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# GENERALIZED HOPF BIFURCATION IN A FREQUENCY DOMAIN FORMULATION

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The multiplicity problem of limit cycles arising from a weak focus is addressed. The proposed methodology is a combination of the frequency domain method to handle some degenerate Hopf bifurcations with the powerful tools of the singularity theory. The frequency domain approach uses the harmonic balance method to study the existence of periodic solutions. On the other hand, the singularity theory provides the conditions and formulas for the classification problem of the unfolding of the singularity in terms of the distinguished and auxiliary parameters. A classical example introduced by Bautin is shown in which the multiplicity of limit cycles is recovered by using this type of hybrid methodology and standard software in the continuation of periodic solutions (LOCBIF and XPPAUT). For small amplitude limit cycles, the proposed methodology gives accurate results.

*Keywords*: Limit cycles; Hopf bifurcation; singularity theory; frequency domain; harmonic balance method.

## 1. Introduction

The appearance of oscillations, also known as limit cycles in the classical theory, has been an active subject of investigations due to its stability implications in engineering systems. The rigorous analytical characterization of the emergence of oscillations through a smooth variation of a distinguished parameter has been possible since the proof of the well-known Hopf bifurcation theorem [Marsden & MacCracken, 1976]. This result gives the necessary and sufficient conditions for the appearance of stable or unstable oscillations when a single pair of complex eigenvalues of the linearized equations cross the imaginary axis and change, normally, the stability of the equilibrium point at criticality. The theorem also provides a local approximation of the periodic solutions which is also very useful for engineering applications when looking for quantitative results. A typical graph considering the amplitude of periodic solutions versus the variation of the distinguished parameter gives a parabolic form which can be located to the left or to the right of the criticality. If the parabolic form is unstable and evolves to the left enclosing a stable equilibrium point, the type of singularity is known as subcritical Hopf bifurcation. On the other hand, if the parabolic form is stable and evolves to the right of criticality enclosing an unstable stationary state, the type of singularity is called a supercritical Hopf bifurcation (see Fig. 1). The subcritical Hopf bifurcation is dangerous in engineering since the basin of attraction of the equilibrium point is affected by the presence of the unstable limit cycle which is a nonlinear phenomenon. On the other hand, the supercritical Hopf bifurcation presents an emergent stable periodic solution immediately after the equilibrium point loses its stability.

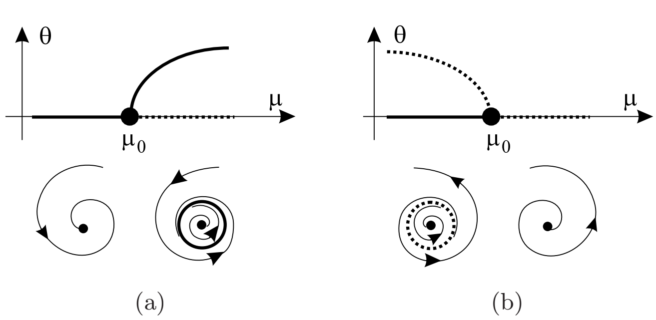


Fig. 1. Hopf bifurcation diagram obtained by varying parameter $\mu$: Solid lines represent stable equilibrium and stable limit cycles, while dashed-lines represent unstable equilibrium and unstable limit cycles (top diagrams). Phase portrait: (a) supercritical and (b) subcritical Hopf bifurcation (bottom diagrams).

In the classical Hopf bifurcation theorem a unique limit cycle is involved but if one of the postulates governing the stability of the periodic solution at criticality fails, then multiple periodic solutions can arise. This failure was studied for the first time in [Bautin, 1952] and [Sibirskii, 1965] in planar systems with nonlinearities of polynomial type of degrees two and three, respectively. Later, other authors studied the multiplicity of limit cycles with other mathematical tools, notably as Takens [1973] did in his now classical work. Furthermore, complete pictures of bifurcation diagrams showing nested limit cycles after the variation of distinguished and auxiliary parameters have been introduced in [Golubitsky & Langford, 1981] using the singularity theory. The number of auxiliary parameters, roughly speaking, is related to the minimum number of independent parameters necessary to describe all the possible bifurcation scenarios obtained by slight perturbations from the degenerate conditions. Furthermore, the mentioned articles put some preliminary results connecting the appearance of multiple periodic solutions with universal normal forms and, at the same time, contributed to enlightening, in certain aspects, of the famous second part of the Hilbert sixteenth problem [Li, 2003]. That problem, which is still unsolved, refers to finding a formula relating the maximum number of limit cycles in planar polynomial-type nonlinear systems and the maximum degree of the polynomial. The connection between this famous problem and the appearance of multiple periodic solutions has been reported in [Farr *et al.*, 1989]. In simple words, the singularity theory gives a powerful methodology to classify different bifurcation diagrams when some hypotheses of the classical Hopf bifurcation theorem fail. The failing postulates are: (a) the complex eigenvalues do not cross transversally the imaginary axis at criticality, (b) some nontrivial coefficients related with the stability of periodic solutions at criticality vanish, and (c) a combination of failures (a) and (b) arises [Golubitsky & Langford, 1981]. In this article, the focus of the investigation is on failures of the type (b) in order to compute the bifurcation diagrams using a formalism enrooted in the so-called frequency domain approach. In this regard, the contribution classifies regions in which one, two, three or more limit cycles appear for a certain combination of system parameters. This classification is significant since the coexistence of multiple limit cycles allow jumps from one stable (maybe small or large) limit cycle to another stable (maybe large or small) limit cycle after perturbing the initial conditions or after small excursions of system parameters [Marzocca *et al.*, 2002]. Furthermore, this classification is important in order to control the appearance of multiplicity (also known as bifurcation control [Chen, 1999; Calandrini *et al.*, 1999]) since it allows to determine the dynamic configuration of the system and then to choose the most appropriate space of system parameters to satisfy suitable operating conditions for engineering systems.

There are several methods to deal with the problem of multiplicity of limit cycles [Farr *et al.*, 1989; Yu & Chen, 2008] and each one of them presents advantages and drawbacks in their implementations. In particular, in [Yu & Chen, 2008] three different methodologies (grouped for simplicity as time-domain methods) are used to compute the leading coefficients to determine the existence and stability of limit cycles in nonlinear planar systems.

In the present exposition an improved higher-order harmonic balance approximation via a frequency domain approach based on [Mees, 1981; Moiola & Chen, 1996] is presented. More specifically, the novel result of this contribution is a particular bifurcation equation for which the methodology presented in [Golubitsky & Schaeffer, 1985] is applied and then an efficient symbolic computation is implemented. By using this methodology it is possible, at least locally, to delimit the regions with different limit cycle multistability in the parameter space and also trace the continuation of periodic solutions. The classical system proposed by Bautin is studied under this hybrid combination of techniques in order to illustrate the main results. Moreover, an independent verification of the bifurcation diagrams is included using two different continuation programs such as LOCBIF and XPPAUT (see [Khibnik *et al.*, 1992; Doedel *et al.*, 2002; Ermentrout, 2002]).

## 2. State-Space Realization and the Frequency Domain Approach

In the present section, a formalism is briefly reviewed in order to deal with a dynamic system $\mathcal{S}$ using a frequency domain approach. In the jargon of nonlinear feedback systems, let us have a plant in the direct path modeled as a linear dynamical system and a feedback path which is static and nonlinear. By using an input–output representation, the plant is described through a transfer function $G(s,\mu) \in \mathbb{C}^{m\times l}$, while the feedback is given by the function $u = -f(y,\mu)$, where $f : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^l$; $u$ is the vector of inputs; $y$ is the vector of outputs and $\mu \in \mathbb{R}$ is the main (or distinguished) bifurcation parameter. It is important to determine for which values of $\mu$ a classical (one limit cycle) or degenerate (multiple limit cycles) Hopf bifurcation arises. The block representation comprises a great variety of autonomous systems such as the Lorenz equations, Rössler oscillator, van der Pol oscillator, etc.

### 2.1. *Dynamical systems in the frequency domain*

Let $\mathcal{D}$ be an $n$-dimensional dynamical nonlinear system described by

$$\dot{x} = F(x,\mu), \tag{1}$$

with $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}$ is the bifurcation parameter and $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a smooth vector field $C^k$, with $k > 3$, with equilibrium point $x = 0$ satisfying $F(0,\mu) = 0$.

By choosing the input and output variables appropriately, it is possible to write the following representation

$$\mathcal{S} : \begin{cases} \dot{x} = A(\mu)x + Bu, \\ y = Cx, \\ u = -f(y,\mu), \end{cases}$$

which verifies that $F(x,\mu) = A(\mu)x - Bf(Cx,\mu)$, $A(\mu) \in \mathbb{R}^{n\times n}$ is not necessarily the linearization around the equilibrium point; $f : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^l$, is $C^k$ with $k > 3$ (it can also contain linear terms), $C \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times l}$.

By applying the Laplace transform, the following representation is obtained $(\mathcal{L}y)(s) = G(s,\mu)(\mathcal{L}u)(s)$, where $G(s,\mu) = C[sI - A(\mu)]^{-1} \times B \in \mathbb{C}^{m\times l}$ is known as the transfer function and $s$ is the Laplace variable.

The realization $\{A(\mu), B, C\}$ associated to $G(s,\mu)$ should be minimal (controllable and observable) and, if possible, with less number of variables and equations than the original system ($m \leq n$, $l \leq n$). The minimal condition assures that both representations, $\mathcal{D}$ and $\mathcal{S}$, have the same dynamical behavior, i.e. they are topological equivalent (see [Mees, 1981; Agamennoni *et al.*, 2008]).

The representation as an autonomous system $\mathcal{S}$ is described by the equation $\mathcal{G}f(y,\mu)+y=0$ where $\mathcal{G}$ is a linear operator between the input and output space with a proper rational transfer function $G(s,\mu)$.

Let $\hat{y}$ be the equilibrium solution of

$$G(0,\mu)f(y,\mu)+y=0.$$

Then, by linearizing around the equilibrium point $\hat{y}$, a gain matrix $J=\frac{\partial f}{\partial y}|_{\hat{y}}$ is obtained and an open-loop transfer function $G(s,\mu)J$ for which the polynomial $\det(\lambda I-G(s,\mu)J)=0$ defines $\lambda_k(s,\mu)$ characteristic functions, $k=1,\ldots,k_0$. Notice that $k_0$ is the minimum number between $m$ and $l$.

It is supposed that the system $\mathcal{S}$ satisfies the following:

$(\mathrm{H1})_{df}$: there is a simple characteristic function of the open-loop transfer matrix $G(i\,\omega,\mu)J$, noted for simplicity as $\hat{\lambda}(\omega,\mu)$, such that for a unique frequency $\omega_0$ and a critical value of the parameter $\mu_0$ the following can be verified

$$\hat{\lambda}(\omega_0,\mu_0)=-1.$$

$(\mathrm{H2})_{df}$: $\langle \boldsymbol{\lambda}_{\partial\mu},\boldsymbol{\lambda}_{\partial\omega}^{\perp}\rangle\neq 0$, where $\boldsymbol{\lambda}_{\partial\mu}=(\frac{\partial\Re\hat{\lambda}}{\partial\mu}(\omega_0,\mu_0),\frac{\partial\Im\hat{\lambda}}{\partial\mu}(\omega_0,\mu_0))$, $\boldsymbol{\lambda}_{\partial\omega}^{\perp}=(\frac{\partial\Im\hat{\lambda}}{\partial\omega}(\omega_0,\mu_0),-\frac{\partial\Re\hat{\lambda}}{\partial\omega}(\omega_0,\mu_0))$, $\Re$ and $\Im$ are the real and imaginary parts and $\langle\cdot,\cdot\rangle$ is the scalar real product.

*Remark.* $(\mathrm{H1})_{df}$ and $(\mathrm{H2})_{df}$ are the classical Hopf bifurcation conditions in the frequency domain. It is supposed here that the rest of characteristic functions do not cross the critical value $-1+i0$ in the complex plane.

## 2.2. *Hopf bifurcation theorem in the frequency domain*

In the following, a version of the Hopf bifurcation theorem using higher-order harmonic balance is presented. Its proof uses an iterative method to determine the necessary vectors for computing the main coefficients of the bifurcation equation up to any order. The previous versions which appeared in [Mees & Chua, 1979; Mees, 1981; Moiola & Chen, 1996] have used general formulas up to order eight. The new algorithm can be implemented simply by using symbolic computations.

By using this methodology that differs sensibly from [Moiola & Chen, 1996], a scalar bifurcation equation is obtained as a corollary relating the amplitude of the periodic solution and the main bifurcation parameter. This relationship gives useful characteristics of the mechanism of Hopf bifurcation. The scalar equation is also equivalent to other bifurcation equations obtained with different methods, such as the center manifold approach [Kuznetsov, 2004; Marsden & MacCracken, 1976; Sotomayor *et al.*, 2006] or the Lyapunov–Schmidt reduction [Farr *et al.*, 1989; Golubitsky & Schaeffer, 1985]. Moreover, this scalar equation has the additional advantage that can be analyzed using the powerful results of the singularity theory [Golubitsky & Schaeffer, 1985] making possible the treatment of different cases of Hopf degeneracies [Golubitsky & Langford, 1981]. Then, the proposed methodology is very useful not only for general results but also for specific examples of interest.

**Theorem 1.** *Let $\mathcal{S}$ be a system that verifies $(\mathrm{H1})_{df}$, then the bifurcation equation of periodic orbits of higher-order in the frequency domain is*

$$\theta\left(\hat{\lambda}(\omega,\mu)+1+\sum_{k=1}^{q}\theta^{2k}\xi_k(\omega,\mu)+\mathcal{O}(\theta^{2q+2})\right)=0. \tag{2}$$

*The nonzero solutions of (2) are one-to-one in correspondence with the periodic solutions of small amplitude $\theta$ of the system $\mathcal{S}$ with period close to $2\pi/\omega_0$.*

*The system $\mathcal{S}$ admits approximate oscillations of the form*

$$y(t)=\hat{y}+\sum_{j=-2q}^{2q}e_j(\omega,\mu)\exp(ij\omega t)+\mathcal{O}(\theta^{2q+2}). \tag{3}$$

*The expressions of $\xi_k(\omega,\mu)$ and $e_j(\omega,\mu)$ are obtained in the proof of the theorem.*

**Observations.** (i) The complete proof requires previous (important) results which were given in [Mees, 1981], as the justification of the harmonic balance method to search for the oscillations and the abandonment of much higher-order harmonics in the approximation of the periodic solutions.

(ii) A general idea, in which the algorithm is based, is provided in the proof below.

*Proof.* An approximation of the periodic solution up to order $2q$ with frequency $\omega$ is given by,

$$\tilde{y}(t) = \hat{y} + \sum_{j=-2q}^{2q} e_j(\omega, \mu) \exp(ij\omega t),$$

and an expansion up to order $2q$ of $f(y, \mu) \approx f(\hat{y}, \mu) + J(y - \hat{y}) + \sum_{j=-2q}^{2q} f_j \exp(ij\omega t)$, is equated using a harmonic balance. From $\sum_{j=-2q}^{2q} f_j \exp(ij\omega t) = \sum_{j=2}^{2q+1} F_j(\tilde{y}(t) - \hat{y}, \tilde{y}(t) - \hat{y}, \ldots, \tilde{y}(t) - \hat{y})/j!$, where $F_j(y - \hat{y}, \ldots, y - \hat{y})$ is a multilinear function of $j$ arguments evaluated in $\tilde{y}(t)$, the expressions of $f_j = f_j(e, \mu)$ have been obtained, $e = (e_{-2q}, \ldots, e_{-1}, e_0, e_1, \ldots, e_{2q})$ where $e_j = e_j(\omega, \mu) \in \mathbb{C}^m$, $e_{-j} = \bar{e}_j$, $\forall\, j$.

The harmonic balance method consists of equating the input of the nonlinear part $f(y, \mu)$ with the output of the linear one $G(s, \mu)$ for each one of the harmonics to be considered in the expansion. Then, it must be solved for all $j$,

$$e_j = -G(ij\omega, \mu)(Je_j + f_j). \tag{4}$$

It is enough to consider only $2q$ equations ($j \neq 1$) and $2q + 1$ variables. Then, the expressions of $e_j$ can be computed for all $j \neq 1$ in terms of $e_1$.

For $j = 1$, the following must be solved

$$(I + G(i\omega, \mu)J)e_1 + G(i\omega, \mu)f_1 = 0. \tag{5}$$

Let $\hat{L} = -\hat{\lambda}(\omega, \mu)I + G(i\omega, \mu)J$ and $L = I + G(i\omega, \mu)J$, which are singular in the bifurcation point. Then, a reduction method must be applied at this stage. Let $v \in \mathrm{Ker}(\hat{L})$ and $\overline{w} \in \mathrm{Range}(\hat{L})^{\perp}$ verifying $w^T v = 1$. Consider

$$\mathbb{C}^m = \mathrm{Ker}(\hat{L}) \oplus M, \quad \mathbb{C}^m = N \oplus \mathrm{Range}(\hat{L}),$$

then it can be written $e_1 = v + v^{\perp}$ with $v \in \mathrm{Ker}(\hat{L})$ where $\dim(\mathrm{Ker}(\hat{L})) = 1$. It is verified also that $v^{\perp} \in M$ such that $\overline{v}^T v^{\perp} = 0$. A projection $Q : \mathbb{C}^m \to \mathrm{Ker}(\hat{L})$ is defined orthogonal to $\mathrm{Range}(\hat{L})$.

Equation (5) is rewritten as

$$\begin{cases} \text{(i)} \quad (I - Q)(L(v + v^{\perp}) \\ \qquad\qquad + G(i\omega, \mu)f_1(e(v + v^{\perp}), \mu)) = 0 \\ \text{(ii)} \quad Q(L(v + v^{\perp}) \\ \qquad\qquad + G(i\omega, \mu)f_1(e(v + v^{\perp}), \mu)) = 0. \end{cases} \tag{6}$$

Since $(I - Q)L : M \to \mathrm{Range}(\hat{L})$ is invertible in $(\omega_0, \mu_0)$, the implicit function theorem can be applied to (6)(i) and it yields $v^{\perp} = v^{\perp}(v, \omega, \mu)$ around $(\omega_0, \mu_0)$. By replacing in the second equation, we get

$$\begin{aligned} &QL(v + v^{\perp}(v, \omega, \mu)) \\ &\quad + QG(i\omega, \mu)f_1(e(v + v^{\perp}(v, \omega, \mu)), \mu) = 0. \end{aligned} \tag{7}$$

**Observations.** (i) $Lv = v + \hat{\lambda}(\omega, \mu)v$. (ii) $(I - Q)Lv = (1 + \hat{\lambda}(\omega, \mu))(I - Q)v = 0$.

The choice of the basis in $\mathrm{Ker}(\hat{L})$ and $\mathrm{Ker}(\hat{L}^T)$ leads to essentially equivalent equations, and then the formulation is translated to other convenient coordinate systems.

Let us suppose that $v_1$ is a base of $\mathrm{Ker}(\hat{L})$ of module 1 and $\overline{w}$ is a base of $\mathrm{Range}(\hat{L})^{\perp} = \mathrm{Ker}(\hat{L}^T)$ verifying $w^T v_1 = 1$ and $\overline{w}^T v_1 = 0$ (biorthogonality of the right and left eigenvectors of $\hat{L}$). Any element of $\mathrm{Ker}(\hat{L})$ can be written in a unique form as $v = \theta v_1$ with $\theta \in \mathbb{R}$ and $Qz = v_1 w^T z$ is defined for $z \in \mathbb{C}^m$.

Then, Eq. (7) in coordinates results in

$$\begin{aligned} &\theta(w^T v_1 + \hat{\lambda}(\omega, \mu)w^T v_1) + w^T v^{\perp}(1 + \hat{\lambda}(\omega, \mu)) \\ &\quad + w^T G(i\omega, \mu)f_1(e(\theta v_1 + v^{\perp}(\theta v_1, \omega, \mu)), \mu) = 0. \end{aligned}$$

The expansions in $\theta$ are obtained in recursive form from $e_j(\omega, \mu) \in \mathbb{C}^m$. It yields $e_0(\omega, \mu) = e_{02}\theta^2 + e_{04}\theta^4 + e_{06}\theta^6 + \cdots + e_{02q}\theta^{2q}$, $e_1(\omega, \mu) = v_1\theta + e_{13}\theta^3 + e_{15}\theta^5 + \cdots + e_{12q-1}\theta^{2q-1}$, $e_2(\omega, \mu) = e_{22}\theta^2 + e_{24}\theta^4 + e_{26}\theta^6 + e_{22q}\theta^{2q}$, $e_3(\omega, \mu) = e_{33}\theta^3 + e_{35}\theta^5 + e_{32q-1}\theta^{2q-1}, \ldots, e_{2q}(\omega, \mu) = e_{2q2q}\theta^{2q}$. Furthermore, by replacing them in $w^T(v^{\perp}(1 + \hat{\lambda}(\omega, \mu)) + G(i\omega, \mu)f_1(e))$, where the coefficients of $\theta^{2k+1}$ give the expressions of $\xi_k(\omega, \mu)$ for $k = 1, 2, \ldots q$, the bifurcation equation (2) is finally obtained. ■

**Corollary 2.1.** *If the conditions* $(\mathrm{H1})_{df}$ *and* $(\mathrm{H2})_{df}$ *are verified, from* (2) *for* $\theta \neq 0$ *sufficiently small, it gives*

$$\mu = \mu_0 + \mu_2\theta^2 + \mu_4\theta^4 + \cdots, \tag{8}$$

$$\omega = \omega_0 + \omega_2\theta^2 + \omega_4\theta^4 + \cdots. \tag{9}$$

*Proof.* The equations which form the real and imaginary parts of (2) verify $(\mathrm{H2})_{df}$. By using implicit differentiation, it yields (8) and (9). ■

**Observations.** (i) Equation (8) is the expression of the bifurcation of periodic solutions since it relates the main bifurcation parameter $\mu$ and the amplitude $\theta$, characterizing the Hopf bifurcation phenomenon. The expression (9) gives the modification of the frequency in terms of the variation of the amplitude.

(ii) For $q = 1$, if the conditions $(\mathrm{H1})_{df}$, $(\mathrm{H2})_{df}$ and $\mu_2 \neq 0$ are verified, the classical conditions of the Hopf bifurcation theorem in the frequency domain are satisfied and the appearance of oscillations follows the typical quadratic relation, at least locally, from the equilibrium.

(iii) It is interesting to notice which is the minimum power of $\theta$ in (8) that characterizes completely the local bifurcation diagrams and the types of singularities organized around a weak focus (in two dimensions) or near a degenerate Hopf bifurcation (in three or more dimensions), i.e. a generalized Hopf bifurcation [Guckenheimer & Holmes, 1983; Takens, 1973; Yu, 1999]. The next sections deal with these problems.

## 3. Generalized Hopf Bifurcation

In this section, an adaptation of the formulas in the frequency domain approach is given in terms of the singularity theory. A brief introduction of singularity theory is also included for the notation and in order to relate the coefficients of the normal form of the generalized Hopf bifurcation using the framework of the frequency domain. In this section, it is shown how to apply the results in order to determine the defining conditions of the singularities. With this information and the application of the singularity theory, the existence and location of multiple cycles can be studied rigorously. This approach unifies the treatment of many diverse problems related with multiple limit cycles, and includes the advantage of its practicality as well as its elegance and rigorousness.

### 3.1. *Normal forms of generalized Hopf bifurcation*

Let us consider a general bifurcation problem given by a scalar equation in $\theta$,

$$g(\theta,\mu) = 0, \quad g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}, \text{ smooth with parameter } \mu \in \mathbb{R}, \text{ and } g(0,\mu_0) = 0. \quad (10)$$

A bifurcation diagram is called to the set $D(g) = \{(\theta,\mu) \in \mathbb{R}\times\mathbb{R} : g(\theta,\mu) = 0\}$. A *normal form* is the simplest polynomial expression *equivalent* to (10), equivalent in the sense of singularity theory, which is noted as $g_p(\theta,\mu) = 0$.

A dynamical system (1) is represented as an input–output system $\mathcal{S}$ which satisfies $(\mathrm{H1})_{df}$ in $(\omega_0,\mu_0)$, and applying the reduction method in the frequency domain, the bifurcation equation of periodic orbits is obtained. (2) Using implicit differentiation to the real and imaginary parts of (2), we get a relationship between the bifurcation parameter and the amplitude of oscillation $\theta$, that is, a single scalar equation $g(\theta,\mu) = 0$ that verifies $g(0,\mu_0) = 0$. In particular $g$ is odd in $\theta$, so it can be written as $g(\theta,\mu) = \theta\, r(z,\mu)$, where $z = \theta^2$, then the scalar equation of bifurcation results in

$$\theta\, r(z,\mu) = 0,$$

where the nonzero solutions correspond exactly to the existence of nontrivial periodic solutions of $\mathcal{S}$.

The singularity theory applied to the study of bifurcations has been developed mainly in [Golubitsky & Schaeffer, 1979, 1985]. In those references the general conditions for an odd scalar bifurcation equation (10) to be $\mathbb{Z}_2$-equivalent to a normal form of generalized Hopf bifurcation $g_p(\theta,\mu) = \theta(\epsilon\, z^q + \delta(\mu-\mu_0)) = 0$, where $z = \theta^2$, have been determined.

Now, we find conditions in $\hat{\lambda}(\omega,\mu)$ and $\xi_k(\omega,\mu)$ with $k = 1,\ldots,q$, coefficients of (2), for which the normal form of $\theta\, r(z,\mu) = 0$ in a neighborhood of $\mu_0$ is equivalent to $\theta(\epsilon\, z^q + \delta(\mu-\mu_0)) = 0$. Let $\boldsymbol{\xi}_k = (\Re\xi_k(\omega_0,\mu_0), \Im\xi_k(\omega_0,\mu_0))$ and $\boldsymbol{\lambda}_{\partial\mu} = (\frac{\partial\Re\hat{\lambda}}{\partial\mu}(\omega_0,\mu_0), \frac{\partial\Im\hat{\lambda}}{\partial\mu}(\omega_0,\mu_0))$, $\boldsymbol{\lambda}^{\perp}_{\partial\omega} = (\frac{\partial\Im\hat{\lambda}}{\partial\omega}(\omega_0,\mu_0), -\frac{\partial\Re\hat{\lambda}}{\partial\omega}(\omega_0,\mu_0))$.

**Lemma 1.** *Let $\mathcal{S}$ be a system that verifies $(\mathrm{H1})_{df}$. In a neighborhood of $\mu = \mu_0$ and $z = 0$, the equation $\theta\, r(z,\mu) = 0$, with $z = \theta^2$, obtained from (2) is $\mathbb{Z}_2$-equivalent to the normal form*

$$\theta(\epsilon\, z^q + \delta(\mu-\mu_0)) = 0, \quad (11)$$

*if it satisfies*

(i) *for $q = 1$, nondegeneracy conditions $\epsilon = \langle\boldsymbol{\xi}_1, \boldsymbol{\lambda}^{\perp}_{\partial\omega}\rangle \neq 0$ and $\delta = \langle\boldsymbol{\lambda}_{\partial\mu}, \boldsymbol{\lambda}^{\perp}_{\partial\omega}\rangle \neq 0$ (classical Hopf bifurcation),*

($\mathrm{ii}_1$) *defining conditions $\langle\boldsymbol{\xi}_k, \boldsymbol{\lambda}^{\perp}_{\partial\omega}\rangle = 0$ and $\langle\boldsymbol{\xi}_k, \boldsymbol{\lambda}^{\perp}_{\partial\mu}\rangle = 0$, $\forall\, k = 1,\ldots,q-1$, and nondegeneracy conditions $\epsilon = \frac{\langle\boldsymbol{\xi}_q, \boldsymbol{\lambda}^{\perp}_{\partial\omega}\rangle}{q!} \neq 0$ and $\delta = \langle\boldsymbol{\lambda}_{\partial\mu}, \boldsymbol{\lambda}^{\perp}_{\partial\omega}\rangle \neq 0$,*

*or*

($\mathrm{ii}_2$) *defining conditions $\langle\boldsymbol{\xi}_k, \boldsymbol{\lambda}^{\perp}_{\partial\omega}\rangle = 0$, $\forall\, k = 1,\ldots,q-1$, and nondegeneracy conditions $\frac{\partial^q r}{\partial z^q}(0,\mu_0) \neq 0$ and $\delta = \langle\boldsymbol{\lambda}_{\partial\mu}, \boldsymbol{\lambda}^{\perp}_{\partial\omega}\rangle \neq 0$.*

*Remarks.* (i) If $q = 1$, $\delta \neq 0$ and $\epsilon \neq 0$ are the classical Hopf bifurcation conditions $(\mathrm{H2})_{df}$ and $(\mathrm{H3})_{df}$, respectively.

(ii) The bifurcation diagrams obtained for $q > 1$ with $\delta \neq 0$ ($(\mathrm{H2})_{df}$), describe the so-called generalized Hopf bifurcation, any slight perturbation on the normal form can change the number of limit cycles in its surroundings. The singularity theory provides the mathematical tools to classify and recover them.

### 3.2. *Singularity theory: Some preliminary definitions*

The *recognition problem* is the procedure which enables to find this specific normal form. It is interesting to know *all* the possible nonequivalent expressions, with different bifurcation diagrams, which can be obtained by perturbing the normal form.

In general, the scalar equation $g(\theta, \mu) = 0$ with $g(0, \mu_0) = 0$ is said to be of finite codimension if there exists a finite set of monomials $\{\theta^l(\mu - \mu_0)^j\}$ such that each perturbation of $g(\theta, \mu)$, which satisfies $g(0, \mu_0) = 0$, is equivalent to one of the following expressions

$$g_p(\theta, \mu) + \sum_{i=1}^{k} \alpha_i \theta^{l_i}(\mu - \mu_0)^{j_i} = 0, \qquad (12)$$

where $g_p(\theta, \mu) = 0$ is a normal form of $g(\theta, \mu) = 0$ and $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$ is close to the origin and the values of $l_i$ and $j_i$ depend on each particular problem.

The minimum number of parameters necessary to describe all the nonequivalent expressions in the unfolding is known as the *codimension*. Equation (12) is known as the *universal unfolding* of $g_p(\theta, \mu) = 0$ if it includes all the nonequivalent perturbations.

In a finite codimension problem, it is possible to enumerate all the possible nonequivalent bifurcation diagrams $D(g_p, \alpha) = \{(\theta, \mu) \in \mathbb{R} \times \mathbb{R} : g_p(\theta, \mu, \alpha) = 0\}$.

The *transition varieties* are subsets in the space of the unfolding parameters which separate different persistent perturbed diagrams. These varieties are typified in a general form [Golubitsky & Schaeffer, 1985]. In that book, for example, the boundaries of different multiplicities of limit cycles are shown, facilitating the classification of diverse regions of qualitatively similar behavior.

In particular, a universal unfolding of the normal form $(\epsilon\theta^{2q} + \delta(\mu - \mu_0)) = 0$ is given by $(\epsilon\theta^{2q} + \delta(\mu - \mu_0) + \alpha_1\theta^2 + \cdots + \alpha_{q-1}\theta^{2(q-1)}) = 0$, where $q - 1$ is the codimension of the problem, and $\alpha = (\alpha_1, \ldots, \alpha_{q-1})$ are the unfolding parameters.

For example, for $q = 3$ all the possible perturbations of the equation $\theta(\epsilon\theta^6 + \delta(\mu - \mu_0)) = 0$, are $\mathbb{Z}_2$-equivalent to the unfolding

$$\theta(\epsilon\theta^6 + \delta(\mu - \mu_0) + \alpha_1\theta^2 + \alpha_2\theta^4) = 0, \qquad (13)$$

where $\delta = \pm 1$ and the two auxiliary parameters agree with the codimension of the problem. It means that it is enough to vary $\alpha_1$ and $\alpha_2$ to obtain all the possible configurations of the limit cycle multiplicity.

The transition varieties delimit the boundaries of the limit cycle multiplicity in the plane of parameters $\alpha_1$ and $\alpha_2$ of the unfolding of the normal form, and they are defined as

$$\mathcal{H}_0 = \{\alpha_1 = 0\},$$

$$\mathcal{H}_1 = \left\{\alpha_1 = \frac{\alpha_2^2}{3}, \alpha_2 \leq 0\right\},$$

$$\mathcal{D} = \left\{\alpha_1 = \frac{\alpha_2^2}{4}, \alpha_2 \leq 0\right\}.$$

In each of these curves the bifurcation diagrams are not persistent, it means that even a small perturbation modifies drastically the shape of the periodic branches (see Fig. 2).

The transition varieties divide the plane $(\alpha_1, \alpha_2)$ in three regions: $R_1 = \{\alpha_1 < 0\}$, a zone in which up to two cycles live; $R_2 = \{\alpha_1 > 0 \cap \alpha_2 > 0 \cup -3\alpha_1 + \alpha_2^2 < 0\}$, a zone in which one cycle appears; and $R_3 = \{\alpha_1 > 0 \cup \alpha_2 < 0 \cup -3\alpha_1 + \alpha_2^2 > 0\}$, a region in which up to three cycles appear (see Fig. 3).

### 3.3. *The bifurcation problem in the frequency domain*

A dynamical system with a bifurcation parameter $\mu$ and auxiliary parameters $\mathcal{P}$, is represented as an input–output system $\mathcal{S}$ which satisfies $(\mathrm{H1})_{df}$ in $(\omega_0, \mu_0)$, and by applying the reduction method in the frequency domain, the bifurcation equation of periodic orbits is obtained

$$\theta\left(\hat{\lambda}(\omega, \mu, \mathcal{P}) + 1 + \sum_{k=1}^{q} \theta^{2k}\xi_k(\omega, \mu, \mathcal{P}) + \mathcal{O}(\theta^{2q+2})\right) = 0. \qquad (14)$$

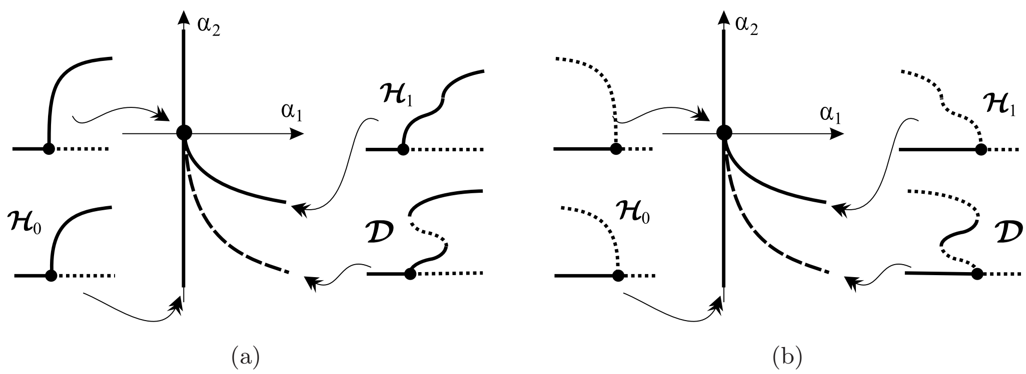


Fig. 2. Nonpersistent diagrams in the transition varieties in the plane $(\alpha_1, \alpha_2)$, with stable equilibrium for $\mu < \mu_0$, (a) $\delta = 1$, (b) $\delta = -1$.

From (14) we get a relationship between the bifurcation parameter and the amplitude of oscillation $\theta$,

$$g(\theta, \mu, \mathcal{P}) = \theta\, r(\theta^2, \mu, \mathcal{P}) = 0 : g(0, \mu_0, \mathcal{P}) = 0.$$

By using this methodology in a particular problem with a bifurcation parameter $\mu$ and auxiliary parameters, all possible perturbing bifurcation diagrams can be achieved, i.e. the expressions of all the coefficients in the universal unfolding as functions of the auxiliary parameters of the original problem are obtained.

Once the defining and nondegeneracy conditions of Lemma 1 are used to characterize $\mathcal{P}_0$, where the normal forms of generalized Hopf bifurcation $g(\theta, \mu, \mathcal{P}_0) = 0$ is $\mathbb{Z}_2$-equivalent to

$$\theta(\epsilon\theta^{2q} + \delta(\mu - \mu_0)) = 0, \tag{15}$$

then from (14), we obtain a universal unfolding

$$\begin{aligned}\epsilon(\mathcal{P})\theta^{2q} + \delta(\mathcal{P})(\mu - \mu_0) + \alpha_1(\mathcal{P})\theta^2 \\ + \alpha_2(\mathcal{P})\theta^4 + \cdots + \alpha_{q-1}(\mathcal{P})\theta^{2(q-1)} = 0,\end{aligned} \tag{16}$$

from which the expressions of the parameters of the universal unfolding $\alpha_k$ are derived in terms of the auxiliary parameters $\mathcal{P}$, verifying $\alpha_k(\mathcal{P}_0) = 0$ for $k = 1, \ldots, q-1$.

The transition varieties determine the relationships among the parameters $\alpha_k(\mathcal{P})$ which separate different behaviors, in particular, the coexistence of multiple limit cycles. Then, the choice of parameters $\mathcal{P}$ will allow to control the possible dynamical behavior of the problem.

**Observation.** The first $q$ in Lemma 1 where the nondegeneracy condition is verified determines

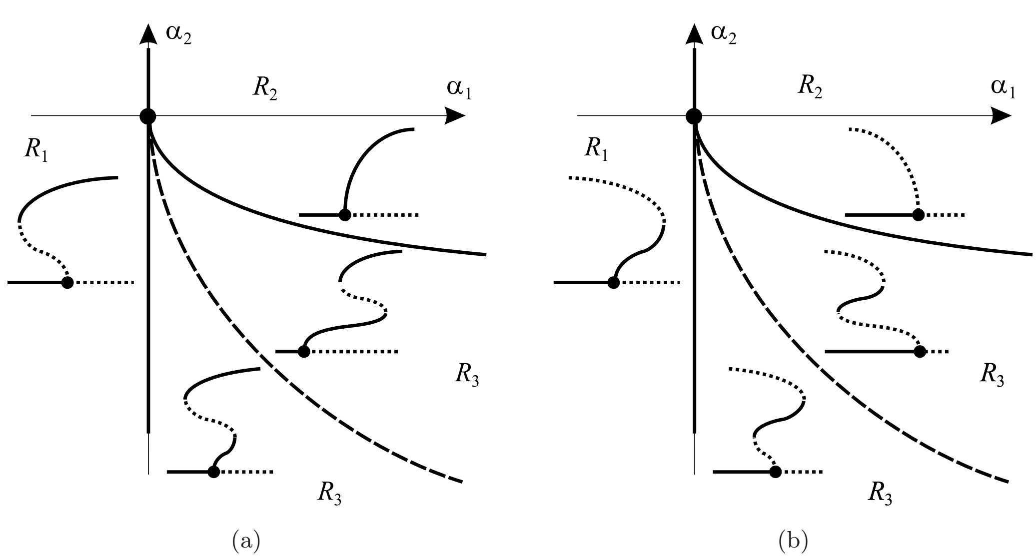


Fig. 3. Regions of persistent diagrams in the plane of $(\alpha_1, \alpha_2)$ with stable equilibrium for $\mu < \mu_0$, (a) $\delta = 1$, (b) $\delta = -1$.

the minimum necessary value in developing the bifurcation equation, and the number $2q$ is the necessary order to approximate the periodic solution by using the Fourier series.

## 4. Applications

In this section, we show how to apply the results presented before in order to determine the defining conditions of the singularities in the Bautin classical planar system [Bautin, 1952], with this information and the application of the singularity theory, the existence and location of multiple cycles can be studied. By the end, the results are compared to those obtained using LOCBIF and XPPAUT packages.

### 4.1. *A classical Bautin planar system*

A dynamical system proposed by Bautin [1952] is considered. It is a planar quadratic system described by

$$\begin{aligned}\dot{x}_1 &= \mu x_1 - x_2 - \lambda_3 x_1^2 + (2\lambda_2 + \lambda_5)x_1x_2 + \lambda_6 x_2^2,\\ \dot{x}_2 &= x_1 + \mu x_2 + \lambda_2 x_1^2 + (2\lambda_3 + \lambda_4)x_1x_2 - \lambda_2 x_2^2,\end{aligned} \tag{17}$$

where $\mu$ is the bifurcation parameter and $\mathcal{P} = (\lambda_2, \ldots, \lambda_6)$ are auxiliary parameters.

(I) *Formulation in the frequency domain*

The following representation $\mathcal{S}$ in the frequency domain is chosen, using the notation of Sec. 2,

$$A(\mu) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$f(y, \mu, \mathcal{P}) = \begin{bmatrix} -(2\lambda_2 + \lambda_5)y_1y_2 - (\mu+1)y_1 + y_2 + \lambda_3 y_1^2 - \lambda_6 y_2^2 \\ -(2\lambda_3 + \lambda_4)y_1y_2 - y_1 - (\mu+1)y_2 - \lambda_2 y_1^2 + \lambda_2 y_2^2 \end{bmatrix}.$$

Let $G(s,\mu) = \left[\begin{smallmatrix} 1/(s+1) & 0 \\ 0 & 1/(s+1) \end{smallmatrix}\right]$, $\hat{y} = (0,0)$ be the equilibrium point and $J = \left[\begin{smallmatrix} -1-\mu & 1 \\ -1 & -1-\mu \end{smallmatrix}\right]$.

The characteristic function of $G(s,\mu)J$ in $s = i\omega$ is given by

$$\hat{\lambda}(\omega, \mu) = \frac{-(1+\mu+\omega)}{1+\omega^2} + i\frac{-1+\omega+\mu\omega}{1+\omega^2},$$

which satisfies $(\mathrm{H1})_{df}$ for $\mu_0 = 0$ and $\omega_0 = 1$, and is independent of the auxiliary parameters $\mathcal{P}$. Let $\hat{L} = -\hat{\lambda}(\omega,\mu)I + G(i\omega,\mu)J$. The system has a trivial stable equilibrium for $\mu < 0$.

(II) *Algorithm*: *expansions in* $\theta$ *of* $e_j$ ($j = 1, \ldots, 2k$), *and* $\xi_k(\omega, \mu, \mathcal{P})$ ($k = 1, \ldots, q$)

Let us suppose that the periodic solution can be expressed by $\tilde{y}(t) = \hat{y} + \sum_{j=-2q}^{2q} e_j \exp(ij\omega t)$ after using a harmonic balance of order $2q$ with frequency $\omega$. Equating the coefficients of $\exp(ij\omega t)$ in $\sum_{j=-2q}^{2q} f_j \exp(ij\omega t) = F_2[\tilde{y}(t), \tilde{y}(t)]/2$ ($F_2$ is a multilinear function defined from the nonlinear terms of $f(y,\mu,\mathcal{P})$), the expressions of $f_j = f_j(e,\mu,\mathcal{P})$ for $j = -2q, \ldots, 2q$ with $e = (e_{-2q}, \ldots, e_{-1}, e_0, e_1, \ldots, e_{2q})$ where $e_{-j} = \bar{e}_j$, are obtained.

We consider $v_1 = (1/\sqrt{2}, -i/\sqrt{2}) \in N(\hat{L})$ and $\overline{w} = (1/\sqrt{2}, -i/\sqrt{2}) \in N(\hat{L}^T) = \mathrm{Range}(\hat{L})^\perp$ then, following the algorithm described before, the expansions in $\theta$ of $e_j$ are obtained.

The general algorithm uses successively the following expression to find the coefficients of the expansion in $\theta$ of $e_j$,

for $j \neq 1$,

$$e_j = -(I + G(ij\omega,\mu)J)^{-1} G(ij\omega,\mu) f_j(e,\mu,\mathcal{P}); \tag{18}$$

for $j = 1$ while it uses the orthogonal projection to find the coefficients of the expansion in $\theta$ of $v^\perp$ ($e_1 = v_1\theta + v^\perp$) as solution of

$$\begin{gathered}(I - Q)(Lv^\perp + G(i\omega,\mu)f_1(e,\mu,\mathcal{P})) = 0,\\ \overline{v}_1^T v^\perp = 0.\end{gathered} \tag{19}$$

Then $\xi_k(\omega,\mu,\mathcal{P})$ is the coefficient of $\theta^{2k+1}$ in

$$w^T(v^\perp(1 + \hat{\lambda}(\omega,\mu)) + G(i\omega,\mu)f_1(e,\mu,\mathcal{P})). \tag{20}$$

The expansion in $\theta$ of each $e_j$ defined afterwards have in bold type the coefficients of the harmonic expressions necessary to compute for each $k$ the corresponding $\xi_k(\omega,\mu,\mathcal{P})$ by using (20). These expressions are successively used in (18) and (19) to find the new coefficients (two larger orders in $\theta$) in the expansion of the harmonics just considered, and to determine the two new harmonics added in each step.

Let $e_1 = v_1\theta$, then

$$A_1 = \{e = (\bar{v}_1\theta, v_1\theta)\}.$$

If $j = 0, 2$, $\mathbf{e_{02}}$ and $\mathbf{e_{22}}$ are the coefficients of $\theta^2$ in the expansion of $e_0$ and $e_2$, respectively, when the right-hand side of (18) is evaluated by considering $A_1$. Let

$$A_2 = \{e = (\bar{\mathbf{e}}_{\mathbf{22}}\theta^2, \bar{v}_1\theta, \mathbf{e_{02}}\,\theta^2, v_1\,\theta, \mathbf{e_{22}}\,\theta^2)\};$$

$\mathbf{e_{13}}$ is the coefficient of $\theta^3$ in the expansion $v^{\perp}$ in the solution of (19) when considering $A_2$. Then

$$A_3 = \{e = (\bar{e}_{22}\,\theta^2, \bar{v}_1\,\theta + \bar{\mathbf{e}}_{\mathbf{13}}\theta^3, e_{02}\theta^2, v_1\theta + \mathbf{e_{13}}\theta^3, e_{22}\,\theta^2)\};$$

$\xi_1(\omega, \mu, \mathcal{P})$ is the coefficient of $\theta^3$ replacing $A_3$ in (20).

If $j = 0, 2, 3, 4$, $\mathbf{e_{04}}$, $\mathbf{e_{24}}$ and $\mathbf{e_{44}}$ are the coefficients of $\theta^4$ in the expansion of $e_0$, $e_2$ and $e_4$, respectively, when the right-hand side of (18) is evaluated by considering $A_3$, and $\mathbf{e_{33}}$ is the coefficient of $\theta^3$ in the expansion of $e_3$. Let

$$\begin{aligned}A_4 = \{e = (&\bar{\mathbf{e}}_{\mathbf{44}}\,\theta^4, \bar{\mathbf{e}}_{\mathbf{33}}\,\theta^3, \bar{e}_{22}\,\theta^2 + \bar{\mathbf{e}}_{\mathbf{24}}\,\theta^4,\\ &\bar{v}_1\,\theta + \bar{e}_{13}\,\theta^3, e_{02}\,\theta^2 + \mathbf{e_{04}}\,\theta^4, v_1\,\theta + e_{13}\,\theta^3,\\ &e_{22}\,\theta^2 + \mathbf{e_{24}}\,\theta^4, \mathbf{e_{33}}\,\theta^3, \mathbf{e_{44}}\,\theta^4)\};\end{aligned}$$

$\mathbf{e_{15}}$ is the coefficient of $\theta^5$ in the expansion of $v^{\perp}$ in the solution of (19) when considering $A_4$. Then

$$\begin{aligned}A_5 = \{e = (&\bar{e}_{44}\,\theta^4, \bar{e}_{33}\,\theta^3, \bar{e}_{22}\,\theta^2 + \bar{e}_{24}\,\theta^4,\\ &\bar{v}_1\,\theta + \bar{e}_{13}\,\theta^3 + \bar{\mathbf{e}}_{\mathbf{15}}\,\theta^5, e_{02}\,\theta^2 + e_{04}\,\theta^4,\\ &v_1\,\theta + e_{13}\,\theta^3 + \mathbf{e_{15}}\,\theta^5, e_{22}\,\theta^2 + e_{24}\,\theta^4,\\ &e_{33}\,\theta^3, e_{44}\,\theta^4)\};\end{aligned}$$

$\xi_2(\omega, \mu, \mathcal{P})$ is the coefficient of $\theta^5$ replacing $A_5$ in (20); and so on.

The described algorithm has been implemented in the symbolic computation framework of Mathematica 6 [Wolfram Research, 2004].

The expressions of $\xi_k(\omega, \mu, \mathcal{P})$ for $k = 1, \ldots, q$ are then obtained in order to determine the bifurcation equation (14).

**Observation.** (i) The solution of the equation for $\theta = 0$ corresponds to the trivial solution while the nonzero solutions are the periodic orbits with approximate period $2\pi$.

(ii) In the following, we can see that if we consider up to the third order, it suffices to enumerate *all* possible bifurcation diagrams of periodic orbits by assuming that they verify $(\mathrm{H1})_{df}$ and $(\mathrm{H2})_{df}$.

(III) *Lemma 1 and* $\mathcal{P}_0$

Using the information of the real and imaginary parts of (14), we obtain $\theta\, r(\theta^2, \mu, \mathcal{P}) = 0$. Since $\boldsymbol{\lambda}_{\partial\mu} = (-\frac{1}{2}, \frac{1}{2})$, $\boldsymbol{\lambda}^{\perp}_{\partial\omega} = (\frac{1}{2}, -\frac{1}{2})$, $\delta = \langle \boldsymbol{\lambda}_{\partial\mu}, \boldsymbol{\lambda}^{\perp}_{\partial\omega}\rangle = -\frac{1}{2} \neq 0$ (independently of the auxiliary parameters), the condition $(\mathrm{H2})_{df}$ is verified, then by using the Lemma 1(i) it is obtained:

(i) If $\epsilon_1 = \langle \boldsymbol{\xi}_1, \boldsymbol{\lambda}^{\perp}_{\partial\omega}\rangle = \frac{-1}{8}\lambda_5(\lambda_3 - \lambda_6) \neq 0$ the normal form is $\mathbb{Z}_2$-equivalent to

$$\theta\left(\epsilon_1\theta^2 - \frac{1}{2}\mu\right) = 0,$$

then the problem results in a classical Hopf bifurcation of codimension zero, i.e. there does not exist multiplicity of limit cycles by small perturbations, at least locally. In the case that the coefficient $\epsilon_1$ is zero, a higher-order expansion should be taken and Lemma 1($\mathrm{ii}_1$) for $q = 2$ should be used.

(ii) Let

$$(dc_1)\begin{cases}\langle \boldsymbol{\xi}_1, \boldsymbol{\lambda}^{\perp}_{\partial\omega}\rangle = \dfrac{-1}{8}\lambda_5(\lambda_3 - \lambda_6)\\ \qquad = 0, \quad \text{then consider } \lambda_5 = 0,\\ \qquad\qquad \lambda_6 - \lambda_3 \neq 0,\\ \langle \boldsymbol{\xi}_1, \boldsymbol{\lambda}^{\perp}_{\partial\mu}\rangle = \dfrac{-1}{24}(-16\lambda_2^2 - 18\lambda_3^2 - 9\lambda_3\lambda_4\\ \qquad - \lambda_4^2 + 12\lambda_3\lambda_6\\ \qquad + \lambda_4\lambda_6 - 10\lambda_6^2)\\ \qquad = 0 \quad (\text{with } \lambda_5 = 0);\end{cases}$$

for $\mathcal{P}_0$ verifying $(dc_1)$ then $\epsilon_2 = \frac{\langle \boldsymbol{\xi}_2, \boldsymbol{\lambda}^{\perp}_{\partial\omega}\rangle}{2} = \frac{1}{24}\lambda_2\lambda_4(\lambda_3 - \lambda_6)(5\lambda_3 + \lambda_4 - 5\lambda_6)$, if $\epsilon_2 \neq 0$ then the normal form is $\mathbb{Z}_2$-equivalent to

$$\theta(\epsilon_2\theta^4 - \frac{1}{2}\mu) = 0,$$

the problem results in a degenerate Hopf bifurcation of codimension one, then the universal unfolding is

$$\theta\left(\epsilon_2(\mathcal{P})\theta^4 - \frac{1}{2}\mu + \epsilon_1(\mathcal{P})\theta^2\right) = 0,$$

and it can have limit cycle multiplicity. In case the coefficient $\epsilon_2$ is zero, a higher-order expansion should be considered, and Lemma 1(ii$_1$) for $q = 3$ should be used.

(iii) Let $(dc_2)\{\langle \boldsymbol{\xi}_2, \boldsymbol{\lambda}^{\perp}_{\partial\omega}\rangle = \frac{1}{12}\lambda_2\lambda_4(\lambda_3 - \lambda_6)(5\lambda_3 + \lambda_4 - 5\lambda_6) = 0$, then consider $\lambda_4 = 5(\lambda_6 - \lambda_3)$; for $\mathcal{P}_0$ verifying $(dc_1)$ and $(dc_2)$ then $\epsilon_3 = \frac{\langle \boldsymbol{\xi}_3, \boldsymbol{\lambda}^{\perp}_{\partial\mu}\rangle}{6} = -\frac{75}{48}\lambda_2(\lambda_3 - \lambda_6)^3(\lambda_2^2 + \lambda_6(-\lambda_3 + 2\lambda_6))$, if $\epsilon_3 \neq 0$ then the normal form is $\mathbb{Z}_2$-equivalent to

$$\theta\left(\epsilon_3\theta^6 - \frac{1}{2}\mu\right) = 0, \tag{21}$$

the problem results in a degenerate Hopf bifurcation of codimension two. The universal unfolding is

$$\theta\left(\epsilon_3(\mathcal{P})\theta^6 - \frac{1}{2}\mu + \epsilon_2(\mathcal{P})\theta^4 + \epsilon_1(\mathcal{P})\theta^2\right) = 0, \tag{22}$$

and it can have limit cycle multiplicity. In the Appendix, the general expressions of $\epsilon_1(\mathcal{P})$, $\epsilon_2(\mathcal{P})$ and $\epsilon_3(\mathcal{P})$ are collected.

For $\mathcal{P}_0$ verifying $(dc_1)$ and $(dc_2)$, $\epsilon_1 = \epsilon_1(\mathcal{P}_0) = \frac{-1}{8}\lambda_5(\lambda_3 - \lambda_6)$, $\epsilon_2 = \epsilon_2(\mathcal{P}_0) = \frac{1}{24}\lambda_2\lambda_4(\lambda_3 - \lambda_6)(5\lambda_3 + \lambda_4 - 5\lambda_6)$, and $\epsilon_3 = \epsilon_3(\mathcal{P}_0) = -\frac{75}{48}\lambda_2(\lambda_3 - \lambda_6)^3(\lambda_2^2 + \lambda_6(-\lambda_3 + 2\lambda_6))$.

**Observations.** (i) If $\langle \boldsymbol{\xi}_3, \boldsymbol{\lambda}^{\perp}_{\partial\mu}\rangle = 0$ then $\langle \boldsymbol{\xi}_k, \boldsymbol{\lambda}^{\perp}_{\partial\mu}\rangle = 0\ \forall\, k \geq 4$. In this case the results developed in [Golubitsky & Langford, 1981] determine that it suffices to study the expansion up to $q = 3$. This is Bautin result which shows that the singularity transforms in a center when $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$ and then it is not possible to find four local limit cycles from a single equilibrium point when the planar system has a quadratic nonlinearity.

(ii) The values $\epsilon_k$, $k = 1, 2, 3$ differ from the ones found by other methods by a positive constant value [Farr *et al.*, 1989; Gaiko, 2003].

(iii) In the case that $\epsilon_1 = \epsilon_2 = 0$ and $\epsilon_3 \neq 0$ the system is said to have a weak focus of third order (the first nonvanishing coefficient is $\theta^6$).

(iv) The sixth order approximation of the periodic solution is enough to study all the perturbations of a weak focus of order 3 [Chow *et al.*, 1994; Farr *et al.*, 1989].

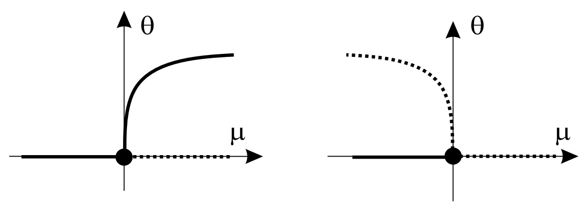


Fig. 4. Bifurcation diagrams for the normal form of a weak focus of third order.

Let us define $\mathcal{P}_0 = \{(\lambda_2, \lambda_3, \lambda_4, 0, \lambda_6) : \lambda_4 = 5(\lambda_6 - \lambda_3), \lambda_2(\lambda_2^2 + \lambda_6(-\lambda_3 + 2\lambda_6)) \neq 0$ and $\lambda_6 - \lambda_3 \neq 0\}$. The equation $\theta(r(\theta^2, \mu, \mathcal{P}_0)) = 0$ is $\mathbb{Z}_2$-equivalent to (21), a normal form of the weak focus of third order. The nonzero solutions of that equation represent the periodic solutions of the original system. In Fig. 4, the two possible bifurcation diagrams are shown with their respective stability. The system has a stable equilibrium for $\mu < 0$, then if $\epsilon_3 > 0$, a stable periodic orbit branch appears while if $\epsilon_3 < 0$, an unstable periodic orbit branch arises.

(IV) *A particular case*: $\lambda_3$ *as the control parameter*

Consider the bifurcation equation obtained with the frequency method (22) as a universal unfolding of (21), for $\epsilon_3 \neq 0$. The explicit expressions of the coefficients in terms of original parameters $\mathcal{P}$, for a particular case, are used to control the existence of multiple cycles. A particular case in which $\lambda_3$ is the control parameter is considered in order to determine the regions where one, two or three limit cycles exist. Let us consider

$$\mathcal{P}_{\lambda_3} = \left(-\frac{36}{100}, \lambda_3, -\frac{30\,665}{10\,000}, -\frac{1}{1000}, \frac{4}{10}\right),$$

with $\epsilon_3(\mathcal{P}_{\lambda_3}) \neq 0$ and $\epsilon_1(\mathcal{P}_{\lambda_3})$ and $\epsilon_2(\mathcal{P}_{\lambda_3})$ being the parameters of the universal unfolding.

By using the information of the transition varieties it is possible to find several intervals of $\lambda_3$ in which there are certain distinctive local diagrams of limit cycles bifurcations which are valid in a neighborhood of $(\mu, \theta) = (0, 0)$. In each of these curves the bifurcation diagrams are not persistent, it means that even a small perturbation modifies drastically the shape of the periodic branches. Some distinctive sets are separated from the following transition varieties:

$$\mathcal{H}_0 = \{\epsilon_1(\mathcal{P}_{\lambda_3}) = 0\},$$

$$\mathcal{H}_1 = \left\{\epsilon_1(\mathcal{P}_{\lambda_3}) = \frac{\epsilon_2(\mathcal{P}_{\lambda_3})^2}{3}, \epsilon_2(\mathcal{P}_{\lambda_3}) \leq 0\right\},$$

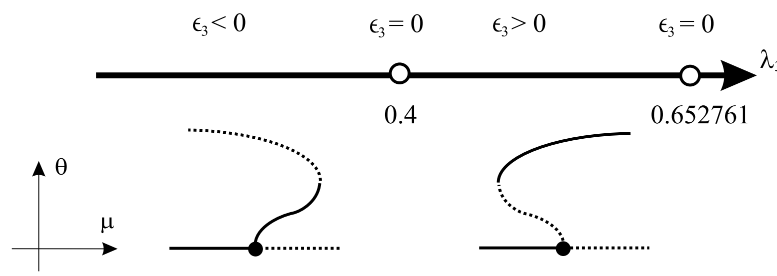


Fig. 5. Persistent diagrams corresponding to parameter $\lambda_3 \in (-4633, 0.652761)$.

$$\mathcal{D} = \left\{\epsilon_1(\mathcal{P}_{\lambda_3}) = \frac{\epsilon_2(\mathcal{P}_{\lambda_3})^2}{4}, \epsilon_2(\mathcal{P}_{\lambda_3}) \leq 0\right\}.$$

The transition varieties and the sign$(\epsilon_3(\mathcal{P}_{\lambda_3}))$ determine intervals of $\lambda_3$,

$$R_1(\lambda_3) = \{\epsilon_1(\mathcal{P}_{\lambda_3}) < 0 \cap (\epsilon_3(\mathcal{P}_{\lambda_3}) > 0 \cup \epsilon_3(\mathcal{P}_{\lambda_3}) < 0)\},$$

an interval in which up to two limit cycles can appear;

$$R_2(\lambda_3) = \{((\epsilon_1(\mathcal{P}_{\lambda_3}) > 0 \cap \epsilon_2(\mathcal{P}_{\lambda_3}) > 0) \cup -3\epsilon_1(\mathcal{P}_{\lambda_3}) + \epsilon_2(\mathcal{P}_{\lambda_3})^2 < 0) \cap (\epsilon_3(\mathcal{P}_{\lambda_3}) > 0 \cup \epsilon_3(\mathcal{P}_{\lambda_3}) < 0)\},$$

a zone in which one cycle appears; and

$$R_3(\lambda_3) = \{(\epsilon_1(\mathcal{P}_{\lambda_3}) > 0 \cap \epsilon_2(\mathcal{P}_{\lambda_3}) < 0 \cap -3\epsilon_1(\mathcal{P}_{\lambda_3}) + \epsilon_2(\mathcal{P}_{\lambda_3})^2 > 0) \cap (\epsilon_3(\mathcal{P}_{\lambda_3}) > 0 \cup \epsilon_3(\mathcal{P}_{\lambda_3}) < 0)\},$$

an interval in which up to three limit cycles can appear.

Then, an induced division of intervals of $\lambda_3$ is obtained according to the number of limit cycles and sign$(\epsilon_3(\mathcal{P}_{\lambda_3}))$, where the diagrams of bifurcation are valid in a neighborhood of $(\mu, \theta) = (0, 0)$. These intervals are given by the following expressions:

$$R_1(\lambda_3) = \{0.4 < \lambda_3 < 0.652761 \cup 1.005999 < \lambda_3 < 1.44686\}_{\epsilon_3(\mathcal{P}_{\lambda_3})>0} \cup \{-4633.09 < \lambda_3 < 0.4\}_{\epsilon_3(\mathcal{P}_{\lambda_3})<0},$$

$$R_2(\lambda_3) = \{1.00479 < \lambda_3 < 1.005999 \cup \lambda_3 > 1.44686\}_{\epsilon_3(\mathcal{P}_{\lambda_3})<0},$$

$$R_3(\lambda_3) = \{0.652761 < \lambda_3 < 1.00479\}_{\epsilon_3(\mathcal{P}_{\lambda_3})<0}.$$

The persistent bifurcation diagrams of $R_1(\lambda_3)$, $R_2(\lambda_3)$ and $R_3(\lambda_3)$, which are valid in a neighborhood of $(\mu, \theta) = (0, 0)$, are shown in Figs. 5–7. The stability of the limit cycles are determined by using topological arguments.

(V) *Some comparisons using FDM, XPPAUT, LOCBIF*

In the following tables the results are obtained using the frequency domain method (FDM), XPPAUT and LOCBIF.

Let us consider $\lambda_3 = 0.5$, then the bifurcation diagram corresponds to two nested limit cycles for certain values of $\mu$. The detection of the saddle-node bifurcation of cycles using FDM and XPPAUT is in good agreement compared to LOCBIF, which clearly fails in obtaining the correct result (see Table 1).

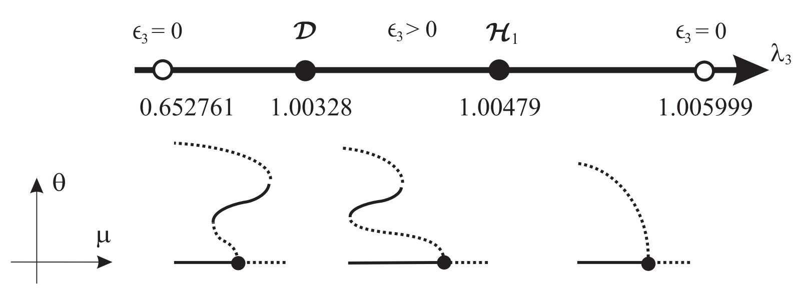


Fig. 6. Persistent diagrams corresponding to parameter $\lambda_3 \in (0.652761, 1.00599)$.

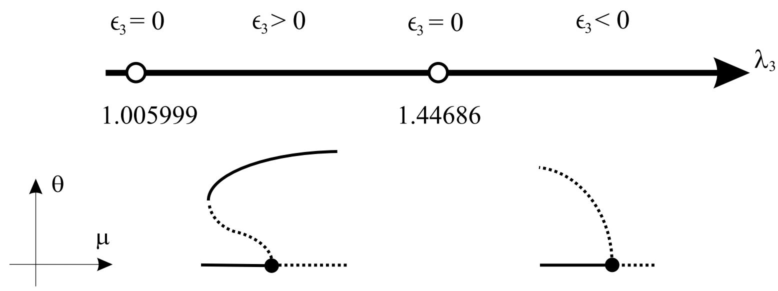


Fig. 7. Persistent diagrams corresponding to parameter $\lambda_3 \in (1.00599, \ldots)$.

Table 1. Values of the saddle-node bifurcation of cycles using FDM, XPPAUT and LOCBIF for $\lambda_3 = 0.5$.

| | FDM | XPPAUT | LOCBIF |
|---|---|---|---|
| $\mu$ | $-6.60874 \times 10^{-9}$ | $-6.61743 \times 10^{-9}$ | $0.143785 \times 10^{-5}$ |
| $\theta$ | 0.022990772026 | — | — |
| $x_{1\max}$ | 0.0324013 | 0.0325279 | 0.08130336 |
| $T$ | 6.28391 | 6.28391 | 6.2888877 |

By moving to region $R_3(\lambda_3)$ for $\lambda_3 = 1$ and $\lambda_3 = 0.98$, the bifurcation diagrams show up to three nested limit cycles for certain values of $\mu$. In this case, the detection of the closest saddle-node bifurcation of cycles to the equilibrium point is in good agreement between FDM and XPPAUT compared to LOCBIF (Tables 2 and 4).

On the other hand for the detection of the farthest saddle-node bifurcation, XPPAUT and LOCBIF results are in good agreement (see Tables 3 and 5) and clearly the FDM gives a mistaken detection due to its local nature.

In all the continuation computations, the results of XPPAUT are considered the most accurate among the used methods. Concerning the approximations of the maximum amplitude of limit cycles, it is well-known that LOCBIF results are dependent on the starting condition so the values of the tables should be considered with this severe constraint. Finally, it is important to recognize that

Table 2. Values of the closest saddle-node bifurcation of cycles using FDM, XPPAUT and LOCBIF for $\lambda_3 = 1$.

| | FDM | XPPAUT | LOCBIF |
|---|---|---|---|
| $\mu$ | $-1.630061 \times 10^{-6}$ | $-1.625689 \times 10^{-6}$ | $-1.3650 \times 10^{-6}$ |
| $\theta$ | 0.151323 | — | — |
| $x_{1\max}$ | 0.211414 | 0.221542 | 0.18239 |
| $T$ | 6.28215 | 6.2821 | 6.28206 |

Table 3. Values of the farthest saddle-node bifurcation of cycles using FDM, XPPAUT and LOCBIF for $\lambda_3 = 1$.

| | FDM | XPPAUT | LOCBIF |
|---|---|---|---|
| $\mu$ | $1.35767 \times 10^{-5}$ | $3.125773 \times 10^{-5}$ | $3.11091 \times 10^{-5}$ |
| $\theta$ | 0.38862 | — | — |
| $x_{1\max}$ | 0.56323 | 0.820185 | 0.81181 |
| $T$ | 6.27888 | 6.2787 | 6.27877 |

Table 4. Values of the closest saddle-node bifurcation of cycles using FDM, XPPAUT and LOCBIF for $\lambda_3 = 0.98$.

| | FDM | XPPAUT | LOCBIF |
|---|---|---|---|
| $\mu$ | $-5.95068 \times 10^{-7}$ | $-5.949686 \times 10^{-7}$ | $0.621585 \times 10^{-8}$ |
| $\theta$ | 0.0909926 | — | — |
| $x_{1\max}$ | 0.127321 | 0.1312512 | 0.1114841 |
| $T$ | 6.28251 | 6.2825 | 6.282310 |

Table 5. Values of the farthest saddle-node bifurcation of cycles using FDM, XPPAUT and LOCBIF for $\lambda_3 = 0.98$.

| | FDM | XPPAUT | LOCBIF |
|---|---|---|---|
| $\mu$ | $0.272415 \times 10^{-3}$ | $0.968583 \times 10^{-3}$ | $0.9684659 \times 10^{-3}$ |
| $\theta$ | 0.5640684 | — | — |
| $x_{1\max}$ | 0.890431 | 2.3086 | 1.511434 |
| $T$ | 6.26838 | 6.2798 | 6.279882 |

the proximity to a weak focus alerts on the use of a standard simulation for verification purposes due to its extremely slow convergence.

## 5. Conclusions

In the current article, an improved higher-order harmonic balance approximation via a frequency domain approach based on [Mees, 1981; Moiola & Chen, 1996] has been presented. The novel result is a particular bifurcation equation for which the methodology presented in [Golubitsky & Schaeffer, 1985] has been applied and an efficient symbolic computation has been implemented. By using this approach, it is possible, at least locally, to delimit the regions with different limit cycle multistability in the parameter space and also trace the continuation of periodic solutions. This approach has also served to approximate with great precision the smallest limit cycles arising in the unfolding of a weak focus in the classical Bautin example and, at the same time, make comparisons with standard bifurcation packages such as XPPAUT and LOCBIF. More specifically, it has been shown from the results of Tables 1, 2 and 4, that LOCBIF fails to predict precisely the location of the closest saddle-node of limit cycles to the equilibrium, while the FDM results agree very well with those of XPPAUT. On the other hand, the detection of the farthest saddle-node bifurcations of cycles (Tables 3 and 5) has been proved to be quite tough when using local approaches such as the FDM. Finally, the present paper completes previous attempts to characterize degenerate Hopf bifurcations made by [Moiola & Chen, 1996] and [Calandrini *et al.*, 1999] using a combination set of analytical and numerical techniques.

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## Appendix

The general expressions for the coefficients of the universal unfolding (22) in terms of the original auxiliary parameters are:

$$\epsilon_1(\mathcal{P}) = -\frac{1}{8}\lambda_5(\lambda_3-\lambda_6),$$

$$\begin{aligned}\epsilon_2(\mathcal{P}) = \frac{-1}{48}(\lambda_3-\lambda_6)(&32\lambda_2^2\lambda_5 - 4\lambda_2(-5\lambda_5^2+\lambda_4(5\lambda_3+\lambda_4-5\lambda_6))\\ &+\lambda_5(12\lambda_3^2+5\lambda_3\lambda_4-\lambda_4^2+3\lambda_5^2+4\lambda_3\lambda_6+3\lambda_4\lambda_6+16\lambda_6^2)),\end{aligned}$$

$$\begin{aligned}\epsilon_3(\mathcal{P}) = \frac{-1}{9216}(\lambda_3-\lambda_6)(&52224\lambda_2^4\lambda_5 + 16\lambda_2^2\lambda_5(5700\lambda_3^2+500\lambda_3\lambda_4-485\lambda_4^2+1755\lambda_5^2-6248\lambda_3\lambda_6\\ &+1340\lambda_4\lambda_6+7076\lambda_6^2)+128\lambda_2^3(497\lambda_5^2+\lambda_4(-580\lambda_3-89\lambda_4+580\lambda_6))\\ &-8\lambda_2(8100\lambda_3^3\lambda_4+98\lambda_4^4-645\lambda_5^4-852\lambda_4^3\lambda_6-9124\lambda_5^2\lambda_6^2+30\lambda_3^2(233\lambda_4^2-282\lambda_5^2-346\lambda_4\lambda_6)\\ &+4\lambda_4\lambda_6(399\lambda_5^2-865\lambda_6^2)+21\lambda_4^2(\lambda_5^2+78\lambda_6^2)+4\lambda_3(391\lambda_4^3-1037\lambda_4\lambda_5^2-1801\lambda_4^2\lambda_6\end{aligned}$$

$$+ 2408\lambda_5^2\lambda_6 + 1435\lambda_4\lambda_6^2)) + \lambda_5(15444\lambda_3^4 - 459\lambda_4^4 + 325\lambda_5^4 + 1678\lambda_4^3\lambda_6 + 108\lambda_3^3(87\lambda_4 - 44\lambda_6)$$
$$+ 9057\lambda_5^2\lambda_6^2 + 25556\lambda_6^4 + 17\lambda_4^2(26\lambda_5^2 - 175\lambda_6^2) + 3\lambda_3^2(-2397\lambda_4^2 + 4259\lambda_5^2 + 2364\lambda_4\lambda_6 + 6312\lambda_6^2)$$
$$+ 2\lambda_3(-2283\lambda_4^3 + 3933\lambda_4\lambda_5^2 + 4883\lambda_4^2\lambda_6 - 6253\lambda_5^2\lambda_6 + 4894\lambda_4\lambda_6^2 - 1480\lambda_6^3)$$
$$+ \lambda_4(-3906\lambda_5^2\lambda_6 + 3164\lambda_6^3))).$$