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Anisotropic BV–L^2 regularization of linear inverse ill-posed problems

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ABSTRACT

During the last two decades several generalizations of the traditional Tikhonov–Phillips regularization method for solving inverse ill-posed problems have been proposed. Many of these variants consist essentially of modifications on the penalizing term, which force certain features in the obtained regularized solution ([11,18]). If it is known that the regularity of the exact solution is inhomogeneous it is often desirable the use of mixed, spatially adaptive methods ([7,12]). These methods are also highly suitable when the preservation of edges is an important issue, since they allow for the inclusion of anisotropic penalizers for border detection ([20]). In this work we propose the use of a penalizer resulting from the convex spatially-adaptive combination of a classic L^2 penalizer and an anisotropic bounded variation seminorm. Results on existence and uniqueness of minimizers of the corresponding Tikhonov–Phillips functional are presented. Results on the stability of those minimizers with respect to perturbations in the data, in the regularization parameter and in the operator are also established. Applications to image restoration problems are shown.

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1. Introduction

A linear inverse problem can be formulated as: find $u \in \mathcal{X}$ such that

$$Tu = v, \tag{1}$$

where T is a bounded linear operator with non-closed range between two infinite dimensional normed spaces $\mathcal{X}$ and $\mathcal{Y}$ (usually $\mathcal{X}$ and $\mathcal{Y}$ are function spaces) and v is the data, which might be exactly or approximately known (with a certain error). Under these hypotheses it is well known that problem (1) is ill-posed, the Moore–Penrose pseudo-inverse of T is unbounded and therefore small errors in the data v may result in arbitrarily large errors in the approximations of u ([19]). Because of this, problem (1) must be "regularized" before any attempt to solve it is made. Regularizing problem (1) essentially means replacing it by a family of "well-posed" problems whose solutions converge (in an appropriate sense) to a solution of the original problem. Undoubtedly, the most usual way of regularizing a linear ill-posed problem is by means of the Tikhonov–Phillips method, which can be formulated as an unconstrained minimization problem. In fact, given a penalizer W with domain $\mathcal{D} \subset \mathcal{X}$, the Tikhonov–Phillips regularized solution of (1) is the global minimizer over $\mathcal{D}$ of the functional

$$F_{\alpha,W}(u) = \|Tu - v\|^2 + \alpha W(u), \tag{2}$$

where $\alpha > 0$ is a constant called regularization parameter. The original method was proposed independently by Phillips and Tikhonov in 1962 and 1963 ([15,21]) using $W(u) = \|u\|^2_{\mathcal{X}}$. Other penalizers can also be used to regularize the problem and in the last few decades, considerable research has been devoted in this direction. For instance, an interesting problem is the study of penalizers which are particularly appropriate for preserving certain known or assumed properties of the exact solution. Sufficient conditions on a general penalizer W guaranteeing existence, uniqueness and stability of the minimizers of (2) under different types of perturbations can be found in [6,9,11,17,18].

In this article we will consider functionals W of mixed L^2–BV type in which the idea of "anisotropic" penalization will be incorporated. Mixed penalizers were previously studied by a few authors (see [5,11,12]). Anisotropy ideas were used in [8,14] and [20] for edge enhancement in problems of image deblurring and image denoising, whereas in [4] for image inpainting. It is also timely to mention that multi-penalty regularization with component-wise penalization was studied by Naumova and Pereverzyev in [13] while the case of multi-parameter Tikhonov regularization with ℓ^0 sparsity constraint was studied by Wang et al. in [22].

2. Preliminaries

In what follows $\Omega \subset \mathbb{R}^2$ will be a bounded open convex set with Lipschitz boundary, $\mathcal{M}(\Omega)$ shall denote the set of all real valued measurable functions defined on Ω and $\widehat{\mathcal{M}}(\Omega)$ the subset of $\mathcal{M}(\Omega)$ formed by those functions with values in $[0, 1]$.

Definition 2.1. Given $\theta \in \widehat{\mathcal{M}}(\Omega)$, a measurable matrix field $A : \Omega \to \mathbb{R}^{2\times 2}$ and $p \in [1, \infty)$ we define the functional $J^{\Omega}_{\theta,A,p}(u)$ with values on the extended reals by

$$J^{\Omega}_{\theta,A,p}(u) \doteq \sup_{\vec{\nu} \in \mathcal{V}^{\Omega}_{\theta,A,p}} \int_{\Omega} -u \operatorname{div}(\theta A\vec{\nu})\, dx, \qquad u \in L^1(\Omega), \tag{3}$$

where $\mathcal{V}^{\Omega}_{\theta,A,p} \doteq \{\vec{\nu} : \Omega \to \mathbb{R}^2$ such that $\theta A\vec{\nu} \in C^1_0(\Omega)$ and $|\vec{\nu}(x)|_{p^*} \leq 1\, \forall\, x \in \Omega\}$ and p^* is the conjugate dual of p. Here, for any $p \in [1, \infty]$, $|\cdot|_p$ denotes the p-norm in $\mathbb{R}^2$.

Remark 2.2. It can be easily shown that for any $u \in L^1(\Omega)$ and any $\Omega^* \subset \Omega$ there holds $J^{\Omega^*}_{\theta,A,p}(u) \leq J^{\Omega}_{\theta,A,p}(u)$ and $\mathcal{V}^{\Omega}_{\theta,A,p} \subset \mathcal{V}^{\Omega^*}_{\theta,A,p}$.

Definition 2.3. The functional J^{Ω}_0 (BV-seminorm) with values on the extended reals is defined as

$$J^{\Omega}_0(u) = \sup_{\vec{\nu}\in\mathcal{V}^{\Omega}} \int_{\Omega} -u \operatorname{div} \vec{\nu}\, dx, \qquad u \in L^1(\Omega) \tag{4}$$

with $\mathcal{V}^{\Omega} \doteq \{\vec{\nu} : \Omega \to \mathbb{R}^2 \text{ such that } \vec{\nu} \in C^1_0(\Omega) \text{ and } |\vec{\nu}(x)|_2 \leq 1 \,\forall\, x \in \Omega\}$.

Remark 2.4. In the case $\theta(x) \equiv 1$, $A(x) \equiv I$ (the identity matrix) and $p = 2$ one clearly has $\mathcal{V}^{\Omega}_{\theta,A,p} = \mathcal{V}^{\Omega}$ and $J^{\Omega}_{\theta,A,p} = J^{\Omega}_0$. For convenience we shall suppress the superscript Ω and unless otherwise specified $J_{\theta,A,p}$, $\mathcal{V}_{\theta,A,p}$ and $\mathcal{V}$ shall denote $J^{\Omega}_{\theta,A,p}$, $\mathcal{V}^{\Omega}_{\theta,A,p}$ and $\mathcal{V}^{\Omega}$, respectively.

When θ, u and A are smooth, the functional $J_{\theta,A,p}$ takes a particular form.

Proposition 2.5. *Let $J_{\theta,A,p}$ be as in (3). If $\theta,\ u \in W^{1,1}(\Omega)$, $A \in W^{1,1}(\Omega;\mathbb{R}^{2\times 2})$ is symmetric and $p \in [1,2]$, then $J_{\theta,A,p}(u) = \|\theta\, |A\nabla u|_p\, \|_{L^1(\Omega)}$.*

Proof. Assume that $\theta,\ u \in C^1(\Omega)$ and $A \in C^1(\Omega;\mathbb{R}^{2\times 2})$ (by standard density arguments, the result can be proved for general $\theta,\ u \in W^{1,1}(\Omega)$ and $A \in W^{1,1}(\Omega;\mathbb{R}^{2\times 2})$).

Let $\vec{n}$ denote the outward unit normal to $\partial\Omega$ and q the conjugate dual of p. Then for all $\vec{\nu} \in \mathcal{V}_{\theta,A,p}$ it follows that

$$\begin{aligned}
\int_{\Omega} -u \operatorname{div}(\theta A\vec{\nu})\, dx &= \int_{\Omega} \nabla u \cdot \theta A\vec{\nu}\, dx - \int_{\partial\Omega} (u\theta A\vec{\nu} \cdot \vec{n})\, dS \\
&= \int_{\Omega} \nabla u \cdot \theta A\vec{\nu}\; dx && (\text{since } \theta A\vec{\nu}|_{\partial\Omega} = 0) \\
&= \int_{\Omega} \theta A\nabla u \cdot \vec{\nu}\; dx && (\text{since } A \text{ is symmetric}) \\
&\leq \int_{\Omega} |\theta A\nabla u|_p\, |\vec{\nu}|_q\; dx && (\text{by Hölder's inequality}) \\
&\leq \int_{\Omega} |\theta A\nabla u|_p\; dx && (\text{since } |\vec{\nu}(x)|_q \leq 1 \,\forall\, x \in \Omega).
\end{aligned}$$

Taking supremum over $\vec{\nu} \in \mathcal{V}_{\theta,A,p}$ it follows that $J_{\theta,A,p}(u) \leq \|\theta\, |A\nabla u|_p\, \|_{L^1(\Omega)}$. For the opposite inequality, let $A\nabla u(x) \doteq (w_1(x),\ w_2(x))^T$ and define

$$\vec{\nu}_*(x) \doteq \begin{cases} |A\nabla u(x)|_p^{1-p} \left(\operatorname{sgn}(w_1(x))\, |w_1(x)|^{p-1},\ \ \operatorname{sgn}(w_2(x))\, |w_2(x)|^{p-1}\right)^T, & \text{if } A\nabla u(x) \neq 0, \\ 0, & \text{if } A\nabla u(x) = 0. \end{cases}$$

Then one has that $|\vec{\nu}_*(x)|_q \leq 1 \,\forall\, x \in \Omega$ and

$$\int_{\Omega} A\nabla u \cdot \theta\vec{\nu}_*\, dx = \int_{\Omega} |\theta A\nabla u|_p\; dx. \tag{5}$$

Now, given any $\varepsilon > 0$ there exists a function $\varphi \in C_0^\infty(\Omega;\mathbb{R}^2)$ such that by convolving $\vec{\nu}_*$ with φ we obtain a function $\vec{\nu}_\varepsilon \in C_0^1(\Omega;\mathbb{R}^2)$ such that $|\vec{\nu}_\varepsilon(x)|_q \leq 1\ \forall x \in \Omega$ (and therefore $\vec{\nu}_\varepsilon \in \mathcal{V}_{\theta,A,p}$ since u and θ are in $C^1(\Omega)$ and $A \in C^1(\Omega;\mathbb{R}^{2\times 2})$) for which $\left|\int_\Omega A\nabla u \cdot \theta(\vec{\nu}_* - \vec{\nu}_\varepsilon)\,dx\right| \leq \varepsilon$. From this and (5) it follows that $\int_\Omega -u\,\mathrm{div}(\theta A\vec{\nu}_\varepsilon)\,dx \geq \|\theta\,|A\nabla u|_p\,\|_{L^1(\Omega)} - \varepsilon$. Since ε is arbitrary, taking supremum over $\vec{\nu} \in \mathcal{V}_{\theta,A,p}$ we conclude that

$$J_{\theta,A,p}(u) \geq \|\theta\,|A\nabla u|_p\,\|_{L^1(\Omega)}.$$

Hence $J_{\theta,A,p}(u) = \|\theta\,|A\nabla u|_p\,\|_{L^1(\Omega)}$, as we wanted to prove. □

In this article we will consider penalizers of the form

$$\begin{aligned} W_{\theta,A,p}(u) &= \alpha_1 \int_\Omega |\sqrt{1-\theta(x)}\,u(x)|^2\,dx + \alpha_2 \sup_{\vec{\nu}\in\mathcal{V}_{\theta,A,p}} \int_\Omega -u\,\mathrm{div}(\theta A\vec{\nu})\,dx \\ &= \alpha_1 \int_\Omega |\sqrt{1-\theta(x)}\,u(x)|^2\,dx + \alpha_2\,J_{\theta,A,p}(u), \end{aligned} \tag{6}$$

where α_1, α_2 are positive constants. In view of Proposition 2.5 if θ, $u \in W^{1,1}(\Omega)$ and $A \in W^{1,1}(\Omega;\mathbb{R}^{2\times 2})$ is symmetric then $W_{\theta,A,p}(u)$ as defined in (6) takes the form

$$W_{\theta,A,p}(u)(u) = \alpha_1 \int_\Omega |\sqrt{1-\theta(x)}\,u(x)|^2\,dx + \alpha_2\,\|\theta|A\nabla u|_p\|_{L^1(\Omega)}.$$

Note that the case $\theta(x) \equiv 0$ corresponds to the penalizer associated to the classical Tikhonov–Phillips method, while $\theta(x) \equiv 1$ corresponds to a pure anisotropic p-BV method, with the classical bounded variation method corresponding to $p = 2$ and $A(x) \equiv I$. The matrix field A is introduced with the objective of allowing anisotropic penalization. There are several ways of constructing this so-called "anisotropy matrix field", either from structural prior information or from the available data (see [4,8]). The construction of this matrix field is a very important matter on which we shall not get any deeper here. The general case can then be thought of as a convex combination of a classical L^2 and an anisotropic p-BV penalizers. The particular case $A(x) \equiv I$ was studied in [12].

3. Main results

In this section we state our main results about existence, uniqueness and stability of minimizers of generalized Tikhonov–Phillips functionals with penalizers involving spatially varying combinations of the L^2-norm and the functional $J_{\theta,A,p}$ under different hypotheses on the function θ and the anisotropy matrix A.

Definition 3.1. Let $p \in [1,2]$, $Q \subset \Omega$, $\theta \in \widehat{\mathcal{M}}(Q)$ with $\frac{1}{\theta} \in L^\infty(Q)$, $A : Q \to \mathbb{R}^{2\times 2}$ a measurable matrix field. We shall say that θ and A satisfy the weight-anisotropy (WA) condition in Q if

$$\|\theta(x)A(x)\|_{\mathcal{L}(\mathbb{R}^{2,p^*})} \geq \frac{1}{\|\theta^{-1}\|_{L^\infty(Q)}},\ \forall\,x \in Q. \tag{7}$$

Here, for a 2×2 matrix B, $\|B\|_{\mathcal{L}(\mathbb{R}^{2,p^*})}$ denotes the norm of B as an operator on $\mathbb{R}^2$ with the p^*-norm.

In order to be able to use certain known coercivity properties of the functional J_0^Ω (see [1]) an inequality of the type $J_{\theta,A,p}(u) \geq C\, J_0^\Omega(u)$ for some constant C is highly desired. The following theorem provides sufficient conditions on θ and A assuring that such an inequality holds.

Theorem 3.2. *Let $p \in [1,2]$ and $\theta \in \widehat{\mathcal{M}}(\Omega)$ with $\frac{1}{\theta} \in L^\infty(\Omega)$, $A : \Omega \to \mathbb{R}^{2\times 2}$ a measurable matrix field such that they satisfy the WA condition in Ω and let $J_{\theta,A,p}$, J_0^Ω be the functionals defined in (3) and (4), respectively. Then $J_0^\Omega(u) \leq \|\frac{1}{\theta}\|_{L^\infty(\Omega)}\, J_{\theta,A,p}(u)$ for all $u \in L^1(\Omega)$.*

Proof. Let $u \in L^1(\Omega)$. Then for all $\vec{\nu} \in \mathcal{V}$

$$\begin{aligned} \int_\Omega -u \operatorname{div} \vec{\nu}\, dx &= \|\theta^{-1}\|_{L^\infty(\Omega)} \int_\Omega -u \operatorname{div}\left(\frac{\theta A A^{-1}\vec{\nu}}{\|\theta^{-1}\|_{L^\infty(\Omega)}\theta}\right) dx \\ &\leq \|\theta^{-1}\|_{L^\infty(\Omega)} \sup_{\vec{\omega}\in\mathcal{V}_{\theta,A,p}} \int_\Omega -u \operatorname{div}(\theta A \vec{\omega})\, dx \\ &= \|\theta^{-1}\|_{L^\infty(\Omega)}\, J_{\theta,A,p}(u), \end{aligned}$$

where the inequality is a consequence of $\frac{A^{-1}\vec{\nu}}{\|\theta^{-1}\|_{L^\infty(\Omega)}\,\theta} \in \mathcal{V}_{\theta,A,p}$ (which follows immediately from the fact that θ and A satisfy the WA condition in Ω and $\vec{\nu} \in \mathcal{V}$). Then, taking supremum for $\vec{\nu} \in \mathcal{V}$ we conclude that $J_0^\Omega(u) \leq \|\theta^{-1}\|_{L^\infty(\Omega)}\, J_{\theta,A,p}(u)$. □

The following lemma will be of fundamental importance for proving several of the upcoming results.

Lemma 3.3. *Let $p \in [1,2]$, $\theta \in \widehat{\mathcal{M}}(\Omega)$ and $A : \Omega \to \mathbb{R}^{2\times 2}$ be a measurable matrix field. Then the functional $J_{\theta,A,p}$ defined by (3) is convex and weakly lower semicontinuous with respect to the $L^q(\Omega)$ topology for every $q \in [1,\infty)$.*

Proof. The convexity of $J_{\theta,A,p}$ is trivial. To prove the weak lower semicontinuity, let $q \in [1,\infty)$, $\{u_n\} \subset L^q(\Omega)$ and $u \in L^q(\Omega)$ be such that $u_n \overset{w-L^q}{\to} u$. We want to show that $J_{\theta,A,p}(u) \leq \liminf_{n\to\infty} J_{\theta,A,p}(u_n)$. Let $\vec{\nu}_* \in \mathcal{V}_{\theta,A,p}$ and q^* be the conjugate dual of q. Since $\theta A \vec{\nu}_* \in C_0^1(\Omega)$ it follows that $\operatorname{div}(\theta A \vec{\nu}_*) \in L^\infty(\Omega) \subset L^{q^*}(\Omega)$. Then $\int_\Omega -u \operatorname{div}(\theta A \vec{\nu}_*)\, dx = \lim_{n\to\infty} \int_\Omega -u_n \operatorname{div}(\theta A \vec{\nu}_*)\, dx \leq \liminf_{n\to\infty} \sup_{\vec{\nu}\in\mathcal{V}_{\theta,A,p}} \int_\Omega -u_n \operatorname{div}(\theta A \vec{\nu})\, dx = \liminf_{n\to\infty} J_{\theta,A,p}(u_n)$, where the first equality follows from the weak convergence of u_n to u in $L^q(\Omega)$. Taking supremum over all $\vec{\nu}_* \in \mathcal{V}_{\theta,A,p}$ it follows that $J_{\theta,A,p}(u) \leq \liminf_{n\to\infty} J_{\theta,A,p}(u_n)$. □

We are now ready to present several results on existence, uniqueness and stability of minimizers of generalized Tikhonov–Phillips functionals with penalizers involving spatially varying combinations of the L^2-norm and of the functional $J_{\theta,A,p}$, under different hypotheses on the function θ and the anisotropy matrix A.

Theorem 3.4. *Let $\mathcal{X} = L^2(\Omega)$, $\mathcal{Y}$ a reflexive Banach space, $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$, $v \in \mathcal{Y}$, α_1, α_2 positive constants, $p \in [1,2]$, $\theta \in \widehat{\mathcal{M}}(\Omega)$ such that $\frac{1}{1-\theta} \in L^1(\Omega)$ and $\frac{1}{\theta} \in L^\infty(\Omega)$, $A : \Omega \to \mathbb{R}^{2\times 2}$ a measurable matrix field such that θ and A satisfy the WA condition in Ω (see Definition 3.1). Then the functional*

$$\begin{aligned} F_{\theta,A,p}(u) &\doteq \|Tu - v\|_{\mathcal{Y}}^2 + \alpha_1 \|\sqrt{1-\theta}\, u\|_{L^2(\Omega)}^2 + \alpha_2\, J_{\theta,A,p}(u) \\ &= \|Tu - v\|_{\mathcal{Y}}^2 + W_{\theta,A,p}(u), \qquad u \in \mathcal{X}, \end{aligned} \tag{8}$$

(where $J_{\theta,A,p}$ and $W_{\theta,A,p}$ are defined in (3) and (6) respectively) has a unique global minimizer $\hat{u} \in BV(\Omega)$.

Proof. We will prove that $W_{\theta,A,p}$ is weakly lower semicontinuous and that $W_{\theta,A,p}$-bounded sets are relatively weakly compact in $\mathcal{X}$. It can be easily verified that these two conditions imply the validity of Assumptions 3.11 and 3.22 in [18] and therefore Proposition 4.1 in [18] holds (for the linear case). To prove the weak lower semicontinuity of $W_{\theta,A,p}$ let $\{u_n\} \subset \mathcal{X}$ such that $u_n \overset{w}{\to} u$. Since $\sqrt{1-\theta} \in L^\infty(\Omega)$ one has $\sqrt{1-\theta}\, u_n \overset{w}{\longrightarrow} \sqrt{1-\theta}\, u$. The condition $\frac{1}{1-\theta} \in L^1(\Omega)$ implies that the functional $\|\sqrt{1-\theta} \cdot \|_{L^2(\Omega)}$ defines a norm in $L^2(\Omega)$ and therefore it is weakly lower semicontinuous. From this and the weak lower semicontinuity of $J_{\theta,A,p}$ in $L^2(\Omega) = \mathcal{X}$ (see Lemma 3.3) we conclude that the functional $W_{\theta,A,p}$ is in fact weakly lower semicontinuous on $\mathcal{X}$.

To prove the relative weak compactness of $W_{\theta,A,p}$-bounded sets, let $\{u_n\} \subset \mathcal{X}$ be such that $W_{\theta,A,p}(u_n) \leq c_1 < \infty\ \forall n$. We want to show that there exists a subsequence $\{u_{n_j}\} \subset \{u_n\}$ and $u \in \mathcal{X}$ such that $u_{n_j} \overset{w}{\to} u$. Since $\{W_{\theta,A,p}(u_n)\}$ is uniformly bounded, there exists $K < \infty$ such that $\|\sqrt{1-\theta}\, u_n\|_{L^2(\Omega)} \leq K\ \forall n$. From this, the hypotheses $\frac{1}{1-\theta} \in L^1(\Omega)$, $\frac{1}{\theta} \in L^\infty(\Omega)$ and Theorem 3.2, it follows easily that $\{u_n\}$ is $BV(\Omega)$-bounded. The existence of a global minimizer of functional (8) belonging to $BV(\Omega)$ then follows from the fact that $BV(\Omega)$-bounded sets are relatively compact in $L^2(\Omega)$ ([2,3]). Finally, note that the condition $\frac{1}{1-\theta} \in L^1(\Omega)$ implies the strict convexity of $\|\sqrt{1-\theta}\, u\|^2_{L^2(\Omega)}$ and, in light of Lemma 3.3, since the other two terms on the RHS of (8) are convex, the uniqueness of the global minimizer of $F_{\theta,A,p}$ follows. □

Remark 3.5. Note that if $\theta(x) \equiv 0$, then $W_{\theta,A,p}(u) = \alpha_1 \|u\|^2_{L^2(\Omega)}$ and $F_{\theta,A,p}$ as defined in (8) is the classical zero-order Tikhonov–Phillips functional, while for $\theta(x) \equiv 1$ a pure anisotropic p-BV penalizer is obtained. Although the hypotheses of Theorem 3.4 clearly exclude both of these cases, the existence of a global minimizer for the first one is well known while for the second one, existence is guaranteed by the next theorem.

Theorem 3.6. *Let $\mathcal{X} = L^1(\Omega)$, $\mathcal{Y}$ be a normed space, $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $v \in \mathcal{Y}$, α a positive constant, $p \in [1, 2]$, $A : \Omega \to \mathbb{R}^{2\times 2}$ a measurable matrix field such that $\|A(x)\|_{\mathcal{L}(\mathbb{R}^{2,p^*})} \geq 1\, \forall x \in \Omega$ and $T\chi_\Omega \neq 0$. Then the functional*

$$F_{A,p}(u) \doteq \|Tu - v\|^2_{\mathcal{Y}} + \alpha \sup_{\vec{\nu} \in \mathcal{V}_{A,p}} \int_\Omega -u\, div(A\vec{\nu})\, dx, \qquad u \in \mathcal{X}, \tag{9}$$

where $\mathcal{V}_{A,p} \doteq \{\vec{\nu} : \Omega \to \mathbb{R}^2$ such that $A\vec{\nu} \in C^1_0(\Omega)$ and $|\vec{\nu}(x)|_{p^*} \leq 1\, \forall x \in \Omega\}$, has a global minimizer. Moreover, if T is injective then the global minimizer is unique.

Proof. By virtue of Theorem 3.1 in [1] it is sufficient to prove that the functional given in (9) is weakly lower semicontinuous and BV-coercive. The weak lower semicontinuity of $F_{A,p}$ follows from Lemma 3.3, the boundedness of T and the weak lower semicontinuity of the norm in $\mathcal{Y}$. For the BV-coercitivity note that the Theorem 3.2 implies that

$$F(u) \doteq \|Tu - v\|^2_{\mathcal{Y}} + \alpha J^\Omega_0(u) \leq F_{A,p}(u). \tag{10}$$

Now, since $T\chi_\Omega \neq 0$, by Lemma 4.1 in [1] the functional $F(\cdot)$ is BV-coercive. From this and inequality (10) it follows that $F_{A,p}(\cdot)$ is also BV-coercive as we wanted to prove.

Finally note that if T is injective then $F_{A,p}(\cdot)$ is strictly convex and therefore its global minimizer is unique. □

It is timely to note that in Theorem 3.4 the function θ cannot assume the extreme values 0 or 1 on a set of positive measure. In some cases a pure anisotropic BV regularization in some regions and a pure L^2 regularization in others may be desired, and therefore such a constraint on the function θ will turn out to be inappropriate. In the next three theorems we introduce different conditions which allow the function θ to take the extreme values on sets of positive measure.

Theorem 3.7. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open convex set with Lipschitz boundary, $\mathcal{X} = L^2(\Omega)$, $\mathcal{Y}$ a reflexive Banach space, $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $v \in \mathcal{Y}$, α_1, α_2 positive constants, $p \in [1, 2]$, $\theta \in \widehat{\mathcal{M}}(\Omega)$ and $\Omega_0 \doteq \{x \in \Omega$ such that $\theta(x) = 0\}$. If $\frac{1}{\theta} \in L^\infty(\Omega_0^c)$, $\frac{1}{1-\theta} \in L^1(\Omega)$ and $A : \Omega \to \mathbb{R}^{2\times 2}$ a measurable matrix field such that θ and A satisfy the WA condition in Ω_0^c. Then the functional $F_{\theta,A,p}$ defined by equation (8) has a unique global minimizer $\hat{u} \in L^2(\Omega) \cap BV(\Omega_0^c)$.*

Proof. Under the hypotheses of the theorem, the functional $F_{\theta,A,p}$ in (8) can be written as $F_{\theta,A,p} = \|Tu - v\|^2_{\mathcal{Y}} + W_{\theta,A,p}(u)$, where now $W_{\theta,A,p}$ takes the form

$$W_{\theta,A,p}(u) \doteq \alpha_1 \|u\|^2_{L^2(\Omega_0)} + \alpha_1 \|\sqrt{1-\theta}\, u\|^2_{L^2(\Omega_0^c)} + \alpha_2 \sup_{\vec{v} \in \mathcal{V}_{\theta,A,p}} \int_{\Omega_0^c} -u|_{\Omega_0^c} \operatorname{div}(\theta A \vec{v})\, dx. \tag{11}$$

Just like in Theorem 3.4, for the existence of a minimizer it suffices to prove that $W_{\theta,A,p}$ is weakly lower semicontinuous and that $W_{\theta,A,p}$-bounded sets are relatively weakly compact in $\mathcal{X}$. The weak lower semicontinuity of $W_{\theta,A,p}$ follows from identical steps to the ones in Theorem 3.4. On the other hand, the relative weak compactness on $W_{\theta,A,p}$-bounded sets can be obtained following similar steps as for the proof of Theorem 2.9 in [12], with the obvious modifications to take into account the anisotropy matrix field A. Finally, uniqueness is a consequence of the fact that the hypothesis $\frac{1}{1-\theta} \in L^1(\Omega)$ implies the strict convexity of $F_{\theta,A,p}$. □

Note that in the previous theorem, the condition $\frac{1}{1-\theta} \in L^1(\Omega)$ implies $\theta \neq 1$ a.e. in Ω. The next theorem includes the case in which θ can be equal to one on a set of positive measure.

Theorem 3.8. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open convex set with Lipschitz boundary, $\mathcal{X} = L^2(\Omega)$, $\mathcal{Y}$ a reflexive Banach space, $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $v \in \mathcal{Y}$, α_1, α_2 positive constants, $p \in [1, 2]$. Also let $\Omega_1 \doteq \{x \in \Omega$ such that $\theta(x) = 1\}$, $A : \Omega \to \mathbb{R}^{2\times 2}$ be a measurable matrix field, $\theta \in \widehat{\mathcal{M}}(\Omega)$ and suppose $\frac{1}{1-\theta} \in L^1(\Omega_1^c)$. Furthermore, assume that there exists $M \subset \Omega$ (M a convex region with Lipschitz continuous boundary) such that $\frac{1}{\theta} \in L^\infty(M)$, $T\chi_M \neq 0$ and θ and A satisfy the WA condition in M. Then the functional defined by (8) has a global minimizer $\hat{u} \in L^2(\Omega) \cap BV(M)$. If moreover, $u \in \mathcal{N}(T)$ and $u \neq 0$ implies $u|_{\Omega_1} \neq 0$, then such a global minimizer is unique.*

Proof. We will prove that under the hypotheses of the theorem, the functional $F_{\theta,A,p}(\cdot)$ defined by (8) is weakly lower semicontinuous with respect to the $L^2(\Omega)$ topology and BV-coercive. First, note that under the hypotheses of the theorem we can write

$$F_{\theta,A,p}(u) = \|Tu - v\|^2_{\mathcal{Y}} + \alpha_1 \|\sqrt{1-\theta}\, u\|^2_{L^2(\Omega_1^c)} + \alpha_2\, J_{\theta,A,p}(u), \quad u \in \mathcal{X}. \tag{12}$$

Since $\frac{1}{1-\theta} \in L^1(\Omega_1^c)$, it follows that $\|\sqrt{1-\theta} \cdot \|_{L^2(\Omega_1^c)}$ is a norm in $L^2(\Omega_1^c)$ and therefore $\|\sqrt{1-\theta}\, u\|^2_{L^2(\Omega_1^c)}$ is weakly lower semicontinuous. The weak lower semicontinuity of $F_{\theta,A,p}(\cdot)$ then follows immediately from this fact, from Lemma 3.3 and from the convexity of $\|Tu - v\|^2_{\mathcal{Y}}$. For the BV-coercivity, note that

$$\begin{aligned} \|Tu - v\|^2_{\mathcal{Y}} + \alpha_2 J_0^M(u) &\leq \|Tu - v\|^2_{\mathcal{Y}} + \alpha_2 \left\| \frac{1}{\theta} \right\|_{L^\infty(M)} J_{\theta,A,p}(u) \quad \text{(from Theorem 3.2 and Remark 2.2)} \\ &\leq \|Tu - v\|^2_{\mathcal{Y}} + \alpha_2 \left\| \frac{1}{\theta} \right\|_{L^\infty(M)} J_{\theta,A,p}(u) + \alpha_1 \|\sqrt{1-\theta}\, u\|^2_{L^2(\Omega_1^c)} \end{aligned}$$

$$\leq \left\| \frac{1}{\theta} \right\|_{L^\infty(M)} F_{\theta,A,p}(u) \qquad (\text{since } \left\|\theta^{-1}\right\|_{L^\infty(M)} \geq 1). \tag{13}$$

Now, since $T\chi_M \neq 0$, by Lemma 4.1 in [1] the functional $\|Tu - v\|^2_{\mathcal{Y}} + \alpha_2 J_0^M(u)$ is BV-coercive. From this and inequality (13) it follows that $F_{\theta,A,p}(\cdot)$ is also BV-coercive. The existence of a global minimizer $\hat{u} \in L^2(\Omega)$ is then obtained from Theorem 3.1 in [1]. The fact that $\hat{u} \in BV(M)$ follows immediately from observing that $\hat{u} \in L^1(M)$ and, by virtue of (13), $J_0^M(\hat{u}) < \infty$ (since $F_{\theta,A,p}(\hat{u}) < \infty$). Finally, since $Tu = 0$ (for $u \neq 0$) implies $u|_{\Omega_1} \neq 0$, it follows that $F_{\theta,A,p}$ is strictly convex and therefore such a global minimizer $\hat{u}$ is unique. □

Note that if in Theorem 3.8 $\Omega_1 = \emptyset$ and $M = \Omega_0^c$ (where Ω_0^c is as in Theorem 3.7) then Theorem 3.8 reduces to Theorem 3.7.

Finally, we present two stability results for the minimizers of functionals of type (8) under perturbations in the data, in the model and in the regularization parameters.

Theorem 3.9. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open convex set with Lipschitz boundary, $\mathcal{X} = L^2(\Omega)$, $\mathcal{Y}$ a reflexive Banach space, $p \in [1,2]$, $\theta \in \widehat{\mathcal{M}}(\Omega)$ such that $\frac{1}{1-\theta} \in L^1(\Omega)$ and $\frac{1}{\theta} \in L^\infty(\Omega)$, $A : \Omega \to \mathbb{R}^{2\times 2}$ a measurable matrix field such that θ and A satisfy the WA condition in Ω. Let also $T, T_n \in \mathcal{L}(\mathcal{X},\mathcal{Y}), v, v_n \in \mathcal{Y}, \alpha_1, \alpha_2, \alpha_1^{(n)}, \alpha_2^{(n)}$ positive constants, for $n = 1, 2, ...$, such that as $n \to \infty$, $\alpha_1^{(n)} \to \alpha_1$, $\alpha_2^{(n)} \to \alpha_2$, $v_n \to v$ and $T_n u \to Tu$ uniformly for u in $W_{\theta,A,p}$-bounded sets, where $W_{\theta,A,p}$ is as in (6). Let $F_{\theta,A,p}$ be as in (8) and define*

$$F^{(n)}_{\theta,A,p}(u) \doteq \|T_n u - v_n\|^2_{\mathcal{Y}} + \alpha_1^{(n)} \|\sqrt{1-\theta}\, u\|^2_{L^2(\Omega)} + \alpha_2^{(n)} J_{\theta,A,p}(u), \qquad u \in L^2(\Omega).$$

If $\hat{u}, u_n$ are the global minimizers of $F_{\theta,A,p}$ and $F^{(n)}_{\theta,A,p}$, respectively, then $u_n \xrightarrow{w} \hat{u}$.

Proof. Note first that by virtue of Theorem 3.4 the functionals $F_{\theta,A,p}$ and $F^{(n)}_{\theta,A,p}$ have unique global minimizers $\hat{u}$ and u_n, respectively. To prove that $u_n \xrightarrow{w} \hat{u}$, we will resort to Theorem 3.3 in [11], for which we need to prove that the following conditions hold: i) the functional $W_{\theta,A,p}$ is uniformly bounded from below; ii) every $W_{\theta,A,p}$-bounded set is relatively compact in $\mathcal{X}$; iii) the functional $F_{\theta,A,p}$ is $W_{\theta,A,p}$ subsequentially weakly lower semicontinuous; iv) the functional $F^{(n)}_{\theta,A,p}$ is $W_{\theta,A,p}$-coercive; and v) $F^{(n)}_{\theta,A,p}$ is $W_{\theta,A,p}$-uniformly consistent for $F_{\theta,A,p}$ (i.e. $F^{(n)}_{\theta,A,p}(u) \to F_{\theta,A,p}(u)$, uniformly for u in $W_{\theta,A,p}$-bounded sets).

Condition i) is trivial while condition ii) was proved in Theorem 3.4. On the other hand, for iii), with a proof analogous to that of Theorem 3.8 (with $\Omega_1 = \emptyset$) it follows that $F_{\theta,A,p}$ is weakly lower semicontinuous (and hence subsequentially weakly lower semicontinuous). For iv), let $\{u_j\} \subset L^2(\Omega)$ such that $W_{\theta,A,p}(u_j) \to \infty$ as $j \to \infty$ and note that $F^{(n)}_{\theta,A,p}(u_j) \geq \alpha_1^{(n)} \|\sqrt{1-\theta}\, u_j\|^2_{L^2(\Omega)} + \alpha_2^{(n)} J_{\theta,A,p}(u_j) \geq c_n W_{\theta,A,p}(u_j)$, where $c_n \doteq \min\left\{\frac{\alpha_1^{(n)}}{\alpha_1}, \frac{\alpha_2^{(n)}}{\alpha_2}\right\} > 0$, for all $n \in \mathbb{N}$. Thus, $F^{(n)}_{\theta,A,p}$ is $W_{\theta,A,p}$-coercive. Finally, v) follows immediately from the convergence of $\alpha_i^{(n)}$ to α_i, as $n \to \infty$, for $i = 1, 2$, from the fact v_n converges to v and from the hypothesis that $T_n u \to Tu$ uniformly for u on $W_{\theta,A,p}$-bounded sets.

The result then follows from Theorem 3.3 in [11]. □

The next stability result corresponds to the existence proof of Theorem 3.6.

Theorem 3.10. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open convex set with Lipschitz boundary, $\mathcal{X} = L^1(\Omega)$, $\mathcal{Y}$ a reflexive Banach space, $p \in [1,2]$, $A : \Omega \to \mathbb{R}^{2\times 2}$ a measurable matrix field such that θ and A satisfy the WA condition in Ω. Let also $T, T_n \in \mathcal{L}(\mathcal{X},\mathcal{Y}), v, v_n \in \mathcal{Y}, \alpha, \alpha_n$ positive constants, for $n = 1, 2, ...$, such that as $n \to \infty$, $\alpha_n \to \alpha$, $v_n \to v$, $\|T_n \chi_\Omega\| \geq \gamma > 0$ and $T_n u \to Tu$ for every u in $\mathcal{X}$. Let $F_{A,p}$ be as in (9) and define*

$$F^{(n)}_{A,p}(u) \doteq \|T_n u - v_n\|^2_{\mathcal{Y}} + \alpha_n \sup_{\vec{\nu} \in \mathcal{V}_{A,p}} \int_\Omega -u\, div(A\vec{\nu})\, dx, \qquad u \in L^1(\Omega),$$

where $\mathcal{V}_{A,p}$ is as defined in Theorem 3.6. If $\hat{u}, u_n$ are the global minimizers of $F_{A,p}$ and $F^{(n)}_{A,p}$, respectively, then $\|u_n - \hat{u}\|_{L^1(\Omega)} \longrightarrow 0$.

Proof. Under the hypotheses of the theorem, the existence of unique global minimizers $\hat{u}, u_n$ of $F_{A,p}$ and $F^{(n)}_{A,p}$, respectively, follows immediately from Theorem 3.6.

To prove the result we will use Theorem 3.2 in [1]. For that, it suffices to prove that: i) the functionals $F_{A,p}$ and $F^{(n)}_{A,p}$ are BV-coercive; ii) $F_{A,p}$ and $F^{(n)}_{A,p}$ are lower semicontinuous; iii) the sequence of functionals $\{F^{(n)}_{A,p}\}$ is uniformly BV-coercive and iv) $\{F^{(n)}_{A,p}\}$ is consistent for $F_{A,p}$, uniformly on BV-bounded sets.

Conditions i) and ii) for $F_{A,p}$ and $F^{(n)}_{A,p}$ follow immediately as in the proof of Theorem 3.6. For the uniform BV-coercivity condition iii) note that for any $u \in L^1(\Omega)$ one has

$$\begin{aligned} F^{(n)}_{A,p}(u) &= \|T_n u - v_n\|^2_{\mathcal{Y}} + \alpha_n \sup_{\vec{\nu}\in\mathcal{V}_{A,p}} \int_\Omega -u \operatorname{div}(A\vec{\nu})\, dx \\ &\geq \|T_n u - v_n\|^2_{\mathcal{Y}} + \alpha_n J_0(u) \quad \text{(by virtue of Theorem 3.2)} \\ &\geq \|T_n u - v_n\|^2_{\mathcal{Y}} + (\alpha - \epsilon) J_0(u) \quad \text{(for some } \epsilon > 0, \text{ since } \alpha_n \to \alpha). \end{aligned} \tag{14}$$

Now, by Theorem 4.2 in [1] it follows that the functional $\|T_n u - v_n\|^2_{\mathcal{Y}} + (\alpha - \epsilon) J_0(u)$ is uniformly BV-coercive. From this and inequality (14) we conclude that $\{F^{(n)}_{A,p}\}$ is uniformly BV-coercive. Condition iv) (consistency) follows immediately as in Theorem 4.2 in [1].

The result then follows from Theorem 3.2 in [1]. □

4. Applications to image restoration

The purpose of this section is to present some applications of the mixed regularization method developed in the previous section, consisting in the simultaneous use of penalizers of L^2 and anisotropic bounded-variation (BV) type, to tackle an image restoration problem. We will show how this mixed method outperforms the pure single ones, more so when the regularity of the exact solution is inhomogeneous and/or anisotropic.

4.1. Modeling

The basic mathematical model for image blurring is given by the following Fredholm integral equation of the first kind:

$$\mathcal{T} f(x, y) = \iint_\Omega k(x, y, x', y') f(x', y') dx' dy' = g(x, y), \tag{15}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain, $f \in \mathcal{X} \doteq L^2(\Omega)$ represents the original image, k is the so called "point spread function" (PSF) and g is the blurred image. For the examples shown below we used a PSF of "atmospheric turbulence" type, i.e. we chose k to be a two-dimensional Gaussian kernel:

$$k(x, y, x', y') = (2\pi\sigma_h\sigma_v)^{-1} \exp\left(-\frac{1}{2\sigma_h^2}(x - x')^2 - \frac{1}{2\sigma_v{}^2}(y - y')^2\right), \tag{16}$$

where σ_h and σ_v are the horizontal and vertical standard deviations, respectively. It is well known ([6]) that with this PSF the operator $\mathcal{T}$ in (15) is compact with non-closed range and therefore $\mathcal{T}^\dagger$, the Moore–Penrose inverse of $\mathcal{T}$, is unbounded and problem (15), i.e.

$$\mathcal{T} f = g \tag{17}$$

is ill-posed.

4.2. Discretization

For the numerical examples that follow, we considered images defined over the domain $\Omega = [0,1] \times [0,1]$, discretized to obtain an M-by-M pixel grid and hence an M-by-M matrix U, whose entries consist of the values of the light intensity function f at the centerpoints of each pixel. Next, we stacked the columns of the matrix U to get a vector $u \in \mathbb{R}^{M^2}$ so that $u_{M(l-1)+m} = U_{m,l}\ \forall\, l, m = 1, 2, \ldots, M$. Denoting with T the matrix associated to the standard discretization of the operator $\mathcal{T}$, the finite dimensional problem corresponding to (17) reads

$$Tu = v,$$

where v is the vector obtained by evaluating g at the centerpoints of the pixels. We further assume that our observations are contaminated with white noise. Hence, our model is finally stated as

$$Tu = v + \epsilon, \tag{18}$$

where $\epsilon \in \mathbb{R}^{M^2}$ is a realization of a random variable with distribution $\mathcal{N}(0, \sigma^2_{noise} I_{M^2})$.

Similarly, the discretized version of functional (8) takes the form

$$F_{\theta,A,p}(u) \doteq \frac{1}{M^2}\,\|Tu - v\|_2^2 + \frac{\alpha_1}{M^2}\|\sqrt{1-\theta}\,u\|_2^2 + \frac{\alpha_2}{M^2}\sum_{m\in\mathfrak{M}} \theta_m \left\| A_m \begin{pmatrix} M(u_m - u_{m+1}) \\ M(u_m - u_{m-M}) \end{pmatrix} \right\|_p, \tag{19}$$

where A_m and θ_m are the values of the matrix field A and of the weighting function θ at the centerpoint of the mth pixel, respectively, and $\mathfrak{M}$ denotes the set of interior pixels. Next, we shall state a suitable method to approximate the minimizer of (19) when $p = 1$. Similar steps lead to the corresponding method for the case $p = 2$. Although we do not delve into details here, a complete explanation for the case $p = 1$ can be found in [10].

4.3. Numerical implementation

We build the anisotropy matrix field $A : \Omega \to \mathbb{R}^{2\times 2}$ following the ideas in [4]. We begin by computing an *a-priori* estimation of the gradient field $\nabla u_p(x, y)$, where u_p is a zero-order Tikhonov–Phillips restoration. Then A is constructed from ∇u_p so as to comply with the following properties:

- $A(x, y)$ is a symmetric positive definite matrix $\forall (x, y) \in \Omega$.
- If $\nabla u_p(x, y) = 0$, $A(x, y) = I$ (the identity matrix).
- If $\nabla u_p(x, y) \neq 0$, $A(x, y)$ has eigenvalues $\sigma_j(x, y)$ and eigenvectors $v_j(x, y)$, $j = 1, 2$, such that

$$v_1(x,y) \parallel \nabla u_p(x,y), \qquad \sigma_1(x,y) = h(|\nabla u_p(x,y)|),$$
$$v_2(x,y) \perp \nabla u_p(x,y), \qquad \sigma_2(x,y) = 1,$$

where the function h above is decreasing, with $0 < h(t) \leq 1\ \forall t \in \mathbb{R}^+$, $h(0) = 1$ and $h(\infty) = 0$.

For our numerical examples, we took $h(t) = [1 + (t/\tau)^\kappa]^{-1}$, where $\tau, \kappa > 0$ are control parameters that can roughly be referred to as the break point from which we infer the image has an edge and the width of the transition region, respectively. With this choice of h, A was constructed as

$$A(x,y) = I - (1 - h(|\nabla u_p(x,y)|)) \left[\frac{\nabla u_p(x,y)}{|\nabla u_p(x,y)|}\right] \left[\frac{\nabla u_p(x,y)}{|\nabla u_p(x,y)|}\right]^{\mathrm{T}}. \tag{20}$$

The weighting function θ was constructed by scaling to $[0,1]$ the norm of ∇u_p; that is

$$\theta(x,y) = \frac{|\nabla u_p(x,y)|}{\max_{(x,y)\in\Omega} |\nabla u_p(x,y)|}. \tag{21}$$

Notice that with this choice of θ, functional (19) resembles the one corresponding to pure Tikhonov–Phillips regularization where $|\nabla u_p|$ is small, while it approaches a pure anisotropic BV functional where $|\nabla u_p|$ is large. It can be shown that θ and A as chosen in (21) and (20), satisfy the WA condition (7) in $\Omega = [0,1]\times[0,1]$ for both $p=1$ and $p=2$.

It now remains to find the minimizer of (19). To accomplish this, we approximate $F_{\theta,A,1}$ by a differentiable functional in order to consider its first order necessary condition. We do so by replacing, for $w \in \mathbb{R}$, the value of $|w|$ by $\phi(w)$, where $\phi : \mathbb{R} \to \mathbb{R}$ is given by $\phi(t) \doteq \sqrt{t^2 + \eta^2} - \eta$, for η sufficiently small. With this choice of ϕ, it can be shown (see §12 of [16]) that there exists a function ψ satisfying the following duality relation:

$$\begin{aligned} \phi(t) &= \inf_{s>0}(st^2 + \psi(s)), \\ \psi(s) &= \sup_{t\in\mathbb{R}}(\phi(t) - st^2), \end{aligned} \tag{22}$$

and therefore

$$\left\| A_m \begin{pmatrix} M(u_m - u_{m+1}) \\ M(u_m - u_{m-M}) \end{pmatrix} \right\|_1 \approx \inf_{s_m\in\mathbb{R}^+} \left(s_m t_{m,1}^2 + \psi(s_m)\right) + \inf_{q_m\in\mathbb{R}^+} \left(q_m t_{m,2}^2 + \psi(q_m)\right),$$

where

$$t_{m,1} = M[a_{1,1}^m(u_m - u_{m-M}) + a_{1,2}^m(u_m - u_{m+1})],$$

and

$$t_{m,2} = M[a_{2,1}^m(u_m - u_{m-M}) + a_{2,2}^m(u_m - u_{m+1})].$$

Define now the four M^2-by-M^2 diagonal matrices $A^{i,j}$, for $i,j = 1,2$, by $A_{m,m}^{i,j} = a_{i,j}^m$ if $m \in \mathfrak{M}$ and $A_{m,m}^{i,j} = 0$ otherwise. In a similar fashion, let $\Theta \doteq diag(\theta_m)_{M^2\times M^2}$, and $S, Q : \mathbb{R}^{M^2} \to \mathbb{R}^{M^2\times M^2}$ defined as $S(s) \doteq diag(s_m)_{M^2\times M^2}$ and $Q(q) \doteq diag(q_m)_{M^2\times M^2}$. Let L_x and L_y be the M^2-by-M^2 first order finite difference approximating matrices for the components of the gradient, and let R_1 and R_2 be the M^2-by-M^2 matrices defined as $R_1 \doteq A^{1,1}L_x + A^{1,2}L_y$ and $R_2 \doteq A^{2,1}L_x + A^{2,2}L_y$. Finally, let I be the M^2-by-M^2 identity matrix, and define the functional $\hat{F}_{\theta,A,1}(u,s,q) : (\mathbb{R}^{M^2})^3 \to \mathbb{R}$ by

$$\begin{aligned} \hat{F}_{\theta,A,1}(u,s,q) = \|Tu - v\|^2 &+ \frac{\alpha_1}{M^2} u^t (I_{M^2} - \Theta) u + \alpha_2 u^t R_1^t \Theta S R_1 u + \alpha_2 u^t R_2^t \Theta Q R_2 u \\ &+ \frac{\alpha_2}{M^2} \sum \theta_m \psi(s_m) + \frac{\alpha_2}{M^2} \sum \theta_m \psi(q_m). \end{aligned} \tag{23}$$

It can be shown ([10]) that

$$\inf_{s,q\in\mathbb{R}^{M^2}} \hat{F}_{\theta,A,1}(u,s,q) = F_{\theta,A,1}(u),$$

and hence, our problem of approximating the minimizer of (19) turns out to be tantamount to minimizing $\hat{F}_{\theta,A,1}$ with respect to u, s and q, simultaneously. Note that the first order necessary condition on $\hat{F}_{\theta,A,1}$ with respect to u can be written as:

$$\left(T^tT + \alpha_1(I_{M^2} - \Theta) + \alpha_2 R_1^t \Theta S R_1 + \alpha_2 R_2^t \Theta Q R_2\right) u = T^t v. \tag{24}$$

In order to minimize $\hat{F}_{\theta,A,1}$ with respect to s and q we resort to (22) to deduce that if $b_m \doteq \arg\min_{s_m \in \mathbb{R}_+} \left\{ s_m t_{m,1}^2 + \psi(s_m) \right\}$, then

$$b_m = \frac{\phi'(t_{m,1})}{2t_{m,1}}. \tag{25}$$

Similarly, if $c_m \doteq \arg\min_{q_m \in \mathbb{R}_+} \left\{ q_m t_{m,2}^2 + \psi(q_m) \right\}$, then

$$c_m = \frac{\phi'(t_{m,2})}{2t_{m,2}}. \tag{26}$$

Finally, the iterative algorithm can be stated as follows:

Step 1 – Initializing. Set $j = 0$, and initialize $u^j = u^0$, $b^j = b^0$ and $c^j = c^0$. ($u^0, b^0, c^0 \in \mathbb{R}^{M^2}$ arbitrarily chosen.)
Step 2 – Counting. Make $j = j + 1$.
Step 3 – Updating b. Update b^j and c^j using equations (25) and (26).
Step 4 – Updating u. Update u^j by solving the linear system (24).
Step 5 – Stopping. If a previously defined convergence criterion is satisfied, the algorithm ends and the global minimizer of (23), $\hat{u}$, is approximated by u^j. Otherwise, the algorithm repeats from step 2.

Next, we show some restoration examples produced with the aforementioned algorithm.

4.4. Numerical results

We shall first consider a 130×130 pixel color image and its corresponding blurred noisy version, obtained by model (18) with $\sigma_h = \sigma_v = 0.025$ and $\sigma_{noise} = 2.5\%$ of the maximum data value.

It is worth mentioning here that the blurring process, noise addition and restorations for the following examples were done separately on the red, green and blue layers of the color images (available in the web version of this article).

Example 1. Fig. 1 depicts the blurred noisy image that constitutes the observation v of the problem, along with a restoration produced with the traditional zero-order Tikhonov–Phillips regularization (from which we later estimated the gradient field to build A and θ). Fig. 2 shows the images restored using both the isotropic and anisotropic BV methods, with $p = 1$. Note that there is a significant difference on the curvature of some of the edges produced by the different methods. Finally, Fig. 3 depicts the results of the isotropic and anisotropic mixed methods.

In order to make an objective performance comparison, we use the *ISNR*, defined as

$$ISNR(\hat{u}) = 10 \log_{10} \left(\|v - u_0\|^2 / \|\hat{u} - u_0\|^2 \right),$$

(a) (b)

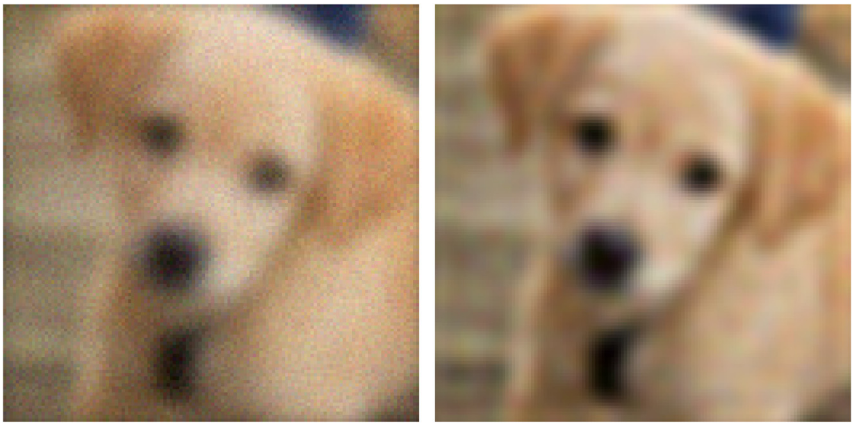

Fig. 1. (a) Blurred noisy image (observation); (b) Tikhonov–Phillips restoration.

(a) (b)

Fig. 2. (a) Isotropic BV restoration; (b) Anisotropic BV restoration.

(a) (b)

Fig. 3. (a) Mixed isotropic restoration; (b) Mixed anisotropic restoration.

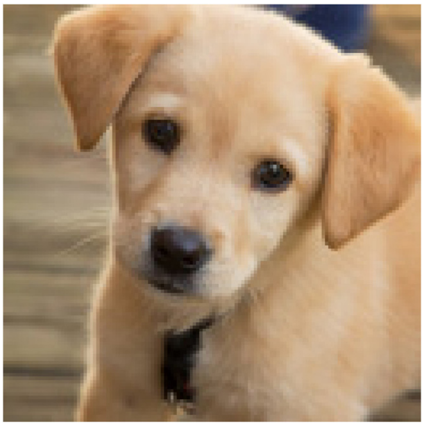

Fig. 4. Original image.

Table 1
ISNR values.

Restoration method	*ISNR*
Tikhonov 0	2.508
Isotropic BV	2.452
Anisotropic BV	2.998
Mixed isotropic	3.211
Mixed anisotropic	3.398

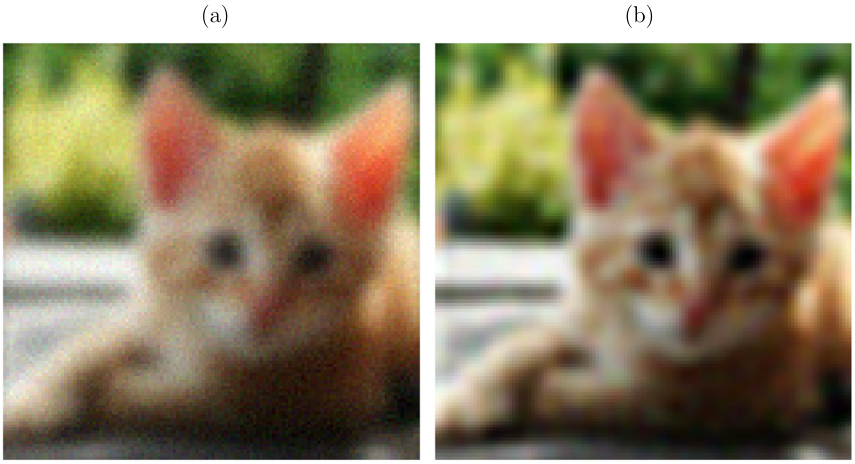

Fig. 5. (a) Blurred noisy image (observation); (b) Tikhonov–Phillips restoration.

where u_0 is the original image (unknown in real-life problems) and $\hat{u}$ is the restored image. The *ISNR* values of the restorations for Example 1 are depicted in Table 1, alongside the original image u_0, in Fig. 4.

In order to illustrate the importance of allowing θ to be space-dependent, the same previous deblurring problem was solved by choosing θ constant (note that in this case the existence and uniqueness of minimizers is well known for the extreme cases $\theta = 1$ and $\theta = 0$, while for $0 < \theta < 1$ well-posedness is given by Theorem 3.8). Both the mixed isotropic and mixed anisotropic methods were run for the image shown in Fig. 1(a) setting $\theta \in (0, 1)$ as a constant. Note that the value of this constant results irrelevant since it is absorbed by the regularization parameters α_1 and α_2 (obtained using Morozov's Discrepancy Principle). The obtained ISNR values were 3.124 and 3.285 for the isotropic and anisotropic mixed regularization models, respectively.

Example 2. We now show another example of a 130×130 pixel color image, blurred with standard deviations $\sigma_h = \sigma_v = 0.02$ and contaminated with 2.5% (of the maximum data value) white additive Gaussian noise. The blurred noisy image v is depicted in Fig. 5 along with the restoration produced with a zero-order Tikhonov–Phillips regularization. In this case, the restorations including BV regularization were done with $p = 2$. Figs. 6 and 7 show the results for the BV and mixed methods.

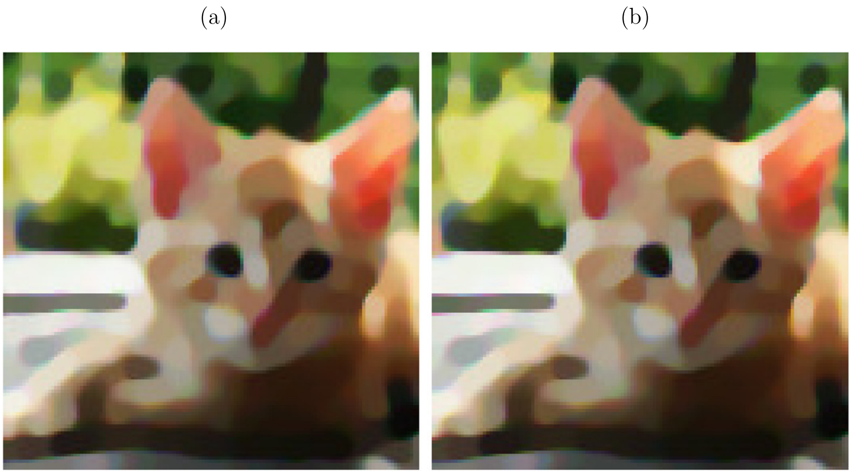

Fig. 6. (a) Isotropic BV restoration; (b) Anisotropic BV restoration.

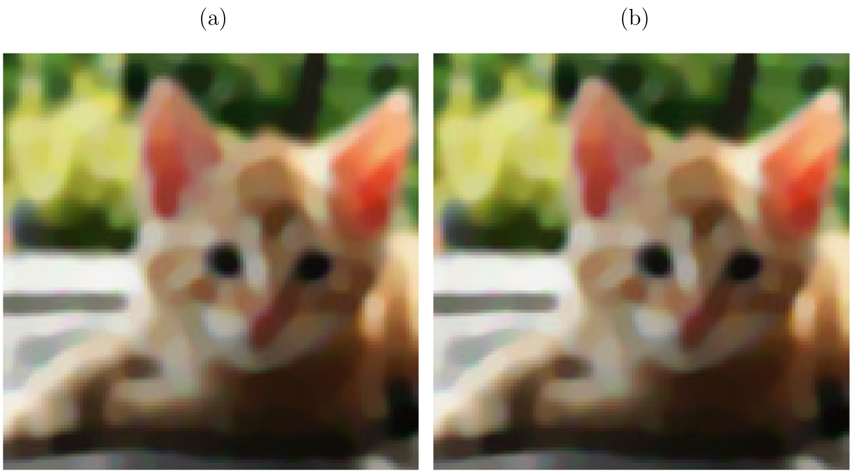

Fig. 7. (a) Mixed isotropic restoration; (b) Mixed anisotropic restoration.

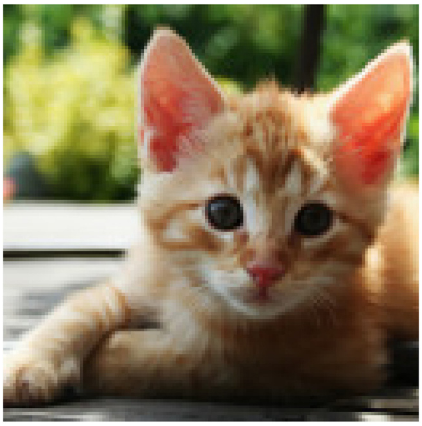

Fig. 8. Original image.

Table 2
ISNR values.

Restoration method	*ISNR*
Tikhonov 0	2.415
Isotropic BV	2.580
Anisotropic BV	2.627
Mixed isotropic	2.845
Mixed anisotropic	2.942

Once again, we computed the *ISNR* values, which are presented in Table 2 along the original image u_0, in Fig. 8.

5. Conclusions

In this work we presented several mathematical results on existence, uniqueness and stability of global minimizers of generalized Tikhonov–Phillips functionals with penalizers given by convex spatially-adaptive combinations of L^2 and anisotropic BV type. These penalizers are conceived so as to capture the benefits of both smooth L^2 regularization and the well known border-preserving properties of total variation penalization. Adaptivity is achieved through a spatially-varying weighting function θ while anisotropy is attained by the inclusion of a matrix field in the BV part of the penalizer. Although both the weighting function and the anisotropy matrix field can be prescribed *a-priori*, we showed how both can be appropriately constructed from a first estimation of the gradient field.

The main stability results (Theorems 3.9 and 3.10) contemplate not only perturbations in the data, but also in the model and in the regularization parameters.

In order to illustrate the performance of the mixed L^2–BV regularization method, some examples of image restoration problems were presented. Through these examples it was shown that the introduction of spatial adaptivity improves the quality of the restoration on images with heterogeneous properties in terms of edges and smooth regions. Furthermore, the introduction of anisotropy in the model was shown to improve restoration of borders in the images. These conclusions are supported by the *ISNR* values on Tables 1 and 2. Although there is undoubtedly much room for further research, these preliminary results indicate that, with appropriate choices of the weighting function θ and of the anisotropy matrix field A, the mixed combined method outperforms all single ones.

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