2$ infinite summations. The following systematic approach results to be rather practical. One first expresses Horn functions in terms of single series of one-variable Gaussian or confluent hypergeometric functions. Then, one makes use of compact expressions obtained previously with a differential equation approach,[34,42] for their nth derivatives with respect to parameters. For the considered 8 Horn functions, and for most parameters, the nth derivatives can be easily written as single sums of generalizations of multivariable Kampé de Fériet functions, noted as ${}_2\Theta_1^{(n)}$ or $\Theta^{(n)}$.

The methodology presented in this contribution can be extended—in the same systematic way—to the study of the derivative with respect to their parameters of other two-variable, or three-variable, hypergeometric functions. The starting point consists in expressing such series as a sum of ${}_2F_1$ (possibly ${}_1F_1$, or even ${}_pF_q$) functions, and then apply the expressions presented in Refs. 34, 42, and 43 for the derivatives with respect to their parameters of these one-variable hypergeometric functions. We have mentioned how the methodology can be applied, for example, to other Horn functions, such as G_1, G_2, and H_4; for other ones, such as G_3, intricate summation indices are difficult to treat in a similar way and need a thorough investigation.

With a study of autoionization of atoms, we have provided a physical illustration in which the successive derivatives of an F_1 function with respect to the first and third parameters appear in a Born-like expansion. The Coulomb interaction of the emitted electron with nuclear charges is built up, step by step, through the expansion coefficients; the latter are obtained from the Appell derivatives evaluated at zero nuclear charges. Such an approach finally provides the interference occurring as a consequence of the scattering of the emitted electron with each individual nucleus.

Besides this application, the mathematical results presented here should be useful in a wide range of physical and mathematical problems in which one or several parameters of two-variable Horn functions play a particular role.

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