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Gravitational waves and magnetic monopoles during inflation with Weitzenböck torsion

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ABSTRACT

We study the variational principle on a Hilbert–Einstein action in an extended geometry with torsion taking into account non-trivial boundary conditions. We obtain an effective energy–momentum tensor that has its source in the torsion, which represents the matter geometrically induced. We explore about the existence of magnetic monopoles and gravitational waves in this torsional geometry. We conclude that the boundary terms can be identified as possible sources for the cosmological constant and torsion as the source of magnetic monopoles. We examine an example in which gravitational waves are produced during a de Sitter inflationary expansion of the universe.

1. Introduction

In the standard treatment of the variational principle over the Hilbert–Einstein action (HE), when a manifold has a boundary $\partial\mathcal{M}$, the action is supplemented by a boundary term which is in general neglected [1]. However, this is not the only manner to study this problem. As was recently demonstrated in [2], it is possible to include the flux around an hypersurface that encloses a physical source without the inclusion of extra terms in the HE action. In that paper was demonstrated that the non-zero flux of the vector metric fluctuations through the closed 3D Gaussian-like hypersurface, is responsible for the gauge-invariance of gravitational waves (GW). However, the torsional contributions were neglected in that paper. In the present paper we extend this analysis on the variational principle, but for an extended geometry with torsion. We obtain an effective energy–momentum tensor with sources only in torsion, which can be viewed as an effective matter tensor in a Riemannian geometry. Such tensor represents matter geometrically induced, but without extra dimensions. In addition, we develop a new manner to obtain GW on a torsional manifold taking into account nontrivial boundary terms. The first contribution to GW with purely torsional nature, was studied in Section 3 A, and the second one based in the boundary term was studied in Section 3 B; in both cases for a general torsion. Also, we present an example in Weitzenböck geometry obtaining the expression for the magnetic density monopoles and a gravitational wave equation for a Friedman-Robertson-Walker (FRW) in Weitzeböck geometry. Finally, in Section 6, we develop some final remarks.

2. Variational principle in torsional geometry

We consider the variational principle in presence of torsion for a HE-like action. We have studied this fundamental problem in [3]. Therefore, we shall expose some results in present section without making a full description. In [3], we have studied the contribution of the new terms which are not present in a Riemannian geometry. The boundary term was studied in [2], but emphasizing the role of non-metricity. Now, we shall consider some gravitational action in an extended geometry (i.e. a non-Riemannian manifold), without the presence of matter

$$I = \frac{1}{2\kappa}\int_M d^4x\sqrt{-g}R, \tag{1}$$

in which $\kappa = 8\pi G$, such that G is the gravitational constant, and

$$R^m_{\ lij} = \Gamma^m_{\ lj,i} - \Gamma^m_{\ li,j} + \Gamma^n_{\ lj}\Gamma^m_{\ ni} - \Gamma^n_{\ li}\Gamma^m_{\ nj}, \tag{2}$$

$$R_{lj} = R^i_{\ lij}, \tag{3}$$

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where $R = R_{nm}g^{nm}$ is the scalar curvature. We have employed the Einstein's convention over repeated indexes. The "," represents a partial derivative and all the indices run between 1 and 4. Furthermore, g_{ab} are the components of the metric tensor and $\sqrt{-g}$ is the volume of the non-Riemannian manifold. The Eq. (2) defines the Riemann curvature tensor, the Eq. (3) give us the Ricci tensor and Eq. (II) is the scalar curvature. We denote with Γ^a_{bc} an arbitrary affine connection, which is defined according to

$$\nabla_{\overrightarrow{e}_a} \overrightarrow{e}_b = \Gamma^n_{ba} \overrightarrow{e}_n, \tag{4}$$

where $\nabla_{\overrightarrow{e}_a}$ denotes the derivative in a-direction of the tangent space $\left\{\overrightarrow{e}_b\right\}$. Here, the up arrow means that the tangent space in the position representation is described by partial derivatives with respect to contravariant coordinates: $\left\{\overrightarrow{e}_b\right\} \equiv \left\{\frac{\partial}{\partial x^b}\right\}$, and the down arrow means that the cotangent space is generated by $\underset{\rightarrow}{e}^b \equiv \left\{dx^b\right\}$, such that $\underset{\rightarrow}{e}^b(\overrightarrow{e}_a) = \delta^b_a$. We wont consider any particular symmetry in the connections. Now we shall make the variation of the action in (1): $\delta I = 0$. Here we must take into account that the scalar R in (II) is related to the connection in (4), which is an abstract connection which is in general non-Riemannian, but fulfils the expression

$$\Gamma^n_{mr} = \left\{{}^{\,n}_{mr}\right\} + K^n{}_{mr}, \tag{5}$$

with $\left\{{}^{\,n}_{mr}\right\}$ the second kind Christoffel symbols representing the Riemannian or Levi-Civita connections, and K^n_{mr} the contortion tensor, which in absence of non-metricity is entirely torsional according to

$$K^a{}_{bc} = -\frac{g^{na}}{2}\{T^s{}_{cn}\, g_{bs} + T^s{}_{bn}\, g_{sc} - T^s{}_{cb}\, g_{sn}\}, \tag{6}$$

with the torsion tensor defined by

$$T^n_{mr} = \Gamma^n_{rm} - \Gamma^n_{mr}, \tag{7}$$

which is a valid expression in a coordinate basis of the four dimensional tangent space to the space–time manifold (TM4). In present work we impose the non-metricity free condition

$$N_{nmr} = g_{nm\,;r} = 0, \tag{8}$$

for an analysis of such contribution to the GW the reader can see [3]. The variation of the Ricci must be related to the variation of the connections obtaining a generalised Palatini identity for torsional geometry

$$g^{mr}\,\delta R_{mr} = W^n{}_{;n} - \frac{1}{2} g^{mr}(\delta\Gamma^n{}_{pr}\, T^p{}_{mn} + \delta\Gamma^n{}_{pm}\, T^p{}_{rn}), \tag{9}$$

with

$$W^n{}_{mr} = \delta\Gamma^n{}_{mr} - \delta\Gamma^k{}_{kr}\,\delta^n{}_m \tag{10}$$

where $W^n = g^{mr} W^n{}_{mr}$. With the use of Eq. (9) in the variation of the action we obtain

$$\begin{aligned}\delta I = &\int_M d^4x\sqrt{-g}\left(R_{ab} - \frac{1}{2}R\, g_{ab}\right)\delta g^{ab} + \int_{\partial M} W^n\, d\Sigma_n \\ &- \frac{1}{2}\int_M d^4x\sqrt{-g}(\delta\Gamma^n_{pr}\, T^p_{mn} + \delta\Gamma^n_{pm}\, T^p_{rn})g^{mr}.\end{aligned} \tag{11}$$

In the first integral we recognize the Einstein tensor for the torsional connection. The second one is due to the boundary term. The third integral is completely originated by the torsion. This is a non-Riemannian contribution.

To finalize this section we must present the explicit form of the $W^n{}_{mr} = W^n{}_{(mr)} + W^n{}_{[mr]}$ tensor, where the symmetric and antisymmetric contributions are, respectively given by

$$\begin{aligned}W^n{}_{(mr)} = &\left[\frac{g^{kn}}{2}\left\{\delta g_{mk\,,r} + \delta g_{kr\,,m} - \delta g_{rm\,,k} - T^t{}_{rk}\,\delta g_{mt}\right.\right. \\ &\left. - T^t{}_{mk}\,\delta g_{tr}\right\} - \frac{\delta g^{kn}}{2}\left\{g_{mk\,,r} + g_{kr\,,m}\right. \\ &\left. - g_{rm\,,k} - T^t{}_{rk}\, g_{mt} - T^t{}_{mk}\, g_{tr}\right\} - \frac{g^{kl}}{4}\left(\delta g_{kl\,,r}\,\delta^n{}_m\right. \\ &\left.\left. + \delta g_{kl\,,m}\,\delta^n{}_r\right) + \frac{\delta g^{kl}}{4}\left(g_{kl\,,r}\,\delta^n{}_m + g_{kl\,,m}\,\delta^n{}_r\right)\right],\end{aligned} \tag{12}$$

$$\begin{aligned}W^n{}_{[mr]} = &\left[\frac{g^{kn}}{2}T^t{}_{rm}\,\delta g_{tk} - \frac{\delta g^{kn}}{2}T^t{}_{rm}\, g_{tk} - \frac{g^{kl}}{4}\left(\delta g_{kl\,,r}\,\delta^n_m\right.\right. \\ &\left.\left. - \delta g_{kl\,,m}\,\delta^n{}_r\right) + \frac{\delta g^{kl}}{4}(g_{kl\,,r}\,\delta^n{}_m - g_{kl\,,m}\,\delta^n{}_r)\right],\end{aligned} \tag{13}$$

such that $W^n = W^n{}_{(mr)}\, g^{mr}$.

3. Physics of the torsional geometry and 4D induced matter

In presence of torsion, but zero non-metricity, the variation of the action takes the form

$$\begin{aligned}\delta I = &\int_M d^4x\sqrt{-g}\left[R_{ab} - \frac{1}{2}R\, g_{ab} - \frac{1}{2}L_{(ab)}\right]\delta g^{ab} \\ &+ \int_M d^4x\sqrt{-g}\, W^n_{\ ;n}\end{aligned} \tag{14}$$

with

$$\begin{aligned}L_{(sd)} = &\left\{\Delta^p{}_{mrsd}\, K^n{}_{pn} - \Delta^p{}_{nrsd}\, K^n{}_{pm} - \Delta^p{}_{nmsd}\, K^n{}_{pr}\right. \\ &\left. + \Delta^n{}_{npsd}\,(K^p{}_{rm} + K^p{}_{mr})\right\} g^{mr}.\end{aligned} \tag{15}$$

Furthermore

$$\begin{aligned}\Delta^p{}_{mrsd} = &\frac{g^{pk}}{2}\{-(g_{ms}\, g_{kd})_{,r} - (g_{ks}\, g_{rd})_{,m} + (g_{ms}\, g_{rd})_{,k} \\ &+ T^l{}_{rk}\, g_{ms}\, g_{ld} + T^l{}_{mk}\, g_{ls}\, g_{rd} - T^l{}_{rm}\, g_{ls}\, g_{rd}\} \\ &- \frac{1}{2}(-T^l{}_{rd}\, g_{ml} - T^l{}_{md}\, g_{lr} + T^l{}_{rm}\, g_{ld})\,\delta^p_s.\end{aligned} \tag{16}$$

The first integral in (14) includes the extended Einstein tensor with the torsional (Weitzenböck) contribution, and the second one includes the boundary contribution. We have obtained the expression (14) in absence of matter. We can distinguish two possible cases.

1. The first case describes infinity manifolds and there are no boundary contributions: $W^n_{;n} = 0$, so that the first integrand in (14) is null:

$$R_{ab} - \frac{1}{2}R\, g_{ab} = \frac{1}{2}L_{(ab)}. \tag{17}$$

After some algebraic manipulation of $L_{(ab)}$, the last assumption leads to a wave equation originated in the presence of torsion:

$$\Box\,(\Delta^n_{prsd}\,\delta g^{sd}\, T^p_{mn}) = 0, \tag{18}$$

where Δ^n_{prsd} is given by (16).

2. The second case describes finite manifolds so that the boundary contributions are significative: $W^n_{;n} \neq 0$. In that case the first integrand must be nonzero in order to $\delta I = 0$:

$$\delta g^{ab}\left[R_{ab} - \frac{1}{2}R\, g_{ab} - \frac{1}{2}L_{(ab)}\right] + W^n_{\ ;n} = 0. \tag{19}$$

In the relativistic formalism without boundary conditions (i.e., when $W^n_{\ ;n} = 0$), the cosmological constant can be added to the Einstein equations as an integration constant. Therefore, in order to recover the dynamics in absence of boundary conditions, it is reasonable to make the interpretation that the second integral can be related to the cosmological constant as

$$W^{n}{}_{;\,n} = \Lambda(x)\, g_{ab}\delta g^{ab}. \tag{20}$$

Here, $\Lambda(x)$ is a function of the proper time on the torsional manifold, and play the role of a dynamical cosmological constant with source in the boundary term. The identification in (17) defines geometrically the physical sources originated in the torsion, without cosmological constant.

With the help of (5) and the contortion tensor (6) in the Eq. (19), we recover the effective equation for the Riemannian part of the Einstein tensor

$${}^{(R)}R_{ab} - \frac{1}{2}\,{}^{(R)}R\, g_{ab} + \Lambda g_{ab} = kT_{ab}, \tag{21}$$

where the supra-index "${}^{(R)}$" in ${}^{(R)}R_{ab}$ and ${}^{(R)}R$, indicates that these objects are Riemannian and constructed entirely with the Levi-Civita connection. The tensor T_{ab} in (21) is an effective and geometrically induced energy–momentum tensor over the Riemannian space–time, which is associated with the variation over the Weitzenböck manifold

$$\begin{aligned} k\,T_{ab} &= -K^{n}{}_{nb\,|a} + K^{n}{}_{ab\,|n} - K^{l}{}_{nb}K^{n}{}_{la} + K^{l}{}_{ab}K^{n}{}_{lm} \\ &\quad + \frac{g^{sd}}{2}\left(K^{n}{}_{nd\,|s} - K^{n}{}_{sd\,|n} + K^{l}{}_{nd}K^{n}{}_{ls} - K^{l}{}_{sd}K^{n}{}_{lm}\right) g_{ab}, \end{aligned} \tag{22}$$

which is of torsional nature in a clear way. We see, from last expression, that the generalized Einstein tensor and the Riemannian one, must be related according to

$$\begin{aligned} G_{ab} &= {}^{(R)}G_{ab} + K^{n}_{nb\,|a} - K^{n}_{ab\,|n} + K^{l}_{nb}K^{n}_{la} - K^{l}_{ab}K^{n}_{lm} \\ &\quad - \frac{g^{sd}}{2}\left(K^{n}{}_{nd\,|s} - K^{n}{}_{sd\,|n} + K^{l}{}_{nd}K^{n}{}_{ls} - K^{l}{}_{sd}K^{n}{}_{lm}\right) g_{ab}. \end{aligned} \tag{23}$$

We must remark that $K^{n}_{nb\,|a}$ denotes the covariant derivative of the torsional contorsion tensor according to the derivative operator of the Riemannian geometry in which the connection is the Christoffel symbol. From (21) we obtain the expression from its symmetric part and a consistence equation for the anti-symmetric contribution. In presence of matter, which is related to the lagrangian density $\mathcal{L}$ with an energy–momentum tensor ${}^{(m)}T_{ab}$, we obtain

$$\begin{aligned} G_{(ab)} + \frac{1}{2}\Lambda\, g_{ab} &= k\,{}^{(m)}T_{(ab)} \;\rightarrow\; {}^{(R)}G_{(ab)} + \frac{1}{2}\Lambda\, g_{ab} \\ &= k\left[{}^{(m)}T_{(ab)} + T_{(ab)}\right], \end{aligned} \tag{24}$$

$$\begin{aligned} G_{[ab]} &= R_{[ab]} = k\,{}^{(m)}T_{[ab]} \;\rightarrow\; {}^{(R)}G_{[ab]} = 0 \\ &= k\left[{}^{(m)}T_{[ab]} + T_{[ab]}\right]. \end{aligned} \tag{25}$$

Equations in the l.h.s. of (24) and (25) before the arrows are Einstein-Cartan-like eqs [4]. We assume a symmetric metric tensor $g_{ab} = g_{(ab)}$. The tensor T_{ab} is induced from torsional geometry in the Weitzenböck representation and is adequate to describe matter with spin or magnetic monopoles. However, ${}^{(m)}T_{ab}$ must be proposed from an additional term in the action. In this work we propose an empty torsional geometry with ${}^{(m)}T_{ab} = 0$, then the r.h.s. of the Eq. (24) can be reduced to

$${}^{(R)}G_{(ab)} + \frac{1}{2}\Lambda\, g_{ab} = k\,T_{(ab)}, \tag{26}$$

and the r.h.s. of (25) is reduced to

$$0 = T_{[ab]}. \tag{27}$$

Notice that the Eq. (26) corresponds to the usual Riemannian Einstein equations, and (27) are the conditions obtained from the anti-symmetric part of the Einstein torsional tensor.

4. Weitzenböck geometry

The Weitzenböck geometry [5] could be formulated from the vierbein, which are coefficients that express the relation between two different basis of TM4, $\{\vec{E}_A\}$ and $\{\vec{e}_a\}$

$$\vec{E}_A = e^{a}_{A}\,\vec{e}_a, \qquad \vec{e}_a = \bar{e}^{A}_{a}\,\vec{E}_A, \tag{28}$$

with the properties [6]

$$e^{a}_{A}\,\bar{e}^{A}_{b} = \delta^{a}_{b}, \qquad e^{b}_{B}\,\bar{e}^{A}_{b} = \delta^{A}_{B}, \tag{29}$$

such that the covariant derivative of basis elements $\bar{e}^{A}_{b}$ to be zero: $\bar{e}^{A}_{b;C} = 0$. If we take into account the Eqs. (28) and (29), one can show that the components of some arbitrary tensor T in $\mathcal{T}^{p}{}_{m}(M)$, transforms according to

$$T^{a_1\ldots a_p}{}_{b_1\ldots b_m} = e^{a_1}_{A_1}\,\ldots\, e^{a_p}_{A_p}\,\bar{e}^{B_1}_{b_1}\,\ldots\,\bar{e}^{B_m}_{b_m}\,T^{A_1\ldots A_p}{}_{B_1\ldots B_m}. \tag{30}$$

In this framework it is possible to define the Weitzenböck connection

$${}^{(We)}\Gamma^{a}_{bc} = e^{a}_{N}\,\vec{e}_c(\bar{e}^{N}_{b}), \tag{31}$$

so that

$${}^{(We)}\Gamma^{A}_{BC} = 0, \tag{32}$$

and therefore ${}^{(We)}R^{A}_{BCD} = 0$. In the usual Weitzenböck basis is proposed that the basis $\{\vec{e}_a\}$ to be a coordinate basis of TM4, with certain metric characterized by g_{ab}, which is of interest. The basis $\{\vec{E}_A\}$ can be non-coordinate but must be chosen in the form that the metric tensor expressed in such basis to be characterized by η_{AB}. In this context the Weitzenböck torsion is

$${}^{(We)}T^{a}_{bc} = e^{a}_{A}\,\bar{e}^{B}_{b}\,\bar{e}^{C}_{c}\,{}^{(We)}T^{A}_{BC} = e^{a}_{A}\,\bar{e}^{B}_{b}\,\bar{e}^{C}_{c}\,C^{A}_{BC}, \tag{33}$$

where C^{A}_{BC} are the structure coefficients of the basis $\{\vec{E}_A\}$. The structure must be taken into account in the Eqs. (2), (3), (II) and (7). The structure coefficients $\{\vec{E}_A\}$ are defined by [7]

$$[\vec{E}_B, \vec{E}_A] = C^{C}_{AB}\,\vec{E}_C. \tag{34}$$

The non-metricity related to the Weitzenböck connection is

$${}^{(We)}N_{abc} = \bar{e}^{A}_{a}\,\bar{e}^{B}_{b}\,\bar{e}^{C}_{c}\,{}^{(We)}N_{ABC} = \bar{e}^{A}_{a}\,\bar{e}^{B}_{b}\,\bar{e}^{C}_{c}\,\eta_{AB\,,C}. \tag{35}$$

In order to obtain (33) and (35), we have used (32). Usually, we choose $\{\vec{E}_A\}$ in order to it be an orthonormal basis, such that $\eta_{AB} = -1, 0, +1$. Therefore one obtains $\eta_{AB\,,C} = 0$, and the Eq. (35) becomes null:

$${}^{(We)}N_{ABC} = 0. \tag{36}$$

The usual Weitzenböck geometry is a torsional geometry with zero non-metricity. Such elements characterize the Weitzenböck connection (5) with a contortion tensor, which is due as a function of the Weitzenböck torsion in (33). The zero non-metricty condition must be removed only if the elements of the basis $\{\vec{E}_A\}$ are chosen such that its inner product is different that a constant.

4.1. Magnetic monopoles with Weitzenböck geometry

With a standard gravito-electromagnetic action for the extended geometry we must obtain an extension of the Maxwell equations:

$$*d(F) = {}^{(m)}J, \tag{37}$$

$$*d(*F) = {}^{(e)}J. \tag{38}$$

where F is the extended Faraday 2-form, d is an exterior covariant derivative and $*$ is the adjunction operation. The source term ${}^{(m)}J$ is a cotangent vector of the magnetic current and ${}^{(e)}J$ is the electric current. In both terms the zero component ${}^{(*)}J_0 = \rho_*$ is the corresponding Hodge-dual of the charge density [8,9]. The expression (37) is different of the usual one $d(F) = 0$, because the torsion of the geometry is the responsible for a non zero current of magnetic monopoles. The adjunction is taken into account in order to match correctly the vectorial order of both sides of the magnetic side in the Eq. (37). The expression in (38) has the usual appearance of the Maxwell ones, but for gravito-electromagnetic currents produced by a torsional geometry.

A p-form is an anti-symmetric tensorial object W of order p

$$W = \frac{1}{p!}\, w_{i_1 \ldots i_p}\, \underset{\rightarrow}{e}^{\,i_1} \wedge \ldots \wedge \underset{\rightarrow}{e}^{\,i_p},$$

in which the wedge product is the anti-symmetrization of the tensor product. The exterior covariant derivative associated with certain covariant derivative denoted by $(;)$, is defined by

$$d(W) = \frac{1}{p!}\, w_{i_1 \ldots i_p ;k}\, \underset{\rightarrow}{e}^{\,k} \wedge \underset{\rightarrow}{e}^{\,i_1} \wedge \ldots \wedge \underset{\rightarrow}{e}^{\,i_p}.$$

The adjunction operation in a manifold of dimension m, is

$$* \, W = \frac{\sqrt{|g|}}{(m-p)!p!}\varepsilon_{j_1 \ldots j_p i_{p+1} \ldots i_n} w^{j_1 \ldots j_p} \underbrace{\underset{\rightarrow}{e}^{\,i_{p+1}} \wedge \ldots \wedge \underset{\rightarrow}{e}^{\,i_m}}_{m-p}, \qquad (39)$$

which takes into account a p-form and gives us a $(m-p)$-form. The Einstein-Faraday 2-form is defined from the exterior covariant derivative of the 1-form, which is the cotangent version of the tetra-vector $A = (\varphi, \overrightarrow{A})$. Here, $\overrightarrow{A}$ is the usual 3-vector potential

$$F = d(A).$$

If the connection is symmetric, then $d(F) = d(d(A)) = 0$, which implies the absence of magnetic monopoles.

In the present work we are dealing with a torsional Weitzenböck geometry, so that we must apply the Eq. (37) for the corresponding connections. Then, the 0-component of the magnetic current will be

$$\begin{aligned}[*d(F)]_0 = \rho_m = -3^{(We)}T^{D}_{21}\overrightarrow{E}_D(A_3) + {}^{(We)}T^{D}_{13}\overrightarrow{E}_D(A_2) \\ + {}^{(We)}T^{D}_{32}\overrightarrow{E}_D(A_1),\end{aligned} \qquad (40)$$

which implies that $\rho_m = \vec{e}^{\,A}_{a=0}\,{}^{(m)}J_A$, in the $\vec{e}_a$ basis.

5. Example: Magnetic monopoles and GW from torsion in a FRW expansion

In this section we shall address the usual Weitzenböck scenario as described in Section 4. In particular, we shall study the case (1), where is absent the cosmological constant and the boundary conditions are trivial. We shall start with an ortho-normalized Lorentzian metric related to a non-coordinate basis: $\{\overrightarrow{E}_A\} = \{\partial_t,\ a(t)\,\partial_a\}$, with the structure coefficients C^A_{BC}

$$C^1_{10} = C^2_{20} = C^3_{30} = \frac{\dot{a}(t)}{a(t)}, \qquad (41)$$

and its counterpart with lower changed indices. The Eq. (41) must be used with (33) in order to obtain the Weitzenböck torsion. The non-zero vierbein are

$$e^{a=0}_{A=0} = 1, \qquad e^{i}_{I} = a(t), \qquad \bar{e}^{A=0}_{a=0} = 1, \qquad \bar{e}^{I}_{i} = a(t)^{-1}, \qquad (42)$$

which are only valid in the case in which $i = I$. Here, I runs over the three space indices of the basis $\vec{E}_A$, and i runs over the three space indices of the basis $\vec{e}_a$. This one is a coordinate basis given by $\{\overrightarrow{e}_a\} = \{\partial_t, \partial_a\}$. The metric tensor g_{ab}, for the basis $\vec{e}_a$, takes the form

$$[g]_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2(t) & 0 & 0 \\ 0 & 0 & -a^2(t) & 0 \\ 0 & 0 & 0 & -a^2(t) \end{pmatrix}, \qquad (43)$$

which describes an isotropic an homogenous spatially flat, FRW expanding universe with scale factor $a(t)$. If we use the (41) and (42) in (33), we obtain that the non-zero components of the Weitzenböck torsion are ${}^{(We)}T^{(i)}_{(i)\,0} = e^{(i)}_{A} e^{B}_{(i)} e^{C}_{c=0}\, {}^{(We)}T^{A}_{BC} = e^{(i)}_{A} e^{B}_{(i)} e^{C}_{c=0}\, {}^{(We)}C^{A}_{BC} = {}^{(We)}C^{A=(i)}_{B=(i)\ C=0} = \frac{\dot{a}(t)}{a(t)}$. Therefore the non-zero torsion components will be

$${}^{(We)}T^{(i)}_{(i)\,0} = \frac{\dot{a}(t)}{a(t)}, \qquad (44)$$

where the indices between parenthesis indicate that such indices are equal and the Einstein notation for repeated indexes is not used. Using the Eq. (44) in (40), we obtain that the density of magnetic monopoles is

$$\rho_m = 0. \qquad (45)$$

This implies a non-trivial absence of magnetic monopoles due to the absence of torsion of spatial nature. On the other hand, we must notice that the wave equation (18) is used in the present case. The expression (18) is simplified because some elements of the torsion are null. From the Eq. (44),we obtain that (18) is reduced to

$$\Box \left[\frac{\dot{a}(t)}{a(t)}\left(\delta g^{IJ} - \delta g^{I0}\right)\right] = 0, \qquad (46)$$

$$\Box\, \delta g^{00} = 0. \qquad (47)$$

Notice that the expression (47) is equivalent to a gauge choice.

5.1. GR in a de Sitter inflation

For the particular case in which $a(t) = e^{H_0 t}$, we see that $\frac{\dot{a}(t)}{a(t)} = H_0$. Thus the expression (46) have particular solutions in the equation

$$\Box\, \delta g^{AB} = 0. \qquad (48)$$

This case is important because describes a de Sitter inflationary expansion with an equation of state: $P/\rho = -1$. Here, P is the isotropic pressure and ρ is the energy density of the universe during the expansion. During a de Sitter expansion ρ remains constant.

On the other hand the Eq. (48) has the solution in the form of a free plane wave for Weitzenböck derivative operators. This is a simple wave equation expressed in terms of the basis $\vec{E}_A$ which is related to null connections. The solution of (48) admits an expansion of the form

$$\begin{aligned}\delta g_{AB}(t,\vec{x}) = \frac{1}{(2\pi)^{3/2}} \sum_{M=+,\times} \int d^3k\; e^{M}_{AB}(\hat{z}) \\ \times \left[A_k\, e^{i\vec{k}.\hat{z}}\, \chi_k(t) + A^{\dagger}_k\, e^{-i\vec{k}.\hat{z}}\, \chi^{*}_k(t)\right],\end{aligned} \qquad (49)$$

where index $M = +, \times$ denote the Transverse-Traceless (TT) polarizations $+, \times$, on the plane orthogonal to $\vec{k}$, and e^{M}_{AB} are the components of the polarization tensor, such that

$$e^{M}_{AB}\, \bar{e}^{AB}_{M'} = \delta^{M}_{M'}. \qquad (50)$$

For the case in which the scale factor is $a(t) = e^{H_0\, t}$, and the Hubble parameter is a constant H_0, the equation of motion for the modes $\chi_k(t)$ is

$$\ddot{\chi}_k(t) + 3H_0\dot{\chi}_k(t) + \left[\frac{k^2}{e^{2H_0 t}}\right]\chi_k(t) = 0. \tag{51}$$

The general solution for the modes in (51) can be written in terms of the first and second kind Hankel functions $\mathcal{H}_\nu^{(1,2)}\left[\frac{ke^{-H_0 t}}{H_0}\right]$

$$\chi_k(t) = e^{-\frac{3}{2}H_0 t}\left\{A\,\mathcal{H}_{3/2}^{(1)}\left[\frac{ke^{-H_0 t}}{H_0}\right] + B\,\mathcal{H}_{3/2}^{(2)}\left[\frac{ke^{-H_0 t}}{H_0}\right]\right\}. \tag{52}$$

Since $g^{AB}g_{AB} = 4$, it is easy to prove that $\delta\left(g^{AB}g_{AB}\right) = 0$, so that we obtain

$$g_{AB}\,\delta g^{AB} + \delta g_{AB}\,g^{AB} = 0. \tag{53}$$

Since the dynamics of δg^{AB} is described by the linear differential equation (48), which describes a wave dynamics on a background curved spacetime, δg^{AB} can be written as a Fourier expansion of a tensor field

$$\delta g^{AB}(t,\vec{x}) = \frac{1}{(2\pi)^{3/2}}\sum_{M=+,\times}\int d^3k\;\bar{e}_M^{AB}(\hat{z}) \times\left[\widetilde{A}_k\,e^{i\vec{k}.\hat{z}}\,\widetilde{\chi}_k(t) + \widetilde{A}_k^{\dagger}\,e^{-i\vec{k}.\hat{z}}\,\widetilde{\chi}_k^{*}(t)\right]. \tag{54}$$

In order to be fulfilled the condition (53), we need require that the annihilation and creation operators comply with

$$A_k\,\chi_k + \widetilde{A}_k\,\widetilde{\chi}_k = 0, \tag{55}$$

$$A_k^{\dagger}\,\chi_k^{*} + \widetilde{A}_k^{\dagger}\,\widetilde{\chi}_k^{*} = 0. \tag{56}$$

This implies that the field $\delta g^{AB}(t,\vec{x})$ must be expanded in terms of the coefficients A_k and $A_k^{\dagger}$, as

$$\delta g^{AB}(t,\vec{x}) = \frac{-1}{(2\pi)^{3/2}}\sum_{M=+,\times}\int d^3k\;\bar{e}_M^{AB}(\hat{z}) \times\left[A_k\,e^{i\vec{k}.\hat{z}}\,\chi_k(t) + A_k^{\dagger}\,e^{-i\vec{k}.\hat{z}}\,\chi_k^{*}(t)\right], \tag{57}$$

where $\bar{e}_M^{AB}$ agrees with (50).

6. Final comments

We have studied the variational principle in presence of a torsional geometry in presence of non-trivial boundary conditions. In the extended Einstein equations here obtained, we have define an effective geometrically induced energy momentum tensor for a Riemannian representation af a torsional (Weitzenböck) one. The energy–momentum tensor here obtained must be viewed as representing geometrically induced matter from Weitzenböck torsion. This is the main difference with other approaches (for instance, the Space-Time-Matter theory [10]), where the energy–momentum tensor is induced from an extra dimensional vacuum. However, in our approach the gravitational wave dynamics on a Weitzenböck manifold the torsion and boundary terms must be taken into account in order to explain the origin of the cosmological constant and GW. Our theory has many similitude to whole of Ferraro-Fiorini [11]. As it was shown in (2) the boundary terms could be responsible for the cosmological constant which is produced by a source inside a 3D Gaussian (closed) hypersurface that encloses that source. In the inflationary example studied in Section 5 magnetic monopoles are absent, due to the globally isotropy and homogeneity of the universe at large (cosmological) scales. This agrees with the absence of magnetic monopoles predicted by inflationary models.

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