BEST $L^p$-APPROXIMANT PAIR ON SMALL INTERVALS

Abstract

In this paper, we study the behavior of best $L^p$-approximations by algebraic polynomial pairs on unions of intervals when the measure of those intervals tends to zero.

1 Introduction

Let $X = \{x_j\}_{j=1}^k \subset \mathbb{R}$, $k \in \mathbb{N}$, and let $\{B_j\}_{j=1}^k$ be pairwise disjoint closed intervals centered at $x_j$ of radius 1. Let $n, m \in \mathbb{N} \cup \{0\}$ and suppose that $n + m + 1 = kq + r$ with $q \in \mathbb{N} \cup \{0\}$, $0 < r < k$. For $s \in \mathbb{N} \cup \{0\}$, we let $C^s(I)$ denote the space of real functions defined on $I := \bigcup_{j=1}^k B_j$, which are continuously differentiable up to order $s$ on $I$. For simplicity, we write $C(I)$ instead of $C^0(I)$.

If $\|\cdot\|$ is a norm defined on $C(I)$ and $h \in C(I)$, then for each $0 < \epsilon \leq 1$, we write $\|h\|_\epsilon = \|h^\epsilon\|$, where $h^\epsilon(x) = h(\epsilon(x - x_j) + x_j)$, $x \in B_j$. We put

$$\|h\| = \left(\int_I |h(x)|^p \frac{dx}{|I|}\right)^{\frac{1}{p}}, \quad 1 < p < \infty,$$

Mathematical Reviews subject classification: Primary: 41A20, 41A21; Secondary: 32A10

Key words: Best approximation, Algebraic polynomial, Padé approximant pair, $L^p$-norm

Received by the editors November 4, 2014

Communicated by: Viktor Kolyada

*This work was supported by Universidad Nacional de Río Cuarto and CONICET.
where $|I|$ is the Lebesgue measure of $I$. For $I_{\epsilon} := \bigcup_{j=1}^{k} [x_j - \epsilon, x_j + \epsilon]$, we observe that $(C(I_{\epsilon}), \| \cdot \|_{\epsilon})$ is a normed space and

$$\|h\|_{\epsilon} = \left( \int_{I_{\epsilon}} |h(x)|^p \frac{dx}{|I_{\epsilon}|} \right)^{\frac{1}{p}}.$$  

We define $\|h\|_{\infty} := \max_{x \in I} |h(x)|$ and $\|h\|_{B_j} := \left( \int_{B_j} |h(x)|^p dx \right)^{\frac{1}{p}}, 1 \leq j \leq k$.

Let $\Pi^n$ be the class of algebraic polynomials with real coefficients of degree at most $n$. We consider the set $H_{nm} := \{ (P, Q) \in \Pi^n \times \Pi^m : \parallel Q \parallel_{\infty} = 1 \}$.

Given $(P, Q), (U, V) \in H_{nm}$, we identify $(P, Q)$ with $(U, V)$ if and only if $P = \lambda U, Q = \lambda V, |\lambda| = 1$. We denote it briefly by $(P, Q) \equiv (U, V)$.

Let $f \in C(I)$ and $0 < \epsilon \leq 1$. We say that $(P_{\epsilon}, Q_{\epsilon}) \in H_{nm}$ is a best approximant pair of $f$ from $H_{nm}$ with respect to $\| \cdot \|_{\epsilon}$ if

$$\|f Q_{\epsilon} - P_{\epsilon}\|_{\epsilon} = \inf_{(P, Q) \in H_{nm}} \|f Q - P\|_{\epsilon}. \tag{1}$$

It is easy to see that the pair $(P_{\epsilon}, Q_{\epsilon})$ always exists.

Given $q > 0, f \in \mathcal{C}^{q-1}(I)$ and $(P, Q) \in \Pi^n \times \Pi^m$, if

$$(f Q - P)^{(s)}(x_j) = 0, \quad 0 \leq s \leq q - 1, \quad 1 \leq j \leq k, \tag{2}$$

then $(P, Q)$ is said to be a Padé approximant pair of $f$ at $X$. If $Q \neq 0$ and

$$\left( f - \frac{P}{Q} \right)^{(s)}(x_j) = 0, \quad 0 \leq s \leq q - 1, \quad 1 \leq j \leq k,$$

then the rational function $\frac{P}{Q}$ is called a Padé approximant of $f$ at $X$.

We define

$$W_{nm}(f, X) := \{ (P, Q) \in H_{nm} : (P, Q) \text{ is a Padé approximant pair of } f \text{ at } X \}.$$  

If $q = 0$, then no constraint over the pair is assumed and $W_{nm}(f, X) = H_{nm}$.

Clearly, $W_{nm}(f, X) \neq \emptyset$. In fact, let $x_{k+1} \in I - X$, and we consider the system (2) with constraints $(f Q - P)^{(s)}(x_{k+1}) = 0, 0 \leq s \leq r - 1$. This system always has a nontrivial solution for $(P, Q)$, since it is a homogeneous system of $n + m + 1$ equations in $n + m + 2$ unknowns. Now, if $Q = 0$, then $P = 0$ because $P \in \Pi^n$, a contradiction. So, $Q \neq 0$ and $\left( \frac{P}{\|Q\|_{\infty}}, \frac{Q}{\|Q\|_{\infty}} \right) \in W_{nm}(f, X)$. 

We say that \((P_0, Q_0) \in \mathcal{W}_m^n(f, X)\) is a best Padé approximant pair of \(f\) at \(X\) if
\[
\sum_{j=1}^k |(fQ_0 - P_0)^{(q)}(x_j)|^p \leq \sum_{j=1}^k |(fQ - P)^{(q)}(x_j)|^p
\]
for all \((P, Q) \in \mathcal{W}_m^n(f, X)\). If \((P, Q) \in \Pi^n \times \Pi^m\), \(Q \neq 0\), then \(\frac{P}{Q}\) is said to be normal if it is irreducible and either \(\deg P = n\) or \(\deg Q = m\). (The null rational function \(0\) is normal if and only if \(\deg Q = 0\).)

In 1934, Walsh proved in [9] that the Taylor polynomial of degree \(n\) for an analytic function \(f\) can be obtained by taking the limit as \(\epsilon \to 0\) of the best (Chebyshev) approximant from \(\Pi^n\) to \(f\) on the disk \(|z| \leq \epsilon\). In [10], the author generalized this result to rational approximation. In [2], Chui, Shisha and Smith proved that the net of best (Chebyshev) approximants pairs on \([0, \epsilon]\), from \(\{(P, Q) \in \Pi^n \times \Pi^m : Q(0) = 1\}\), converges to the Padé approximant pair in the origin as \(\epsilon \to 0\). Similar results for the \(L_2\)-norm can be seen in [3]. The case of a unique point in several variables was treated in [1] with the \(L^p\)-norms. Finally, the case of \(L^p\)-approximation on \(k\) disjoint intervals, where \(n + m + 1\) is divisible by \(k\), was investigated in [6].

In Section 2, we show that there exists at least a best Padé approximant pair of \(f\) at \(X\). In Section 3, we prove that, any cluster point of best approximant pairs \(\{(P_\epsilon, Q_\epsilon)\}\) as \(\epsilon \to 0\) is a best Padé approximant pair of \(f\) at \(X\).

## 2 Existence of best Padé approximant pairs

Now, we establish an existence theorem of best Padé approximant pairs.

**Theorem 1.** Let \(f \in C^q(I)\). Then there exists at least one best Padé approximant pair of \(f\) at \(X\).

**Proof.** Let \(\{(P_l, Q_l)\} \subset \mathcal{W}_m^n(f, X)\) be a sequence satisfying
\[
\lim_{l \to \infty} \sum_{j=1}^k |(fQ_l - P_l)^{(q)}(x_j)|^p = \inf_{(P,Q) \in \mathcal{W}_m^n(f, X)} \sum_{j=1}^k |(fQ - P)^{(q)}(x_j)|^p =: E. \quad (4)
\]

If \(q > 0\), then \((fQ_l - P_l)^{(i)}(x_j) = 0, 0 \leq i \leq q - 1, 1 \leq j \leq k\). According to (4), there is constant \(M > 0\) such that
\[
|(fQ_l - P_l)^{(i)}(x_j)| \leq M, \quad 0 \leq i \leq q, \quad 1 \leq j \leq k, \quad l \in \mathbb{N}. \quad (5)
\]
We observe that if \(q = 0\), (5) is true also, by (4).
Let \((S, T) \in W^m_m(f, X)\). Since \(SQ_l - TP_l = T(fQ_l - P_l) - Q_l(fT - S)\), by the Leibniz rule for the \(i\)th derivative of a product of two factors,

\[
|SQ_l - TP_l|^{(i)}(x_j) | \leq N, \quad 0 \leq i \leq q, \quad 1 \leq j \leq k, \quad l \in \mathbb{N},
\]

for some constant \(N > 0\). As \(\|P\| := \max_{0 \leq i \leq q, 1 \leq j \leq k} |P^{(i)}(x_j)|\) is a norm on \(\Pi^{k(q+1)-1}\), the equivalence of the norms in \(\Pi^{k(q+1)-1}\) implies that \(\{SQ_l - TP_l\}\) is uniformly bounded on \(I\), and consequently \(\{TP_l\}\) is uniformly bounded on \(I\). Since \(\|P\|_T := \max_{t \in I} |TP(t)|\) is a norm on \(\Pi^n\), we get that \(\{P_l\}\) is uniformly bounded on \(I\). So, there is a subsequence of \(\{(P_l, Q_l)\}\), which we denote the same way, and \((P_0, Q_0) \in \Pi^n \times \Pi^m\) such that \(P_l \to P_0\) and \(Q_l \to Q_0\) uniformly on \(I\). By (4), it is obvious that \(\sum_{j=1}^{k} |(fQ_0 - P_0)^{(q)}(x_j)| = E\). On the other hand, \((P_0, Q_0) \in W^m_m(f, X)\) because \((P_l, Q_l) \in W^m_m(f, X)\) for all \(l\). So, \((P_0, Q_0)\) is a best Padé approximant pair of \(f\) at \(X\).

**Remark 2.** We observe that if \((P, Q)\) is a best Padé approximant pair of \(f\) at \(X\), then so is \((-P, -Q)\).

### 3 Convergence of best approximant pairs

Let \(q > 0, f \in C^q(I)\) and \((S, T) \in W^m_m(f, X)\). We denote by \(M_{p,q} \in \Pi^{q-1}\) the best approximant of \(x^q\) from \(\Pi^{q-1}\) with respect to the norm

\[
\|h\|_p = \left(\int_{-1}^{1} |h(t)|^p dt\right)^{\frac{1}{p}}.
\]

If \(x^q - M_{p,q}(x) = \prod_{s=0}^{q-1} (x - t_s)\), it is well known that \(t_s \in (-1, 1), 0 \leq s \leq q - 1\) and \(t_s \neq t_c\) if \(s \neq c\); see [7, §5.10]. We put \(K_{pq} = \|x^q - M_{p,q}\|_p\). Let

\[
z_{s,s}^j = \epsilon t_s + x_j \in [x_j - \epsilon, x_j + \epsilon], \quad 0 \leq s \leq q - 1, \quad 1 \leq j \leq k, \quad 0 < \epsilon \leq 1,
\]

and let \(y_1, ..., y_r \notin I\) be such that \(y_v \neq y_w\) if \(v \neq w\) and \(T(y_v) \neq 0, 1 \leq v \leq r\).

**Lemma 3.** Under the above assumptions, for each \(0 < \epsilon \leq 1\), there exists \((P_\epsilon, Q_\epsilon) \in \mathcal{H}^n_m\) such that

\[
P_\epsilon(z_v) = f(z_v)Q_\epsilon(z_v^\epsilon) - \epsilon^{q'}, \quad 0 \leq s \leq q - 1, \quad 1 \leq j \leq k
\]

\[
P_\epsilon(y_v) = \frac{S}{T}(y_v)Q_\epsilon(y_v^\epsilon) - \epsilon^{q'}, \quad 1 \leq v \leq r.
\]
Proof. Let $0 < \epsilon \leq 1$. Clearly, there exists a nontrivial $(U_\epsilon, V_\epsilon) \in \Pi^n \times \Pi^n$ such that

$$U_\epsilon (z_{js}^\epsilon) = f (z_{js}^\epsilon) V_\epsilon (z_{js}^\epsilon), \quad 0 \leq s \leq q - 1, \quad 1 \leq j \leq k$$

$$U_\epsilon (y_v) = \frac{S}{T} (y_v) V_\epsilon (y_v), \quad 1 \leq v \leq r.$$  \hspace{1cm} (7)

In fact, (7) is a homogeneous system of $n + m + 1$ equations in $n + m + 2$ unknowns and therefore always has a nontrivial solution. We observe that if $V_\epsilon = 0$, then $U_\epsilon = 0$, a contradiction; so $V_\epsilon \neq 0$. Now, taking $P_\epsilon = \frac{V_\epsilon}{\|V_\epsilon\|_\infty} - \epsilon^q$ and $Q_\epsilon = \frac{V_\epsilon}{\|V_\epsilon\|_\infty}$, we conclude that $(P_\epsilon, Q_\epsilon) \in H_m^n$ satisfies (6). \hfill \square

Lemma 4. Let $\{(P_\epsilon, Q_\epsilon)\} \subset H_m^n$ be the net of Lemma 3. Then $\{P_\epsilon\}$ and $\{Q_\epsilon\}$ are uniformly bounded on compact sets as $\epsilon \to 0$. Moreover, if $\{P_\epsilon\}$ and $\{Q_\epsilon\}$ are subsequences convergent to $P_*$ and $Q_*$ respectively, then $P_* T - Q_* S = 0$.

Proof. Since $\|Q_\epsilon\|_\infty = 1, 0 < \epsilon \leq 1$, the net $\{Q_\epsilon\}$ is uniformly bounded on compact sets.

Let $0 \leq i \leq q - 1, 1 \leq j \leq k$ and $1 \leq v \leq r$. From (6), we get $(fQ_\epsilon - P_\epsilon)^r (z_{ji}^1) = \epsilon^q, 0 < \epsilon \leq 1$, and therefore,

$$|(fQ_\epsilon - P_\epsilon)^r (z_{ji}^1)| = O(\epsilon^q) \quad \text{as} \quad \epsilon \to 0.$$  \hspace{1cm} (8)

As $(S, T) \in W_m^n (f, X)$, we have $(fT - S)^{(i)} (x_j) = 0$. Expanding $(fT - S)^r$ by its Taylor polynomial at $x_j$, $1 \leq j \leq k$, up to order $q - 1$, for each $x \in B_j$, there exists $\xi(x) \in [x_j - \epsilon, x_j + \epsilon]$ such that

$$(fT - S)^r (x) = \frac{\epsilon^q}{q!} (fT - S)^{(q)} (\xi(x))(x - x_j)^q,$$

and consequently,

$$|(fT - S)^r (z_{ji}^1)| = O(\epsilon^q) \quad \text{as} \quad \epsilon \to 0.$$  \hspace{1cm} (9)

But

$$|(P_* T - Q_* S)^r (z_{ji}^1)| \leq |T^r (z_{ji}^1)| \left| (fQ_\epsilon - P_\epsilon)^r (z_{ji}^1) \right| + |Q_\epsilon^r (z_{ji}^1)| \left| (fT - S)^r (z_{ji}^1) \right|,$$

so according to (8) and (9), we have $|(P_* T - Q_* S)(z_{ji}^1)| = |(P_* T - Q_* S)^r (z_{ji}^1)| = O(\epsilon^q)$ as $\epsilon \to 0$.

On the other hand, (6) implies $|(P_* T - Q_* S)(y_v)| = O(\epsilon^q)$ as $\epsilon \to 0$, $1 \leq v \leq r$. So, there exist $0 < \epsilon_0 \leq 1$ and $N > 0$, independent of $i, j$ and $v$, such that

$$|(P_* T - Q_* S)(z_{ji}^1)| \leq \epsilon^q N \quad \text{and} \quad |(P_* T - Q_* S)(y_v)| \leq \epsilon^q N,$$  \hspace{1cm} (10)
0 ≤ i ≤ q - 1, 1 ≤ j ≤ k, 1 ≤ v ≤ r and 0 < ε ≤ ε₁.

Let \( wₚ(x) = \prod_{l=1}^{q-1} (x - zₗᵢ) \prod_{u=1}^{r} (x - y_u) \). It is easy to check that

\[
w'_p(z) = \begin{cases} 
\epsilon^{q-1} \prod_{i \in j} (z_{j₁} - z_{cᵢ}) \prod_{s \neq i} (t_s - t_r) \prod_{u=1}^{r} (z_{jᵢ} - y_u) & \text{if } z = z_{jᵢ}^p \\
\epsilon^{q-1} \prod_{i \in j} (y_v - z_{cᵢ}) \prod_{u=1}^{r} (y_v - y_u) & \text{if } z = y_v 
\end{cases}
\]

and for \( x \in I \),

\[
wₚ(x) = \begin{cases} 
\epsilon^{q-1} \prod_{i \in j} (x - z_{cᵢ}) \prod_{s \neq i} (x - z_{jᵢ}) \prod_{u=1}^{r} (x - y_u) & \text{if } z = z_{jᵢ}^p \\
\epsilon^{q-1} \prod_{i \in j} (x - z_{cᵢ}) \prod_{u=1}^{r} (x - y_u) & \text{if } z = y_v 
\end{cases}
\]

Therefore, there is \( M > 0 \), independent of \( i, j \) and \( v \), satisfying

\[
\lim_{\epsilon \to 0} \frac{|w'_p(z_{jᵢ}^p)|}{\epsilon^{q-1}} = \prod_{i \in j} |x_j - x_c| \prod_{s \neq i} |t_s - t_r| \prod_{u=1}^{r} |x_j - y_u| \geq \frac{1}{M},
\]

\[
\lim_{\epsilon \to 0} |w'_p(y_v)| = \prod_{i \in j} |y_v - x_c| \prod_{u=1}^{r} |y_v - y_u| \geq \frac{1}{M},
\]

and \( \frac{|wₚ(x)|}{x - z} \leq M, x \in I \) and \( z = z_{jᵢ}^p, y_v \). Hence, (10) implies that there exists \( 0 < \epsilon₁ \leq \epsilon_0 \) such that

\[
\left| \frac{(P.T - Q.S)(z_{jᵢ}^p)wₚ(x)}{w'_p(z_{jᵢ}^p)(x - z_{jᵢ}^p)} \right| \leq \epsilon NM^2 \quad \text{and} \quad \left| \frac{(P.T - Q.S)(y_v)wₚ(x)}{w'_p(y_v)(x - y_v)} \right| \leq \epsilon NM^2,
\]

\( x \in I, 0 < i \leq q - 1, 1 \leq j \leq k, 1 \leq v \leq r \) and \( 0 < \epsilon \leq \epsilon₁ \). Now, using the Lagrange interpolation formula,

\[
\left| (P.T - Q.S)(x) \right| = \left| \sum_{j=1}^{k} \sum_{i=0}^{q-1} \frac{(P.T - Q.S)(z_{jᵢ})wₚ(x)}{w'_p(z_{jᵢ})(x - z_{jᵢ})} + \sum_{u=1}^{r} \frac{(P.T - Q.S)(y_v)wₚ(x)}{w'_p(y_v)(x - y_v)} \right| \leq \epsilon NM^2
\]
for \( x \in I, \, 0 < \varepsilon \leq \varepsilon_1 \). Since
\[
\|P_\varepsilon\|_T \leq NM^2 + \|S\|_\infty, \quad 0 < \varepsilon \leq \varepsilon_1,
\]
from the equivalence of the norms in \( \Pi^n \), we conclude that \( \{P_\varepsilon\} \) is uniformly bounded on compact sets as \( \varepsilon \to 0 \).

Finally, if \( \{P_\varepsilon\} \) and \( \{Q_\varepsilon\} \) are subsequences convergent to \( P_* \) and \( Q_* \) respectively, by (11) we get
\[
P_* - Q_* = 0.
\]

**Lemma 5.** Let \( \{(P_\varepsilon, Q_\varepsilon)\} \subset \mathcal{H}_m^n \) be the net of Lemma 3. If \( \frac{S}{\Pi} \) is normal, then there exist \( \alpha \in \{1, -1\} \) and a subsequence of \( \{(P_\varepsilon, Q_\varepsilon)\} \), which we denote the same way, such that \( \lim_{\varepsilon \to 0} P_\varepsilon = \alpha S \) and \( \lim_{\varepsilon \to 0} Q_\varepsilon = \alpha T \) uniformly on \( I \). Moreover,
\[
[(fQ_\varepsilon - P_\varepsilon)^i] [z_{j_0}^1, z_{j_1}^1, \cdots, z_{js}^1] = 0
\]
for \( 0 \leq s \leq q - 1, \, 1 \leq j \leq k, \, 0 < \varepsilon < 1 \).

**Proof.** By Lemma 4, there is a subsequence of \( \{(P_\varepsilon, Q_\varepsilon)\} \), which we denote the same way, and \( P_0 \in \Pi^n, \, Q_0 \in \Pi^m \) such that \( P_\varepsilon \to P_0 \) and \( Q_\varepsilon \to Q_0 \) uniformly on \( I \) as \( \varepsilon \to 0 \). Moreover,
\[
P_0 T = Q_0 S. \tag{13}
\]

For \( 1 \leq j \leq k \), let \( K_j = \{i: 0 \leq i \leq m \text{ and } Q_0^{(i)}(x_j) \neq 0\} \). Since \( \|Q_0\|_\infty = 1, \, 0 < \varepsilon \leq 1 \), we have \( \|Q_0\|_\infty = 1 \), and thus \( K_j \neq \emptyset \). Set \( k_j = \min(K_j) \). By hypothesis, \( T(x_j) \neq 0 \), so (13) implies that there are \( P_1 \in \Pi^n \) and \( Q_1 \in \Pi^m \) satisfying
\[
P_0(x) = \prod_{c=1}^{k} (x-x_c)^{k_c} P_1(x), \quad Q_0(x) = \prod_{c=1}^{k} (x-x_c)^{k_c} Q_1(x) \quad \text{and} \quad Q_1(x_c) \neq 0, \tag{14}
\]
x \in I, \, 1 \leq c \leq k. Using (13) again, we obtain
\[
P_1 T = Q_1 S.
\]

Since \( \frac{S}{\Pi} \) is normal, either \( \deg T = m \) or \( \deg S = n \). If \( \deg T = m \), then \( \deg P_1 \leq \deg S \). But \( \frac{S}{\Pi} \) is irreducible, so \( \deg P_1 = \deg S \) and \( \deg Q_1 = \deg T \). Therefore, there exists \( \alpha \neq 0 \) such that \( P_1 = \alpha S \) and \( Q_1 = \alpha T \). Now, according to (14), we have \( k_c = 0, \, 1 \leq c \leq k \), and consequently, \( P_0 = \alpha S \) and \( Q_0 = \alpha T \). But \( \|Q_0\|_\infty = \|T\|_\infty = 1 \), so \( |\alpha| = 1 \) and
\[
P_0 = \alpha S \quad \text{and} \quad Q_0 = \alpha T. \tag{15}
\]
If \( \deg S = n \), in the same manner we can see (15).

Finally, let \( 0 \leq s \leq q - 1 \), \( 1 \leq j \leq k \) and \( \epsilon > 0 \). From Lemma 3,

\[
e^q = (fQ_\epsilon - P_\epsilon)(z_{js}^q) = (fQ_\epsilon - P_\epsilon)(\epsilon(z_{js}^q - x_j) + x_j) = (fQ_\epsilon - P_\epsilon)'(z_{js}^q). \quad (16)
\]

So, (12) immediately follows. \( \square \)

**Lemma 6.** Suppose that \( \frac{S}{T} \) is normal, and let \( \{(P_\epsilon, Q_\epsilon)\} \subset \mathcal{H}_m^n \) be the subsequence of Lemma 4. Then for each \( \epsilon > 0 \) and \( x \in B_j, \ 1 \leq j \leq k, \ x \neq z_{js}^1, \ 0 \leq s \leq q - 1 \), there exists \( \xi_\epsilon(x) \in (x_j - \epsilon, x_j + \epsilon) \) satisfying

\[
\frac{1}{e^q}(fQ_\epsilon - P_\epsilon)'(x) = \frac{1}{q!}(fQ_\epsilon - P_\epsilon)^{(q)}(\xi_\epsilon(x)) \prod_{l=0}^{q-1} (x - z_{jl}^1). \quad (17)
\]

**Proof.** Let \( \epsilon > 0 \). It is well known that the \((q - 1)\)th Lagrange interpolation polynomial for \((fQ_\epsilon - P_\epsilon)'\) with respect to \(z_{j0}^1, z_{j1}^1, \cdots , z_{j(q-1)}^1\) can be expressed as

\[
W_\epsilon(x) = \sum_{s=0}^{q-1} [(fQ_\epsilon - P_\epsilon)'][z_{j0}^1, z_{j1}^1, \cdots , z_{j(q-1)}^1, x] \prod_{l=0}^{q-1} (x - z_{jl}^1).
\]

By Lemma 5, we have \( W_\epsilon = 0 \). Let \( x \in B_j, \ 1 \leq j \leq k, \ x \neq z_{js}^1, \ 0 \leq s \leq q - 1 \). From \([8, Th. 3, p. 309]\), we get

\[
(fQ_\epsilon - P_\epsilon)'(x) = [(fQ_\epsilon - P_\epsilon)'][z_{j0}^1, z_{j1}^1, \cdots , z_{j(q-1)}^1, x] \prod_{l=0}^{q-1} (x - z_{jl}^1). \quad (18)
\]

Since \( f \in \mathcal{C}^q(I) \), \([8, Th. 4, p. 310]\) implies that there exists \( \zeta_\epsilon(x) \in (x_j - 1, x_j + 1) \) such that

\[
[(fQ_\epsilon - P_\epsilon)'][z_{j0}^1, z_{j1}^1, \cdots , z_{j(q-1)}^1, x] = \frac{1}{q!}((fQ_\epsilon - P_\epsilon)'(\zeta_\epsilon(x)))
\]

\[
= \frac{\epsilon^q}{q!}(fQ_\epsilon - P_\epsilon)^{(q)}(\epsilon(\zeta_\epsilon(x) - x_j) + x_j).
\]

So, according to (18), we have (17) with \( \xi_\epsilon(x) = \epsilon(\zeta_\epsilon(x) - x_j) + x_j \). \( \square \)

**Theorem 7.** Let \( q > 0, \ f \in \mathcal{C}^q(I) \) and let \( (S, T) \in \mathcal{W}_m^n(f, X) \) be such that \( \frac{S}{T} \) is normal. Then there exists a sequence \( \{(P_\epsilon, Q_\epsilon)\} \subset \mathcal{H}_m^n \) such that

\[
\lim_{\epsilon \to 0} \frac{1}{e^q} \|(fQ_\epsilon - P_\epsilon)'\|_{B_j} = \frac{1}{q!} \|(fT - S)^{(q)}(x_j)\|_{K_{pq}}, \quad 1 \leq j \leq k. \quad (19)
\]
Since (20) implies that \( \lim_{\epsilon \to 0} P_\epsilon = \alpha S \) uniformly on \( I \). From Lemma 6, for each \( \epsilon \) and \( x \in B_j, 1 \leq j \leq k, x \neq z_{js}^1, 0 \leq s \leq q - 1 \), there is \( \xi_\epsilon(x) \in (x_j - \epsilon, x_j + \epsilon) \) satisfying

\[
\frac{1}{\epsilon^q}(f Q_\epsilon - P_\epsilon)^\prime(x) = \frac{1}{q!}(f Q_\epsilon - P_\epsilon)^{(q)}(\xi_\epsilon(x)) \prod_{i=0}^{q-1} (x - z_{js}^1).
\]  

(21)

Since (20) implies that \( \lim_{\epsilon \to 0} (f Q_\epsilon - P_\epsilon)^{(q)}(\xi_\epsilon(x)) = \alpha(f T - S)^{(q)}(x_j) \), we have

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon^q}(f Q_\epsilon - P_\epsilon)^\prime(x) = \frac{\alpha}{q!}(f T - S)^{(q)}(x_j) \prod_{i=0}^{q-1} (x - z_{js}^1),
\]  

(22)

\( x \in B_j, 1 \leq j \leq k, x \neq z_{js}^1, 0 \leq s \leq q - 1 \).

On the other hand, by (20) we see that \( \{P_\epsilon\} \) and \( \{Q_\epsilon\} \) are uniformly bounded on \( I \) as \( \epsilon \to 0 \). Hence, there exist \( M > 0 \) and \( \epsilon_1 > 0 \) such that \( |(f Q_\epsilon - P_\epsilon)^{(q)}(x)| \leq q! M, x \in I, 0 < \epsilon < \epsilon_1. \) So, from (21) we deduce that

\[
\left| \frac{1}{\epsilon^q}(f Q_\epsilon - P_\epsilon)^\prime(x) \right| \leq 2^q M, \quad x \in B_j, \quad x \neq z_{js}^1, \quad 0 < \epsilon < \epsilon_1.
\]

According to (22) and the Lebesgue Convergence Theorem, we get

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon^q} \|(f Q_\epsilon - P_\epsilon)^\prime\|_{B_j} = \frac{1}{q!}\|(f T - S)^{(q)}(x_j)\| \prod_{i=0}^{q-1} (x - z_{js}^1) \|_{B_j}.
\]

Now, substituting \( x - x_j \) by \( t \) into the above equality gives

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon^q} \|(f Q_\epsilon - P_\epsilon)^\prime\|_{B_j} = \frac{1}{q!}\|(f T - S)^{(q)}(x_j)\| t^q - M_{pq}(t) \|_p = \frac{1}{q!}\|(f T - S)^{(q)}(x_j)\| K_{pq}.
\]

\[\square\]

**Theorem 8.** Let \( f \in C^q(I) \) and let \( \{(S_{\epsilon}, T_{\epsilon})\} \subset H^m_\alpha \) be a net of best approximant pairs of \( f \) from \( H^m_\alpha \) with respect to \( \| \cdot \|_\epsilon \). Then \( \{S_{\epsilon}\} \) and \( \{T_{\epsilon}\} \) are uniformly bounded on compact sets as \( \epsilon \to 0 \).
PROOF. Since $\|T_\epsilon\|_\infty = 1$, $0 < \epsilon \leq 1$, the net $\{T_\epsilon\}$ is uniformly bounded on compact sets.

Let $(S, T) \in W^m_n(f, X)$. Then for each $1 \leq j \leq k$,

$$
\frac{\|(S, T - T_\epsilon S)\|_{B_j}}{\epsilon^q} = \frac{1}{\epsilon^q} \|(-T(fT_\epsilon - S_\epsilon) + T_\epsilon(fT - S))\|_{B_j} \\
\leq \frac{(2k)^{1/p}}{\epsilon^q} \|T(fT_\epsilon - S_\epsilon) + T_\epsilon(fT - S)\|_\epsilon \\
\leq \frac{(2k)^{1/p}}{\epsilon^q} (\|T(fT_\epsilon - S_\epsilon)\|_\epsilon + \|T_\epsilon(fT - S)\|_\epsilon) \\
\leq \frac{(2k)^{1/p}}{\epsilon^q} (\|fT_\epsilon - S_\epsilon\|_\epsilon + \|fT - S\|_\epsilon) \\
\leq \frac{2(2k)^{1/p}}{\epsilon^q} \|fT - S\|_\epsilon.
$$

If $q = 0$, then

$$
\|(S, T - T_\epsilon S)(x_j)\| = O(1) \quad \text{as} \quad \epsilon \to 0, \quad 1 \leq j \leq k. \quad (24)
$$

In otherwise, as $(fT - S)^{(i)}(x_j) = 0$, $0 \leq i \leq q - 1$, $1 \leq j \leq k$, expanding $(fT - S)^{(i)}$ by its Taylor polynomial at $x_j$ up to order $q - 1$, it follows that for each $x \in B_j$, there exists $\xi(x) \in [x_j - \epsilon, x_j + \epsilon]$ such that

$$(fT - S)^{(i)}(x) = \frac{\partial^i}{\partial x^i} (fT - S)^{(q)}(\xi(x))(x - x_j)^q.$$ 

So, $\|(fT - S)^{(i)}\|_{B_j} = O(\epsilon^q)$ as $\epsilon \to 0$, and consequently

$$\|fT - S\|_\epsilon = O(\epsilon^q) \quad \text{as} \quad \epsilon \to 0. \quad (25)$$

Therefore, by (23), we get $\|(S, T - T_\epsilon S)^{(i)}\|_{B_j} = O(\epsilon^q)$ as $\epsilon \to 0$, $1 \leq j \leq k$.

Since $(S, T - T_\epsilon S)^{(i)} \in \Pi^{n+m}$ on $B_j$, according to Lemma 2.2 in [5], we have

$$\|(S, T - T_\epsilon S)^{(i)}(x_j)\| = O(\epsilon^{q-i}) \quad \text{as} \quad \epsilon \to 0, \quad (26)$$

$1 \leq j \leq k$, $0 \leq i \leq q$. Since $n + m + 1 < k(q + 1)$, from (24) and (26) we show that $\{S, T - T_\epsilon S\} \subset \Pi^{n+m}$ is uniformly bounded on $I$ as $\epsilon \to 0$; that is, there exist $M > 0$ and $\epsilon_1 > 0$ such that

$$\|(S, T - T_\epsilon S)(x)\| \leq M, \quad x \in I, \quad 0 < \epsilon < \epsilon_1.$$

As $|T_\epsilon S(x)| \leq \|S\|_\infty$, $x \in I$, $0 < \epsilon < \epsilon_1$, we have $|S_\epsilon|_T = \max_{x \in I} |(S, T)(x)| \leq \|S\|_\infty + M, \quad 0 < \epsilon < \epsilon_1$. Finally, by the equivalence of the norms in $\Pi^n$, we conclude that $\{S_\epsilon\}$ is uniformly bounded on compact sets as $\epsilon \to 0$. \qed
**Theorem 9.** Let \( f \in C^q(I) \) and let \( \{ (S_\epsilon, T_\epsilon) \} \) be a net of best approximant pairs of \( f \) from \( \mathcal{H}_{nm} \) with respect to \( \| \cdot \| \_p \). Suppose that there exists a best Padé approximant pair of \( f \) at \( X \), say \( (S, T) \), such that \( \frac{S}{T} \) is normal. Then any cluster point of \( \{ (S_\epsilon, T_\epsilon) \} \) as \( \epsilon \to 0 \) is a best Padé approximant pair of \( f \) at \( X \).

**Proof.** According to Theorem 8, it follows that the set of cluster points of the net \( \{ (S_\epsilon, T_\epsilon) \} \) as \( \epsilon \to 0 \) is nonempty. Now, it is sufficient to prove that if \( (S^*, T^*) \) is the limit point of \( \{ (S_{\epsilon_l}, T_{\epsilon_l}) \} \) as \( \epsilon_l \to 0 \), then \( (S^*, T^*) \) is a best Padé approximant pair of \( f \) at \( X \). If \( q = 0 \), then the result is obvious, because

\[
\sum_{j=1}^{k} |(fT_s - S_s)(x_j)|^p = \lim_{\epsilon_l \to 0} k \| fT_{\epsilon_l} - S_{\epsilon_l} \|_{\epsilon_l}^p \\
\leq \lim_{\epsilon_l \to 0} k \| fT - S \|_{\epsilon_l}^p \\
= \sum_{j=1}^{k} |(fT - S)(x_j)|^p.
\]

Now assume \( q > 0 \). Let \( 1 \leq j \leq k \), \( 0 \leq i \leq q - 1 \). As in the proof of Theorem 8, we have

\[
| (S_{\epsilon_l}T - T_{\epsilon_l}S)^{(i)}(x_j) | = O(\epsilon_l^{q-i}) \quad \text{as} \quad \epsilon_l \to 0. \tag{27}
\]

Therefore, \( (S,T) - T_sS)^{(i)}(x_j) = 0 \). Since

\[ S_sT - T_sS = -T(fT_s - S_s) + T_s(fT - S) \]

and \( (S,T) \in W_{nm}^{(f,X)} \), using the Leibniz rule we get \( (T(fT_s - S_s))^{(i)}(x_j) = 0 \), and thus

\[
(fT_s - S_s)^{(i)}(x_j) = 0, \tag{28}
\]

because \( T(x_j) \neq 0 \). As \( i \) and \( j \) are arbitrary, \( (S_\epsilon, T_\epsilon) \) is a Padé approximant pair of \( f \) at \( X \), and so \( (S_\epsilon, T_\epsilon) \in W_{nm}^{(f,X)} \) since \( \| T_\epsilon \|_\infty = 1 \).

Expanding \( (S_{\epsilon_l}T - T_{\epsilon_l}S)^{\epsilon_l} \) and \( T_{\epsilon_l}(fT - S)^{\epsilon_l} \) by their Taylor polynomials at \( x_j \) up to order \( q - 1 \), it follows that for each \( x \in B_j \), there exist
\( \xi_\epsilon(x), \eta_\epsilon(x) \in [x_j - \epsilon, x_j + \epsilon] \) such that

\[
\begin{align*}
T(x) \frac{1}{\epsilon} (f_{T\epsilon} - S\epsilon)^{\epsilon} (x) &= \frac{1}{\epsilon} \left( T(f_{T\epsilon} - S\epsilon)^{\epsilon} (x) \right) \\
&= \frac{1}{\epsilon} \left( - (S\epsilon T - T\epsilon S)^{\epsilon} (x) + (T\epsilon (fT - S)^{\epsilon} (x) \right) \\
&= - \sum_{i=0}^{q-1} \epsilon_{\epsilon} (S\epsilon T - T\epsilon S)^{(i)} (x_j) (x - x_j)^i \\
&- \frac{(S\epsilon T - T\epsilon S)^{(q)} (\xi_\epsilon(x))}{q!} (x - x_j)^q + \frac{1}{q!} \sum_{s=0}^{q} \left( \frac{q}{s} \right) (fT - S)^{(s)} (\eta_\epsilon(x)) T^{(q-s)}(\xi_\epsilon(x)) (x - x_j)^q. 
\end{align*}
\]

As \( T(x_j) \neq 0 \), from (27) there exist a subsequence of \( \{\epsilon\} \), which we denote the same way, and \( a_{ij} \in \mathbb{R}, 0 \leq i \leq q - 1, 1 \leq j \leq k \), such that

\[
\lim_{\epsilon \to 0} \epsilon_{\epsilon} (S\epsilon T - T\epsilon S)^{(i)} (x_j) = T(x_j) a_{ij}. 
\]

According to Theorem 8, we have

\[
\lim_{\epsilon \to 0} \frac{(S\epsilon T - T\epsilon S)^{(q)} (\xi_\epsilon(x))}{q!} = \frac{(S*T - T*S)^{(q)}(x_j)}{q!), 
\]

so (28)–(31) imply that

\[
\begin{align*}
\lim_{\epsilon \to 0} \frac{1}{\epsilon} (f_{T\epsilon} - S\epsilon)^{\epsilon} (x) \\
&= \frac{1}{q! T(x_j)} \left( (S*T - T*S)^{(q)}(x_j) + (fT - S)^{(q)}(x_j) \right) (x - x_j)^q \\
&- \sum_{i=0}^{q-1} a_{ij} (x - x_j)^i \\
&= \frac{1}{q!} (fT - S)^{(q)}(x_j) (x - x_j)^q - \sum_{i=0}^{q-1} a_{ij} (x - x_j)^i 
\end{align*}
\]

uniformly on \( B_j \). Therefore, substituting \( x - x_j \) by \( t \) into the following ineqaul-
ity gives
\[
\lim_{\epsilon_l \to 0} \frac{1}{\epsilon_l} \| (fT_{\epsilon_l} - S_{\epsilon_l})^t \|_{B_j} = \left\| \frac{1}{q!} (fT_J - S_J) (q) (x) - \sum_{i=0}^{q-1} a_{ij} (x - x_j)^i \right\|_{B_j}
\geq \frac{1}{q!} \| (fT_J - S_J) (q) (x) \|_{t^q - M_{pq}(t)} p
= \frac{1}{q!} | (fT_J - S_J) (q) (x) | K_{pq} ,
\]
(32)

1 \leq j \leq k. Since \( S \) is normal, from Theorem 7, there exists a subsequence of \( \{ \epsilon_l \} \), which we denote the same way again, such that \( \{ (P_{\epsilon_l}, Q_{\epsilon_l}) \} \subset \mathcal{H}^n_m \) and
\[
\lim_{\epsilon_l \to 0} \frac{1}{\epsilon_l} \| (fQ_{\epsilon_l} - P_{\epsilon_l})^t \|_{B_j} = \frac{1}{q!} | (fT - S) (q) (x) | K_{pq},
\]
(33)

1 \leq j \leq k. But \( \{ (S_{\epsilon_l}, T_{\epsilon_l}) \} \) is a net of best approximant pairs of \( f \) from \( \mathcal{H}^n_m \), so (32) and (33) imply
\[
\sum_{j=1}^{k} | (fT_J - S_J) (q) (x) | p \leq \sum_{j=1}^{k} | (fT - S) (q) (x) | p .
\]

Finally, by (28), \( (S, T) \) is a best \( \text{Padé} \) approximant pair.

We say that the best \( \text{Padé} \) approximant pair of \( f \) at \( X \) is unique if, whenever \( (P, Q), (U, V) \in \mathcal{W}^n_m(f, X) \) satisfy (3), then \( (P, Q) \equiv (U, V) \).

The next corollary immediately follows.

**Corollary 10.** Let \( f \in C^q(I) \), \( q > 0 \), and suppose that there exists a unique best \( \text{Padé} \) approximant pair of \( f \) at \( X \), say \( (S, T) \), such that \( \frac{S}{T} \) is normal. Then \( \frac{S}{T} \) is a \( \text{Padé} \) approximant of \( f \) at \( X \). In addition, if \( \{ (S_\epsilon, T_\epsilon) \} \) is a net of best approximant pairs of \( f \) from \( \mathcal{H}^n_m \) with respect to \( \| \cdot \|_\epsilon \), then \( \frac{S_\epsilon}{T_\epsilon} \) converges to \( \frac{S}{T} \) uniformly on some neighborhood of \( X \) as \( \epsilon \to 0 \).

**References**


