Link and knot invariants from non-abelian Yang-Baxter 2-cocycles

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Abstract

We define a knot/link invariant using set theoretical solutions (X, σ) of the Yang-Baxter equation and non commutative 2-cocycles. We also define, for a given (X, σ), a universal group $U_{nc}(X)$ governing all 2-cocycles in X, and we exhibit examples of computations.

Introduction and preliminaries

The first part of this work consist of a generalization to biquandles and the notion of non-commutative 2-cocycle given in [AG] for quandles. It is also a generalization to the non-commutative case of part of the work in [CEGS] for commutative cocycles. In this way, we obtain in principle new invariants for biquandles that do not come from quandles, admitting non-commutative 2-cocycles, that is, whose universal group (see section 2 or 3) is non abelian.

In the second section we define a universal group governing all 2-cocycles for a given biquandle X, that is, a group $U_{nc}(X)$ together with a 2-cocycle $\pi : X \times X \to U_{nc}(X)$ such that if $f : X \times X \to G$ is a non commutative 2-cocycles with values in a group G, then there is a unique group homomorphism $\widetilde{f} : U_{nc}(X) \to G$ such that $f = \widetilde{f}\pi$. For instance, if $U_{nc}(X)$ is the trivial group, then every 2-cocycle is trivial. On the opposite, if $U_{nc}(X)$ is nontrivial, this universal property says that it carries all information that any group could give using non commutative 2-cocycles.

In the third section, a reduced version of U_{nc} is given, it depends on a map $\gamma : X \to U_{nc}(X)$. The constructed group is called $U_{nc}^{\gamma}(X)$, in particular it is a group and there is given a 2-cocycle $\pi_\gamma : X \times X \to U_{nc}^{\gamma}(X)$ with the following property (Theorem 24): if $f : X \times X \to G$ is a 2-cocycle, then there exists a cohomologous (see definition 7) 2-cocycle $f_\gamma : X \times X \to G$ and a group homomorphism $\widetilde{f}_\gamma$ such that $f_\gamma = \widetilde{f}_\gamma \pi_\gamma$. Since the invariant defined in section 1 is unchanged for cohomologous cocycles (Proposition

14), the invariant produced with f is the same as the one coming from f_γ, so we see that all invariants are governed by the group U_{nc}^γ, which is, in general, smaller than U_{nc}.

In section 4 we exhibit some examples of computations. Most of them were performed using [GAP2015]. We wish to heartily thank Leandro Vendramin for teaching us the basic facts on syntax and GAP programing, and helping us with our first (and not so firsts) steps implementing the algorithms we needed. An interesting observation on examples is that, if (X, σ) is a solution of the Yang-Baxter equation, then also is $\overline{\sigma} := \sigma^{-1}$, and if σ makes X into a biquandle (see definition below), then also $\overline{\sigma}$ gives a biquandle structure. One may suspect that $\overline{\sigma}$ is, in a sense, equivalent to σ and probably gives no new information, but this is not the case: one may have $U_{nc}^\gamma(\sigma) = 1$ (so that σ gives always trivial invariants for any 2-cocycle f) but $\overline{\sigma}$ may give non trivial invariants. Section 5 end with concluding remarks.

Before going into section 1 we recall the notion of biquandle and quandle:

Definition 1. A set theoretical solution of the Yang-Baxter equation is a pair (X, σ) where $\sigma : X \times X \to X \times X$ is a bijection satisfying

$$(\text{Id} \times \sigma)(\sigma \times \text{Id})(\text{Id} \times \sigma) = (\sigma \times \text{Id})(\text{Id} \times \sigma)(\sigma \times \text{Id})$$

Notation: $\sigma(x, y) = (\sigma^1(x, y), \sigma^2(x, y))$ and $\sigma^{-1}(x, y) = \overline{\sigma}(x, y)$.

Definition 2. A **rack** is a pair $(X, \triangleleft)$ where $\triangleleft : X \to X$ verifies

1. for all $x \in X$, the map $- \triangleleft x : X \to X$ $(y \mapsto y \triangleleft x)$ is bijective, and

2. $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$

A rack is called a **quandle** if $x \triangleleft x = x$ for all $x \in X$.

One can easyly see that $\sigma(x, y) = (y, x \triangleleft y)$ is a solution of the YBeq if and only if $(X, \triangleleft)$ is a rack. A solution (X, σ) is called non-degenerated, or **birack** if in addition:

1. (*left invertibility*) for any $x, z \in X$ there exists a unique y such that $\sigma^1(x, y) = z$,

2. (*right invertibility*) for any $y, t \in X$ there exists a unique x such that $\sigma^2(x, y) = t$.

Given a birack (X, σ), a remarkable map $\Phi : X \times X \to X \times X$ can be defined in the following way (see [S] or [LYZ]). Introduce the notation $\sigma(x, y) = (g_x(y), f_y(x))$, and define Φ by $\Phi(x, y) := (x, g_x y)$. Notice that the birack condition means that both g_x and f_x are bijective maps for any x, in particular Φ is bijective. This map was considered in [S], [LYZ], and several places after. One of its properties is that the operation called **derived** rack structure on X given by

$$x \triangleleft y := x \triangleleft_\sigma y = g_y(f_{g_x^{-1}(y)}(x))$$

is actually a rack structure and, moreover, the map Φ makes the following diagram commutative

$$
\begin{array}{ccc}
X \times X & \xrightarrow{\Phi} & X \times X \\
\downarrow{\sigma} & & \downarrow{c_\triangleleft} \\
X \times X & \xrightarrow{\Phi} & X \times X
\end{array}
$$

where $c_\triangleleft(x, y) = (y, x \triangleleft y)$, that is, the standard braiding associated to a rack. Using this map one can see the following:

Lemma 3. *If X is a finite set and σ a birack solution of the YBeq then the following assertions are equivalent*

    1) For all $y_0 \in X$ there exists a unique $x_0 =: S(y_0)$ such that $\sigma(x_0, y_0) = (x_0, y_0)$.

   1)* There exists a function $S : X \to X$ such that $(X \times X)^\sigma = \{(S(y), y) : y \in X\}$.

    2) For all $x_0 \in X$ there exists a unique $y_0 =: s(x_0)$ such that $\sigma(x_0, y_0) = (x_0, y_0)$.

   2)* There exists a function $s : X \to X$ such that $(X \times X)^\sigma = \{(x, s(x)) : x \in X\}$.

    3) The derived rack $(X, \lhd)$ is a quandle, that is, $x \lhd x = x$ for all $x \in X$

If one of the above equivalent conditions holds, we say that (X, σ) is a **biquandle**. Notice that if this is the case, then function s is necesarily bijective (with inverse S).

Proof. It is clear that 1 is equivalent to 1*, and similarly for 2⇔2*. Let us prove $2 \Rightarrow 3$. Using the above commutative diagram on can conclude that Φ induces a bijection

$$\Phi : (X \times X)^\sigma \xrightarrow{\;\cong\;} (X \times X)^{c_\lhd}$$

But if $\sigma(x_0, y_0) = (x_0, y_0)$ then $\Phi(x_0, y_0) = (x_0, x_0)$. So, for every $x \in X$, $c_\lhd(x, x) = (x, x \lhd x) = (x, x)$, that is, $x \lhd x = x$ and hence $(X, \lhd)$ is a quandle.

    $3 \Rightarrow 2)$. Assume $(X, \lhd)$ is a quandle. As before, we know that Φ restricts to a bijection $(X \times X)^\sigma \to (X \times X)^{c_\lhd} = \{(x, x) : x \in X\}$. But Φ is bijective, with inverse $\Phi^{-1}(a, b) = (a, g_a^{-1}(b))$, so

$$(X \times X)^\sigma = \Phi^{-1}(\{(x, x) : x \in X\}) = \{(x, g_x^{-1}x) : x \in X\}$$

that is, $(X \times X)^\sigma$ is of the form $\{(x, s(x)) : x \in X\}$, with s the function $s(x) := g_x^{-1}(x)$. The proof of $3 \Leftrightarrow 1$ is analogous. $\qquad\square$

    The biquandle condition is crutial in relation with the first Reidemeister move; in section 2 we review some examples of biquandles.

1   Non-abelian 2-cocycles

Let (X, σ) be a biquandle and H a (not necessarily abelian) group.

Definition 4. A function $f : X \times X \to H$ is a *braid non-commutative 2-cocycle* if

- $f(x_1, x_2) f(\sigma^2(x_1, x_2), x_3) = f(x_1, \sigma^1(x_2, x_3)) f(\sigma^2(x_1, \sigma^1(x_2, x_3)), \sigma^2(x_2, x_3))$, and

- $f(\sigma^1(x_1, x_2), \sigma^1(\sigma^2(x_1, x_2), x_3)) = f(x_2, x_3)$

are satisfied for any $x_1, x_2, x_3 \in X$.

Definition 5. If f further satisfies $f(x, s(x)) = 1$ for all $x \in X$ then it is called of *type I*.

Remark 6. If f is a braided non commutative 2-cocycle and $\lambda : X \to H$ is an arbitrary function, then
$$f'(x, y) = \lambda(x) f(x, y) \lambda^{-1}(\sigma^2(x, y))$$
is also a braided non commutative 2-coycle. If moreover f is of type I, and $\lambda(x) = \lambda(s(x))$ for all $x \in X$, then f' is also of type I.

Definition 7. Two cocycles f, f' are cohomologous ($f \sim f'$) if there is a function $\gamma : X \to H$ such that $\gamma(x) = \gamma(s(x))$ and
$$f'(x, y) = \gamma(x) f(x, y) \gamma^{-1}(\sigma^2(x, y)), \ \forall x, y \in X.$$

Remark 8. It is easy to see that $\sim$ is an equivalence relation.

An equivalence class is called a cohomology class. The *set* of cohomology classes is denoted by $H^2_{NC}(X, H)$. This definitions, in case $(X, \lhd)$ is a quandle and considering $\sigma(x, y) = (y, x \lhd y)$, agree with the ones in [CEGS], since in this case the second condition of definition 4 is trivial. As in the rack/quandle case, if H is not commutative, $H^2_{NC}(X, H)$ need not to be a group, it is just a set.

Remark 9. If H happens to be commutative and $f : X \times X \to H$ is a 2-cocycle in the non commutative sense, then f is necessarily a (special type of) 2-cocycle with trivial coefficients in the sense of [CES], but our definition is more restrictive, because we ask for a set of equations of the form $ab = a'b'$ and $c = c'$ (plus being type I), while in the usual abelian 2-cocycles the equation is of the form $abc = a'b'c'$ (plus being type I).

Remark 10. The first condition of definition 4 is invariant under inverting σ, namely, f satisfies it for σ if and only if f does it for σ^{-1}. On the other hand, the second condition is not invariant under inverting σ. For example, if $(X, \lhd)$ is a rack and $\sigma(x, y) = (y, x \lhd y)$, then the second condition is trivially satisfied for any function f (and hence, this definition is equivalent, in this setting, to the one given in [CEGS]), while for $\overline{\sigma}(x, y) = (y \lhd^{-1} x, x)$ means that f must be invariant under the action of the Inner group associated to the rack X.

1.1   Weights

Let X be a biquandle, H a group, $f : X \times X \to H$ a non-abelian 2-cocycle. Let $L = K_1 \cup \cdots \cup K_r$ be a classical oriented link diagram on the plane, where $K_1, \ldots, K_r$ are connected components, for some positive integer r. A *coloring* of L by X is a rule that assigns an element of X to each semi-arc of L, in such a way that for every crossing

we have $(z, t) = \sigma(x, y)$ if the crossing is positive, and $(c, d) = \sigma^{-1}(a, b)$ if the crossing is negative.

Let $\mathcal{C} \in Col_X(L)$ be a coloring of L by X and $(b_1, \ldots, b_r)$ a set of base points on the components $(K_1, \ldots, K_r)$. Let $\tau^{(i)}$, for $i = 1, \ldots, r$ the set of crossings such that the

under-arc is from the component i. Let $(\tau_1^{(i)}, \ldots, \tau_{k_{(i)}}^{(i)})$ be the crossings in $\tau^{(i)}$, $i = 1, \ldots, r$ such that appear in this order when one travels K_j in the given orientation.

At a positive crossing τ, let x_τ, y_τ be the color on the incoming arcs. The *Boltzmann weight* at τ is $B_f(\tau, \mathcal{C}) = f(x_\tau, y_\tau)$. At a negative crossing τ, denote $\sigma(x_\tau, y_\tau)$ the colors on the incoming arcs. The *Boltzmann weight* at τ is $B_f(\tau, \mathcal{C}) = f^{-1}(x_\tau, y_\tau)$

$$B_{f,\tau} = f(x_\tau, y_\tau): \qquad x_\tau \qquad \qquad y_\tau \qquad \sigma^1(x_\tau, y_\tau) \qquad \sigma^2(x_\tau, y_\tau) \; : B_{f,\tau} = f(x_\tau, y_\tau)^{-1}$$

$$\sigma^1(x_\tau, y_\tau) \qquad \sigma^2(x_\tau y_\tau) \qquad x_\tau \qquad \qquad y_\tau$$

We will show that a convenient product of these weights is invariant under Reidemeister moves.

1.2 Reidemeister type I moves

First notice that $\sigma(x, s(x)) = (x, s(x))$ implies $\sigma^{-1}(x, s(x)) = (x, s(x))$, so, adding any orientation to the diagram

$$x \qquad \bigcirc \; s(x) \qquad \qquad x$$
$$x \qquad \qquad \qquad x$$
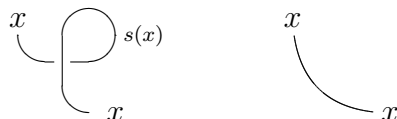
the condition $f^{\pm 1}(x, s(x)) = 1$ assures that the factor due to this crossing do not count.

1.3 Reidemeister type II moves

We consider several cases:

Case 1:

$$\sigma^1(x, y) \qquad \sigma^2(x, y) \qquad\qquad \sigma^1(x, y) \qquad \sigma^2(x, y)$$

$$x \qquad y$$

$$\sigma^1(x, y) \qquad \sigma^2(x, y) \qquad\qquad \sigma^1(x, y) \qquad \sigma^2(x, y)$$

Case 2:

$$x \qquad y \qquad\qquad\qquad x \qquad y$$

$$\sigma^2(y,x) \quad \sigma^1(y,x)$$

$$x \qquad y \qquad\qquad\qquad x \qquad y$$

Case 3: In this case and the following, start naming the top arcs of the diagrams on the left, the rest of the arcs are known as X is a biquandle.

Case 4:

The product of weights corresponding to the diagrams on the left in cases 1 and 3 is $f^{-1}(x,y)f(x,y) = 1$, in cases 2 and 4 is $f(x,y)f^{-1}(x,y) = 1$.

1.4 Reidemeister type III moves

While there are eight oriented Reidemeister type III moves, only four of them are different.

Case 1: Start by naming the incoming-arcs x_1, x_2, x_3. In case 1, as well as in the rest of the cases, once chosen three arcs in both diagrams the remaining arcs are respectively equal as σ is a solution of YBeq.
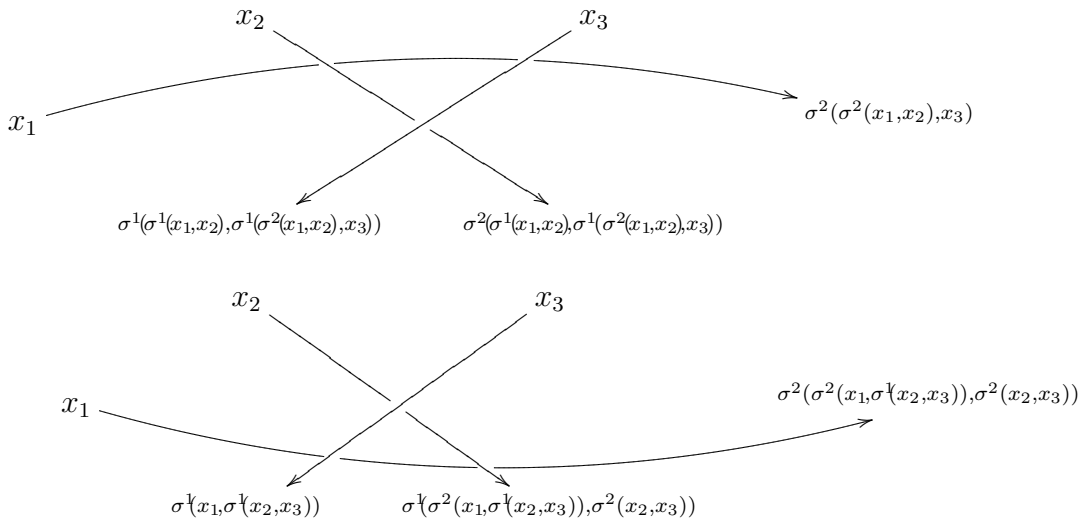
The product of the weights following the horizontal under-arc, in the first diagram, is:

$$I = f(x_1, x_2)f(\sigma^2(x_1, x_2), x_3)$$

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and in the second, is:

$$II = f(x_1, \sigma^1(x_2, x_3))f(\sigma^2(x_1, \sigma^1(x_2, x_3)), \sigma^2(x_2, x_3))$$

$I = II$ is one of the equalities defining 2-cocycle, the other equation defining 2-cocycle affirms that the weights given to the *other* crossings are the same. Notice that in the quandle coloring this condition is trivial, but in the biquandle coloring it is not.

Case 2: Start by naming the arcs $\sigma^2(x_1, x_2)$, x_2 and $\sigma^1(x_2, x_3)$ in both diagrams. The remaining arcs are known using the fact that X is a biquandle and
$\sigma^1\left(\sigma^1(x_1, x_2), \sigma^1(\sigma^2(x_1, x_2), x_3)\right) = \sigma^1(x_1, \sigma^1(x_2, x_3))$ (due to the braid equation).
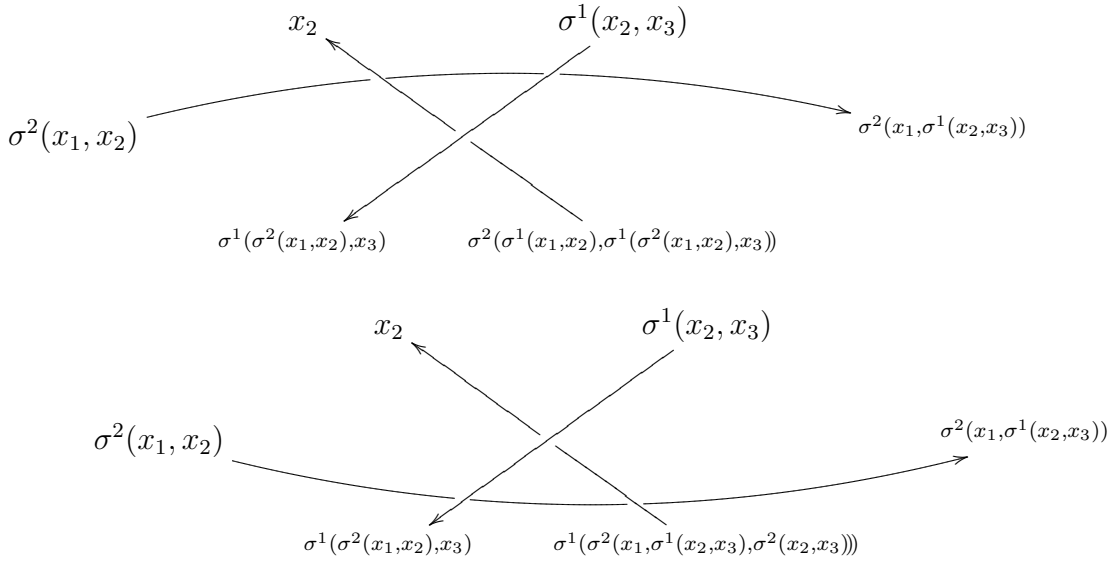
The product of weights for the horizontal line in the first diagram is

$$I = f^{-1}(x_1, x_2)f(x_1, \sigma^1(x_2, x_3))$$

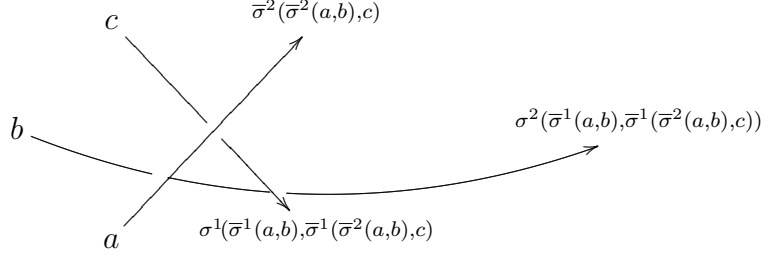and for the second diagram is

$$II = f(\sigma^2(x_1, x_2), x_3)f^{-1}(\sigma^2(x_1, \sigma^1(x_2, x_3)), \sigma^2(x_2, x_3))$$

The remaining weights in both diagrams are $a = f^{-1}(\sigma^1(x_1, x_2), \sigma^1(\sigma^2(x_1, x_2), x_3))$ and $b = f^{-1}(x_2, x_3)$. As f is a 2-cocycle, $a = b$.

Case 3: Name the incoming arcs by a, b and c.

Remark 11. YBeq is equivalent to the following equation, which explains the equality of the out-coming arcs in both diagrams.

$$(\sigma \times 1)(1 \times \overline{\sigma})(\overline{\sigma} \times 1) = (1 \times \overline{\sigma})(\overline{\sigma} \times 1)(1 \times \sigma) \tag{1}$$

$$\overline{\sigma}^2(\overline{\sigma}^2(a,\sigma^1(b,c)),\sigma^2(b,c))$$

c

b

$\overline{\sigma}^1(\overline{\sigma}^2(a,\sigma^1(b,c)),\sigma^2(b,c))$

$\overline{\sigma}^1(a,\sigma^1(b,c))$

a

$\overline{\sigma}^2(\overline{\sigma}^2(a,b),c)$

c

b

$\sigma^2(\overline{\sigma}^1(a,b),\overline{\sigma}^1(\overline{\sigma}^2(a,b),c))$

$\sigma^1(\overline{\sigma}^1(a,b),\overline{\sigma}^1(\overline{\sigma}^2(a,b),c)$

a
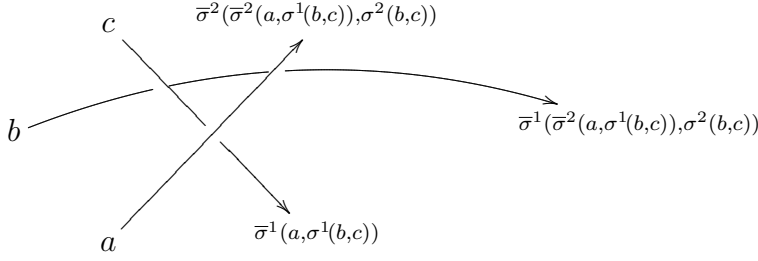
The product of weights for the horizontal line in the first diagram is

$$I = f(b,c)f^{-1}\left(\overline{\sigma}(\overline{\sigma}^2(a,\sigma^1(b,c)),\sigma^2(b,c))\right)$$

and for the second diagram is

$$II = f^{-1}(\overline{\sigma}(a,b))f\left(\overline{\sigma}^1(a,b),\overline{\sigma}^1(\overline{\sigma}^2(a,b),c)\right).$$

Using (1) in I:

$$I = f(b,c)f\left((\sigma^2(\overline{\sigma}^1(a,b),\overline{\sigma}^1(\overline{\sigma}^2(a,b),c)),\overline{\sigma}^2(\overline{\sigma}^2(a,b),c))\right)$$

Take te changes of variables $(x_1,d) = \overline{\sigma}(a,b)$ and $(x_2,x_3) = \overline{\sigma}(d,c)$. Then

$$I = f(\sigma^2(x_1,\sigma^1(x_2,x_3)),\sigma^2(x_2,x_3))f^{-1}(\sigma^2(x_1,x_2),x_3)$$

$$II = f^{-1}(x_1,\sigma^1(x_2,x_3))f(x_1,x_2)$$

We see that if f is a non-commutative 2 cocycle then $I = II$.

The weights that correspond to the other crossings are:

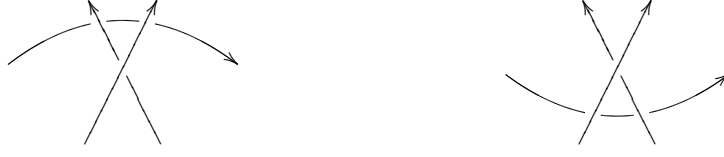$$III = f^{-1}(\overline{\sigma}(a,\sigma^1(b,c))), \ \ IV = f^{-1}(\overline{\sigma}(\overline{\sigma}^2(a,b),c)),$$

changing variables and composing (1) with $1 \times \delta$:

$$III = f^{-1}(\sigma^1(x_1,x_2),\sigma^1(\sigma^2(x_1,x_2),x_3)), \ \ IV = f^{-1}(x_2,x_3)$$

$III = IV$ is verified as f is a 2-cocycle.

Case 4: We only exhibit the diagram corresponding to this case, the computations are similar to the previous case.

This shows, not only, that the product of the weights does not change under Reidemeister moves but the remaining weights stay the same.

For a group element $h \in H$, denote $[h]$ denote the conjugacy class to which h belongs.

Definition 12. The set of conjugacy classes

$$\overrightarrow{\Psi}(L, f) = \overrightarrow{\Psi}_{(X,f)}(L) = \{[\Psi_i(L, \mathcal{C}, f)]\}_{\substack{1 \leq i \leq r \\ \mathcal{C} \in \overline{Col}_X(L)}}$$

where $\Psi_i(L, \mathcal{C}, f) = \prod_{j=1}^{k(i)} B_f(\tau_j^{(i)}, \mathcal{C})$ (the order in this product is following the orientation of the component) is called the *conjugacy biquandle cocycle invariant* of the link.

Theorem 13. *The conjugacy biquandle cocycle invariant Ψ is well defined.*

Proof. The fact that Ψ does not change under Reidemeister moves for fixed base points was proven earlier. A change of base points causes cyclic permutations of Boltzmann weights, and hence the invariant is defined up to conjugacy. $\square$

Proposition 14. *If f, g are two cohomologous non-commutative 2-cocyle functions then $[\Psi_i(L, \mathcal{C}, f)] = [\Psi_i(L, \mathcal{C}, g)]$.*

Proof. Let us suppose $f(x_1, x_2) = \gamma(x_1)g(x_1, x_2)\gamma^{-1}(\sigma^2(x_1, x_2))$. There are four cases to analyse:

(1)

(2)

(3)

(4)
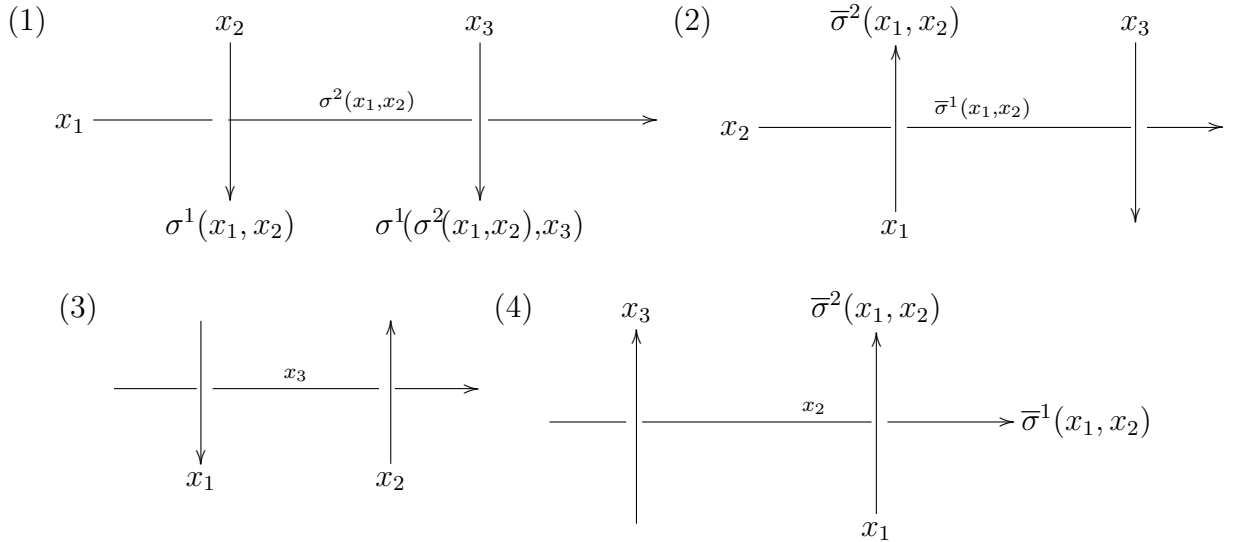
In case 1), the product of weights for the horizontal line is: $f(x_1, x_2)f(\sigma^2(x_1, x_2), x_3) =$

$$= \gamma(x_1)g(x_1, x_2)\gamma^{-1}(\sigma^2(x_1, x_2))\gamma(\sigma^2(x_1, x_2)g(\sigma^2(x_1, x_2), x_3)\gamma^{-1}(\sigma^2(\sigma^2(x_1, x_2), x_3))$$

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In case 2): $f^{-1}(\overline{\sigma}(x_1, x_2))f(\overline{\sigma}^1(x_1, x_2), x_3) =$

$\gamma(\sigma^2(\overline{\sigma}(x_1, x_2)))g^{-1}(\overline{\sigma}(x_1, x_2))\gamma^{-1}(\overline{\sigma}^1(x_1, x_2))\gamma(\overline{\sigma}^1(x_1, x_2))g(\overline{\sigma}^1(x_1, x_2), x_3)\gamma^{-1}(\sigma^2(\overline{\sigma}^1(x_1, x_2), x_3))$

In case 3): $f(\overline{\sigma}(x_1, x_3))f^{-1}(\overline{\sigma}(x_2, x_3)) =$

$\gamma(\overline{\sigma}^1(x_1, x_2))g(\overline{\sigma}(x_1, x_3))\gamma^{-1}(\sigma^2(\overline{\sigma}(x_1, x_3)))\gamma(\sigma^2(\overline{\sigma}(x_2, x_3)))g(\overline{\sigma}(x_2, x_3))\gamma^{-1}(\overline{\sigma}(x_2, x_3))$

And finally in case 4): $f^{-1}(x_2, x_3)f^{-1}(\overline{\sigma}(x_1, x_2)) =$

$\gamma(\sigma^2(x_2, x_3))g^{-1}(x_2, x_3)\gamma^{-1}(x_2)\gamma(\sigma^2(\overline{\sigma}(x_1, x_2)))g^{-1}(\overline{\sigma}(x_1, x_2))\gamma^{-1}(\overline{\sigma}^1(x_1, x_2))$

$\square$

2   Universal non commutative 2-cocycle

Given a biquandle (X, σ) and a group H, recall a non commutative 2-cocycle is a function $f : X \times X \to H$ satisfying

$$f(x, y)f(\sigma^2(x, y), z) = f(x, \sigma^1(y, z))f(\sigma^2(x, \sigma^1(y, z)), \sigma^2(y, z))$$

and

$$f(\sigma^1(x, y), \sigma^1(\sigma^2(x, y), z)) = f(y, z)$$

for any $x, y, z \in X$, and is called type I if in addition $f(x, s(x)) = 1$.

Definition 15. We define $U_{nc} = U_{nc}(X, \sigma)$, the Universal biquandle 2-cocycle group, as the group freely generated by symbols $(x, y) \in X \times X$ with relations

(Unc1) $(x, y)(\sigma^2(x, y), z) = (x, \sigma^1(y, z))(\sigma^2(x, \sigma^1(y, z)), \sigma^2(y, z))$

(Unc2) $(\sigma^1(x, y), \sigma^1(\sigma^2(x, y), z)) = (y, z)$

(Unc3) $(x, s(x)) = 1$

The following is immediate from the definitions:

Proposition 16. *Let X be a biquandle:*

- *Denote $[x, y]$ the class of (x, y) in U_{nc}. The map*

$$\begin{aligned}\pi \colon X \times X &\to U_{nc} \\ (x, y) &\mapsto [x, y]\end{aligned}$$

  is a type I non commutative 2-cocycle.

- *Let H be a group and $f : X \times X \to H$ a type I non commutative 2-cocycle, then there exists a unique group homomorphism $\overline{f} : U_{nc} \to H$ such that $f = \overline{f}\pi$.*

$$\begin{array}{ccc} X \times X & \xrightarrow{\ f\ } & H \\ {\scriptstyle \pi}\downarrow & \nearrow & \\ U_{nc} & {\scriptstyle \exists! \ \overline{f}} & \end{array}$$

In particular, given (X, σ), there exists non trivial 2-cocycles if and only if U_{nc} is a nontrivial group.

Proposition 17. *U_{nc} is functorial. That is, if $\phi : (X, \sigma) \to (Y, \tau)$ is a morphism of set theoretical solutions of the YBeq, namely ϕ satisfy*

$$(\phi \times \phi)\sigma(x, x') = \tau(\phi x, \phi x')$$

then, ϕ induces a (unique) group homomorphism $U_{nc}(X) \to U_{nc}(Y)$ satisfying

$$[x, x'] \mapsto [\phi x, \phi x']$$

Proof. One need to prove that the assignment $(x, x') \mapsto (\phi x, \phi x')$ is compatible with the relations defining $U_{nc}(X)$ and $U_{nc}(Y)$ respectively, and that is clear since $(\phi \times \phi) \circ \sigma = \tau \circ (\phi \times \phi)$. $\qquad\square$

Remark 18. In order to produce an invariant of a knot or link, given a solution (X, σ), we need to produce a coloring of the knot/link by X, and then find a non commutative 2-cocycle, but since U_{nc} is functorial, given X we always have the universal 2-cocycle $X \times X \to U_{nc}(X)$, and hence, we only need to consider all different colorings.

Also, if $\phi : X \to X$ is a bijection commuting with σ, then, given a coloring and its invariant calculated with the universal cocycle, we may apply ϕ to each color and get another coloring, and this will produce the same invariant pushed by ϕ in U_{nc}.

Remark 19. Given a link L of two strands colored using (both) colors $\{1, 2\}$, the invariant obtained is $\Psi_i(L, \mathcal{C}, f) = (i, j)^{ln(i,j)}$ $i, j \in \{1, 2\}$ and $i \neq j$, where $ln(i, j)$ is the linking number between the two strands.

Proof. First notice that every component must be colored by a single color.

To every underarc of the component i with itself will correspond a $(i, i) = 1$ as weight. Then these crossings will not change the product. Then one can think that each component is unknoted with itself. It is well known that any two closed curves in space, if allowed to pass through themselves but not each other, can be moved a concatenation of the following standard positions:
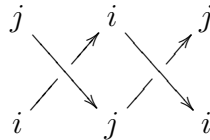
This diagram will contribute a factor $(i, j)^{\mathbf{1}}$ to Ψ_i ((j, i) to Ψ_j) and if trying to calculate the linking number will add $\mathbf{1}$ for each pair of crossings. Analogously in the next diagram:
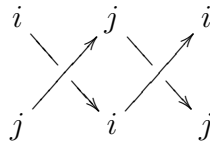
so, the invariant Ψ_i will be $(i, j)^{\frac{a-b}{2}} = (i, j)^{ln(i,j)}$ where a, b are the total amount of positive and negative crossings $\qquad\square$

Example 20. The Whitehead link have linking number cero, the same happens taking the link consisting of two unknots. If you paint these links using $Wada(\mathbb{Z}_3)$ (see example below), Whitehead has only 3 possibilities, while there are 9 ways to paint the pair of unknots.

2.1 Some examples of biquandles of small cardinality

We first list some well-known general constructions generating biquandle solutions:

1. If $(X, \lhd)$ is a rack, one may consider two different solutions of the YBeq:

$$\sigma(x, y) = (y, x \lhd y), \quad \text{and} \quad \overline{\sigma}(x, y) := \sigma^{-1}(x, y) = (y \lhd^{-1} x, x)$$

   these solutions are biquandles if and only if $(X, \lhd)$ is a quandle, namely $x \lhd x = x$ for all $x \in X$, in this case the function s is the identity: $s(x) = x$.

   When considering n.c. 2-cocycles, condition Unc2 is not preserved (in general) if one changes σ with σ^{-1}, so it is relevant to see σ and σ^{-1} as different biquandles.

2. Let $\tau : X \times X \to X \times X$ denote the flip, namely $\tau(x, y) = (y, x)$. Let $\mu, \nu : X \to X$ be two bijections of X. then

$$(\mu \times \nu)\tau(x, y) = (\mu(y), \nu(x))$$

   satisfies YBeq if and only if $\mu\nu = \nu\mu$, and this solution is a biquandle if and only if $\nu = \mu^{-1}$, in this case, the function $s : X \to X$ is equal to μ^{-1}. In this way, the set of bijections of X maps injectively into the set of biquandle structures on X, each conjugacy class of a given bijection maps into an isomorphism class of biquandle structures. Notice that every biquandle structure obtained in this way is involutive, namely $\sigma = (\mu \times \mu^{-1}) \circ \tau$ verifies $\sigma^2 = \text{Id}$.

3. Wada: if G is a group, then the formula $\sigma(x, y) = (xy^{-1}x^{-1}, xy^2)$ is a biquandle, with $s(x) = x^{-1}$. As a particular case, if G is abelian and with additive notation we have $\sigma(x, y) = (-y, x + 2y)$.

4. Alexander biquandle or Alexander switch:

   Let R be a ring, $s, t \in R$ two commuting units, and M an R-module, then

$$\sigma(x, y) = (s \cdot y, t \cdot x + (1 - st) \cdot y), \quad (x, y) \in M \times M$$

   is a biquandle, with function $s(x) = (s^{-1}) \cdot x$. In the particular case $s = -1$, $t = 1$ one gets the abelian Wada's solution. If $s = 1$ then one gets the solution induced by the Alexander rack.

These constructions give a lot of examples, but there are much more. If $|X| = 2$, call $X = \{0, 1\}$, one have the flip, satisfying $s(1) = 1$ and $s(2) = 2$, and this condition fully characterize this solution. If $s(0) \neq 0$ then $s(0) = 1$ and necessarily $s(1) = 0$; this forces $\sigma(0, 0) = (1, 1)$ and $\sigma(1, 1) = (0, 0)$. This is actually a biquandle coming from the bijection construction

$$\sigma(x, y) = (y + 1, x - 1) : x, y \in \mathbb{Z}/2\mathbb{Z}$$

We will call this solution the **antiflip**.

If $|X| = 3$, we call the elements $X = \{0, 1, 2\}$ and identify $X = \mathbb{Z}/3\mathbb{Z}$. The above constructions give the following list:

1. There are three isomorphism classes of quandles of 3 elements:

   (a) the trivial quandle ($x \triangleleft y = x$ for all x, y), this gives the flip solution.

   (b) D_3: $x \triangleleft y = 2y - x$, for $x, y \in \mathbb{Z}/3\mathbb{Z}$, which gives two solutions

   $$\sigma(x, y) = (y, x \triangleleft y) = (y, 2y - x)$$

   and its inverse
   $$\overline{\sigma}(x, y) = (x \triangleleft^{-1} y, x) = (2x - y, x)$$

   (c) another quandle which we call Q_3, with operation given by $- \triangleleft 0 = (12)$ (the permutation $1 \leftrightarrow 2$), and $- \triangleleft 1 = - \triangleleft 2 = \mathrm{Id}$. The solution

   $$\sigma(x, y) = (y, x \triangleleft y)$$

   behaves like the flip for $x, y = 1, 2$, but

   $$\sigma(0, 1) = (1, 0), \ \sigma(1, 0) = (0, 2)$$

   $$\sigma(0, 2) = (2, 0), \ \sigma(2, 0) = (0, 1)$$

   One can check that this equalities can be achieved with the formula

   $$\sigma(x, y) = (y, -x - xy^2) = (y, -x(1 + y^2)) : x, y \in \mathbb{Z}/3\mathbb{Z}$$

   We also have the inverse solution.

2. If $X = \{0\} \coprod \{1, 2\}$, with $\sigma(0, i) = (i, 0)$ and $\sigma(i, 0) = (0, i)$, then the flip on $\{1, 2\}$ produces again the flip on three elements, but the other solution produce a new solution of the YBeq:

   $$\sigma(1, 2) = (1, 2), \ \sigma(2, 1) = (2, 1)$$
   $$\sigma(1, 1) = (2, 2), \ \sigma(2, 2) = (1, 1)$$
   $$\sigma(0, i) = (i, 0), \ \sigma(i, 0) = (0, i)$$

   One may check that this equalities are given by $\sigma(x, y) = (y + x^2 y, x + y^2 x)$.

3. Wada's construction for $\mathbb{Z}_3$ gives the example $\sigma(x, y) = (-y, x - y)$ and its inverse: $\overline{\sigma}(x, y) = (y - x, -x)$.

4. Bijection biquandles:

   (a) Using the bijection $\mu(x) = -x$ we have the solution $\sigma(x, y) = (-y, -x)$,

   (b) if $\mu(x) = x + 1$ then we have the solution $\sigma(x, y) = (y + 1, x - 1)$. One can check that all bijections $\neq \mathrm{Id}$ are conjugated to one of these.

For $M = R = \mathbb{Z}_3$, the units of R are ± 1: the Alexander biquandle gives Wada's for $s = t = -1$, the Dihedral quandle solution for $s = 1$ and $t = -1$, the flip for $s = t = 1$, and the bijection solution $\sigma(x, y) = (-y, -x)$ when $s = t = -1$, so we have no new solution in this small cardinality considering the Alexander biquandle.

In this way, we obtain 10 solutions of the YBeq that are biquandles, three of them (flip, Q_3 and D_3) are quandle solutions. In A. Bartholomew and R. Fenn's classification list (see [BF]) there are 7 biquandles that are not quandles, but we remark that, in Bartholomew and Fenn's list, if a solution σ is listed, then $\overline{\sigma}$ is not listed, even thought σ may not be isomorphic to $\overline{\sigma}$, as solution of the Yang-Baxter equation. For instance, for every quandle $(X, \triangleleft)$, the solution with "name" X is $\sigma(x, y) = (y, x \triangleleft y)$ but the inverse solution $\sigma(x, y) = (x \triangleleft^{-1} y, x)$ do not appear in the list. We use the notation BQ_i^3, $i = 1, \ldots, 10$ for the biquandles solutions of Bartholomew and Fenn, and we denote BQ_i^{3*} the inverse solution with respect to BQ_i^3.

2.2 Computations of U_{nc}

We begin with an explicit computation of the group U_{nc} for a particular example; Wada: $\sigma(x, y) = (-y, x - y)$, which is also BiAlexander with $M = \mathbb{Z}/3\mathbb{Z}, s = -1, t = 1$.

The fixed points are $(x, s(x)) = (x, -x)$, that is $(0, 0), (1, 2), (2, 1)$, so $(0, 0) = (1, 2) = (2, 1) = 1$. Conditions (Unc1-2) are:

$$(x, y)(x - y, z) = (x, -z)(x + z, y - z)$$

$$(-y, -z) = (y, z)$$

From the second equality we get generators $a = (0, 1) = (0, -1)$, $b = (1, 0) = (-1, 0)$ and $c = (1, 1) = (-1, -1)$.

From the first cocycle equation

$$(x, y)(x - y, z) = (x, -z)(x + z, y - z)$$

if $y = sx$ (using $(x, sx) = 1$ and also using Unc2) we get a trivial equality. If $y = sz$ we also get trivial equality, so in principle we have $3^3 - 9 - 9 + 3 = 12$ equations. If we write them *all* in terms of a, b and c we get

$$1 = 1, \ a = a, \ ab = a, \ ac = ac, \ b^2 = b^2, \ bc = c,$$
$$1 = 1, \ c = c, \ b = b, \ 1 = 1, \ ca = ca, \ ca = b$$

of course we can see trivial equations, if we exclude them we get

$$ab = a, \ bc = c, \ ca = b$$

from the first (and also the second) equation we can see that te generator b is trivial. If we write again all the equations with the replacement $b = 1$ we get

$$ca = 1$$

We conclude $U_{nc} = Free(a, c)/(ac = 1) \cong Free(a)$, but also we have described a procedure that can be implemented in a computer program:

1. Add to the set $X \times X$ a new element "1" and begin to define an equivalence relation $(x, s(x)) \sim 1$.

2. from the second condition, add $(y, z) \sim (\sigma^1(x, y), \sigma^1(\sigma^2(x, y), z))$ to the equivalence relation.

   More precisely, given a list of subsets of $(X \times X) \coprod \{1\}$ whose union is $(X \times X) \coprod \{1\}$ (if this is not the case we add the sets $\{(x, y)\}$ to the list) one can easily give an algorithm producing the partition of $(X \times X) \cup \{1\}$ corresponding to the equivalence relation generated by the list of subsets: for each pair of subsets of the list, with nontrivial intersection, we replace these two subsets by their union, run over all different pairs, and iterate until saturate. We call *classes* this list of subsets.

3. From the data *classes*, choose representatives (if the list of subsets is ordered and their members are ordered, just pick the first member for each element of the list). Write down *all* cocycle equations, in terms of these representatives.

4. Eliminate the trivial equations, and

   - for any cocycle equation where 1 appears, in case one found $a.1$, replace it by $1.a$, so we do not count twice the same equation.

   - For any cocycle equation of the form $ac = bc$ or $ca = cb$, add $a \sim b$ and recalculate the equivalence relation that it generates.

   With the new data *classes* go to step 3, and iterate the process until it stabilizes.

   The set of *classes* containing 1 is called S, this is a list of trivial elements in U_{nc}. A set of representatives of the others element of *classes* give a set of generators of U_{nc}. The remaining nontrivial 2-cocycle equations, written in terms of these representatives, give a set of relations. This algorithm produce a relatively small set of generators, and all the relations between them. For instance, in the example above will produce $U_{nc} = Free(a, b)/(ab = 1)$. We have implemented this algorithm in G.A.P, and it produces the following:

   For the dihedral quandle D_3:
Set of generators: $\{f_1 = [1, 2], f_2 = [1, 3], f_3 = [2, 1], f_4 = [2, 3], f_5 = [3, 1], f_6 = [3, 2]\}$.
Trivial elements S: $1 = [1, 1] = [2, 2] = [3, 3]$. Relations:

$$f_1 f_5 = f_2, \ f_2 f_3 = f_1, \ f_3 f_6 = f_4, \ f_4 f_1 = f_3, \ f_5 f_4 = f_6, \ f_6 f_2 = f_5$$

For Wada, the set of generators is $\{f_1 = [1, 2] = [1, 3], f_2 = [2, 2] = [3, 3]\}$. Trivial elements: $1 = [1, 1] = [2, 1] = [2, 3] = [3, 1] = [3, 2]$. Relations:

$$f_2 f_1 = 1$$

An important remark on notation: G.A.P. always gives a numbering of the elements of its objects (and in particular, one can always order them), for instance, $\mathbb{Z}/3\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}\}$ has three elements, that G.A.P. number as $[1, 2, 3]$, where 1 is the first element, 2 is the second and so on; in order to identify the element one has to see the label, and

in this case 1 corresponds to $\overline{0}$, 2 corresponds to $\overline{1}$ and 3 to $\overline{2} = \overline{-1}$, so the equation $1 = [1,1] = [2,1] = [2,3] = [3,1] = [3,2]$ means

$$1 = (\overline{0}, \overline{0}) = (\overline{1}, \overline{0}) = (\overline{1}, \overline{2}) = (\overline{2}, \overline{0}) = (\overline{2}, \overline{1})$$

The new element "1" that we add to $X \times X$ is called [ ], so for example the equality $1 = [1,1] = [2,1] = [2,3] = [3,1] = [3,2]$ comes from the fact that the list (of lists) *classes* contains the element $[[\ ], [1,1], [2,1], [2,3], [3,1], [3,2]]$.

In the following we use the notation as in G.A.P.

For the inverse solution to Wada's: generators $\{f_1 = [1,2] = [2,1] = [3,3], f_2 = [1,3] = [2,2] = [3,1]\}$, trivial elements: $1 = [1,1] = [2,3] = [3,2]$, relations:

$$f_2 f_2 = f_1, \ f_1 f_1 = f_2$$

For the flip on 2 elements $\{1,2\}$: generators $\{f_1 = [1,2], f_2 = [2,1], \}$, trivial elements: $1 = [1,1] = [2,2]$, and no equations at all.

For the flip in 3 elements $\{1,2,3\}$: generators: $\{f_1 = [1,2], f_2 = [1,3], f_3 = [2,1], f_4 = [2,3], f_5 = [3,1], f_6 = [3,2]\}$, trivial elements: $1 = [1,1] = [2,2] = [3,3]$, relations: $f_2 f_1 = f_1 f_2, \ f_4 f_3 = f_3 f_4, \ f_6 f_5 = f_5 f_6$.

Taking the list of biquandles of cardinality 3 from Bartholomew and Fenn's list, adding the inverse solutions (when they are not isomorphic), we obtain the table below.

We remark that the procedure gives not only the number of generators, but the full equivalence class, we omit the full data in the table just for space considerations. We also add to the table the order of σ, and the number of fixed points on the diagonal $\Delta := \{(x,x) : x \in X\}$, for instance, $\Delta^\sigma = \Delta$ if X is a quandle.

name	σ	generators of U_{nc}	equations	order of σ	$\#\Delta^\sigma$
flip	BQ_1^3	6	$f_2 f_1 = f_1 f_2, f_4 f_3 = f_3 f_4,$ $f_6 f_5 = f_5 f_6,$	2	3
a-flip $\cup$ $\{1\}$	BQ_2^3	3	$f_3 f_2 = f_2 f_3,$	2	1
	BQ_3^3	3	−	4	1
	BQ_3^{3*}	3	−	4	1
$Wada(\mathbb{Z}_3)$	BQ_4^3	2	$f_2 f_1 = 1,$	3	1
inv. $Wada(\mathbb{Z}_3)$	BQ_4^{3*}	2	$f_1 f_1 = f_2, \ f_2 f_2 = f_1$	3	1
	BQ_5^3	3	$f_2 f_1 = f_1 f_2,$	2	3
Q_3	BQ_6^3	3	−	4	3
inverse Q_3	BQ_6^{3*}	3	−	4	3
$(x,y) \mapsto (-y,-x)$	BQ_7^3	3	$f_3 f_2 = f_2 f_3,$	2	1
D_3	BQ_8^3	6	$f_1 f_5 = f_2, f_2 f_3 = f_1, f_3 f_6 = f_4,$ $f_4 f_1 = f_3, f_5 f_4 = f_6, f_6 f_2 = f_5,$	3 3	3
inverse D_3	BQ_8^{3*}	0	−,	3	3
	BQ_9^3	2	$f_2 f_2 = f_1, f_1 f_1 = f_2,$	3	0
	BQ_9^{3*}	0	−	3	0
involutive$(\mathbb{Z}_3)$	BQ_{10}^3	2	$f_1 f_2 = f_2 f_1$	2	0

We remark that for some cases (i.e. $BQ_{4,8,9}$) the invariant U_{nc} distinguishes between σ and $\overline{\sigma}$. For BQ_3^3, the generators are the same in the strong sense that the equivalent

classes of generators (as equivalent classes in $X \times X$) are the same, the relations are also the same (no relation at all), so they will give the same knot/link invariant, even though σ and $\overline{\sigma}$ are non isomorphic biquandle solutions.

For most of the cases there is no much more to say in order to describe U_{nc} as a group, for instance $U_{nc}(flip) = U_{nc}(BQ_1^3) = \mathbb{Z}^2 * \mathbb{Z}^2 * \mathbb{Z}^2$: the free product of three copies of $\mathbb{Z}^2$, $U_{nc}(BQ_3^3) \cong U_{nc}(BQ_6^3) \cong F_3$: the free group on 3 generators, $U_{nc}(BQ_2^3) \cong U_{nc}(BQ_5^3) \cong U_{nc}(BQ_7^3) \cong Free(a,b,c)/(bc = cb)$, $U_{nc}(BQ_10^3) \cong \mathbb{Z}^2$,. On the other hand, there are some simplifications for the remaining cases:

$$U_{nc}(BQ_4^3) \cong Free(a,b)/(ab = 1) = Free(a) \cong \mathbb{Z}$$

$$U_{nc}(BQ_9^3) \cong BQ_4^{3*} \cong Free(a,b)/(a^2 = b, \ b^2 = a) \cong Free(a)/(a^3 = 1)$$

$$U_{nc}(D_3) U_{nc}(BQ_8^3) = F_6/(f_1 f_5 = f_2, f_2 f_3 = f_1, f_3 f_6 = f_4, f_4 f_1 = f_3, f_5 f_4 = f_6, f_6 f_2 = f_5)$$

Call $a := f_1$, $b := f_5$ and $c := f_4$, we have

$$ab = f_2, f_2 f_3 = a, f_3 f_6 = c, ca = f_3, bc = f_6, f_6 f_2 = b$$

in particular, we can solve f_2, f_3 and f_6 in terms of a, b, c, so U_{nc} is generated by a, b, c. In order to know the relations, we replace $f_2 = ab$, $f_3 = ca$ and $f_6 = bc$ in the above equations and get

$$abca = a, cabc = c, bcab = b$$

or equivalently

$$abc = 1, cab = 1, bca = 1$$

whose solution is $c = (ab)^{-1}$. We conclude $U_{nc}(D_3) = Free(a,b)$.

The computer program gives the set of generators and relations in a reasonable human time for biquandles of cardinality 12 or less. As a matter of numerical experiment, the groups associated to the inverse solution of biAlexander solution on $\mathbb{Z}_m$, for $s = -1$, and $t = 1$, are cyclic of order m (in a non trivial way) if $m = 3, 5, 7, 11, 13$ (and much more complicated groups for $m = 4, 6, 8, 9, 10, 12$). We don't know if this is a general fact for all primes p. There are nevertheless some general results that can be proven without computer:

Inverse quandle solutions

If $(X, \lhd)$ is a quandle then $\sigma(x,y) = (y, x \lhd y)$ is a biquandle, and condition Unc2 is trivial. But if we consider the *inverse* solution: $\overline{\sigma}(x,y) = (y \lhd^{-1} x, x)$ then condition (Unc2) is not trivial, we have the relations

$$(x,y)(x,z) = (x, z \lhd^{-1} y)(x,y)$$

$$(y \lhd^{-1} x, z \lhd^{-1} x) = (y,z)$$

$$(x,x) = 1$$

We see that, in presence of the second identity, the first one can be modified into

$$(x,y)(x,z) = (x \lhd y, z)(x,y) \qquad (\text{Unc}\overline{Q})$$

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or also $(x \triangleleft z, y \triangleleft z)(x, z) = (x \triangleleft y, z)(x, y)$. Notice that $(\text{Unc}\overline{Q})$, with $x = z$, says

$$(x, y)(x, x) = (x \triangleleft y, x)(x, y) \Rightarrow (x, y) = (x \triangleleft y, x)(x, y) \Rightarrow 1 = (x \triangleleft y, x)$$

This equation, for $x = y$, gives $1 = (x \triangleleft x, x) = (x, x)$. That is, $1 = (x \triangleleft y, x)$ implies the type I condition. So, we may list a set of relations for $U_{nc}(\overline{\sigma})$ in the following way

$$\begin{cases} (x, y)(x, z) & = & (x \triangleleft y, z)(x, y) & (Unc\overline{Q}1) \\ (x \triangleleft y, x) & = & 1 & (Unc\overline{Q}2) \\ (x \triangleleft z, y \triangleleft z) & = & (x, y) & (Unc\overline{Q}3) \end{cases}$$

Corollary 21. *Let Q be a quandle and consider the biquandle solution $\overline{\sigma}(a, b) = (b \triangleleft^{-1} a, a)$. If Q is such that for every $z \in Q$ there exists y with $z = x \triangleleft y$, then $U_{nc}(\overline{\sigma}) = 1$.*

Proof. Given (z, x), let y be such that $z = x \triangleleft y$, then $(z, x) = (x \triangleleft y, x) = 1$. $\qquad \square$

Example 22. If $(X, \triangleleft) = D_n = (\mathbb{Z}/n\mathbb{Z}, x \triangleleft y = 2y - x)$ with n is odd then $U_{nc}(\overline{\sigma}) = 1$.

3 The reduced U_{nc}

We recall that if $f : X \times X \to G$ is a (type I) cocycle and $\gamma : X \to G$ is a function satisfying $\gamma(x) = \gamma(sx)$, then $f_\gamma(x, y) := \gamma(x)f(x, y)\gamma(\sigma^2(x, y))^{-1}$ is also a (type I) 2-cocycle, and the knot/link invariant produce by f is the same as the one produced by f. In particular, one can consider the universal 2-cocycle $\pi : X \times X \to U_{nc}$ and try to see if there is a cohomologous one, simpler that π. This procedure leads to a construction that we call *reduced* universal group:

Definition 23. Let $\gamma : X \to U_{nc}$ be a (set theoretical) map such that $\gamma(x) = \gamma(s(x))$ and $\pi_\gamma : X \times X \to U_{nc}$ given by

$$\pi_\gamma(x, y) = \gamma(x)(x, y)\gamma(\sigma^2(x, y))^{-1}$$

Define $S = \{(x, y) \in X \times X : \pi_\gamma(x, y) = 1 \in U_{nc}\} \subseteq X \times X$ and consider the group U_{nc}^γ defined by

$$U_{nc}^\gamma := U_{nc}/ < \pi(x, y)/(x, y) \in S >$$

Denote $\overline{[x, y]} \in U_{nc}^\gamma$ the class of (x, y) and $p : X \times X \to U_{nc}^\gamma$ the map $p(x, y) = \overline{[x, y]}$.

Theorem 24. *With notations as in the above definition, The map $p : X \times X \to U_{nc}^\gamma$ has the following universal property:*

- *p is a 2-cocycle.*

- *for any group G and 2-cocycle $f : X \times X \to G$, there exists a cohomologous map f^Γ and a group homomorphism $f^\Gamma : U_{nc}^\gamma \to G$ such that f^Γ factorizes through p, that is $f^\Gamma = f^\gamma p$.*

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Proof. The fact that p is a 2-cocycle is immediate. By (16) we obtain the existence of the unique group morphism $\overline{f}$ such that

$$\begin{array}{ccc} X \times X & \xrightarrow{\ f\ } & G \\ {\scriptstyle \pi}\downarrow & \nearrow & \\ U_{nc} & {\scriptstyle \exists!\ \overline{f}} & \end{array}$$

commutes. Define $f^\Gamma := \overline{f} \circ \pi_\gamma$; in diagram:
$$\begin{array}{ccc} X \times X & \xrightarrow{\ \pi_\gamma\ } & U_{nc} \\ & {\scriptstyle f^\Gamma}\searrow & \downarrow{\scriptstyle \overline{f}} \\ & & G \end{array}$$

We have

$$f^\Gamma(x,y) = \overline{f} \circ \pi_\gamma(x,y) = \overline{f} \circ \gamma(x)\overline{f} \circ \pi(x,y)\left(\overline{f} \circ \gamma(\sigma^2(x,y))\right)^{-1}$$

so f^Γ and $\overline{f} \circ \pi = f$ are cohomologous. Using again the universal property of U_{nc}, f^Γ factorizes through U_{nc}, hence there exists a group homomorphism $\overline{f^\Gamma} : U_{nc} \to G$ such that $f^\Gamma = \overline{f^\Gamma} \circ \pi$.

On the other hand, since $\pi_\gamma(S) = 1$ we have $f^\Gamma(S) = \overline{f}(\pi_\gamma(S)) = \overline{f}(1) = 1$, but also $f^\Gamma(S) = \overline{f^\Gamma}(\pi(S))$, so the group homomorphism $\overline{f^\Gamma} : U_{nc} \to G$ induces a map $f^\gamma U_{nc}^\gamma = U_{nc}/\pi(S) \to G$ such that, if $p' : U_{nc} \to U_{nc}/\pi(S)$ is the canonical group projection to the quotient ($p = p' \circ \pi$), then $\overline{f^\Gamma} = f^\gamma \circ p'$. In diagram:

$$\begin{array}{ccccc} & & \overset{p}{\overbrace{\qquad\qquad}} & & \\ X \times X & \xrightarrow{\ \pi\ } & U_{nc} & \xrightarrow{\ p'\ } & U_{nc}/\pi(S) == U_{nc}^\gamma \\ & {\scriptstyle f^\Gamma}\searrow & \downarrow{\scriptstyle \exists!\overline{f^\Gamma}} & {\scriptstyle \exists f^\gamma}\swarrow & \\ & & G & & \end{array}$$

Clearly $f^\Gamma = f^\gamma \circ p$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

For a given γ, the associated U_{nc}^γ is called the *reduced* universal group.

Corollary 25. *Given a biquandle X, if there exists $\gamma : X \to U_{nc}$ such that $U_{nc}^\gamma = 1$ then every 2-cocycle in X is trivial.*

A general example of the above situation is given by some Alexander biquandles.

3.1   The Alexander biquandle

Let $A = \mathbb{Z}[s, t, s^{-1}, t^{-1}]$, X an A-module and $\sigma : X \times X \to X \times X$ given by the matrix

$$\begin{pmatrix} 0 & t \\ s & (1-st) \end{pmatrix}$$

equivalently $\sigma(x,y) = (sy, tx + (1-st)y)$. The condition of being a fixed point is $x = sy$:

$$\sigma(sy, y) = ((sy), t(sy) + (1-st)y) = (sy, y)$$

Cocycle conditions are

$$\begin{cases} (x, y)(tx + (1 - st)y, z) &=& (x, sz)(tx + (1 - st)sz, ty + (1 - st)z) \\ (sy, sz) &=& (y, z) \\ (sy, y) &=& 1 \end{cases}$$

Following M. Graña, we can adapt to the biquandle situation the proof for the quandle case (see Lemma 6.1 of [G]). Consider $\gamma : X \to U_{nc}$ given by $\gamma(x) = (0, cx)$, where $c = (1 + st)^{-1}$. Notice that c is an endomorphism commuting with s, and $(sy, sz) = (y, z) \in U_{nc}$, so

$$\gamma(x) = (0, cx) = (s0, scx) = (0, csx) = \gamma(sx)$$

hence, we can use γ in order to get another 2-cocycle, cohomologous to π. Recall

$$\pi_\gamma(x, y) := \gamma(x)(x, y)\gamma(\sigma^2(x, y))^{-1}$$

where $\sigma(x, y) = (sy, tx + (1 - st)y)$, so

$$\pi_\gamma(x, y) = (0, cx)(x, y)\big(0, c(tx + (1 - st)y)\big)^{-1}$$

in particular

$$\pi_\gamma(0, y) = (0, 0)(0, y)(0, c(1 - st)y)^{-1} = (0, y)(0, y)^{-1} = 1$$

so, the class of $(0, y)$ is trivial in U_{nc}^γ.

Lemma 26. *Let X be an Alexander birack such that $(1 - st)$ is invertible in $\mathrm{End}(X)$. If we define γ as above, then, the following identities hold in U_{nc}^γ:*

1. $(x, 0) = 1$ for all x, and

2. $(a, b) = (a, b + a)$ for all $a, b \in X$.

Proof. 1. From the cocycle condition,

$$(x, y)(tx + (1 - st)y, z) = (x, sz)(tx + (1 - st)sz, ty + (1 - st)z)$$

taking $x = 0 = z$ we get

$$(0, y)((1 - st)y, 0) = (0, 0)(0, ty)$$

but we know that $(0, *) = 1$ in U_{nc}^γ, so $((1 - st)y, 0) = 1$, and because $(1 - st)$ is a unity we conclude $(x, 0) = 1$ for all x.

2. Using the cocycle condition

$$(x, y)(tx + (1 - st)y, z) = (x, sz)(tx + (1 - st)sz, ty + (1 - st)z)$$

and clear z from $tx + (1 - st)sz = 0$, that is, set $z = \frac{-t}{(1-st)s}x$, then

$$(x, y)\left(tx + (1 - st)y, \frac{-t}{(1 - st)s}x\right) = \left(x, s\frac{-t}{(1 - st)s}x\right)(0, ty + (1 - st)z)$$

or

$$(x, y)\left(tx + (1 - st)y, \frac{-t}{(1 - st)s}x\right) = \left(x, \frac{-t}{(1 - st)}x\right) \qquad (*)$$

clearing $y = \frac{-t}{1 - st}x$, get

$$\left(x, \frac{-t}{1 - st}x\right)(0, z) = (x, sz)\left(tx + (1 - st)sz, t\frac{-t}{1 - st}x + (1 - st)z\right)$$

or

$$\left(x, \frac{-t}{1 - st}x\right) = (x, sz)\left(tx + (1 - st)sz, \frac{-t^2}{1 - st}x + (1 - st)z\right) \qquad (\dagger)$$

in particular, using RHS of $(*)$ = LHS of $(\dagger)$ with $y = sz$ we get

$$(x, sz)\left(tx + (1 - st)sz, \frac{-t}{(1 - st)s}x\right) = (x, sz)\left(tx + (1 - st)sz, \frac{-t^2}{1 - st}x + (1 - st)z\right)$$

so

$$\left(tx + (1 - st)sz, \frac{-t}{(1 - st)s}x\right) = \left(tx + (1 - st)sz, \frac{-t^2}{1 - st}x + (1 - st)z\right)$$

Now we simply change variables. Call $a = tx + (1 - st)sz$, then $(1 - st)z = a - \frac{t}{s}x$, replacing

$$\left(a, \frac{-t}{(1 - st)s}x\right) = \left(a, \frac{-t^2}{1 - st}x + a - \frac{t}{s}x\right)$$

or

$$\left(a, \frac{-t}{(1 - st)s}x\right) = \left(a, \frac{-t}{(1 - st)s}x + a\right)$$

Call $b = \frac{-t}{(1 - st)s}x$ (notice that $(x, y) \mapsto (a, b)$ is bijective) and get $(a, b) = (a, b + a)$. $\qquad \square$

Inductively $(a, b) = (a, b + na) \ \forall n \in \mathbb{N}$; if a generates X additively then

$$(a, b) = (a, 0) = 1 \ \forall b \in X$$

Corollary 27. *If p is an odd prime, $X = \mathbb{F}_p$, and $s^{-1} \neq t \in \mathbb{F}_p \setminus \{0\}$, then every cocycle in the Alexander's birack in X is cohomologous to the trivial one. In other words, the reduced Universal group U_{nc}^γ is trivial. In particular every 2-cocycle in D_3 is trivial.*

Remark 28. This generalizes the result of Graña in [G] where he proves the quandle case, that is, the case $s = 1$.

A biquandle example that is not a quandle is the following:

Example 29. If X=Wada($\mathbb{Z}_3$)=biAlexander($M = \mathbb{Z}_3$, $s = t = -1$), then $1 - st = -1 \in \mathbb{Z}_3$ so $U_{nc}^\gamma = 1$ and every non commutative 2-cocycle is trivial. In particular, for any coloring with this biquandle, the corresponding element in U_{nc} is trivial.

Remark 30. In the previous corollary, the hypothesis $|X|$ being prime was essential, the smallest case where it fails is $X = \mathbb{Z}_4$, as an example of computation we calculate the reduced universal group for $X = \mathbb{Z}_4$ and for Z_8 with $s = -1$ and $t = 1$.

3.2   Reduced U_{nc} in computer

It is clear that the procedure that computes U_{nc} in the computer can be trivially adapted for the reduced version, just adding as input a given set $S_0 \subset X \times X$, and begin with $S = S_0 \cup \{(x, sx) : x \in S\}$, instead of simply $S = \{(x, sx) : x \in S\}$. The procedure will actually compute a list of generators and relations of the quotient group $U_{nc}/(S_0)$, so it could be also used to produce other quotients, not only U_{nc}^γ. The advantage of U_{nc}^γ is that it gives the same knot/link invariant as U_{nc}, so in order to find suitable S_0's one can do the following:

- In order to produce functions $\gamma : X \to U_{nc}$ with $\gamma(x) = \gamma(sx)$, consider the equivalence relation on X induced by s, that is the equivalence relation generated by $x \sim s(x)$. Denote $\overline{x}$ the class of x modulo s.

- for all pairs $(x, y) \in X \times X$, consider the coboundary relation

$$f_\gamma(x, y) = \gamma(x) f(x, y) \gamma(\sigma^2(x, y))^{-1}$$

  if $\overline{x} = \overline{\sigma^2(x, y)}$ then $f(x, y)$ is conjugated to $f_\gamma(x, y)$, so $f(x, y) = 1 \iff f_\gamma(x, y) = 1$, so it is clear that (x, y) can not be included in S because of γ.

- if $\overline{x} \neq \overline{\sigma^2(x, y)}$ then we can choose $\gamma : X \to U_{nc}$ such that $\gamma(z) = \gamma(sz)$ for all z and $\gamma(x) f(x, y) = \gamma(\sigma^2(x, y))$.

By the above considerations, it is useful to list all tuples

$$(\overline{x}, (x, y), \overline{\sigma^2(x, y)})$$

with $\overline{x} \neq \overline{\sigma^2(x, y)}$. For any of these elements, add the pair $(\overline{x}, \overline{\sigma^2(x, y)})$ to a set f "used" elements, so we continue with the others with $(\overline{x}, (x, y), \overline{\sigma^2(x, y)})$ with $\overline{x} \neq \overline{\sigma^2(x, y)}$ but $(\overline{x}, \overline{\sigma^2(x, y)})$ not "used". This procedure is easily implemented in G.A.P. For example, for the Dihedral quandle gives

$$[0, [0, 1], 2], \ [0, [0, 2], 1], \ [1, [1, 0], 2]]$$

so we can choose $\gamma(0) = 1, \gamma(2) = [0, 1], \gamma(1) = [0, 2]$ and hence define $S_0 := \{[0, 1], [0, 2]\}$. With this entry, the procedure computing U_{nc}^γ gives $S = X \times X$, that is $U_{nc}^\gamma(D_3)$ is trivial, in agreement with Corollary 27. We give the list of generators and relations of U_{nc}^γ for biquandles of cardinality 3, with the corresponding S_0.

name	σ	generators of U_{nc}^{γ}	equations	S_0
flip	BQ_1^3	6	$f_2f_1 = f_1f_2, f_4f_3 = f_3f_4,$ $f_6f_5 = f_5f_6,$	−
a-flip$\{2,3\} \cup \{1\}$	BQ_2^3	3	$f_3f_2 = f_2f_3,$	−
	BQ_3^3	3	−	−
	BQ_3^{3*}	3	−	−
$Wada(\mathbb{Z}_3)$	BQ_4^3	0	−	$\{[1,2]\}$
inv. $Wada(\mathbb{Z}_3)$	BQ_4^{3*}	1	$f_1^3 = 1$	−
	BQ_5^3	2	−	$\{[1,3]\}$
Q_3	BQ_6^3	2	−	$\{[2,1]\}$
inverse Q_3	BQ_6^{3*}	3	−	−
$(x,y) \mapsto (-y,-x)$	BQ_7^3	3	$f_3f_2 = f_2f_3,$	−
D_3	BQ_8^3	0	−	$\{[1,2],[1,3]\}$
inverse D_3	BQ_8^{3*}	0	−	−
	BQ_9^3	1	$f_1^3 = 1,$	−
	BQ_9^{3*}	0	−	−
involutive$(\mathbb{Z}_3)$	BQ_{10}^3	2	$f_1f_2 = f_2f_1$	−

We exhibit another example of computation using this algorithm:

3.3 Reduced group for Alexander biquandle in $\mathbb{Z}_4$

In order to choose a possible γ, we list as above elements of the form $(\overline{x}, (x,y), \overline{\sigma^2(x,y)})$ with $\overline{x} \neq \overline{\sigma^2(x,y)}$. without repeating $(\overline{x}, (x,y), \overline{\sigma^2(x,y)})$, this gives only one element

$$[0, [0,1], 2]]$$

so we compute U_{nc}^{γ} with $S_0 = \{(0,1)\}$. The G.A.P. answer is

4 generators: $\{f_1=[2,1]=[4,1], f_2=[2,2]=[4,4], f_3=[2,3]=[4,3], f_4=[3,2]=[3,4]\}$, the trivial elements are

$$1 = [1,1] = [1,2] = [1,3] = [1,4] = [2,4] = [3,1] = [3,3] = [4,2]$$

and relations

$$f_1 = f_2f_3, \ f_3 = f_2f_1, \ f_1f_2 = f_3, \ f_3f_2 = f_1,$$
$$f_2f_1 = f_1f_2, \ f_2f_3 = f_3f_2, \ f_3f_1 = f_1f_3$$

Notice that f_4 does not appear in the list of relations. Calling $a = f_2$, $b = f_3$, we get

$$f_1 = ab, \ b = af_1, f_1a = b, \ ba = f_1$$
$$bf_1 = f_1b, \ af_1 = f_1a, \ ab = ba$$

Replacing f_1 by ab we get

$$b = aab, aba = b, \ ba = ab$$

$$bab = abb, \ aab = aba,$$

whose solution is

$$a^2 = 1, \ ab = ba$$

so $U_{nc}^\gamma = Free(a, b, f_4)/(a^2 = 1, ab = ba) \cong (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}) * \mathbb{Z}$.

3.4 Reduced Universal group of 4-cycles in S_4

Another example of application of U_{nc}^γ is the following: consider the quandle

$$Q = \{(1, 2, 3, 4), (1, 2, 4, 3), (1, 3, 4, 2), (1, 3, 2, 4), (1, 4, 3, 2), (1, 4, 2, 3)\}$$

that is, 4-cycles in S_4, with quandle operation $x \triangleleft y = y^{-1}xy$. Recall that $f : Q \times Q \to U_{nc}(Q)$ is cohomologous to f_γ if there exists a function $\gamma : Q \to U_{nc}$ such that

$$f_\gamma(x, y) = \gamma(x)f(x, y)\gamma(x \triangleleft y)^{-1}$$

If we list $(x, (x, y), \sigma^2(x, y))$ without repeating "used" pairs $(x, x \triangleleft y)$, we get

$$[1, [1, 2], 4], [1, [1, 3], 6], [1, [1, 4], 3], [1, [1, 6], 2], [2, [2, 1], 6], [2, [2, 5], 4],$$

$$[2, [2, 6], 5], [3, [3, 1], 4], [3, [3, 4], 5], [3, [3, 5], 6], [4, [4, 2], 5], [5, [5, 2], 6]$$

If we define $\gamma(1) = 1$, $\gamma(4) = [1, 2]$, $\gamma(6) = [1, 3]$, $\gamma(3) = [1, 4]$, $\gamma(2) = [1, 6]$, $\gamma(5) = \gamma(2)[2, 6]$ then $S_0 = \{[1, 2], [1, 3], [1, 4], [1, 6], [2, 6]\}$. If we compute U_{nc} using our algorithm, it gives 30 generators with with 108 equations, while U_{nc}^γ has only 5 generators with 20 equations

$$1 = f_1 f_3, f_2 f_4 = 1, f_3 f_1 = 1, f_5 f_1 = 1, f_1 f_5 = 1,$$

$$f_1 f_1 = f_2, f_1 f_1 = f_4, f_1 f_1 = f_3 f_5, f_5 = f_1 f_4, f_1 = f_2 f_5,$$

$$f_1 = f_3 f_4, f_2 f_1 = f_3, f_4 f_3 = f_1, f_4 f_1 = f_5, f_5 f_2 = f_1,$$

$$f_1 f_2 = f_3, f_1 f_2 = f_2 f_1, f_1 f_3 = f_3 f_1, f_1 f_4 = f_4 f_1, f_1 f_5 = f_5 f_1,$$

Call $a := f_1$, then $f_3 = f_5 = a^{-1}$, $f_2 = f_4 = a^2$, and replacing these values into the 20 equations, the only remaining condition is $a^4 = 1$, we conclude $U_{nc}^\gamma = \langle a : a^4 = 1 \rangle$

This quandle is interesting because it distinguish (using U_{nc}^γ and its canonical cocycle) the trefoil from its mirror image: there are 30 colorings, 6 of them give trivial invariant both for the trefoil and its mirror (these are the 6 constant colorings), but the other 24 colorings gives a^{-1} for the trefoil and a for its mirror, and clearly $a \neq a^{-1}$ in $\langle a : a^4 = 1 \rangle$.

4 Some knots/links and their n.c. invariants

There are 3 quandles of size 3, *none of them* give nontrivial invariant for knots up to 11 crossings. On the other hand, using the biquandle BQ_2^3=aflip$\coprod\{1\}$, from the list of 84 knots with less or equal to 10 crossings, all of them have exactly 3 different colorings, but there are 44 with nontrivial invariant. For instance, figure eight has nontrivial invariant for tree biquandles of size 3: BQ_2^3=aflip$\coprod\{1\}$, BQ_7^3: $\sigma(x, y)=(-y, -x)$, and BQ_9^3.

We illustrate in next table the number of colorings (denoted by c) and nontrivial invariants, for knots up to 6 crossings and biquandles from Bartholomew and Fenn's list:

	3_1	4_1		5_1		5_2	6_1	6_2	6_3	
	c	c		c		c	c	c	c	
BQ_1^3	3	3		3		3	3	3	3	
BQ_2^3	3	3	f_3,f_3	3	f_3^{-1},f_3^{-1}	3	3	3	3	$f_3,f_3,$
BQ_3^3	3	3		3		3	3	3	3	
BQ_4^3	9	1		1		3	9	3	1	
BQ_5^3	3	3		3		3	3	3	3	
BQ_6^3	3	3		3		3	3	3	3	
BQ_7^3	3	3	f_3,f_3	3	f_3^{-1},f_3^{-1}	3	3	3	3	f_3,f_3
BQ_8^3	9	3		3		3	9	3	3	
BQ_9^3	9	3	f_1,f_1,f_1	3	$f_1^{-1},f_1^{-1},f_1^{-1}$	3	9	3	3	f_1,f_1,f_1
BQ_{10}^3	3	0		0		3	3	3	0	

We see that using only number of colorings we can separate this list of knots in 3 groups: $\{3_1,6_1\}$, $\{4_1,5_1,6_3\}$ and $\{5_2,6_2\}$, and using the invariant we can also distinguish 5_1 from all others. One interesting remark is that, using only quandles, there are always the trivial constant colorings, but using biquandles it may happens that a knot admit no coloring at all, as we see with biquandle BQ_{10}^3.

4.1 Alexander biquandle on $\mathbb{Z}_4$ and $\mathbb{Z}_8$

The Borromean link has trivial linking number, but has only 3 colorings using D_3, so we distinguish from three separated unknots. The Unc invariant are trivial for all biquandles of size 3.

On the other hand, for the biAlexander biquandle on $\mathbb{Z}_4$ with $s=-1$ and $t=1$, even though there are 64 colorings, they give non trivial invariants:

Recall Unc=$Free(a,b,f_4)/(b^2=1, ab=ba)$, the invariant for the Borromean link is trivial in 40 colorings, but gives twice $(\alpha,\alpha,1)$, $(\alpha,1,\alpha)$, $(1,\alpha,\alpha)$, $(1,\alpha,\alpha^{-1})$, $(\alpha,1,\alpha^{-1})$, $(\alpha,\alpha^{-1},1)$ on the others, with $\alpha=a$ and $\alpha=a^{-1}$.,

In a similar way, Whitehead's link has trivial linking number, give trivial invariant for all biquandles of size 3 (even though non-trivial number of colorings), with bialexander on $\mathbb{Z}_4$ also give trivial invariant, but with with biAlexander on $\mathbb{Z}_8$ one has non trivial invariants. First we compute U_{nc}^γ for $\mathbb{Z}_8$, $t=1$, $s=-1$, with subset $S_0=\{[1,2],[1,3],[2,2]\}$ (it may be seen that this is a subset corresponding to a convenient γ). The algorithm gives as answer that U_{nc}^γ has 4 generators:

$$f_1=(2,1)=(2,7)=(4,1)=(4,7)=(6,1)=(6,3)=(8,1)=(8,3),$$

$$f_2=(2,3)=(2,5)=(4,3)=(4,5)=(6,5)=(6,7)=(8,5)=(8,7),$$

$$f_3=(2,4)=(2,6)=(4,2)=(4,4)=(6,6)=(6,8)=(8,4)=(8,6),$$

$$f_4=(3,2)=(3,4)=(3,6)=(3,8)=(7,2)=(7,4)=(7,6)=(7,8),$$

Trivial elements are

$$1 = [1,1] = [1,2] = [1,3] = [1,4] = [1,5] = [1,6] = [1,7] = [1,8] = [2,2] = [2,8] = [3,1]$$

$$= [3,3] = [3,5] = [3,7] = [4,6] = [4,8] = [5,1] = [5,2] = [5,3] = [5,4] = [5,5] = [5,6]$$

$$= [5,7] = [5,8] = [6,2] = [6,4] = [7,1] = [7,3] = [7,5] = [7,7] = [8,2] = [8,8]$$

with relations

$$f_1 f_3 = f_2, \ f_3 f_1 = f_2, \ f_3 f_2 = f_1, \ f_3 f_3 = 1,$$

$$f_2 f_1 = f_1 f_2, \ f_2 f_2 = f_1 f_1, \ f_2 f_3 = f_1, \ f_3 f_1 = f_1 f_3, \ f_3 f_2 = f_2 f_3$$

Calling $a := f_1$, $b := f_3$, we get

$$ab = f_2, f_2 a = a f_2, f_2 f_2 = aa, f_2 b = a, ba = f_2, ba = ab, b f_2 = a, b f_2 = f_2 b, bb = 1$$

so $b^2 = 1$, and $f_2 = ab = ba$. we conclude $U_{nc}^\gamma = Free(a, b, f_4)/(b^2 = 1, ab = ba)$.

If we use this biquandle with Whitehead's link we get 64 colorings, 32 of them give trivial invariant, 16 colorings give $(b,1)$ and 16 colorings give as invariant $(1,b)$.

5   Final comments

In the examples we saw, very often the group U_{nc}^γ is non commutative, but we haven't found a knot/link with genuine non commutative invariant, that is, for example a commutator of two non commuting elements of U_{nc}^γ. Also, sometimes U_{nc}^γ have pairs of commuting elements and other non commuting, for instance, $U_{nc}^\gamma(biAlex(Z_8)) = Free(a, b, f_4)/(b^2 = 1, ab = ba)$, but using this biquandle, computing the invariants for knots and links with less than 11 crossings, the elements a and b do not "mix" with f_4. We don't know if this is a general fact or not, that is, if the invariant obtained is the same if we use the abelianization of U_{nc}^γ.

If U_{nc}^γ happens to be abelian, then the information we get with the non commutative invariant is essentially the state-sum invariant for the canonical cocycle $\pi_\gamma : X \times X \to U_{nc}^\gamma$. If this is the case (or if one consider the abelianization of U_{nc}^γ), then our construction can be seen as a natural and nontrivial way to produce interesting 2-cocycles, so that sate-sum invariant becomes a procedure with input only a biquandle, and not a biquandle *plus* a 2-cocycle, because a natural 2-cocycle is always present when one gives a biquandle.

Another natural question about state-sum invariant for biquandles is how to generalize it for 2-cocycles with values in nontrivial coefficients, which is known for quandles, but unknown for biquandles. In order to answer this question, it should be convenient to have an action of some group (to be defined) into the abelian group of coefficients where the 2-cocycle takes values, and if one imitates the quandle case, one should define, for each crossing, an exponent (in this group) that twist the value of the cocycle at that crossing. If the exponent is well-define, that is, for instance it remains unchanged under Reidemeister moves of other crossings, then essentially it must be a non commutative 2-cocycle. The group U_{nc} was the candidate, and in fact this was origin of the present work. In the quandle case there is a natural map $U_{nc}(X) \to G_X$, where G_X is the group generated by X with relations $xy = zt$ if $\sigma(x,y) = (z,t)$; the map $U_{nc}(X) \to G_X$ is simply determined by $(x_1, x_2) \mapsto x_2$. So, for quandles, G_X-modules are natural candidates for

coefficients (see [CEGS]), or also $U_{nc}(X)$-modules, or quandle-modules as considered in [AG]. We hope 2-cocycles with values in U_{nc}-modules will allow to define more general state-sum invariants, but at the moment we don't know how, we end remarking that for biquandles, there is no general well-defined map $U_{nc}(X) \to G_X$, and $U_{nc}(X)$ sometimes is the trivial group.

In [FG] one can found the GAP programs computing colorings, U_{nc}^γ, and invariants for knots and links given as planar diagrams.

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