ON FINITE DIMENSIONAL JACOBIAN ALGEBRAS

SONIA TREPODE AND YADIRA VALDIVIESO-DÍAZ

Abstract. We show that Jacobian algebras arising from a sphere with n-punctures, with $n \geq 5$, are finite dimensional algebras. We consider also a family of cyclically oriented quivers and we prove that, for any primitive potential, the associated Jacobian algebra is finite dimensional.

1. Introduction

A *potential* S in a quiver Q is an element of the path algebra of Q such that S is a linear combination of cyclic paths. The *Jacobian algebra* $\mathcal{P}(A, S)$ associated with the quiver to a potential is the quotient of the complete path algebra by the *Jacobian ideal*, where $J(S)$ is generated by the partial derivatives of S with respect to the arrows of Q

Quivers with potentials play an important role in Physics and Mathematics. Derksen, Weyman y Zelevinsky, in [5], introduced quivers with potentials in the context of cluster algebras. In the same year, Fomin, Shapiro y Thurston, in [6] gave a class of cluster algebras arising from ideal triangulations of surfaces. Later, a link between this papers was established by Labardini-Fragoso in [7], he considered surfaces with a non-empty boundary and he gave a potential associated with an ideal triangulation such that its corresponding Jacobian algebra is finite dimensional.

In the firts part of this work, we study Jacobian Algebras associated with ideal triangulations of a sphere with $n \geq 5$. Our main result is the following:

Theorem 1. *Let* $(\mathbb{S}, \mathbb{M})$ *be a sphere with n-punctures, where* $n \geq 5$*. For every ideal triangulation* τ *of* $(\mathbb{S}, \mathbb{M})$*, the Jacobian algebra* $\mathcal{P}(A(\tau), S(\tau))$ *is finite dimensional.*

An sphere with 4-punctures was studied by Barot and Geiss (in [2], Section 5), the algebra associated to this surface is a tubular cluster algebra.

In the Theory of cluster algebras, primitive potentials, which are a linear combination of all the chordless cycles in a quiver Q, appear in many contexts, for example in cluster tilted algebras of Dynkin type

([5], Section 9). Also, it follows from [4] that cluster tilted algebras with cyclically oriented quivers have a primitive potential (see definition of a cluster tilted algebra in Section 4).

In the proof of our main Theorem 1, we use a particular ideal triangulation τ of a sphere with n-punctures such that the quiver associated to τ is cyclically oriented but its associated potential is not primitive.

In the second part of this work, we give a class of cyclically oriented quivers such that any primitive potential induce a finite dimensional Jacobian algebra.

The paper is organized as follows: In Section 2, we recall some definitions of quivers with potentials, path algebras, Jacobian algebras and ideal triangulations of surfaces. In Section 3, we prove that every Jacobian algebra associated with an ideal triangulation of a sphere with $n \geq 5$ are finite dimensional. Finally, in Section 4, we give a combinatorial description of a quiver Q such that any of its primitive potentials induce a finite dimensional Jacobian algebra.

Remark. While we were finishing this manuscript, we were aware of the recent paper [8], where Labardini studies properties of the potential associated with surfaces with an empty boundary and some particular genus, and the author asks if the Jacobian algebras associated with triangulations of surfaces with an empty boundary are finite dimensional algebras. Theorem 1 answers affirmatively the question of Labardini for surfaces of genus zero.

2. Preliminaries

2.1. **Quivers and potentials.** In this section, we fix notations for a path algebra and the complete path algebra, and recall basic definitions of quivers with potential (cf. [5]).

Let Q be a finite quiver and k be a field. We denote by R the k-vector space k^{Q_0}, by A the k-vector space k^{Q_1} and, for each nonnegative integer d by A^d the R-bimodule $\underbrace{A \otimes_R \cdots \otimes_R A}_{d}$.

With this notation, the *path algebra* of Q is the k-algebra defined as the (graded) tensor algebra

$$R\langle A \rangle = \bigoplus_{d=0}^{\infty} A^d$$

and the *complete path algebra* of Q is the k-vector space defined by

$$R\langle\langle A \rangle\rangle = \prod_{d=0}^{\infty} A^d.$$

Also, $R\langle\langle A\rangle\rangle$ is a topological k-algebra with the $\mathfrak{m}$-adic topology, where $\mathfrak{m}$ is the ideal $\prod_{d=1}^{\infty} A^d$.

Notice that the elements of $R\langle\langle A\rangle\rangle$ are (possibly infinite) k-linear combinations of paths in Q.

Denote by $R\langle\langle A\rangle\rangle_{\mathrm{cyc}}$ the k-subspace of $R\langle\langle A\rangle\rangle$ whose element are k-linear combinations of cycles in Q.

Definition 1. ([5], Definition 3.1)

- A potential S is an element of the k-subspace $R\langle\langle A\rangle\rangle_{\mathrm{cyc}}$.
- For every $\xi \in Q_1$, we define the *cyclic derivative* ∂_ξ as the continuous k-linear map
$$R\langle\langle A\rangle\rangle_{\mathrm{cyc}} \to R\langle\langle A\rangle\rangle$$
acting on paths by
$$\partial_\xi(a_1 \cdots a_d) = \sum_{k=1}^{d} \delta_{\xi a_k} a_{k+1} \cdots a_d a_1 \cdots a_{k-1}$$
- The Jacobian ideal $J(S)$ of a potential S is the closure of the ideal
$$\mathrm{I}(S) = \langle \partial_\xi(S) \mid \xi \in Q_1\rangle$$
in $R\langle\langle A\rangle\rangle$.
- The Jacobian algebra $\mathcal{P}(A, S)$ of S is the closure of the quotient $R\langle\langle A\rangle\rangle / J(S)$.

2.2. **Triangulations of surfaces.** In this section, we review some definitions and theorems of triangulations of surfaces (cf. [[6])]).

Definition 2. ([6], Definition 2.1)

A *bordered surface with marked points* is a pair $(\mathbb{S}, \mathbb{M})$, where $\mathbb{S}$ is a connected oriented 2-dimensional Riemann surface with a (possibly empty) boundary and $\mathbb{M}$ is a finite and non-empty set of points in $\mathbb{S}$, called *marked points*, such that there is at least one marked point on each connected component of the boundary of $\mathbb{S}$.

Marked points in the interior of $\mathbb{S}$ are called *punctures*, we denote by P the set of punctures.

In this paper, we study spheres with n punctures, $n \geq 5$, but for the following definitions, in order to avoid surfaces that cannot be triangulated or with "not-good" properties, we need to exclude:

- spheres with one or two punctures;
- unpunctured or once-punctured monogons;

- unpunctured digons; and
- unpunctured triangles

Definition 3. ([6], Definition 2.2, 2.4) A (simple) *arc* γ in $(\mathbb{S}, \mathbb{M})$ is a curve in $\mathbb{S}$ such that:

- the endpoints of γ are marked points in $\mathbb{M}$;
- γ does not intersect itself, except that its endpoints may coincide;
- γ is not contractible into $\mathbb{M}$ or onto the boundary of $\mathbb{S}$;
- γ does not cut out an unpunctured monogon or an unpunctured digon.

Two arcs are *compatible* if there are arcs in their respective isotopy classes whose relative interiors do not intersect.

An arc whose endpoints coincide is called a *loop*.

Definition 4. ([6], Definition 2.6)

An *ideal triangulation* of $(\mathbb{S}, \mathbb{M})$ is any maximal collection of pairwise compatible arcs whose relative interiors do not intersect each other.

The arcs of the triangulation cut the surface $\mathbb{S}$ into *ideal triangles*. The three sides of an ideal triangle do not have to be distinct, i.e., we allow *self-folded* triangles.

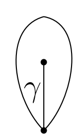


FIGURE 1. Self-folded ideal triangle

An ideal triangulation of an n-punctured sphere, is easy to calculate, it consists of $3n$ arcs.

3. Jacobian algebras arising from a sphere with n-punctures

The algebra arising from a sphere with punctures was studied for first time by Barot and Geiss in [2]. They prove that the tubular cluster algebra of type (2,2,2,2) corresponds to a sphere with 4-punctures (see definition of a cluster tilted algebra in Section 4). In this section, we study the Jacobian algebras arising from a sphere with n-punctures, where $n \geq 5$.

It is well known that the finite dimension property is preserved via mutations, then, in order to prove the Theorem 1, it is enough to prove that there exists an ideal triangulation τ such that the Jacobian algebra

$\mathcal{P}(A(\tau), S(\tau))$ is finite dimensional. For that reason below we give a particular ideal triangulation with this property.

Consider the ideal triangulation τ described in Figure 2. For notational convenience we label the punctures on the poles with p_{n+1} and p_{n+2}, so we will consider the sphere with $(n+2)$-punctures, where $n \geq 3$.

The quiver associated to the ideal triangulation τ is the quiver described in Figure 3.

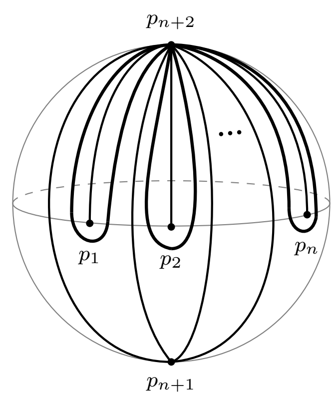


FIGURE 2. Sphere with $(n+2)$-punctures

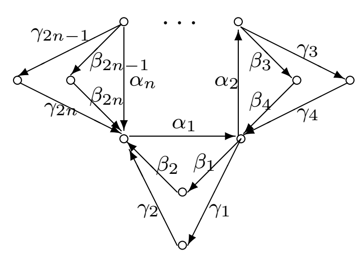


FIGURE 3. Quiver associated with the ideal triangulation τ

For each puncture $p_i \in P$ in the sphere, we choose a non-zero scalar $x_i \in k$. By Definition 23 in [7], the potential $S(\tau)$ associated with the ideal triangulation τ given in Figure 2, according to the label in the arrows of the quiver in Figure 3 is:

$$\begin{aligned} S(\tau) &= x_{n+1}\alpha_1 \ldots \alpha_n + x_{n+2}\delta_{2n-1}\delta_{2n} \ldots \delta_1\delta_2 \\ &+ \textstyle\sum_{i=1}^{n} \alpha_i\beta_{2i-1}\beta_{2i} + \sum_{i=1}^{n} -\frac{1}{x_i}\alpha_i\delta_{2i-1}\delta_{2i} \end{aligned}$$

We denote by $S = S(\tau)$ and by $A = A(\tau)$. Before proving Theorem 1, we establish some useful identities in the Jacobian algebra $\mathcal{P}(A, S)$.

Lemma 1. *The following identities hold in the Jacobian algebra $\mathcal{P}(A, S)$:*

$$\beta_{2i-1}\beta_{2i}\delta_{2(i-1)-1}\delta_{2(i-1)} = x_{i-1}\beta_{2i-1}\beta_{2(i-1)}\beta_{2(i-1)-1}\beta_{2(i-1)} \tag{1}$$

$$= \frac{x_{i-1}}{x_i}\delta_{2i-1}\delta_{2i}\beta_{2(i-1)-1}\beta_{2(i-1)} \tag{2}$$

$$= \frac{1}{x_i}\delta_{2i-1}\delta_{2i}\delta_{2(i-1)-1}\delta_{2(i-1)} \tag{3}$$

Proof. The first and the second identity follow in the same way. We prove the first one.

Since

$$\partial_{\alpha_{i-1}}(S) = \beta_{2(i-1)-1}\beta_{2(i-1)} + x_{n+1}\alpha_i \dots \alpha_{i-2} - x_{i-1}^{-1}\delta_{2(i-1)-1}\delta_{2(i-1)},$$

then in the Jacobian algebra $\mathcal{P}(A, S)$ we have the identity

$$\beta_{2i-1}\beta_{2i}\delta_{2(i-1)-1}\delta_{2(i-1)} = x_{n+1}x_{i-1}\beta_{2i-1}\beta_{2i}\alpha_i \dots \alpha_{i-2}$$
$$+x_{i-1}\beta_{2i-1}\beta_{2i}\beta_{2(i-1)-1}\beta_{2(i-1)}$$

The first term on the right hand is in the Jacobian ideal, because it contains the factor $\beta_{2i}\alpha_i = \partial_{\beta_{2i-1}}(S)$. Then

$$\beta_{2i-1}\beta_{2i}\delta_{2(i-1)-1}\delta_{2(i-1)} = x_{i-1}\beta_{2i-1}\beta_{2i}\beta_{2(i-1)}\beta_{2(i-1)-1}\beta_{2(i-1)}$$

Now we prove the third identity. By the relations induced by $\partial_{\alpha_{i-1}}(S)$, we have:

$$\beta_{2i-1}\beta_{2i}\delta_{2(i-1)-1}\delta_{2(i-1)} = -x_{n+1}\alpha_{i+1} \dots \alpha_{i-1}\delta_{2(i-1)-1}\delta_{2(i-1)}$$
$$+\frac{1}{x_{i-1}}\delta_{2i-1}\delta_{2i}\delta_{2(i-1)-1}\delta_{2(i-1)}$$

Denote by $\rho = \alpha_{i+1} \dots \alpha_{i-2}\alpha_{i-1}\delta_{2(i-1)-1}\delta_{2(i-1)}$. Notice that ρ is a path of length $n+1$. We claim that ρ is in the Jacobian ideal.

Using $\partial_{\delta_{2(i-1)}}(S)$, we have the following identity

$$\alpha_{i-1}\delta_{2(i-1)-1} = x_{i-1}x_{n+2}\delta_{2(i-2)-1}\delta_{2(i-2)} \dots \delta_1\delta_2$$

.

Then, replacing $\alpha_{i-1}\delta_{2(i-1)-1}$ in ρ, we have

$$\rho = x_{n+2}x_{i-1}\alpha_{i+1} \dots \alpha_{i-2}\delta_{2(i-2)-1}\delta_{2(i-2)} \dots \delta_1\delta_2\delta_{2(i-1)},$$

that is a path of length $3n$ in the Jabobian algebra.

Replacing $\delta_{2(i-2)-1}\delta_{2(i-2)}$ by the relation induced by $\partial_{\alpha_{i-2}}(S)$, the path ρ is a path of length $4n-3$. Iterating this process and using the topology of the Jacobian algebra ρ is in the Ideal Jacobian.

Then, $\beta_{2i-1}\beta_{2i}\delta_{2(i-1)-1}\delta_{2(i-1)} = \frac{1}{x_{i-1}}\delta_{2i-1}\delta_{2i}\delta_{2(i-1)-1}\delta_{2(i-1)}$. □

Lemma 2. *Then the following identities hold in the Jacobian algebra $\mathcal{P}(A,S)$.*

$$\alpha_i\delta_{2i-1}\delta_{2i} = x_ix_{i-1}\delta_{2(i-1)-1}\delta_{2(i-1)}\alpha_{i-1} \tag{4}$$

$$\alpha_i\alpha_{i+1}\delta_{2(i+1)-1} = \delta_{2(i-1)}\alpha_i\alpha_{i+1} = 0 \tag{5}$$

Proof. The identity (4) follows as the first two identities in Lemma 1. We proof the second one.

Notice that

$$\partial_{\delta_{2(i+1)}}(S) = -x_{i+1}^{-1}\alpha_{(i+1)}\delta_{2(i+1)-1} + x_{n+2}\delta_{2(i)-1}\delta_{2i}\ldots\delta_{2(i+1)},$$

then we have the identity

$$\alpha_i\alpha_{i+1}\delta_{2(i+1)-1} = x_{i+1}x_{n+2}\alpha_i\delta_{2i-1}\delta_{2i}\ldots\delta_{2(i+1)}.$$

By the identity (3) in Lemma 1 the term on the right hand is equal to

$$x_{i+1}x_{n+2}\alpha_i\beta_{2(i)-1}\beta_{2i}\delta_{2(i-1)-1}\ldots\delta_{2(i+1)},$$

which is in the Jacobian ideal because it contains a factor $\alpha_1\beta_{2i-1} = \partial_{\beta_{2i}}(S)$.

Then $\alpha_i\alpha_{i+1}\delta_{2(i+1)-1} = 0$ in the Jacobian algebra $\mathcal{P}(A,S)$. □

Now, we can prove our main result.

Remark 1. Notice that every path that contains the factor $\alpha_i\beta_{2i-1}$ or $\beta_{2i}\alpha_i$ is a zero in $\mathcal{P}(A,S)$, because $\alpha_i\beta_{2i-1} = \partial_{\beta_{2i}(S)}$ and $\beta_{2i}\alpha_i = \partial_{\beta_{2i-1}}(S)$. Moreover, by the identities in Lemma 1, every path that contains a factor $\alpha_i\delta_{2i-1}\delta_{2i}\delta_{2(i-1)-1}\delta_{2(i-1)}$ or $\alpha_i\delta_{2i-1}\delta_{2i}\beta_{2(i-1)-1}\beta_{2(i-1)}$ is zero in $\mathcal{P}(A,S)$.

Proof of Theorem 1. It is enough to prove that $\mathcal{P}(A,S)$ is finite dimensional. We will prove that every path of length at least $2n+4$ is an element of the Jacobian ideal $J(S)$.

Let ρ be a path of length at least $2n+4$. By the Remark 1 it is enough to analyze if ρ contains only β_i and δ_i or only α_i and δ_i in its factors.

Consider the first case. Without loss of generality we can assume that ρ starts with β_{2n-1} or δ_{2n-1}.

We notice that if the length of the path is at least 6 then we have the following possible factors:

I. $\beta_{2i-1}\beta_{2i}\delta_{2(i-1)-1}\delta_{2(i-1)}$,
II. $\delta_{2i-1}\delta_{2i}\delta_{2(i-1)-1}\delta_{2(i-1)}$,
III. $\beta_{2i-1}\beta_{2i}\beta_{2(i-1)-1}\beta_{2(i-1)}$ or
IV. $\delta_{2i-1}\delta_{2i}\beta_{2(i-1)-1}\beta_{2(i-1)}$.

By Lemma 1, $\rho = x\delta_{2n-1}\delta_{2n}\ldots\delta_1\delta_2\delta_{2n-1}\delta_{2n}\rho'$, where ρ' is the rest of the path ρ and $x \in k$ is the product certain scalars x_j.

Hence, by the relation induced of the partial derivative

$$\partial_{\delta_2}(W) = x_{n+2}\delta_{2n-1}\delta_{2n}\ldots\delta_1 + x_1\alpha_1\delta_1,$$

we have that

$$\rho = -xx_1\alpha_1\delta_1\delta_2\delta_{2n-1}\delta_{2n}\rho'.$$

Then ρ is zero in $\mathcal{P}(A, S)$ (see Remark 1).

Now suppose ρ has only elements α_i and δ_i. Then ρ contains the following possible factors:

I'. $\alpha_i\alpha_{i+1}\delta_{2(i+1)-1}$ (or $\delta_{2i}\alpha_i\alpha_{i+1}$)
II'. $\alpha_i\delta_{2i-1}\delta_{2i}\alpha_i\delta_{2i-1}\delta_{2i}$
III'. $\alpha_i\ldots\alpha_{i-1}\alpha_i\alpha_{i+1}\alpha_{i+2}$

It follows from the identity (5) in the Lemma 2, that ρ is zero in the case where it contains the factor I'.

By the identity (4) in Lemma 2, the second possible factor is equal to

$$x_{i-1}^{-1}x_i^{-1}\alpha_i\alpha_{i+1}\delta_{2(i+1)-1}\delta_{2(i+1)}\delta_{2(i-1}\delta_{2i)}$$

then ρ is zero.

Finally, using the relation $\partial_{\alpha_i}(S)$ and $\partial_{\beta_{2i}}(S)$ the last factor is equal to

$$\alpha_i\delta_{2i_1}\delta_{2i}\alpha_i\alpha_{i+1}\alpha_{i+2}.$$

Since the walk ρ contains the possible factor I', then it is zero. □

4. Potentials in a class of cyclically oriented Quivers

We have shown that the Jacobian algebras arising from a sphere with n-punctures are finite dimensional. We observe that the quiver Q in Figure 3 associated with the ideal triangulation in Figure 2 is cyclically oriented and its associated potential is not primitive. It is not difficult to prove that the Jacobian algebras, given by the quiver Q with any primitive potential, are not finite dimensional.

In this section, we consider a particular family of cyclically oriented quivers and we prove that, for any primitive potential, the associated Jacobian algebra is finite dimensional.

We start by recalling the definition of cyclically oriented quiver.

Definition 5. ([4], Definition 3.1) A *walk of length* p in a quiver Q is a $(2p+1)$-tuple

$$w = (x_p, \alpha_p, x_{p-1}, \alpha_{p-1}, \dots, x_1, \alpha_1, x_0)$$

such that for all i we have $x_i \in Q_0$, $\alpha \in Q_1$ and $\{s(\alpha_i), e(\alpha_i)\} = \{x_p, x_{p-1}\}$. The walk w is *oriented* if either $s(\alpha_i) = x_{p-1}$ and $e(\alpha_i) = x_p$ for all i or $s(\alpha_i) = x_p$ and $e(\alpha_i) = x_{p-1}$ for all i. Furthermore, w is called a *cycle* if $x_0 = x_p$. A cycle of length 1 is called a *loop*. We often omit the vertices and abbreviate w by $\alpha_p \cdots \alpha_1$. An oriented walk is also called *path*.

A cycle $c = (x_p, \alpha_p, \dots, x_1, \alpha_1, x_p)$ is called *non-intersecting* if its vertices $x_1, \dots, x_p$ are pairwise distinct. A non-intersecting cycle of length 2 is called 2-cycle. If c is a non-intersecting cycle then any arrow $\beta \in Q \setminus \{\alpha_1, \dots, \alpha_p\}$ with $\{s(\beta), e(\beta)\} \subseteq \{x_1, \dots, x_p\}$ is called a *chord* of c. A cycle c is called *chordless* if it is non-intersecting and there is no chord of c.

A quiver Q without loop and 2-cycle is call *cyclically oriented* if each chordless cycle is oriented. Note that this implies that there are no multiple arrows in Q. A quiver without oriented cycle is called *acyclic* and an algebra whose quiver is acyclic is called *triangular*.

Let A be a finite-dimensional algebra over an algebraically closed field k. We denote by $\mathcal{D}^b(A)$ the bounded derived category of the category of finite-dimensional (left) A-modules, by τ the Auslander-Reiten translation and by S the suspension of $\mathcal{D}^b(A)$. Let $\mathcal{C}_A$ be the hull triangulated category of $\mathcal{D}^b(A)/\tau^{-1}S$.

Amiot showed in [1] that if the global dimension of A is less or equal than two, and the functor $\tau^{-1}S$ is nilpotent, then the category $\mathcal{C}_A$ is Hom-finite, and the image of A in $\mathcal{C}_A$ is a *cluster-tilting object*. In this case, the category $\mathcal{C}_A$ is called *cluster category*.

In the hereditary case, the endomorphism algebra $\mathrm{End}_{\mathcal{C}_A}(A)$ is called *cluster tilted algebra*.

In [3] cluster algebras (or cluster categories in the hereditary case) with cyclically oriented quivers were studied in order to decide if a cluster algebra is of finite type. Later in [4], it was given an explicit description of the minimal relations in cluster tilted algebras with this kind of quivers, and it follows from this result that the potential associated with this kind of algebras is primitive. As we observed at the beginning of the section, the quiver Q in Figure 3 associated with the ideal triangulation in Figure 2 is cyclically oriented and its associated potential is not primitive, showing that the previous result does not extend to Jacobian algebras.

In the following theorem, we give a combinatorial description of the quiver such that the Jacobian algebra with a primitive potential is a finite dimensional algebra.

Theorem 2. *Let Q be a quiver such that:*

i) Q is cyclically oriented;
ii) for every not-minimal path ρ there exists at least one arrow with at most 2 antiparallel minimal paths.

Then the Jacobian algebra $\mathcal{P}(A, S)$, where S is a primitive potential, is a finite dimensional algebra.

We need some definitions and lemmas before proving Theorem 2.

Definition 6. ([4], Definition 3.3) A path γ which is antiparallel to an arrow η in a quiver Q is a *shortest path* if the full subquiver generated by the induced oriented cycle $\eta\gamma$ is chordless. A path $\gamma = (x_0 \xrightarrow{\gamma_1} x_1 \xrightarrow{x} _2 \rightarrow \cdots \rightarrow x_L)$ is called *shortest directed path* if there exists no arrow $x_i \rightarrow x_j$ in Q with $1 \leq i+1 < j \leq L$. A walk $\gamma = (x_0 — x_1 — x_2 — \cdots — x_L)$ is called a *shortest walk* if there is no edge joining x_i with x_j with $1 \leq i+1 < j \leq L$ and $(i,j) \neq (0, L)$ (we write a horizontal line to indicate an arrow oriented in one of the two possible ways).

For each quiver Q, and η an arrow in a cycle C, we construct a sequence of arrows $\{\alpha_0, \ldots, \alpha_k, \ldots\}$ in the following way:

Step 0 We denote by $\alpha_0 = \eta$ and choose an antiparallel shortest path ρ_0 to α_0;

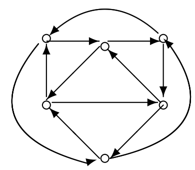
Step 1 We denote by α_1 the arrow in the path ρ such that $t(\alpha_0) = s(\alpha_1)$ and choose an antiparallel shortest path ρ_1 to α_1, such that the cycle $\alpha_0\rho_0$ is different from the cycle $\alpha_1\rho_1$;

Step 2 We denote by α_2 the arrow in the path ρ_1 such that $s(\alpha_1) = t(\alpha_2)$ and choose an antiparallel shortest path ρ_2 to α_2, such that the cycle $\alpha_1\rho_1$ is different from the cycle $\alpha_2\rho_2$;

$\vdots$

Step k We denote by α_k the arrow in the path ρ_{k-1} such that
- If k is even $t(\beta_i) = s(\beta_{i+1})$ or $s(\beta_i) = t(\beta_{i+1})$ if i is odd;
- The cycle $\alpha_k\rho_k$ is different from the cycle $\alpha_{k-1}\rho_{k-1}$

Remark 2. In the following quiver, which is associated to a triangulation of a sphere with 4 punctures, this process generates an infinite sequence of arrows.



Lemma 3. *Let Q be a quiver cyclically oriented and η be an arrow the cycle C. Then there exist a finite sequence of arrows $\{\alpha_0, \dots, \alpha_m\}$ constructed as above, that means, α_m has only one antiparallel shortest path.*

Proof. Suppose that such sequence does not exist. Let $\{\alpha_0, \dots, \alpha_k, \dots\}$ be a sequence of arrow constructed as above, then is infinite. Let j be the minimal natural such the elements of the subsequence $\{\alpha_0, \dots, \alpha_{j-1}\}$ are different. With out lost of generality suppose that $\alpha_0 = \alpha_j$, in this situation j is even.

Let $\{\rho_0, \dots, \rho_j\}$ be the subsequence of the antiparallel shortest path corresponding to the subsequence $\{\alpha_0, \dots, \alpha_j\}$. To fix notation, for each arrow α_k, let $x_i = s(\alpha_k)$ $y_i = t(\alpha_k)$.

Denote by $\zeta_k = (y_k \to \cdots \to x_{k+1})$ the subpath of ρ_k if k is even or $\zeta_k = (y_{k+1} \to \cdots \to x_k)$ the subpath of ρ_k if k is odd. Consider the not oriented walk

$$\zeta = \zeta_0\zeta_1 \dots \zeta_{j-1} = (y_0 \to \cdots \to x_1 \leftarrow \cdots \leftarrow y_2 \to \cdots \leftarrow y_j).$$

We show that ζ is a shortest walk, contradiction the hypothesis. Suppose ζ is not chordless. Let $\beta = x \to y$ be an chord of ζ. Suppose that x belongs to ζ_k and y belongs to $\zeta_{k'}$ and without lost of generality $k \leq k'$ and k, k' are even, moreover consider β such that

$$\zeta' = (x — \ \dots — x_k — \ \dots — y'_k — \ \dots — y — x)$$

is a chordless walk, then by hypothesis ζ' is an oriented chordless cyle, hence $k = k'$. Hence β is a chord in $\alpha_k\rho_k$, a contradiction of the construction of the sequence $\{\alpha_0, \dots, \alpha_k, \dots\}$. □

In the proof of Theorem 2, we use the same notation for the sequences.

Proof of Theorem 2. Let $C = \beta_0 \dots \beta_n$ be a minimal oriented cycle. We only give the argument of the proof in the case that each vertex in Q has at most 2 antiparallel paths. If a vertex has more than 2 antiparallel paths, we can apply recursively the same idea to each antiparallel path.

By Lemma 3, there exists a finite sequence of arrows $\{\alpha_0, \ldots, \alpha_m\}$ and the corresponding finite sequence of antiparallel shortest directed paths $\{\rho_0, \ldots, \rho_m\}$ such that $\beta_0 = \alpha_0$ and α_m has only an antiparallel path. As in Lemma 3, we denote by $x_i = s(\alpha_i)$, $y_i = t(\alpha_i)$ and by $\zeta_i = (y_i \to \cdots \to x_{i+1})$ the subpath of ρ_i if i is even or $\zeta_i = (y_{i+1} \to \cdots \to x_i)$ the subpath of ρ_i if i is odd. Using this notation, we have $C = \beta_0 \ldots \beta_n = \alpha_0\zeta_0\alpha_1$.

Notice that $\partial_{\alpha_i}(S) = k_i\alpha_{i+1}\zeta_i + k_{i-1}\alpha_{i-1}\zeta_{i-1}$ if i is odd or $\partial_{\alpha_i}(S) = k_{i-1}\zeta_{i-1}\alpha_{i-1} + k_i\zeta_i\alpha_{i+1}$ if i is even for every $i \neq m$, where $k_i, k_{i-1} \in k$. Then using the relations induced by $\partial_{\alpha_i}(S)$ we have the following equalities:

$$\begin{aligned} C &= \alpha_0\zeta_0\alpha_1 = -\frac{k_1}{k_2}\alpha_2\zeta_1\alpha_1 = \left(-\frac{k_1}{k_2}\right)\left(-\frac{k_2}{k_1}\right)\alpha_2\zeta_2\alpha_3 = \\ &= \cdots = \left(-\frac{k_{2j-1}}{k_{2j}}\right)\alpha_{2j}\zeta_{2j-1}\alpha_{2j-1} = \alpha_{2j}\zeta_{2j}\alpha_{2j+1}. \end{aligned}$$

Since the arrow α_m has only one antiparallel path, hence $\partial_{\alpha_m}(S) = k_{m-1}\alpha_{m-1}\zeta_{m-1}$ if m is odd or $\partial_{\alpha_m}(S) = k_{m-1}\zeta_{m-1}\alpha_{m-1}$ if m is even. Then the factor $\alpha_{m-1}\zeta_{m-1}$ or $\zeta_{m-1}\alpha_{m-1}$ is zero, then the cycle C is in the Jacobian ideal.

Now we consider C not to be a chordless cycle in Q. By hypothesis, there exists an arrow α in C such that α has at most 2 antiparallel paths. To fix notation denote by $C = (x_1 \overset{\alpha_1}{\to} x_2 \overset{\alpha_2}{\to} \ldots x_n \overset{\alpha_n}{\to} x_1)$ where $\alpha = \alpha_1$. Since C is not a chordless cycle, then there exists a chord $\beta : x_i \to x_j$ with vertices in C such that $C_1 = \beta_1\alpha_j \ldots \alpha_i$ is a chordless cycle. Since Q is cyclically oriented, then the not oriented walk $\zeta_1 = \beta\alpha_{i+1} \ldots \alpha_{j-1}$ is also not chordless. Applying recursively this argument to each not chordless walk ζ_i, we find a chordless cycle C' such that its vertices are also vertices in C. To fix a notation let $C_k = \beta_k\alpha_{j_k} \ldots \alpha_{i_k}$ be the chordless cycle that was found in the k recursive step and $C' = \beta_1 \ldots \beta_m$ be the chordless cycle.

Let β_k be the arrow in C' such that α_1 is an arrow in C_k. Then

$$\partial_{\beta_1}(S) = m_0\alpha_{j_k} \ldots \alpha_{i_k} + m_1\beta_{k+1} \ldots \beta_{k-1} + \sum_{i=2}^{t} m_i\rho_i\gamma_i,$$

where $\rho_i\gamma_i$ are the other antiparallel paths to β_k and m_i are non zero scalars. Hence, the cycle C is equal to

$$-m_0^{-1}m_1\alpha_1 \ldots \alpha_{j_k-1}\beta_{k+1} \ldots \beta_{k-1}\alpha_{i_k+1} \ldots \alpha_n - \sum_{i=2}^{t} m_0^{-1}m_i\rho_i\gamma_i,$$

whose summands have a chordless cycle as a factor. Then, by the first part of the proof, C is zero. □

Acknowledgments

This article is part of the Ph.D. thesis of the second author under the supervision of Sonia Trepode. The second author thanks Professor Michael Barot for discussing some central ideas for this article. She thanks Professor Chrisof Geiss for pointing out some important results on surfaces with non-empty boundary. She also thanks Daniel Labardini-Fragoso for clarifying ideas of his article [7].

References

[1] Claire Amiot. Cluster categories for algebras of global dimension 2 and quivers with potential. *Ann. Inst. Fourier (Grenoble)*, 59(6):2525–2590, 2009.

[2] Michael Barot and Christof Geiss. Tubular cluster algebras i: categorification. *to appear in Math. Z*, page arXiv:1002.4842, 2009.

[3] Michael Barot, Christof Geiss, and Andrei Zelevinsky. Cluster algebras of finite type and positive symmetrizable matrices. *J. London Math. Soc. (2)*, 73(3):545–564, 2006.

[4] Michael Barot and Sonia Trepode. Cluster tilted algebras with cyclically oriented quiver. *to appear in Comm. in algebra*, page arXiv:0905.0028v3, 2010.

[5] Harm Derksen, Jerzy Weyman, and Andrei Zelevinsky. Quivers with potentials and their representations. I. Mutations. *Selecta Math. (N.S.)*, 14(1):59–119, 2008.

[6] Sergey Fomin, Michael Shapiro, and Dylan Thurston. Cluster algebras and triangulated surfaces. I. Cluster complexes. *Acta Math.*, 201(1):83–146, 2008.

[7] Daniel Labardini-Fragoso. Quivers with potentials associated to triangulated surfaces. *Proc. Lond. Math. Soc. (3)*, 98(3):797–839, 2009.

[8] Daniel Labardini-Fragoso. Quivers with potentials associated to triangulated surfaces, part iv: Removing boundary assumptions. page arXiv:1206.1798v1, 2012.

Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Funes 3350, Universidad Nacional de Mar del Plata, 7600 Mar del Plata, Argentina

E-mail address: strepode@mdp.edu.ar

Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Funes 3350, Universidad Nacional de Mar del Plata, 7600 Mar del Plata, Argentina.

E-mail address: ariday83@gmail.com