ON A COUNTEREXAMPLE RELATED TO WEIGHTED WEAK TYPE ESTIMATES FOR SINGULAR INTEGRALS

MARCELA CALDARELLI, ANDREI K. LERNER, AND SHELDY OMBROSI

ABSTRACT. We show that the Hilbert transform does not map $L^1(M_\Phi w)$ to $L^{1,\infty}(w)$ for every Young function Φ growing more slowly than $t\log\log(\mathrm{e}^{\mathrm{e}}+t)$. Our proof is based on a construction of M.C. Reguera and C. Thiele.

1. INTRODUCTION

Let H be the Hilbert transform. One of open questions in the one-weighted theory of singular integrals is about the optimal Young function Φ for which the weak type inequality

$$w\{x\in\mathbb{R}:|Hf(x)|>\lambda\}\le\frac{c}{\lambda}\int_{\mathbb{R}}|f|M_\Phi w\,dx\quad(\lambda>0) \tag{1.1}$$

holds for every weight (i.e., non-negative measurable function) w and any $f\in L^1(M_\Phi w)$, where M_Φ is the Orlicz maximal operator defined by

$$M_\Phi f(x)=\sup_{I\ni x}\inf\left\{\lambda>0:\frac{1}{|I|}\int_I\Phi\left(\frac{|f(y)|}{\lambda}\right)dy\le 1\right\}.$$

If $\Phi(t)=t$, then $M_\Phi=M$ is the standard Hardy-Littlewood maximal operator. If $\Phi(t)=t^r, r>1$, denote $M_\Phi f=M_rf$.

C. Fefferman and E.M. Stein [6] showed that if H is replaced by the maximal operator M, then the corresponding inequality holds with $\Phi(t)=t$. Next, A. Córdoba and C. Fefferman [1] proved (1.1) with $\Phi(t)=t^r, r>1$. This result was improved by C. Pérez [8] who showed that (1.1) holds with $\Phi(t)=t\log^{\varepsilon}(\mathrm{e}+\mathrm{t}),\varepsilon>0$ (see also [7] for a different proof of this result).

2010 *Mathematics Subject Classification.* 42B20,42B25.

Key words and phrases. Hilbert transform, weights, weak-type estimates.

The second author was supported by the Israel Science Foundation (grant No. 953/13).

Very recently, C. Domingo-Salazar, M.T. Lacey and G. Rey [5] obtained a further improvement; their result states that (1.1) holds whenever Φ satisfies
$$\int_1^\infty \frac{\Phi^{-1}(t)}{t^2\log(\mathrm{e}+\mathrm{t})}dt<\infty.$$
For example, one can take $\Phi(t)=t\log\log^\alpha(\mathrm{e}^\mathrm{e}+t), \alpha>1$ or
$$\Phi(t)=t\log\log(\mathrm{e}^\mathrm{e}+t)\log\log\log^\alpha(\mathrm{e}^{\mathrm{e}^\mathrm{e}}+t)\quad(\alpha>1)$$
etc.

A question whether (1.1) is true with $\Phi(t)=t$ has become known as the Muckenhoupt-Wheeden conjecture. This conjecture was disproved by M.C. Reguera and C. Thiele [10] (see also [9] and [2] for dyadic and multidimensional versions of this result).

Denote $\Psi(t)=t\log\log(\mathrm{e}^\mathrm{e}+t)$. It was conjectured in [7] that (1.1) holds with $\Phi=\Psi$. The above mentioned result in [5] establishes (1.1) for essentially every Φ growing faster than Ψ.

The main result of this note is the observation that the Reguera-Thiele example [10] actually shows that (1.1) does not hold for every Φ growing more slowly than Ψ.

Theorem 1.1. *Let Φ be a Young function such that*
$$\lim_{t\to\infty}\frac{\Phi(t)}{t\log\log(\mathrm{e}^\mathrm{e}+t)}=0.$$
Then for every $c>0$, there exist f,w and $\lambda>0$ such that
$$w\{x\in\mathbb{R}:|Hf(x)|>\lambda\}>\frac{c}{\lambda}\int_{\mathbb{R}}|f|M_\Phi w\,dx.$$

This theorem along with the main result in [5] emphasizes that the case of $\Phi=\Psi$ is critical for (1.1). However, the question whether (1.1) holds with $\Phi=\Psi$ remains open.

We mention briefly the main ideas of the Reguera-Thiele example [10] and, in parallel, our novel points. First, it was shown in [10] that given $k\in\mathbb{N}$ sufficiently large, there is a weight w_k supported on $[0,1]$ satisfying $Hw_k\ge ckw_k$ and $Mw_k\le cw_k$ on some subset $E\subset[0,1]$. In Section 2, we show that the latter "A_1 property" can be slightly improved until $M_rw_k\le cw_k$ with $r>1$ depending on k. The second ingredient in [10] was the extrapolation argument of D. Cruz-Uribe and C. Pérez [3]. This argument says that assuming (1.1) with Mw on the right-hand side, one can deduce a certain weighted L^2 inequality for H. It is not clear how to extrapolate in a similar way starting with a general Orlicz maximal function M_Φ in (1.1). In Section 3, we obtain a substitute of the argument in [3] for $M_rw, r>1$, instead of Mw.

2. The Reguera-Thiele construction

We describe below the main parts of the example constructed by M.C. Reguera and C. Thiele [10].

An interval I of the form $[3^j n, 3^j(n+1)), j, n \in \mathbb{Z}$, is called a triadic interval.

Fix $k \in \mathbb{N}$ large enough. Given a triadic interval $I \subset [0,1)$, denote $I^\Delta = \frac{1}{3}I$, namely, I^Δ is the interval with the same center as I and of one third of its length; further, denote by $P(I)$ a triadic interval adjacent to I^Δ and such that $|P(I)| = \frac{1}{3^k}|I|$. Observe that $P(I)$ can be situated either on the left or on the right of I^Δ, and we will return to this point a bit later.

Set now $J^1 = [0,1)$ and $I_{1,1} = P(J^1)$. Next, we subdivide $(J^1)^\Delta$ into 3^{k-1} triadic intervals of equal length, and denote them by $J^2_m, m = 1, 2, \dots, 3^{k-1}$. Set correspondingly $I_{2,m} = P(J^2_m)$. Notice that $|J^2_m| = \frac{1}{3^k}$ and $|I_{2,m}| = \frac{1}{3^{2k}}$ for $m = 1, 2, ...3^{k-1}$. Observe also that the intervals $I_{1,1}$ and $I_{2,m}$ are pairwise disjoint.

Proceeding by induction, at l-th stage, we subdivide each interval $(J^{l-1}_m)^\Delta$ into 3^{k-1} triadic intervals of equal length, and denote all obtained intervals by $J^l_m, m = 1, 2, \dots, 3^{(k-1)(l-1)}$. Set $I_{l,m} = P(J^l_m)$. Then $|J^l_m| = \frac{1}{3^{(l-1)k}}$ and $|I_{l,m}| = \frac{1}{3^{lk}}$, and the intervals $\{I_{l,m}\}$ are pairwise disjoint.

Denote by $\mathcal{I}_l$ and $\mathcal{J}_l$ the families of all intervals $\{I_{l,m}\}$ and $\{J^l_m\}$, respectively, and set $\Omega_l = \cup_{I \in \mathcal{I}_l} I$. Define the weight w_k such that $w_k([0,1]) = 1$, w_k is a constant on Ω_l, and for every $I \in \mathcal{I}_l$ and $J \in \mathcal{J}_{l+1}$, $w_k(I) = w_k(J)$ (we use the standard notation $w_k(E) = \int_E w_k$).

It was proved in [10] that one can specify the situation of the intervals $\{I_{l,m}\}$ such that if $k > 3000$ and $x \in \cup_{l,m} I^\Delta_{l,m}$, then

$$|Hw_k(x)| \geq (k/3)w_k(x); \tag{2.1}$$

moreover,

$$Mw_k(x) \leq 7w_k(x) \quad (x \in \cup_{l,m} I^\Delta_{l,m}),$$

irrespective of the precise configuration of $\{I_{l,m}\}$.

We will show that the latter estimate can be improved by means of replacing Mw_k on the left-hand side by a larger operator $M_r w_k$ with $r > 1$ depending on k. In order to do that, we need a more constructive description of w_k.

Lemma 2.1. *We have,*

$$w_k(x) = \sum_{l=1}^{\infty} \left(\frac{3^k}{3^{k-1}+1} \right)^l \chi_{\Omega_l}(x). \tag{2.2}$$

Proof. Assume that $w_k = \alpha_l$ on Ω_l. Let $J \in \mathcal{J}_l$ and take $I \in \mathcal{I}_l$ such that $I \subset J$. Then

$$w_k(J) = w_k(I) + w_k(J^\Delta) = w_k(I) + \sum_{J' \in \mathcal{J}_{l+1} : J' \subset J^\Delta} w_k(J'). \tag{2.3}$$

Let $I' \in \mathcal{I}_{l-1}$. Then

$$w_k(J) = w_k(I') = \alpha_{l-1}|I'| = \alpha_{l-1}|J|.$$

Similarly, $w_k(J') = \alpha_l |J'|$, and also $w_k(I) = \alpha_l |I| = \alpha_l \frac{|J|}{3^k}$. Hence, (2.3) implies

$$\alpha_{l-1}|J| = \alpha_l \frac{|J|}{3^k} + \alpha_l \sum_{J' \in \mathcal{J}_{l+1} : J' \subset J^\Delta} |J'| = \alpha_l \frac{|J|}{3^k} + \alpha_l \frac{|J|}{3}.$$

From this, $\alpha_l = \frac{3^k}{3^{k-1}+1}\alpha_{l-1}$, and therefore $\alpha_l = \left(\frac{3^k}{3^{k-1}+1}\right)^l \gamma$ for some $\gamma > 0$.

From the condition $w_k([0,1]) = 1$, we obtain

$$\begin{aligned} 1 &= w_k([0,1]) = \gamma \sum_{l=1}^{\infty} \left(\frac{3^k}{3^{k-1}+1}\right)^l |\Omega_l| \\ &= \gamma \sum_{l=1}^{\infty} \left(\frac{3^k}{3^{k-1}+1}\right)^l \frac{3^{(k-1)(l-1)}}{3^{kl}} = \gamma \frac{1}{3^{k-1}} \sum_{l=1}^{\infty} \left(\frac{3^{k-1}}{3^{k-1}+1}\right)^l = \gamma, \end{aligned}$$

and therefore the lemma is proved. ∎

Lemma 2.2. *Let $r = 1 + \frac{1}{3^{k+1}}$. Then for every $I \in \mathcal{I}_l, l \in \mathbb{N}$, and for all $x \in I^\Delta$,*

$$M_r w_k(x) \le 21 w_k(x).$$

Proof. Let $I \in \mathcal{I}_l$, and let $x \in I^\Delta$. Take an arbitrary interval R containing x. If $R \subset I$, then

$$\left(\frac{1}{|R|}\int_R w_k^r(y)dy\right)^{1/r} = \left(\frac{3^k}{3^{k-1}+1}\right)^l = w_k(x).$$

Assume that $R \not\subset I$. Then $|R| \ge |I|/3$. Denote by $\mathcal{F}$ the family of all triadic intervals $I' \subset [0,1)$ such that $|I'| = |I|$ and $I' \cap R \ne \emptyset$. There are at most two intervals $I' \in \mathcal{F}$ not containing in R, and therefore,

$$\sum_{I' \in \mathcal{F}} |I'| \le |R| + \sum_{I' \in \mathcal{F} : I' \not\subset R} |I'| \le |R| + 2|I| \le 7|R|. \tag{2.4}$$

We claim that if $r = 1 + \frac{1}{3^{k+1}}$, then for every $I' \in \mathcal{F}$,

$$\left(\frac{1}{|I'|}\int_{I'} w_k^r(y)dy\right)^{1/r} \le 3w_k(x). \tag{2.5}$$

This property would conclude the proof since then, by (2.4),

$$\frac{1}{|R|}\int_R w_k^r(y)dy \le \sum_{I'\in\mathcal{F}} \frac{|I'|}{|R|}\frac{1}{|I'|}\int_{I'} w_k^r(y)dy \le 7(3w_k(x))^r.$$

To show (2.5), one can assume that I' has a non-empty intersection with the support of w_k. If $I' \neq J$ for some $J \in \mathcal{J}_{l+1}$, then $I' \subset L$, where $L \in \mathcal{I}_\nu, \nu \le l$, and hence

$$\left(\frac{1}{|I'|}\int_{I'} w_k^r(y)dy\right)^{1/r} = \left(\frac{3^k}{3^{k-1}+1}\right)^\nu \le w_k(x).$$

It remains to consider the case when $I' = J$ for some $J \in \mathcal{J}_{l+1}$. Using that for every $j \ge l+1$, $J \in \mathcal{J}_{l+1}$ contains $3^{(k-1)(j-l-1)}$ intervals $I \in \mathcal{I}_j$, we obtain

$$\begin{aligned}\frac{1}{|I'|}\int_{I'} w_k^r(y)dy &= 3^{lk}\sum_{j=l+1}^\infty \sum_{I\in\mathcal{I}_j : I\subset I'} \int_I w_k^r(y)dy\\ &= \sum_{j=l+1}^\infty 3^{(k-1)(j-l-1)}3^{(l-j)k}\left(\frac{3^k}{3^{k-1}+1}\right)^{jr}\\ &= \frac{1}{3^{k-1}}\sum_{j=1}^\infty 3^{-j}\left(\frac{3^k}{3^{k-1}+1}\right)^{(j+l)r}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{|I'|}\int_{I'} w_k^r(y)dy &= \frac{1}{3^{k-1}}\left(\sum_{j=1}^\infty 3^{-j}\left(\frac{3^k}{3^{k-1}+1}\right)^{jr}\right) w_k(x)^r\\ &\le \frac{1}{3^{k-1}}\frac{3}{3-(3^k/(3^{k-1}+1))^r} w_k(x)^r,\end{aligned}$$

whenever $\left(\frac{3^k}{3^{k-1}+1}\right)^r < 3$.

If $r = 1 + \frac{1}{3^{k+1}}$, then

$$\begin{aligned}\left(\frac{3^k}{3^{k-1}+1}\right)^{1+\frac{1}{3^{k+1}}} \le 3^{\frac{1}{3^{k+1}}}\frac{3^k}{3^{k-1}+1} &\le \left(1+\frac{1}{3^k}\right)\frac{3^k}{3^{k-1}+1}\\ &= 3 - \frac{2}{3^{k-1}+1}.\end{aligned}$$

Hence,

$$\frac{1}{|I'|}\int_{I'} w_k^r(y)dy \le \frac{3}{2}\frac{3^{k-1}+1}{3^{k-1}} w_k(x)^r \le 3w_k(x)^r,$$

which completes the proof. □

3. Extrapolation

Here we follow the extrapolation argument of D. Cruz-Uribe and C. Pérez [3], with some modifications.

Lemma 3.1. *Assume that for every weight w and for all $f \in L^1(M_r w)$,*

$$\|Hf\|_{L^{1,\infty}(w)} \le A_r \|f\|_{L^1(M_r(w))} \quad (1<r<2).$$

Let $\alpha_r = \frac{r}{2-r}$. There is $c>0$ such that for any weight w supported in $[0,1]$ one has

$$\int_0^1 \left(\frac{|Hw|}{(M_{\alpha_r} w)^{\alpha_r/r}}\right)^2 w^{\alpha_r} dx \le cA_r^2 \int_0^1 w\,dx \quad (1<r<2).$$

Proof. Denote $\beta_r = \frac{r(r-1)}{2-r}$. The numbers α_r and β_r are chosen in such a way in order to satisfy $\alpha_r - \beta_r = r$ and $\alpha_r - \frac{2\beta_r}{r} = 1$.

Let $g \ge 0$. Since

$$\frac{1}{|I|}\int_I (gw)^r = \left(\frac{1}{w^{\alpha_r}(I)}\int_I (g^r/w^{\beta_r})w^{\alpha_r}\right)\frac{w^{\alpha_r}(I)}{|I|},$$

we get

$$M_r(gw)(x) \le 2\Big(M^c_{w^{\alpha_r}}(g^r/w^{\beta_r})(x) M_{\alpha_r}(w)(x)^{\alpha_r}\Big)^{1/r}, \tag{3.1}$$

where M^c_v means the centered weighted maximal operator with respect to a weight v.

Using the initial assumption on H along with (3.1), and applying Hölder's inequality along with the boundedness of M^c_v on $L^p(v), p>1$, we obtain

$$\begin{aligned}
&\int_{\{|Hf|>1\}} gw \le A_r \|f\|_{L^1(M_r(gw))} \\
&\le 2A_r \int_{\mathbb{R}} \Big(|f| M_{\alpha_r}(w)^{\frac{\alpha_r}{r}} \frac{1}{w^{\alpha_r/2}}\Big)\Big(M^c_{w^{\alpha_r}}(g^r/w^{\beta_r})^{\frac{1}{r}} w^{\alpha_r/2}\Big)dx \\
&\le 2A_r \|f\|_{L^2\left((M_{\alpha_r}w)^{\frac{2\alpha_r}{r}}/w^{\alpha_r}\right)} \|M^c_{w^{\alpha_r}}(g^r/w^{\beta_r})^{\frac{1}{r}}\|_{L^2(w^{\alpha_r})} \\
&\le cA_r \|f\|_{L^2\left((M_{\alpha_r}w)^{\frac{2\alpha_r}{r}}/w^{\alpha_r}\right)} \|g\|_{L^2(w)}.
\end{aligned}$$

Taking here the supremum over all $g \ge 0$ with $\|g\|_{L^2(w)} = 1$ yields

$$\|Hf\|_{L^{2,\infty}(w)} \le cA_r \|f\|_{L^2\left((M_{\alpha_r}w)^{\frac{2\alpha_r}{r}}/w^{\alpha_r}\right)}.$$

By duality, the latter inequality is equivalent to

$$\|Hf\|_{L^2\left(w^{\alpha_r}/(M_{\alpha_r}w)^{\frac{2\alpha_r}{r}}\right)} \le cA_r \|f/w\|_{L^{2,1}(w)},$$

where $L^{2,1}(w)$ is the weighted Lorentz space. It remains to take here $f = w$ and use that

$$\|\chi_{[0,1]}\|_{L^{2,1}(w)} = \int_0^{w([0,1])} t^{-1/2}dt = 2w([0,1])^{1/2}.$$

□

4. Proof of Theorem 1.1

Our goal is to use the extrapolation Lemma 3.1, assuming (1.1) with a general Orlicz maximal function M_Φ. Hence, we need a relation between M_Φ and M_r with possibly good dependence of the corresponding constant on r when $r \to 1$. Such a relation was recently obtained in [4] (see Lemma 6.2 and inequality (6.4) there). For the reader convenience we include a proof here.

Lemma 4.1. *For all $x \in \mathbb{R}$,*

$$M_\Phi f(x) \le \left(2 \sup_{t \ge \Phi^{-1}(1/2)} \frac{\Phi(t)}{t^r}\right)^{1/r} M_r f(x) \quad (r > 1). \tag{4.1}$$

Proof. For any interval $I \subset \mathbb{R}$,

$$\begin{aligned}\int_I \Phi\left(\frac{|f|}{\lambda}\right) &= \int_{\{x\in I:|f|<\Phi^{-1}(1/2)\lambda\}} \Phi\left(\frac{|f|}{\lambda}\right) + \int_{\{x\in I:|f|\ge\Phi^{-1}(1/2)\lambda\}} \Phi\left(\frac{|f|}{\lambda}\right)\\ &\le \frac{|I|}{2} + c_r \int_I (|f|/\lambda)^r dx,\end{aligned}$$

where $c_r = \sup_{t\ge\Phi^{-1}(1/2)} \frac{\Phi(t)}{t^r}$. Therefore, setting $\lambda_0 = \left(\frac{2c_r}{|I|}\int_I |f|^r\right)^{1/r}$, we obtain $\frac{1}{|I|}\int_I \Phi(|f|/\lambda_0)dx \le 1$, which proves (4.1). □

It follows easily from (4.1) that

$$M_\Phi(x) \le c\left(\sup_{t\ge 1} \frac{\Phi(t)^{1/r}}{t}\right) M_r f(x) \quad (r > 1), \tag{4.2}$$

where c may depend on Φ but it does not depend on r.

Proof of Theorem 1.1. Suppose, by contrary, that (1.1) holds. Then combining (4.2) with Lemma 3.1, we obtain

$$\int_0^1 \left(\frac{|Hw|}{(M_{\alpha_r} w)^{\alpha_r/r}}\right)^2 w^{\alpha_r} dx \le c\left(\sup_{t\ge 1} \frac{\Phi(t)^{1/r}}{t}\right)^2 \int_0^1 w dx \quad (1 < r < 2).$$

Set here $r = r_k = 1 + \frac{1}{2\cdot 3^{k+1}+1}$, and $w = w_k$ as constructed in Section 2. Then $\alpha_{r_k} = \frac{r_k}{2-r_k} = 1 + \frac{1}{3^{k+1}}$. Applying (2.1) along with Lemma 2.2 yields

$$\begin{aligned}\int_0^1 \left(\frac{|Hw_k|}{(M_{\alpha_r}w_k)^{\alpha_r/r}}\right)^2 w_k^{\alpha_r}dx &\geq \frac{k^2}{9\cdot 27^{\frac{2}{2-r_k}}}\int_{\cup_{l\in\mathbb{N}}\cup_{I\in\mathcal{I}_l}I^{\Delta}} w_k\\ &= \frac{k^2}{27^{1+\frac{2}{2-r_k}}}\int_0^1 w_k,\end{aligned}$$

and we obtain

$$k \leq c\sup_{t\geq 1}\frac{\Phi(t)^{1/r_k}}{t}. \tag{4.3}$$

It remains to estimate the right-hand side of (4.3). Write $\Phi(t) = t\log\log(\mathrm{e}^{\mathrm{e}}+t)\phi(t)$, where $\lim_{t\to\infty}\phi(t) = 0$. If $t > \mathrm{e}^{r'}$, then

$$\log\log t = \log(r') + \log\log t^{1/r'} \leq \log(r') + t^{1/r'},$$

and hence

$$\frac{\Phi(t)^{1/r}}{t} = \frac{\big(\log\log(\mathrm{e}^{\mathrm{e}}+t)\phi(t)\big)^{1/r}}{t^{1/r'}} \leq c(\log r')^{1/r}(\sup_{t\geq \mathrm{e}^{r'}}\phi(t))^{1/r}.$$

On the other hand, if $0<\delta<1$, then

$$\begin{aligned}\sup_{1\leq t\leq e^{r'}}\frac{\Phi(t)^{1/r}}{t} &\leq \sup_{1\leq t\leq \mathrm{e}^{\mathrm{e}^{(\log r')^{\delta}}}}\big(\log\log(\mathrm{e}^{\mathrm{e}}+t)\phi(t)\big)^{1/r}\\ &+ \sup_{\mathrm{e}^{\mathrm{e}^{(\log r')^{\delta}}}\leq t\leq e^{r'}}\big(\log\log(\mathrm{e}^{\mathrm{e}}+t)\phi(t)\big)^{1/r}\\ &\leq c\Big((\log r')^{\delta/r} + (\log r')^{1/r}\sup_{t\geq \mathrm{e}^{\mathrm{e}^{(\log r')^{\delta}}}}\phi(t)^{1/r}\Big).\end{aligned}$$

Setting $\beta_k = \sup_{t\geq \mathrm{e}^{\mathrm{e}^{(\log r'_k)^{\delta}}}}\phi(t)^{1/r_k}$ and combining both cases, we obtain

$$\begin{aligned}\sup_{t\geq 1}\frac{\Phi(t)^{1/r_k}}{t} &\leq c((\log r'_k)^{\delta/r_k} + \beta_k(\log r'_k)^{1/r_k})\\ &\leq c(k^{\delta}+\beta_k k).\end{aligned}$$

Since $\beta_k \to 0$ as $k\to\infty$, we arrive to a contradiction with (4.3), and therefore the theorem is proved. □

Remark 4.2. The following inequality is contained implicitly in [7]:

$$\lambda w\{x\in\mathbb{R} : |Hf(x)| > \lambda\} \leq c\log(r')\|f\|_{L^1(M_r w)} \quad (r>1).$$

The proof of Theorem 1.1 shows that $\log(r')$ here is optimal, namely, it cannot be replaced by $\varphi(r')$ for any increasing φ such that $\lim_{t\to\infty}\frac{\varphi(t)}{\log t} = 0$.

References

[1] A. Córdoba and C. Fefferman, *A weighted norm inequality for singular integrals*, Studia Math. **57** (1976), 97-101.

[2] A. Criado and F. Soria, *Muckenhoupt-Wheeden conjectures in higher dimensions*, preprint. Available at http://arxiv.org/abs/1312.5255

[3] D. Cruz-Uribe and C.Pérez, *Two-weight extrapolation via the maximal operator*, J. Funct. Anal., **174** (2000), 1–17.

[4] F. Di Plinio and A.K. Lerner, *On weighted norm inequalities for the Carleson and Walsh-Carleson operator*, J. Lond. Math. Soc. (2), **90** (2014), no. 3, 654-674.

[5] C. Domingo-Salazar, M.T. Lacey and G. Rey, *Borderline weak type estimates for singular integrals and square functions*, preprint. Available at http://arxiv.org/abs/1505.01804.

[6] C. Fefferman and E.M. Stein, *Some maximal inequalities*, Amer. J. Math., **93** (1971), 107–115.

[7] T. Hytönen and C. Pérez, *The $L(logL)^\varepsilon$ endpoint estimate for maximal singular integral operators*, J. Math. Anal. Appl. **428** (2015), no. 1, 605–626.

[8] C. Pérez, *Weighted norm inequalities for singular integral operators*, J. London Math. Soc., **49** (1994), 296–308.

[9] M.C. Reguera, *On Muckenhoupt-Wheeden conjecture*, Adv. Math., **227** (2011), no. 4, 1436-1450.

[10] M.C. Reguera and C. Thiele, *The Hilbert transform does not map $L^1(Mw)$ to $L^{1,\infty}(w)$*, Math. Res. Lett. **19** (2012), no. 1, 1–7.

Departamento de Matemática, Universidad Nacional del Sur, Bahía Blanca, 8000, Argentina
E-mail address: marcela.caldarelli@uns.edu.ar

Department of Mathematics, Bar-Ilan University, 5290002 Ramat Gan, Israel
E-mail address: lernera@math.biu.ac.il

Departamento de Matemática, Universidad Nacional del Sur, Bahía Blanca, 8000, Argentina
E-mail address: sombrosi@uns.edu.ar