Form invariant transformations between n– and m– dimensional flat Friedmann–Robertson–Walker cosmologies

Mauricio Cataldo*
Departamento de Física, Facultad de Ciencias, Universidad del Bío–Bío, Avenida Collao 1202, Casilla 5-C, Concepción, Chile.

Luis P. Chimento†
Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria Pabellón I, 1428 Buenos Aires, Argentina.
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We illustrate how the group of symmetry transformations, which preserve the form of the n–dimensional flat Friedmann–Robertson–Walker cosmologies satisfying Einstein equations, acts in any dimension. This group relates the energy density and the isotropic pressure of the cosmic fluid to the expansion rate. The freedom associated with the dimension of the space time yields assisted inflation even when the energy density of the fluid is a dimensional invariant and enriches the set of duality transformations leading to phantom cosmologies.

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I. INTRODUCTION

Over the past several years there has been much interest in examining cosmology in higher dimensions to see if the standard four–dimensional Friedmann–Robertson–Walker (FRW) cosmology can be recovered. The idea that our physical four–dimensional Universe is embedded in a higher–dimensional spacetime has also also attracted the attention of particle physicists and astrophysicists. Theoretical motivation for such attempts can be found within the framework of many theories of unification, among them string, superstring and M theory, require extra spatial dimensions to be consistent. Until today a number of important solutions of Einstein equations in higher dimensions have been obtained and studied, and they have led to important generalizations and wider understanding of gravitational fields. In this respect, of interest are the works on n-dimensional black holes[1], Kaluza–Klein inflationary cosmologies [2], circularly symmetric perfect fluids [3], black holes on branes [4], and recently, contributions on braneworld scenarios [5, 6, 7].

It is interesting to note that some authors have also considered phenomenological analysis in higher dimensional cosmology. For example, the phenomenological analysis of five-dimensional cosmology was stimulated by the work of Binetruy, Deffayet, and Langlois [8], and subsequently by the Randall-Sundrum model [6].

On the other hand, scalar fields play a crucial role in describing cosmological models. In the standard big-bang theory such fields are included for solving most of the problems found at very early times in the evolution of the universe, and are called the "inflaton" scalar field [9, 10, 11]. This scalar field is characterized by its scalar potential.

At the same time, measurements of the luminosity–redshift relations observed for the discovered type Ia supernovae with redshift $z > 0.35$ [12, 13], indicate that at present the universe is expanding with an accelerated fashion suggesting a net negative pressure for the universe. One plausible explanation of this astronomical observation is based on the introduction of a scalar field, which is called the "quintessence" or "dark energy" scalar field.

Although these scalar fields are quite different in nature, there are authors who think that the "inflaton" and the "quintessence" fields might be of the same nature, in which a very specific scalar potential form is used [14].

In Ref. [15] it was shown that in several physical problems the Einstein field equations for flat FRW cosmological models and Bianchi I-type metric containing a scalar field can be linearized and solved by writing them in invariant form. In all these cases explicit use has been made of the non-local transformation group. The symmetry transformations that preserve the form of the Einstein equations introduce an alternative concept of equivalence between different physical problems [16]. Cosmological models are equivalent when the corresponding dynamical equations are form invariant under the action of that group. Hence, it will be interesting to investigate the consequences of this group when the dimension of the space time is taken to be a free parameter of the theory. Notice that the multidimensional point of view has been used in general relativity to extract information or to endow with properties fields and/or physical systems belonging to spaces of different dimensions.

In this sense, the purpose of the present work is to il-

*mcataldo@ubiobio.cl
†chimento@df.uba.ar

lustrate how a group of symmetry transformations acts on n– and m– dimensional flat FRW cosmologies which satisfy Einstein equations. This group relates the energy density and the isotropic pressure of the cosmic fluid *(source variables)* to the expansion rate *(geometrical variable)* linking two different cosmologies, one of which could be accelerated. Hence, even when the energy density is a dimensional invariant we can get assisted inflation [16]-[17] driven by the freedom associated with the dimension of the space time. In the case of requiring the condition (10) of Ref. [18] the linked cosmologies become identical, they share the same scale factor, or there is a duality between contracting and superaccelerated expanding scenarios associated with phantom cosmologies [19]-[20], i.e. the scale factor of one of them is the inverse of that of the other. The above formulation also can be applied to a self–interacting scalar field using its conventional perfect fluid description.

The outline of the present paper is as follows: In Sec. II we review the well known Einstein equations for the FRW metrics in n– and m– dimensional gravities coupled to a perfect fluid. The case for constant bariotropic indices is discussed in detail. In Sec. III we briefly review the field equations for the FRW metrics coupled to a scalar field. In Sec IV some conclusions are given.

II. DIMENSIONAL FORM INVARIANCE SYMMETRY IN FLAT FRW SPACETIMES

We shall assume the spherically symmetric flat FRW metric of an n–dimensional spacetime given by

$$ds^2 = -dt^2 + a_n(t)^2 \left(dr^2 + r^2 d\Omega_{n-2}^2\right), \tag{1}$$

where the spherical sector, related to $n-2$ angular variables θ_i, with i running from 1 to $(n-2)$, is determined to be $d\Omega_{n-2}{}^2 = d\theta_1{}^2 + \sin^2\theta_1 \mathrm{d}\theta_2{}^2 + ... + \sin^2\theta_1 ... \sin^2\theta_{n-3} \mathrm{d}\theta_{n-2}{}^2$, for $n \geq 3$. The Einstein equations for an n–dimensional spacetime are given by

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{R}{2} g_{\alpha\beta} = \kappa_n T_{\alpha\beta},$$

where Greek indices run from 1 to n, and κ_n stands for the multidimensional gravitational constant.

The independent Einstein equations for the n–dimensional FRW metric (1) filled with a perfect fluid are:

$$-G_t{}^t = \frac{(n-1)(n-2)}{2} \frac{\dot{a}_n^2}{a_n^2} = \kappa_n \rho_n, \tag{2}$$

$$-G_r{}^r = (n-2)\frac{\ddot{a}_n}{a_n} + \frac{(n-2)(n-3)}{2} \frac{\dot{a}_n^2}{a_n^2} = -\kappa_n p_n, \tag{3}$$

where κ_n, a_n, ρ_n and p_n are the gravitational constant, scale factor, energy density and the pressure in an n–dimensional spacetime respectively. Dots denote differentiation with respect to t. The dependent Einstein equations are related as $G_{\theta_{n-2}}{}^{\theta_{n-2}} = ... = G_{\theta_1}{}^{\theta_1} = G_r{}^r$.

We can replace Eq. (3) by the conservation equation:

$$\dot{\rho}_n + (n-1)\frac{\dot{a}_n}{a_n}(\rho_n + p_n) = 0, \tag{4}$$

which, as is well known, is derivable from the equation $T^{\alpha\beta}{}_{;\beta} = 0$. Thus the Einstein equations for an n–dimensional flat FRW cosmology are given by Eq (2) and Eq (4), which we shall rewrite in the form:

$$\alpha H^2 = \kappa \rho, \qquad \dot{\rho} + \beta H(\rho + p) = 0, \tag{5}$$

where $\alpha = (n-1)(n-2)/2$, $H = \dot{a}/a$, $\beta = (n-1)$ and we have omitted the subindex n.

For a different m–dimensional flat FRW cosmology the Einstein equations are given by

$$\bar{\alpha}\bar{H}^2 = \bar{\kappa}\bar{\rho}, \qquad \dot{\bar{\rho}} + \bar{\beta}\bar{H}(\bar{\rho} + \bar{p}) = 0, \tag{6}$$

where $\bar{\alpha} = (m-1)(m-2)/2$, $\bar{H} = \dot{\bar{a}}/\bar{a}$, $\bar{\beta} = (m-1)$, and $\bar{a}$, $\bar{\kappa}$, $\bar{\rho}$ and $\bar{p}$ are the scale factor, gravitational constant, energy density and the pressure in an m–dimensional spacetime respectively.

By "invariant form" we shall mean that the system of equations (5) transform into Eqs. (6) under the symmetry transformations:

$$\bar{\rho} = \bar{\rho}(\rho), \tag{7}$$

$$\bar{H} = \pm\theta\sqrt{\frac{\bar{\rho}}{\rho}}\, H, \tag{8}$$

$$\bar{\rho} + \bar{p} = \pm\frac{\beta}{\bar{\beta}\theta}\sqrt{\frac{\bar{\rho}}{\rho}}\frac{d\bar{\rho}}{d\rho}(\rho + p), \tag{9}$$

where $\theta = (\alpha\bar{\kappa}/\bar{\alpha}\kappa)^{1/2}$ and $\bar{\rho} = \bar{\rho}(\rho)$ is an invertible function. Notice that always $\theta^2 = \alpha\bar{\kappa}/\bar{\alpha}\kappa \geq 0$ since $n \geq 3$ and $m \geq 3$, and that these form invariant transformations are defined without imposing any restriction on the cosmic fluid. When the dimension of both cosmologies coincides, then we have $\alpha = \bar{\alpha}$, $\beta = \bar{\beta}$, $\theta = 1$, and these transformations reduce to that of Ref. [16] and are independent of the dimension where the cosmic fluid "lives".

The invariant quantities associated with the set of transformations (7)-(9) are

$$\frac{\bar{\alpha}\bar{H}^2}{\bar{\kappa}\bar{\rho}} = \frac{\alpha H^2}{\kappa\rho}, \tag{10}$$

$$\frac{d\bar{\rho}}{\bar{\beta}\bar{H}(\bar{\rho} + \bar{p})} = \frac{d\rho}{\beta H(\rho + p)}. \tag{11}$$

The first invariant expresses that the expansion of the universe is proportional to the multidimensional gravitational constant and to the energy density contained in the universe. However, the expansion dims with the dimension of the space time because it is proportional to the factor $1/\alpha$. The second invariant expresses the fact that the transformations do not modify the cosmic time.

In the case of considering perfect fluids with equations of state $p = (\gamma - 1)\rho$ and $\bar{p} = (\bar{\gamma} - 1)\bar{\rho}$ in n and m–dimensional spacetimes respectively, we conclude that

the bariotropic indices γ and $\bar{\gamma}$ transform as

$$\bar{\gamma} = \frac{\bar{\rho}+\bar{p}}{\bar{\rho}} = \pm\frac{\beta}{\bar{\beta}\,\theta}\left(\frac{\rho}{\bar{\rho}}\right)^{3/2}\frac{d\bar{\rho}}{d\rho}\,\gamma \tag{12}$$

under the symmetry transformations (7)–(9). In what follows, the upper and the lower signs will be referred to as the (+) and (−) branches respectively.

These general form–invariant transformations relate cosmologies in two different dimensions. For instance, they can be used for generating a new m–dimensional FRW cosmology from a given cosmology in (3+1)–dimensions (with $n = 4$), or in (2+1)–dimensions (with $n = 3$), where a lot of them are known.

In this direction we investigate the consequences of a simple example generated by the following transformation between energy densities:

$$\bar{\rho} = b^2\rho, \tag{13}$$

with b a positive constant. Inserting the latter in (8) and (12) we find that a and $\bar{a}$ are related to each other by

$$\bar{a} = a^{\pm b\theta}, \qquad b\theta = \pm\frac{\beta\gamma}{\bar{\beta}\bar{\gamma}}, \tag{14}$$

where without loss of generality the constant of proportionality has been set equal to unity. Hence, the deceleration parameter $q(t) = -H^{-2}\ddot{a}/a$ transforms as

$$\bar{q} = -1 \pm \frac{1}{b\theta}(q+1). \tag{15}$$

When the energy density is a dimensional invariant, i.e. for the condition $\bar{\rho} = \rho$ or $b = 1$, we get the relation $\theta = \pm\beta\gamma/(\bar{\beta}\bar{\gamma})$ which may be interpreted as a constraint for the bariotropic indices $\bar{\gamma}$ and γ, since the pressures are not the same in both dimensions [21]. In this case an expanding universe with a positive deceleration parameter, (+) branch, transforms into an accelerated one if θ is taken to be large enough. This means that by adequately selecting the dimension of the space time we can get assisted inflation. For instance, for constant γ and $\bar{\gamma}$, the Einstein equations lead to power law solutions:

$$\bar{a} = t^{2/\bar{\beta}\bar{\gamma}} = t^{\pm 2\theta/\beta\gamma} = a^{\pm\theta},$$

after using Eq. (14) for $b = 1$. Then, for the (+) branch and $2\theta > \beta\gamma$ we obtain an accelerated expansion.

It is interesting to investigate the choice $b\theta = 1$, because from (14) we have $\bar{a} = a^{\pm 1}$. Now we pay attention to the (−) branch since in this case the symmetry transformation $a \to a^{-1}$ (duality) maps the initial singularity at $t = 0$, $a(0) = 0$, into other kind of singularity $\bar{a}(0) = \infty$, i.e, the scale factor $\bar{a}$ and the scalar curvature $\bar{R}$ diverge at a finite time. In particular, for $\gamma > 0$ the (−) branch of the power law solution $\bar{a} = (-t)^{-2/\beta\gamma}$ defined for $t < 0$ diverges in the future at $t = 0$. This kind of singularity dubbed "big rip" is a characteristic of phantom or ghost cosmologies. Hence, using the condition $\beta\gamma = -\bar{\beta}\bar{\gamma}$, we obtain the relation between the n– and m– dimensional flat FRW cosmologies:

$$\frac{\bar{\gamma}\bar{\kappa}}{m-2} = -\frac{\gamma\kappa}{n-2}, \tag{16}$$

defining the phantom sector of our model.

The case of considering structural invariance of the scale factors $a(t)$ in n– and m–dimensional FRW cosmologies, i.e. dimensional invariance of the scale factor as it was assumed in Ref. [18], corresponds to selecting the (+) branch of the transformations (7)–(9) generated by Eq. (13) with $b\theta = 1$. Thus, the energy densities corresponding to n– and m–dimensional FRW cosmologies are related by $\bar{\rho} = \rho/\theta^2$ and bariotropic indexes transform as

$$(n-1)\,\gamma \quad \rightleftharpoons (m-1)\,\bar{\gamma}, \tag{17}$$

where we have used the notation of Ref. [18]. Note that the results we have obtained by applying the transformations (7)–(9) enlarge those of Ref. [18] and add the duality between contracting and expanding cosmologies through the (−) branch of the transformations which was not considered in the previous paper.

Finally, when the dimension of both cosmologies coincides we have $\alpha = \overline{\alpha}$, $\beta = \overline{\beta}$, $\theta = 1$, and Eqs. (7)–(9) (or Eqs. (6)–(8) of Ref. [16]) are independent of the dimension where the cosmic fluid "lives".

III. THE SCALAR FIELD CASE

Let us consider a self–interacting scalar field ϕ driven by a potential $V(\phi)$ having an associated perfect fluid energy tensor with energy density and pressure given by

$$\rho = \frac{1}{2}\,\dot{\phi}^2 + V(\phi), \qquad p = \frac{1}{2}\,\dot{\phi}^2 - V(\phi), \tag{18}$$

$$\bar{\rho} = \frac{1}{2}\,\dot{\bar{\phi}}^2 + \bar{V}(\bar{\phi}), \qquad \bar{p} = \frac{1}{2}\,\dot{\bar{\phi}}^2 - \bar{V}(\bar{\phi}), \tag{19}$$

in n–and m–dimensional FRW space respectively. Now, Eqs. (18)-(19) along with Eqs. (7)-(9) give the rules of transformation for ϕ and V

$$\dot{\bar{\phi}}^2 = \bar{\rho} + \bar{p} = \pm\frac{\beta}{\bar{\beta}\,\theta}\sqrt{\frac{\rho}{\bar{\rho}}}\,\frac{d\bar{\rho}}{d\rho}\,\dot{\phi}^2, \tag{20}$$

$$\bar{V} = \bar{\rho} \mp \frac{1}{2}\,\frac{\beta}{\bar{\beta}\,\theta}\sqrt{\frac{\rho}{\bar{\rho}}}\,\frac{d\bar{\rho}}{d\rho}\,\dot{\phi}^2. \tag{21}$$

To illustrate an application of the latter we assume the transformation (13)-(14), then we obtain

$$\dot{\bar{\phi}}^2 = \pm\frac{\beta b}{\bar{\beta}\,\theta}\,\dot{\phi}^2, \tag{22}$$

$$\bar{V} = \frac{b}{2}\left[b \mp \frac{\beta}{\bar{\beta}\,\theta}\right]\dot{\phi}^2 + b^2 V(\phi). \tag{23}$$

In addition, the scale factor transforms according to Eq. (14). Notice that the (+) branch gives the m–dimensional analog of the n–dimensional original cosmological model, while the (−) branch leads us, as in the previous section, to phantom cosmologies. The transformed scalar fields are related to the original one by a dimensional generalization of the transformation considered in Ref. [19]. It represents a generalization of the Wick rotation.

There is an interesting case to be investigated, for instance, let us consider the restricted group of transformations defined by the condition $\bar{V} \propto V$. Then, from Eq. (23) we get $V \propto \dot{\phi}^2$ and $\rho \propto \dot{\phi}^2$. In this case Eq. (5) can be solved by assuming a power law scale factor with a scalar field of the form $\phi \propto \ln t$. The final solution is

$$a = t^{2/\beta\gamma}, \tag{24}$$

$$V = \frac{(2-\gamma)\phi_0^2}{2\gamma}\, e^{-2\phi/\phi_0}, \quad \phi = \phi_0 \ln t, \tag{25}$$

where $\phi_0 = (2/\beta)(\alpha/\kappa\gamma)^{1/2}$. These equations represent the dimensional generalization of the ordinary exponential potential and its associated power law solutions. The respective solution in the m-dimensional flat FRW space time is obtained by inserting the above n-dimensional solution (25) into the transformations (22) and (23); so a straightforward calculation gives

$$\bar{V} = \frac{(2-\bar{\gamma})\bar{\phi_0}^2}{2\bar{\gamma}}\, e^{-2\bar{\phi}/\bar{\phi}_0}, \qquad \phi = \bar{\phi}_0 \ln t, \tag{26}$$

where we have used Eq. (14) and $\bar{\phi}_0 = b\phi_0(\bar{\gamma}/\gamma)^{1/2}$. The scale factor is given by Eqs. (14) and (24):

$$\bar{a} = t^{2/\bar{\beta}\bar{\gamma}} = t^{\pm 2b\theta/\beta\gamma} = a^{\pm b\theta}. \tag{27}$$

This example shows that the form invariant transformations can be used to generate new cosmological solutions in an m-dimensional gravity from a seed one in an n-dimensional gravity.

IV. CONCLUSIONS

The main goal of the present work is to illustrate how a group of symmetry transformations acts on n– and m–dimensional flat FRW cosmologies which satisfy Einstein equations. Cosmological models are equivalent since the corresponding dynamical equations become form invariant under the action of this group. For two different cosmologies, i.e. n– and m– dimensional flat FRW metrics, this group relates their energy densities, isotropic pressures and the scale factors to generic dimensional parameters α, β, $\bar{\alpha}$ and $\bar{\beta}$. If the dimension of both cosmologies coincides, i.e. $n = m$, then the group of symmetry transformations relates their energy densities, isotropic pressures and the scale factors only. In addition, a form invariant symmetry transformation which violates the dominant energy condition induces a duality between contracting and superaccelerated expanding scenarios generating phantom cosmologies. All these multidimensional considerations can also be formulated for the scalar field associating a perfect fluid description with the stress energy tensor.

Finally, these general form–invariance transformations can be considered as an algorithm for generating a new m–dimensional FRW cosmology from a known n–dimensional cosmology. For instance, we can use as a seed solution one given known cosmology in (3+1)–dimensional gravity, where there exist a lot of solutions.

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[1] R.C. Myers and M.J. Perry, Ann. Phys. (N.Y.) **172**, 304 (1986).
[2] R.B. Abbott, S.M. Barr, and S.D. Ellis, Phys. Rev. D **30**, 720 (1984).
[3] A.A. García and C. Campuzano, Phys. Rev. D **67**, 064014 (2003).
[4] R. Emparan, G.T. Horowitz, and R.C. Myers, J. High Energy Phys. **01**, 007 (2000); **01**, 021 (2000).
[5] A. Lukas, B. Ovrut, K. Stelle, and D. Waldram, Phys. Rev. D **59**, 086001 (1999); Nucl. Phys. B **552**, 246 (1999).
[6] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999); **83**, 4690 (1999).
[7] Y. Himemoto and M. Sasaki, Phys. Rev. D **63**, 044015 (2001); Y. Himemoto, T. Tanaka, and M. Sasaki, ibid. **65**, 104020 (2002).
[8] P. Binetruy, C. Deffayet, and D. Langlois, Nucl. Phys. B **565**, 269 (2000).
[9] A.H. Guth, Phys. Rev. D **23**, 347 (1981);
[10] A.D. Linde, Phys. Lett. B **108**, 389 (1982); Phys. Lett. B **129**, 177 (1983).
[11] A. Albrecht, P.J. Steinhardt, Phys. Rev. D **48**, 220 (1982).
[12] S. Perlmutter *et al*, Nature **391**, 51 (1998); S. Perlmutter *et al*, Ap. J. **517**, 565 (1999).

[13] P. M. Garnavich *et al*, Astrophys. J.**509**, 74 (1998).
[14] P. J. E. Peebles and A. Vilenkin, Phys. Rev. D**59**, 063505 (2000).
[15] L. P. Chimento J. Math. Phys. **38**, 2565, (1997).
[16] L.P. Chimento, Phys. Rev. D **65**, 063517 (2002).
[17] Aguirregabiria J. M., Chimento L. P., Jakubi A. S. and Lazkoz R. Phys. Rev. D **67**, 083518 (2003).
[18] A.A. García, A. García–Quiroz, M. Cataldo and S. del Campo, Phys. Rev. D **69**, 041302 (2004).
[19] L.P. Chimento and R. Lazkoz, Phys. Rev. Lett. **91**, 301 (2003). L.P. Chimento and R. Lazkoz, International Journal of Modern Physics D (in press), astro-ph/0405518. L.P. Chimento, gr-qc/0503049.
[20] S. Nojiri and S. Odintsov, Phys. Lett. B **571**, 1 (2003); D.J. Liu and X.Z. Li, Phys. Rev. D **68**, 067301 (2003); Y.S. Piao and E. Zhou, Phys. Rev. D **68**, 083515 (2003); M.P. Dąbrowski, T. Stachowiak, and M. Szydłowski, Phys. Rev. D **68**, 103519 (2003); P.F. González-Díaz, Phys. Lett. B **586**, 1 (2004); A. Feinstein and S. Jhingan, Mod. Phys. Lett A **19**, 457 (2004); H. Stefancic, astro-ph/0310904; Y.S. Piao and E. Zhou, astro-ph/0401231; G. Calcagni, hep-ph/0402126; S. Nojiri and S.D. Odintsov, astro-ph/0403622; J.G. Hao and X.Z. Li, astro-ph/0404154; Z. K. Guo, Y.S. Piao and Y.Z. Zhang, Phys. Lett. B **594**, 247 (2004); F. Piazza and S. Tsujikawa, hep-th/0405054; P.X. Wu and H. W. Yu, astro-ph/0407424; S. Tsujikawa and M. Sami Phys. Lett. B 603, 113 (2004) Y.S. Piao and Y.Z. Zhang, Phys. Rev. D **70**, 043516 (2004); Z. K. Guo, R.G. Cai and Y. Z. Zhang, astro-ph/0412624; E. Babichev, V. Dokuchaev and Yu. Eroshenko, Phys. Rev. Lett. **93**, 021102 (2004); Z.K. Guo, Y.S. Piao, X.M. Zhang and Y.Z. Zhang, Phys. Lett. B **608**, 177 (2005); Z.K. Guo and Y.Z. Zhang, Phys. Rev. D **71**, 023501 (2005); S. Parampreet, gr-qc/0502086; R. Curbelo, T. González and I. Quirós, astro-ph/0502141; Y.S. Piao, Phys. Lett. B **606**, 245 (2005).
[21] If we want to make them dimensionally invariant, we must require that the bariotropic indices be equal, i.e. $\bar{\gamma} = \gamma$. Thus, the scale factors are related to each other by $\bar{a} = a^{\pm\beta/\bar{\beta}}$.