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BEST L^p-APPROXIMANT PAIR ON SMALL **INTERVALS**

Abstract

In this paper, we study the behavior of best L^p -approximations by algebraic polynomial pairs on unions of intervals when the measure of those intervals tends to zero.

1 Introduction

Let $X = \{x_j\}_{j=1}^k \subset \mathbb{R}, k \in \mathbb{N}$, and let $\{B_j\}_{j=1}^k$ be pairwise disjoint closed intervals centered at x_j of radius 1. Let $n, m \in \mathbb{N} \cup \{0\}$ and suppose that n + m + 1 = kq + r with $q \in \mathbb{N} \cup \{0\}, 0 < r < k$. For $s \in \mathbb{N} \cup \{0\}$, we let $\mathcal{C}^{s}(I)$ denote the space of real functions defined on $I := \bigcup_{j=1}^{k} B_{j}$, which are continuously differentiable up to order s on I. For simplicity, we write $\mathcal{C}(I)$ instead of $\mathcal{C}^0(I)$.

If $\|\cdot\|$ is a norm defined on $\mathcal{C}(I)$ and $h \in \mathcal{C}(I)$, then for each $0 < \epsilon \leq 1$, we write $||h||_{\epsilon} = ||h^{\epsilon}||$, where $h^{\epsilon}(x) = h(\epsilon(x - x_i) + x_i), x \in B_i$. We put

$$||h|| = \left(\int_{I} |h(x)|^{p} \frac{dx}{|I|}\right)^{\frac{1}{p}}, \quad 1$$

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where |I| is the Lebesgue measure of I. For $I_{\epsilon} := \bigcup_{j=1}^{k} [x_j - \epsilon, x_j + \epsilon]$, we observe that $(\mathcal{C}(I_{\epsilon}), \|\cdot\|_{\epsilon})$ is a normed space and

$$\|h\|_{\epsilon} = \left(\int_{I_{\epsilon}} |h(x)|^p \frac{dx}{|I_{\epsilon}|}\right)^{\frac{1}{p}}$$

We define $||h||_{\infty} := \max_{x \in I} |h(x)|$ and $||h||_{B_j} := \left(\int_{B_j} |h(x)|^p dx\right)^{\frac{1}{p}}, 1 \le j \le k.$

Let Π^n be the class of algebraic polynomials with real coefficients of degree at most n. We consider the set

$$\mathcal{H}_m^n := \{ (P, Q) \in \Pi^n \times \Pi^m : \|Q\|_\infty = 1 \}.$$

Given $(P,Q), (U,V) \in \mathcal{H}_m^n$, we identify (P,Q) with (U,V) if and only if $P = \lambda U, Q = \lambda V, |\lambda| = 1$. We denote it briefly by $(P,Q) \equiv (U,V)$.

Let $f \in \mathcal{C}(I)$ and $0 < \epsilon \leq 1$. We say that $(P_{\epsilon}, Q_{\epsilon}) \in \mathcal{H}_m^n$ is a best approximant pair of f from \mathcal{H}_m^n with respect to $\|\cdot\|_{\epsilon}$ if

$$\|fQ_{\epsilon} - P_{\epsilon}\|_{\epsilon} = \inf_{(P,Q)\in\mathcal{H}_{m}^{n}} \|fQ - P\|_{\epsilon}.$$
 (1)

It is easy to see that the pair $(P_{\epsilon}, Q_{\epsilon})$ always exists.

Given $q > 0, f \in \mathcal{C}^{q-1}(I)$ and $(P,Q) \in \Pi^n \times \Pi^m$, if

$$(fQ - P)^{(s)}(x_j) = 0, \quad 0 \le s \le q - 1, \quad 1 \le j \le k,$$
 (2)

then (P,Q) is said to be a *Padé approximant pair of* f at X. If $Q \neq 0$ and

$$\left(f - \frac{P}{Q}\right)^{(s)}(x_j) = 0, \quad 0 \le s \le q - 1, \quad 1 \le j \le k,$$

then the rational function $\frac{P}{Q}$ is called a *Padé approximant of f* at X. We define

 $\mathcal{W}_m^n(f,X):=\left\{(P,Q)\in\mathcal{H}_m^n:(P,Q)\text{ is a Padé approximant pair of }f\text{ at }X\right\}.$

If q = 0, then no constraint over the pair is assumed and $\mathcal{W}_m^n(f, X) = \mathcal{H}_m^n$.

Clearly, $\mathcal{W}_m^n(f, X) \neq \emptyset$. In fact, let $x_{k+1} \in I - X$, and we consider the system (2) with constraints $(fQ - P)^{(s)}(x_{k+1}) = 0, \ 0 \leq s \leq r - 1$. This system always has a nontrivial solution for (P,Q), since it is a homogeneous system of n+m+1 equations in n+m+2 unknowns. Now, if Q = 0, then P = 0 because $P \in \Pi^n$, a contradiction. So, $Q \neq 0$ and $\left(\frac{P}{\|Q\|_{\infty}}, \frac{Q}{\|Q\|_{\infty}}\right) \in \mathcal{W}_m^n(f, X)$.

We say that $(P_0, Q_0) \in \mathcal{W}_m^n(f, X)$ is a best Padé approximant pair of f at X if

$$\sum_{j=1}^{k} \left| \left(fQ_0 - P_0 \right)^{(q)} (x_j) \right|^p \le \sum_{j=1}^{k} \left| \left(fQ - P \right)^{(q)} (x_j) \right|^p \tag{3}$$

for all $(P,Q) \in \mathcal{W}_m^n(f,X)$. If $(P,Q) \in \Pi^n \times \Pi^m$, $Q \neq 0$, then $\frac{P}{Q}$ is said to be *normal* if it is irreducible and either deg P = n or deg Q = m. (The null rational function $\frac{0}{Q}$ is normal if and only if deg Q = 0.)

In 1934, Walsh proved in [9] that the Taylor polynomial of degree n for an analytic function f can be obtained by taking the limit as $\epsilon \to 0$ of the best (Tchebychev) approximant from Π^n to f on the disk $|z| \leq \epsilon$. In [10], the author generalized this result to rational approximation. In [2], Chui, Shisha and Smith proved that the net of best (Tchebychev) approximants pairs on $[0, \epsilon]$, from $\{(P, Q) \in \Pi^n \times \Pi^m : Q(0) = 1\}$, converges to the Padé approximant pair in the origin as $\epsilon \to 0$. Similar results for the L^2 -norm can be seen in [3]. The case of a unique point in several variables was treated in [1] with the L^p -norms. Finally, the case of L^{ϕ} -approximation on k disjoint intervals, where n + m + 1is divisible by k, was investigated in [6].

In Section 2, we show that there exists at least a best Padé approximant pair of f at X. In Section 3, we prove that, any cluster point of best approximant pairs $\{(P_{\epsilon}, Q_{\epsilon})\}$ as $\epsilon \to 0$ is a best Padé approximant pair of f at X.

2 Existence of best Padé approximant pairs

Now, we establish an existence theorem of best Padé approximant pairs.

Theorem 1. Let $f \in C^q(I)$. Then there exists at least one best Padé approximant pair of f at X.

PROOF. Let $\{(P_l, Q_l)\} \subset \mathcal{W}_m^n(f, X)$ be a sequence satisfying

$$\lim_{l \to \infty} \sum_{j=1}^{k} |(fQ_l - P_l)^{(q)}(x_j)|^p = \inf_{(P,Q) \in \mathcal{W}_m^n(f,X)} \sum_{j=1}^{k} |(fQ - P)^{(q)}(x_j)|^p =: E.$$
(4)

If q > 0, then $(fQ_l - P_l)^{(i)}(x_j) = 0, 0 \le i \le q - 1, 1 \le j \le k$. According to (4), there is constant M > 0 such that

$$|(fQ_l - P_l)^{(i)}(x_j)| \le M, \quad 0 \le i \le q, \quad 1 \le j \le k, \quad l \in \mathbb{N}.$$
 (5)

We observe that if q = 0, (5) is true also, by (4).

Let $(S,T) \in \mathcal{W}_m^n(f,X)$. Since $SQ_l - TP_l = T(fQ_l - P_l) - Q_l(fT - S)$, by the Leibniz rule for the *i*th derivative of a product of two factors,

$$|(SQ_l - TP_l)^{(i)}(x_j)| \le N, \ 0 \le i \le q, \ 1 \le j \le k, \ l \in \mathbb{N},$$

for some constant N > 0. As $||P|| := \max_{0 \le i \le q} \max_{1 \le j \le k} |P^{(i)}(x_j)|$ is a norm on $\Pi^{k(q+1)-1}$, the equivalence of the norms in $\Pi^{k(q+1)-1}$ implies that $\{SQ_l - TP_l\}$ is uniformly bounded on I, and consequently $\{TP_l\}$ is uniformly bounded on I. Since $||P||_T := \max_{t \in I} |TP(t)|$ is a norm on Π^n , we get that $\{P_l\}$ is uniformly bounded on I. So, there is a subsequence of $\{(P_l, Q_l)\}$, which we denote the same way, and $(P_0, Q_0) \in \Pi^n \times \Pi^m$ such that $P_l \to P_0$ and $Q_l \to Q_0$ uniformly on I. By (4), it is obvious that $\sum_{j=1}^k |(fQ_0 - P_0)^{(q)}(x_j)|^p = E$. On the other hand, $(P_0, Q_0) \in \mathcal{W}_m^n(f, X)$ because $(P_l, Q_l) \in \mathcal{W}_m^n(f, X)$ for all l. So, (P_0, Q_0) is a best Padé approximant pair of f at X.

Remark 2. We observe that if (P,Q) is a best Padé approximant pair of f at X, then so is (-P, -Q).

3 Convergence of best approximant pairs

Let $q > 0, f \in \mathcal{C}^q(I)$ and $(S,T) \in \mathcal{W}^n_m(f,X)$. We denote by $M_{p,q} \in \Pi^{q-1}$ the best approximant of x^q from Π^{q-1} with respect to the norm

$$||h||_p = \left(\int_{-1}^1 |h(t)|^p dt\right)^{\frac{1}{p}}.$$

If $x^q - M_{p,q}(x) = \prod_{s=0}^{q-1} (x - t_s)$, it is well known that $t_s \in (-1, 1), 0 \le s \le q-1$ and $t_s \ne t_c$ if $s \ne c$; see [7, §5.10]. We put $\mathcal{K}_{pq} = \|x^q - M_{p,q}\|_p$. Let

$$z_{js}^{\epsilon} = \epsilon t_s + x_j \in [x_j - \epsilon, x_j + \epsilon], \quad 0 \le s \le q - 1, \quad 1 \le j \le k, \quad 0 < \epsilon \le 1,$$

and let $y_1, ..., y_r \notin I$ be such that $y_v \neq y_w$ if $v \neq w$ and $T(y_v) \neq 0, 1 \leq v \leq r$.

Lemma 3. Under the above assumptions, for each $0 < \epsilon \leq 1$, there exists $(P_{\epsilon}, Q_{\epsilon}) \in \mathcal{H}_m^n$ such that

$$P_{\epsilon}\left(z_{js}^{\epsilon}\right) = f\left(z_{js}^{\epsilon}\right)Q_{\epsilon}\left(z_{js}^{\epsilon}\right) - \epsilon^{q}, \quad 0 \le s \le q-1, \quad 1 \le j \le k$$
$$P_{\epsilon}\left(y_{v}\right) = \frac{S}{T}(y_{v})Q_{\epsilon}\left(y_{v}\right) - \epsilon^{q}, \quad 1 \le v \le r$$
(6)

PROOF. Let $0 < \epsilon \leq 1$. Clearly, there exists a nontrivial $(U_{\epsilon}, V_{\epsilon}) \in \Pi^n \times \Pi^m$ such that

$$U_{\epsilon}\left(z_{js}^{\epsilon}\right) = f\left(z_{js}^{\epsilon}\right)V_{\epsilon}\left(z_{js}^{\epsilon}\right), \quad 0 \le s \le q-1, \quad 1 \le j \le k$$
$$U_{\epsilon}\left(y_{v}\right) = \frac{S}{T}(y_{v})V_{\epsilon}\left(y_{v}\right), \quad 1 \le v \le r$$

$$(7)$$

In fact, (7) is a homogeneous system of n + m + 1 equations in n + m + 2unknowns and therefore always has a nontrivial solution. We observe that if $V_{\epsilon} = 0$, then $U_{\epsilon} = 0$, a contradiction; so $V_{\epsilon} \neq 0$. Now, taking $P_{\epsilon} = \frac{U_{\epsilon}}{\|V_{\epsilon}\|_{\infty}} - \epsilon^{q}$ and $Q_{\epsilon} = \frac{V_{\epsilon}}{\|V_{\epsilon}\|_{\infty}}$, we conclude that $(P_{\epsilon}, Q_{\epsilon}) \in \mathcal{H}_{m}^{n}$ satisfies (6).

Lemma 4. Let $\{(P_{\epsilon}, Q_{\epsilon})\} \subset \mathcal{H}_m^n$ be the net of Lemma 3. Then $\{P_{\epsilon}\}$ and $\{Q_{\epsilon}\}$ are uniformly bounded on compact sets as $\epsilon \to 0$. Moreover, if $\{P_{\epsilon}\}$ and $\{Q_{\epsilon}\}$ are subsequences convergent to P_* and Q_* respectively, then $P_*T - Q_*S = 0$.

PROOF. Since $||Q_{\epsilon}||_{\infty} = 1$, $0 < \epsilon \leq 1$, the net $\{Q_{\epsilon}\}$ is uniformly bounded on compact sets.

Let $0 \leq i \leq q - 1$, $1 \leq j \leq k$ and $1 \leq v \leq r$. From (6), we get $(fQ_{\epsilon} - P_{\epsilon})^{\epsilon} (z_{ji}^{1}) = \epsilon^{q}, 0 < \epsilon \leq 1$, and therefore,

$$\left| \left(fQ_{\epsilon} - P_{\epsilon} \right)^{\epsilon} \left(z_{ji}^{1} \right) \right| = O(\epsilon^{q}) \quad \text{as} \quad \epsilon \to 0.$$
(8)

As $(S,T) \in \mathcal{W}_m^n(f,X)$, we have $(fT-S)^{(i)}(x_j) = 0$. Expanding $(fT-S)^{\epsilon}$ by its Taylor polynomial at $x_j, 1 \leq j \leq k$, up to order q-1, for each $x \in B_j$, there exists $\xi(x) \in [x_j - \epsilon, x_j + \epsilon]$ such that

$$(fT-S)^{\epsilon}(x) = \frac{\epsilon^{q}}{q!}(fT-S)^{(q)}(\xi(x))(x-x_{j})^{q},$$

and consequently,

$$\left| (fT - S)^{\epsilon} (z_{ji}^1) \right| = O(\epsilon^q) \quad \text{as} \quad \epsilon \to 0.$$
(9)

But

$$\left| (P_{\epsilon}T - Q_{\epsilon}S)^{\epsilon}(z_{ji}^{1}) \right| \leq \left| T^{\epsilon}(z_{ji}^{1}) \right| \left| (fQ_{\epsilon} - P_{\epsilon})^{\epsilon}(z_{ji}^{1}) \right| + \left| Q_{\epsilon}^{\epsilon}(z_{ji}^{1}) \right| \left| (fT - S)^{\epsilon}(z_{ji}^{1}) \right|,$$

so according to (8) and (9), we have $|(P_{\epsilon}T - Q_{\epsilon}S)(z_{ji}^{\epsilon})| = |(P_{\epsilon}T - Q_{\epsilon}S)^{\epsilon}(z_{ji}^{1})| = O(\epsilon^{q})$ as $\epsilon \to 0$.

On the other hand, (6) implies $|(P_{\epsilon}T - Q_{\epsilon}S)(y_v)| = O(\epsilon^q)$ as $\epsilon \to 0, 1 \le v \le r$. So, there exist $0 < \epsilon_0 \le 1$ and N > 0, independent of i, j and v, such that

$$\left| (P_{\epsilon}T - Q_{\epsilon}S)(z_{ji}^{\epsilon}) \right| \le \epsilon^{q}N \quad \text{and} \quad \left| (P_{\epsilon}T - Q_{\epsilon}S)(y_{v}) \right| \le \epsilon^{q}N, \tag{10}$$

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 $0 \leq i \leq q-1, 1 \leq j \leq k, 1 \leq v \leq r$ and $0 < \epsilon \leq \epsilon_0$. Let $w_{\epsilon}(x) = \prod_{c=1}^{k} \prod_{l=0}^{q-1} (x - z_{cl}^{\epsilon}) \prod_{u=1}^{r} (x - y_u)$. It is easy to check that

$$w_{\epsilon}'(z) = \begin{cases} \epsilon^{q-1} \prod_{\substack{c=1\\c\neq j}}^{k} \prod_{l=0}^{q-1} (z_{ji}^{\epsilon} - z_{cl}^{\epsilon}) \prod_{\substack{s=0\\s\neq i}}^{q-1} (t_i - t_s) \prod_{u=1}^{r} (z_{ji}^{\epsilon} - y_u) & \text{if } z = z_{ji}^{\epsilon} \\ \prod_{\substack{c=1\\c=1\\l=0}}^{k} \prod_{l=0}^{q-1} (y_v - z_{cl}^{\epsilon}) \prod_{\substack{u=1\\u\neq v}}^{r} (y_v - y_u) & \text{if } z = y_v \end{cases}$$

and for $x \in I$,

$$\frac{w_{\epsilon}(x)}{x-z} = \begin{cases} \prod_{\substack{c=1\\c\neq j}}^{k} \prod_{\substack{l=0\\c\neq j}}^{q-1} (x-z_{cl}^{\epsilon}) \prod_{\substack{s=0\\s\neq i}}^{q-1} (x-z_{js}^{\epsilon}) \prod_{u=1}^{r} (x-y_{u}) & \text{if } z = z_{ji}^{\epsilon} \\ \prod_{c=1}^{k} \prod_{\substack{l=0\\c\neq j}}^{q-1} (x-z_{cl}^{\epsilon}) \prod_{\substack{u=1\\u\neq v}}^{r} (x-y_{u}) & \text{if } z = y_{v} \end{cases}$$

Therefore, there is M > 0, independent of i, j and v, satisfying

$$\begin{split} \lim_{\epsilon \to 0} \frac{\left| w_{\epsilon}'(z_{ji}^{\epsilon}) \right|}{\epsilon^{q-1}} &= \prod_{\substack{c=1 \\ c \neq j}}^{k} |x_j - x_c|^q \prod_{\substack{s=0 \\ s \neq i}}^{q-1} |t_i - t_s| \prod_{u=1}^{r} |x_j - y_u| \ge \frac{1}{M},\\ \lim_{\epsilon \to 0} |w_{\epsilon}'(y_v)| &= \prod_{c=1}^{k} |y_v - x_c|^q \prod_{\substack{u=1 \\ u \neq v}}^{r} |y_v - y_u| \ge \frac{1}{M}, \end{split}$$

and $\left|\frac{w_{\epsilon}(x)}{x-z}\right| \leq M, x \in I$ and $z = z_{ji}^{\epsilon}, y_v$. Hence, (10) implies that there exists $0 < \epsilon_1 \leq \epsilon_0$ such that

$$\left|\frac{(P_{\epsilon}T - Q_{\epsilon}S)(z_{ji}^{\epsilon})w_{\epsilon}(x)}{w_{\epsilon}'(z_{ji}^{\epsilon})(x - z_{ji}^{\epsilon})}\right| \le \epsilon NM^2 \text{ and } \left|\frac{(P_{\epsilon}T - Q_{\epsilon}S)(y_v)w_{\epsilon}(x)}{w_{\epsilon}'(y_v)(x - y_v)}\right| \le \epsilon^q NM^2,$$

 $x\in I,\,0\leq i\leq q-1,\,1\leq j\leq k,\,1\leq v\leq r$ and $0<\epsilon\leq \epsilon_1.$ Now, using the Lagrange interpolation formula,

$$\begin{aligned} |(P_{\epsilon}T - Q_{\epsilon}S)(x)| \\ &= \left| \sum_{j=1}^{k} \sum_{i=0}^{q-1} \frac{(P_{\epsilon}T - Q_{\epsilon}S)(z_{ji}^{\epsilon})w_{\epsilon}(x)}{w_{\epsilon}'(z_{ji}^{\epsilon})(x - z_{ji}^{\epsilon})} + \sum_{v=1}^{r} \frac{(P_{\epsilon}T - Q_{\epsilon}S)(y_{v})w_{\epsilon}(x)}{w_{\epsilon}'(y_{v})(x - y_{v})} \right| \qquad (11) \\ &\leq \epsilon NM^{2} \end{aligned}$$

for $x \in I$, $0 < \epsilon \leq \epsilon_1$. Since

$$||P_{\epsilon}||_T \le NM^2 + ||S||_{\infty}, \quad 0 < \epsilon \le \epsilon_1,$$

from the equivalence of the norms in Π^n , we conclude that $\{P_{\epsilon}\}$ is uniformly bounded on compact sets as $\epsilon \to 0$.

Finally, if $\{P_{\epsilon}\}$ and $\{Q_{\epsilon}\}$ are subsequences convergent to P_* and Q_* respectively, by (11) we get $P_*T - Q_*S = 0$.

Lemma 5. Let $\{(P_{\epsilon}, Q_{\epsilon})\} \subset \mathcal{H}_m^n$ be the net of Lemma 3. If $\frac{S}{T}$ is normal, then there exist $\alpha \in \{1, -1\}$ and a subsequence of $\{(P_{\epsilon}, Q_{\epsilon})\}$, which we denote the same way, such that $\lim_{\epsilon \to 0} P_{\epsilon} = \alpha S$ and $\lim_{\epsilon \to 0} Q_{\epsilon} = \alpha T$ uniformly on I. Moreover,

$$\left[\left(fQ_{\epsilon} - P_{\epsilon}\right)^{\epsilon}\right] \left[z_{j0}^{1}, z_{j1}^{1}, \cdots, z_{js}^{1}\right] = 0$$
(12)

for $0 \leq s \leq q-1$, $1 \leq j \leq k$, $0 < \epsilon < 1$.

PROOF. By Lemma 4, there is a subsequence of $\{(P_{\epsilon}, Q_{\epsilon})\}$, which we denote the same way, and $P_0 \in \Pi^n$, $Q_0 \in \Pi^m$ such that $P_{\epsilon} \to P_0$ and $Q_{\epsilon} \to Q_0$ uniformly on I as $\epsilon \to 0$. Moreover,

$$P_0 T = Q_0 S. \tag{13}$$

For $1 \leq j \leq k$, let $K_j = \{i : 0 \leq i \leq m \text{ and } Q_0^{(i)}(x_j) \neq 0\}$. Since $||Q_{\epsilon}||_{\infty} = 1$, $0 < \epsilon \leq 1$, we have $||Q_0||_{\infty} = 1$, and thus $K_j \neq \emptyset$. Set $k_j = \min(K_j)$. By hypothesis, $T(x_j) \neq 0$, so (13) implies that there are $P_1 \in \Pi^n$ and $Q_1 \in \Pi^m$ satisfying

$$P_0(x) = \prod_{c=1}^k (x - x_c)^{k_c} P_1(x), \quad Q_0(x) = \prod_{c=1}^k (x - x_c)^{k_c} Q_1(x) \quad \text{and} \quad Q_1(x_c) \neq 0,$$
(14)

 $x \in I, 1 \leq c \leq k$. Using (13) again, we obtain

$$P_1T = Q_1S$$

Since $\frac{S}{T}$ is normal, either deg T = m or deg S = n. If deg T = m, then deg $P_1 \leq \deg S$. But $\frac{S}{T}$ is irreducible, so deg $P_1 = \deg S$ and deg $Q_1 = \deg T$. Therefore, there exists $\alpha \neq 0$ such that $P_1 = \alpha S$ and $Q_1 = \alpha T$. Now, according to (14), we have $k_c = 0, 1 \leq c \leq k$, and consequently, $P_0 = \alpha S$ and $Q_0 = \alpha T$. But $||Q_0||_{\infty} = ||T||_{\infty} = 1$, so $|\alpha| = 1$ and

$$P_0 = \alpha S$$
 and $Q_0 = \alpha T$. (15)

If deg S = n, in the same manner we can see (15).

Finally, let $0 \le s \le q - 1$, $1 \le j \le k$ and $\epsilon > 0$. From Lemma 3,

$$\epsilon^{q} = (fQ_{\epsilon} - P_{\epsilon})(z_{js}^{\epsilon}) = (fQ_{\epsilon} - P_{\epsilon})(\epsilon(z_{js}^{1} - x_{j}) + x_{j}) = (fQ_{\epsilon} - P_{\epsilon})^{\epsilon}(z_{js}^{1}).$$
(16)
So, (12) immediately follows.

So, (12) immediately follows.

Lemma 6. Suppose that $\frac{S}{T}$ is normal, and let $\{(P_{\epsilon}, Q_{\epsilon})\} \subset \mathcal{H}_m^n$ be the subsequence of Lemma 4. Then for each $\epsilon > 0$ and $x \in B_j$, $1 \le j \le k$, $x \ne z_{is}^1$, $0 \leq s \leq q-1$, there exists $\xi_{\epsilon}(x) \in (x_j - \epsilon, x_j + \epsilon)$ satisfying

$$\frac{1}{\epsilon^q}(fQ_\epsilon - P_\epsilon)^\epsilon(x) = \frac{1}{q!}(fQ_\epsilon - P_\epsilon)^{(q)}(\xi_\epsilon(x))\prod_{l=0}^{q-1}(x - z_{jl}^1).$$
 (17)

PROOF. Let $\epsilon > 0$. It is well known that the (q-1)th Lagrange interpolation polynomial for $(fQ_{\epsilon}-P_{\epsilon})^{\epsilon}$ with respect to $z_{j0}^1, z_{j1}^1, \cdots, z_{j(q-1)}^1$ can be expressed as

$$W_{\epsilon}(x) = \sum_{s=0}^{q-1} [(fQ_{\epsilon} - P_{\epsilon})^{\epsilon}] [z_{j0}^{1}, z_{j1}^{1}, \cdots, z_{js}^{1}] \prod_{l=0}^{s-1} (x - z_{jl}^{1})$$

By Lemma 5, we have $W_{\epsilon} = 0$. Let $x \in B_j$, $1 \le j \le k$, $x \ne z_{js}^1$, $0 \le s \le q-1$. From [8, Th. 3, p. 309], we get

$$(fQ_{\epsilon} - P_{\epsilon})^{\epsilon}(x) = [(fQ_{\epsilon} - P_{\epsilon})^{\epsilon}] [z_{j0}^{1}, z_{j1}^{1}, \cdots, z_{j(q-1)}^{1}, x] \prod_{l=0}^{q-1} (x - z_{jl}^{1}).$$
(18)

Since $f \in \mathcal{C}^q(I)$, [8, Th. 4, p. 310] implies that there exists $\zeta_{\epsilon}(x) \in (x_j - 1, x_j + 1)$ such that

$$[(fQ_{\epsilon} - P_{\epsilon})^{\epsilon}][z_{j0}^{1}, z_{j1}^{1}, \cdots, z_{j(q-1)}^{1}, x] = \frac{1}{q!}((fQ_{\epsilon} - P_{\epsilon})^{\epsilon})^{(q)}(\zeta_{\epsilon}(x))$$
$$= \frac{\epsilon^{q}}{q!}(fQ_{\epsilon} - P_{\epsilon})^{(q)}(\epsilon(\zeta_{\epsilon}(x) - x_{j}) + x_{j}).$$

So, according to (18), we have (17) with $\xi_{\epsilon}(x) = \epsilon(\zeta_{\epsilon}(x) - x_j) + x_j$.

Theorem 7. Let q > 0, $f \in C^q(I)$ and let $(S,T) \in W^n_m(f,X)$ be such that $\frac{S}{T}$ is normal. Then there exists a sequence $\{(P_{\epsilon}, Q_{\epsilon})\} \subset \mathcal{H}_m^n$ such that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^q} \| (fQ_\epsilon - P_\epsilon)^\epsilon \|_{B_j} = \frac{1}{q!} |(fT - S)^{(q)}(x_j)| \mathcal{K}_{pq}, \quad 1 \le j \le k.$$
(19)

PROOF. By Lemma 5, there exist $\alpha \in \{1, -1\}$ and a sequence $\{(P_{\epsilon}, Q_{\epsilon})\} \subset$ \mathcal{H}_m^n such that

$$\lim_{\epsilon \to 0} P_{\epsilon} = \alpha S \quad \text{and} \quad \lim_{\epsilon \to 0} Q_{\epsilon} = \alpha T \quad \text{uniformly on } I.$$
 (20)

From Lemma 6, for each ϵ and $x \in B_j$, $1 \leq j \leq k$, $x \neq z_{js}^1$, $0 \leq s \leq q-1$, there is $\xi_{\epsilon}(x) \in (x_j - \epsilon, x_j + \epsilon)$ satisfying

$$\frac{1}{\epsilon^q} (fQ_\epsilon - P_\epsilon)^\epsilon(x) = \frac{1}{q!} (fQ_\epsilon - P_\epsilon)^{(q)} (\xi_\epsilon(x)) \prod_{l=0}^{q-1} \left(x - z_{jl}^1\right).$$
(21)

Since (20) implies that $\lim_{\epsilon \to 0} (fQ_{\epsilon} - P_{\epsilon})^{(q)}(\xi_{\epsilon}(x)) = \alpha(fT - S)^{(q)}(x_j)$, we have

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^q} (fQ_{\epsilon} - P_{\epsilon})^{\epsilon}(x) = \frac{\alpha}{q!} (fT - S)^{(q)}(x_j) \prod_{l=0}^{q-1} (x - z_{jl}^1), \qquad (22)$$

 $x \in B_j, 1 \le j \le k, x \ne z_{js}^1, 0 \le s \le q-1.$ On the other hand, by (20) we see that $\{P_{\epsilon}\}$ and $\{Q_{\epsilon}\}$ are uniformly bounded on I as $\epsilon \to 0$. Hence, there exist M > 0 and $\epsilon_1 > 0$ such that $|(fQ_{\epsilon} - P_{\epsilon})^{(q)}(x)| \leq q!M, x \in I, 0 < \epsilon < \epsilon_1$. So, from (21) we deduce that

$$\left|\frac{1}{\epsilon^q}(fQ_{\epsilon} - P_{\epsilon})^{\epsilon}(x)\right| \le 2^q M, \quad x \in B_j, \quad x \neq z_{js}^1, \quad 0 < \epsilon < \epsilon_1.$$

According to (22) and the Lebesgue Convergence Theorem, we get

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^q} \| (fQ_\epsilon - P_\epsilon)^\epsilon \|_{B_j} = \frac{1}{q!} |(fT - S)^{(q)}(x_j)| \left\| \prod_{l=0}^{q-1} (x - z_{jl}^1) \right\|_{B_j}.$$

Now, substituting $x - x_j$ by t into the above equality gives

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^{q}} \| (fQ_{\epsilon} - P_{\epsilon})^{\epsilon} \|_{B_{j}} = \frac{1}{q!} | (fT - S)^{(q)}(x_{j})| \| t^{q} - M_{pq}(t) \|_{p}$$
$$= \frac{1}{q!} | (fT - S)^{(q)}(x_{j})| \mathcal{K}_{pq}.$$

Theorem 8. Let $f \in C^q(I)$ and let $\{(S_{\epsilon}, T_{\epsilon})\} \subset \mathcal{H}_m^n$ be a net of best approximant pairs of f from \mathcal{H}_m^n with respect to $\|\cdot\|_{\epsilon}$. Then $\{S_{\epsilon}\}$ and $\{T_{\epsilon}\}$ are uniformly bounded on compact sets as $\epsilon \to 0$.

PROOF. Since $||T_{\epsilon}||_{\infty} = 1$, $0 < \epsilon \leq 1$, the net $\{T_{\epsilon}\}$ is uniformly bounded on compact sets.

Let $(S,T) \in \mathcal{W}_m^n(f,X)$. Then for each $1 \leq j \leq k$,

$$\frac{\|(S_{\epsilon}T - T_{\epsilon}S)^{\epsilon}\|_{B_{j}}}{\epsilon^{q}} = \frac{1}{\epsilon^{q}} \|(-T(fT_{\epsilon} - S_{\epsilon}) + T_{\epsilon}(fT - S))^{\epsilon}\|_{B_{j}}$$

$$\leq \frac{(2k)^{1/p}}{\epsilon^{q}} \|-T(fT_{\epsilon} - S_{\epsilon}) + T_{\epsilon}(fT - S)\|_{\epsilon}$$

$$\leq \frac{(2k)^{1/p}}{\epsilon^{q}} (\|T(fT_{\epsilon} - S_{\epsilon})\|_{\epsilon} + \|T_{\epsilon}(fT - S)\|_{\epsilon}) \qquad (23)$$

$$\leq \frac{(2k)^{1/p}}{\epsilon^{q}} (\|fT_{\epsilon} - S_{\epsilon}\|_{\epsilon} + \|fT - S\|_{\epsilon})$$

$$\leq \frac{2(2k)^{1/p}}{\epsilon^{q}} \|fT - S\|_{\epsilon}.$$

If q = 0, then

$$|(S_{\epsilon}T - T_{\epsilon}S)(x_j)| = O(1) \quad \text{as} \quad \epsilon \to 0, \quad 1 \le j \le k.$$
(24)

In otherwise, as $(fT-S)^{(l)}(x_j) = 0, \ 0 \le l \le q-1, \ 1 \le j \le k$, expanding $(fT-S)^{\epsilon}$ by its Taylor polynomial at x_j up to order q-1, it follows that for each $x \in B_j$, there exists $\xi_{\epsilon}(x) \in [x_j - \epsilon, x_j + \epsilon]$ such that $(fT-S)^{\epsilon}(x) = \frac{\epsilon^q}{q!} (fT-S)^{(q)} (\xi_{\epsilon}(x))(x-x_j)^q$. So, $\|(fT-S)^{\epsilon}\|_{B_j} = O(\epsilon^q)$ as $\epsilon \to 0$, and consequently

$$\|fT - S\|_{\epsilon} = O(\epsilon^q) \quad \text{as} \quad \epsilon \to 0.$$
 (25)

Therefore, by (23), we get $||(S_{\epsilon}T - T_{\epsilon}S)^{\epsilon}||_{B_j} = O(\epsilon^q)$ as $\epsilon \to 0, 1 \le j \le k$. Since $(S_{\epsilon}T - T_{\epsilon}S)^{\epsilon} \in \Pi^{n+m}$ on B_j , according to Lemma 2.2 in [5], we have

$$\left| \left(S_{\epsilon}T - T_{\epsilon}S \right)^{(i)}(x_j) \right| = O(\epsilon^{q-i}) \quad \text{as} \quad \epsilon \to 0,$$
(26)

 $1 \leq j \leq k, 0 \leq i \leq q$. Since n + m + 1 < k(q + 1), from (24) and (26) we show that $\{S_{\epsilon}T - T_{\epsilon}S\} \subset \Pi^{n+m}$ is uniformly bounded on I as $\epsilon \to 0$; that is, there exist M > 0 and $\epsilon_1 > 0$ such that

$$\left| \left(S_{\epsilon}T - T_{\epsilon}S \right)(x) \right| \le M, \quad x \in I, \quad 0 < \epsilon < \epsilon_1.$$

As $|T_{\epsilon}S(x)| \leq ||S||_{\infty}$, $x \in I$, $0 < \epsilon < \epsilon_1$, we have $||S_{\epsilon}||_T = \max_{x \in I} |(S_{\epsilon}T)(x)| \leq ||S||_{\infty} + M$, $0 < \epsilon < \epsilon_1$. Finally, by the equivalence of the norms in Π^n , we conclude that $\{S_{\epsilon}\}$ is uniformly bounded on compact sets as $\epsilon \to 0$.

Theorem 9. Let $f \in C^q(I)$ and let $\{(S_{\epsilon}, T_{\epsilon})\}$ be a net of best approximant pairs of f from \mathcal{H}^n_m with respect to $\|\cdot\|_{\epsilon}$. Suppose that there exists a best Padé approximant pair of f at X, say (S,T), such that $\frac{S}{T}$ is normal. Then any cluster point of $\{(S_{\epsilon}, T_{\epsilon})\}$ as $\epsilon \to 0$ is a best Padé approximant pair of f at X.

PROOF. According to Theorem 8, it follows that the set of cluster points of the net $\{(S_{\epsilon}, T_{\epsilon})\}$ as $\epsilon \to 0$ is nonempty. Now, it is sufficient to prove that if (S_*, T_*) is the limit point of $\{(S_{\epsilon_l}, T_{\epsilon_l})\}$ as $\epsilon_l \to 0$, then (S_*, T_*) is a best Padé approximant pair of f at X. If q = 0, then the result is obvious, because

$$\sum_{j=1}^{k} \left| \left(fT_* - S_* \right) (x_j) \right|^p = \lim_{\epsilon_l \to 0} k \| fT_{\epsilon_l} - S_{\epsilon_l} \|_{\epsilon_l}^p$$
$$\leq \lim_{\epsilon_l \to 0} k \| fT - S \|_{\epsilon_l}^p$$
$$= \sum_{j=1}^{k} \left| \left(fT - S \right) (x_j) \right|^p.$$

Now assume q > 0. Let $1 \le j \le k$, $0 \le i \le q - 1$. As in the proof of Theorem 8, we have

$$\left| \left(S_{\epsilon_l} T - T_{\epsilon_l} S \right)^{(i)} (x_j) \right| = O(\epsilon_l^{q-i}) \quad \text{as} \quad \epsilon_l \to 0.$$
⁽²⁷⁾

Therefore, $(S_*T - T_*S)^{(i)}(x_j) = 0$. Since

$$S_*T - T_*S = -T(fT_* - S_*) + T_*(fT - S)$$

and $(S,T) \in \mathcal{W}_m^n(f,X)$, using the Leibniz rule we get $(T(fT_* - S_*))^{(i)}(x_j) = 0$, and thus

$$(fT_* - S_*)^{(i)}(x_j) = 0, (28)$$

because $T(x_j) \neq 0$. As *i* and *j* are arbitrary, (S_*, T_*) is a Padé approximant pair of *f* at *X*, and so $(S_*, T_*) \in \mathcal{W}_m^n(f, X)$ since $||T_*||_{\infty} = 1$.

Expanding $(S_{\epsilon_l}T - T_{\epsilon_l}S)^{\epsilon_l}$ and $T_{\epsilon_l}(fT - S)^{\epsilon_l}$ by their Taylor polynomials at x_j up to order q - 1, it follows that for each $x \in B_j$, there exist

 $\xi_{\epsilon_l}(x), \eta_{\epsilon_l}(x) \in [x_j - \epsilon_l, x_j + \epsilon_l]$ such that

$$T^{\epsilon_{l}}(x)\frac{1}{\epsilon_{l}^{q}}\left(fT_{\epsilon_{l}}-S_{\epsilon_{l}}\right)^{\epsilon_{l}}(x) = \frac{1}{\epsilon_{l}^{q}}\left(T(fT_{\epsilon_{l}}-S_{\epsilon_{l}})\right)^{\epsilon_{l}}(x)$$

$$= \frac{1}{\epsilon_{l}^{q}}\left(-\left(S_{\epsilon_{l}}T-T_{\epsilon_{l}}S\right)^{\epsilon_{l}}(x) + \left(T_{\epsilon_{l}}(fT-S)\right)^{\epsilon_{l}}(x)\right)$$

$$= -\sum_{i=0}^{q-1}\frac{\epsilon_{l}^{i-q}\left(S_{\epsilon_{l}}T-T_{\epsilon_{l}}S\right)^{(i)}\left(x_{j}\right)}{i!}(x-x_{j})^{i}$$

$$-\frac{\left(S_{\epsilon_{l}}T-T_{\epsilon_{l}}S\right)^{(q)}\left(\xi_{\epsilon_{l}}(x)\right)}{q!}(x-x_{j})^{q}$$

$$+\frac{1}{q!}\sum_{s=0}^{q}\binom{q}{s}\left(fT-S\right)^{(s)}\left(\eta_{\epsilon_{l}}(x)\right)T_{\epsilon_{l}}^{(q-s)}(\eta_{\epsilon_{l}}(x))(x-x_{j})^{q}.$$
(29)

As $T(x_j) \neq 0$, from (27) there exist a subsequence of $\{\epsilon_l\}$, which we denote the same way, and $a_{ij} \in \mathbb{R}$, $0 \leq i \leq q-1$, $1 \leq j \leq k$, such that

$$\lim_{\epsilon_l \to 0} \epsilon_l^{i-q} \left(S_{\epsilon_l} T - T_{\epsilon_l} S \right)^{(i)} (x_j) = T(x_j) a_{ij}.$$
(30)

According to Theorem 8, we have

$$\lim_{\epsilon_l \to 0} \frac{(S_{\epsilon_l} T - T_{\epsilon_l} S)^{(q)} (\xi_{\epsilon_l}(x))}{q!} = \frac{(S_* T - T_* S)^{(q)}(x_j)}{q!},$$
(31)

so (28)-(31) imply that

$$\lim_{\epsilon_{l} \to 0} \frac{1}{\epsilon_{l}^{q}} \left(fT_{\epsilon_{l}} - S_{\epsilon_{l}} \right)^{\epsilon_{l}} (x)$$

$$= \frac{1}{q! T(x_{j})} \left(-(S_{*}T - T_{*}S)^{(q)}(x_{j}) + (fT - S)^{(q)}T_{*}(x_{j}) \right) (x - x_{j})^{q}$$

$$- \sum_{i=0}^{q-1} a_{ij}(x - x_{j})^{i}$$

$$= \frac{1}{q!} \left(fT_{*} - S_{*} \right)^{(q)} (x_{j}) (x - x_{j})^{q} - \sum_{i=0}^{q-1} a_{ij}(x - x_{j})^{i}$$

uniformly on B_j . Therefore, substituting $x - x_j$ by t into the following inequal-

ity gives

$$\begin{split} \lim_{\epsilon_{l}\to 0} \frac{1}{\epsilon_{l}^{q}} \left\| (fT_{\epsilon_{l}} - S_{\epsilon_{l}})^{\epsilon_{l}} \right\|_{B_{j}} \\ &= \left\| \frac{1}{q!} \left(fT_{*} - S_{*} \right)^{(q)} (x_{j}) \left(x - x_{j} \right)^{q} - \sum_{i=0}^{q-1} a_{ij} (x - x_{j})^{i} \right\|_{B_{j}} \\ &\geq \left| \frac{1}{q!} \left(fT_{*} - S_{*} \right)^{(q)} (x_{j}) \right| \left\| t^{q} - M_{pq}(t) \right\|_{p} \\ &= \frac{1}{q!} \left| \left(fT_{*} - S_{*} \right)^{(q)} (x_{j}) \right| \mathcal{K}_{pq}, \end{split}$$
(32)

 $1 \leq j \leq k$. Since $\frac{S}{T}$ is normal, from Theorem 7, there exists a subsequence of $\{\epsilon_l\}$, which we denote the same way again, such that $\{(P_{\epsilon_l}, Q_{\epsilon_l})\} \subset \mathcal{H}_m^n$ and

$$\lim_{\epsilon_l \to 0} \frac{1}{\epsilon_l^q} \left\| \left(fQ_{\epsilon_l} - P_{\epsilon_l} \right)^{\epsilon_l} \right\|_{B_j} = \frac{1}{q!} \left| \left(fT - S \right)^{(q)} (x_j) \right| \mathcal{K}_{pq}, \tag{33}$$

 $1 \leq j \leq k$. But $\{(S_{\epsilon_l}, T_{\epsilon_l})\}$ is a net of best approximant pairs of f from \mathcal{H}_m^n , so (32) and (33) imply

$$\sum_{j=1}^{k} \left| \left(fT_* - S_* \right)^{(q)} (x_j) \right|^p \le \sum_{j=1}^{k} \left| \left(fT - S \right)^{(q)} (x_j) \right|^p.$$

Finally, by (28), (S_*, T_*) is a best Padé approximant pair.

We say that the best Padé approximant pair of f at X is unique if, whenever $(P,Q), (U,V) \in \mathcal{W}_m^n(f,X)$ satisfy (3), then $(P,Q) \equiv (U,V)$. The next corollary immediately follows

The next corollary immediately follows.

Corollary 10. Let $f \in C^q(I)$, q > 0, and suppose that there exists a unique best Padé approximant pair of f at X, say (S,T), such that $\frac{S}{T}$ is normal. Then $\frac{S}{T}$ is a Padé approximant of f at X. In addition, if $\{(S_{\epsilon}, T_{\epsilon})\}$ is a net of best approximant pairs of f from \mathcal{H}^n_m with respect to $\|\cdot\|_{\epsilon}$, then $\frac{S_{\epsilon}}{T_{\epsilon}}$ converges to $\frac{S}{T}$ uniformly on some neighborhood of X as $\epsilon \to 0$.

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