Monadic BL-algebras: the equivalent algebraic semantics of Hájek's monadic fuzzy logic

Diego Castaño[a*] , Cecilia Cimadamore[a], José Patricio Díaz Varela[a], Laura Rueda[a]

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[a] Departamento de Matemática, Universidad Nacional del Sur, Bahía Blanca 8000, Argentina. INMABB - CONICET, Bahía Blanca 8000, Argentina.

Abstract

In this article we introduce the variety of monadic BL-algebras as BL-algebras endowed with two monadic operators $\forall$ and $\exists$. After a study of the basic properties of this variety we show that this class is the equivalent algebraic semantics of the monadic fragment of Hájek's basic predicate logic. In addition, we start a systematic study of the main subvarieties of monadic BL-algebras, some of which constitute the algebraic semantics of well-known monadic logics: monadic Gödel logic and monadic Łukasiewicz logic. In the last section we give a complete characterization of totally ordered monadic BL-algebras.

Keywords: Mathematical Fuzzy Logic; Monadic Logic; BL-algebras.

1 Introduction

In his book [17] Hájek introduced BL-algebras as the algebraic semantics of his basic fuzzy logic, which is a common framework for Łukasiewicz, Gödel and product logics. Afterwards in [10] it was shown that Hájek's Basic Logic was the logic of continuous t-norms (see also [11]). Subsequently BL-algebras were studied in great depth, see e.g. [3, 8]. BL-algebras were also seen to be the subvariety of bounded integral commutative divisible residuated lattices generated by chains (see [17]).

In [17] Hájek also introduced the basic many-valued predicate logic and proved its strong completeness with respect to its (linear) general semantics, that is, the semantics based on structures whose truth values lie on a BL-chain. A brief description of the monadic fragment of this calculus is also presented. Recall that the monadic fragment consists of the formulas with unary predicates and just one object variable. In addition, Hájek introduced an S5-like modal fuzzy logic and showed that it is equivalent to the monadic basic predicate logic. He also proposed a set of axioms and inference rules for the monadic logic and proved its strong completeness with respect to its (linear) general semantics in [18].

Monadic algebras have been studied since Halmos introduced monadic Boolean algebras in [19]. Monadic versions of other algebraic structures have been also greatly studied since then. The two most important examples are monadic MV-algebras and monadic Heyting algebras. The former were first studied by Rutledge in [23] and then by Di Nola, Grigolia, Cimadamore and Díaz Varela in [13, 12]. The latter were introduced by Monteiro and Varsavsky in [21] and deeply studied by Bezhanishvili in [4]. Monadic Łukasiewicz-Moisil algebras were also studied by Abad in [1] and Heyting algebras with one quantifier were the research topic of Rueda in [22].

*Corresponding author. E-mail addresses: diego.castano@uns.edu.ar (D. Castaño), crcima@criba.edu.ar (C. Cimadamore), usdiavar@criba.edu.ar (J. P. Díaz Varela), larueda@criba.edu.ar (L. Rueda)

In this article we will introduce the variety of monadic BL-algebras and make a standard study of their basic properties, which includes the characterization of their congruences and subdirectly irreducible algebras. This will be the main topic of Section 2. In Section 3 we will give a complete characterization of the range of the monadic operators: m-relatively complete subalgebras. This characterization will be useful to produce the most important examples of monadic BL-algebras which are functional monadic BL-algebras. In the next section, Section 4, we will show that this variety is the equivalent algebraic semantics of Hájek's monadic basic fuzzy logic in the sense of Blok and Pigozzi [6] as well as simplify the original axioms proposed by Hájek. In Section 5 we will see that monadic BL-algebras contain as subvarieties the variety of monadic MV-algebras and monadic Gödel-algebras, the latter being monadic prelinear Heyting algebras that satisfy the equation $\forall(\exists x \vee y) \approx \exists x \vee \forall y$. We will also introduce the subvariety of monadic product algebras and give a special characterization of its subdirectly irreducible members. In addition, in each of these three main subvarieties will give a complete characterization of their totally ordered members. Moreover, we devote Section 6 to study totally ordered monadic BL-algebras in depth. Specifically we will show how to define all possible quantifiers on a given BL-chain. Finally, we conclude the paper describing some of the problems about this variety that constitute our current work.

Throughout this article we assume that the reader is familiar with propositional as well as first order basic logic and with structural properties of BL-algebras.

2 Monadic BL-algebras: definition and representation theorems

We start this section with the definition of the variety $\mathbb{MBL}$ of monadic BL-algebras. We develop the basic algebraic properties and prove that the image of the quantifier is a subalgebra. We also introduce the notion of monadic filter and show that they correspond to congruences. As a corollary, we derive a characterization for subdirectly irreducible algebras and discuss a special BL-subdirect representation for them. We refer the reader to [17] for the definition and basic properties of BL-algebras.

Definition 2.1. An algebra $\mathbf{A} = \langle A, \vee, \wedge, *, \to, \exists, \forall, 0, 1\rangle$ of type $(2, 2, 2, 2, 1, 1, 0, 0)$ is called a *monadic BL-algebra* (an MBL-algebra for short) if $\langle A, \vee, \wedge, *, \to, 0, 1\rangle$ is a BL-algebra and the following identities are satisfied:

(M1) $\forall x \to x \approx 1$.

(M2) $\forall(x \to \forall y) \approx \exists x \to \forall y$.

(M3) $\forall(\forall x \to y) \approx \forall x \to \forall y$.

(M4) $\forall(\exists x \vee y) \approx \exists x \vee \forall y$.

(M5) $\exists(x * x) \approx \exists x * \exists x$.

For brevity, if $\mathbf{A}$ is a BL-algebra and we enrich it with a monadic structure, we denote the resulting algebra by $\langle \mathbf{A}, \exists, \forall\rangle$. We denote by $\mathbb{MBL}$ the variety of MBL-algebras.

Remark 2.2. We would like to point out that two other definitions of monadic BL-algebras are already present in the literature, one by Grigolia (see [16]) and another by Drăgulici (see [14, 15]). The varieties introduced in those papers are the first attempt at algebraizing the monadic fragment of Basic Logic. However, they are not strict enough to be the equivalent algebraic semantics of this logic. Indeed, we will show an example of an algebra that satisfies all the axioms in Grigolia's and Drăgulici's articles, but that does not satisfy axiom (M5).

We recall here the axioms of the structures defined in the aforementioned articles:

Grigolia's axioms:

(E1) $p \to \exists p \approx 1$.

(A1) $\forall p \to p \approx 1$.

(E2) $\forall(p \to \exists q) \approx \exists p \to \exists q$.

(A2) $\forall(\exists p \to q) \approx \exists p \to \forall q$.

(A3) $\forall(p \vee \exists q) \approx \forall p \vee \exists q$.

Drăgulici's axioms:

(Q1) $\forall p \leq p$.

(Q2) $p \leq \exists p$.

(Q3) $\forall(\forall p \to q) = \forall p \to \forall q$.

(Q4) $\forall(p \to \forall q) = \exists p \to \forall q$.

(Q5) $\forall(p \vee \exists q) = \forall p \vee \exists q$.

(Q6) $\forall(\forall p * \forall q) = \forall p * \forall q$.

(Q7) $\forall 1 = 1$.

Consider now the three-element MV-algebra $\mathbf{A}$ over the universe $A = \{0, \frac{1}{2}, 1\}$ and the algebra $\mathbf{B} = \langle \mathbf{A}, \exists, \forall \rangle$, where the quantifiers are given by $\forall 0 = \forall \frac{1}{2} = \exists 0 = 0$ and $\forall 1 = \exists \frac{1}{2} = \exists 1 = 1$. It is straightforward to verify that $\mathbf{B}$ satisfies the two sets of axioms above, but it does not satisfy axiom (M5), since $\exists(\frac{1}{2} * \frac{1}{2}) = \exists 0 = 0$, whereas $\exists \frac{1}{2} * \exists \frac{1}{2} = 1 * 1 = 1$.

Axiom (M5) is the algebraic version of an axiom already noted by Hájek in his book (see [17, Remark 8.3.16 ($\Diamond$3)]). This axiom is valid in the monadic fragment of Basic Logic because the first order sentence in one variable $(\exists x)(\varphi(x) \,\&\, \varphi(x)) \equiv (\exists x)\varphi(x) \,\&\, (\exists x)\varphi(x)$ is valid in the basic predicate logic (see [17, Theorem 5.1.18]). Therefore, the previous example is an indication that Grigolia's algebras and Drăgulici's algebras are not an algebraic semantics for the monadic fragment of Hájek's Basic Logic, but wider classes of algebras.

Note also that this example shows that axiom (M5) is independent of the rest of the axioms.

The next lemma collects some of the basic properties that hold true in any MBL-algebra.

Lemma 2.3. *Let $\mathbf{A} \in \mathbb{MBL}$ and $a, b \in A$.*

(M6) $\forall\exists a = \exists a$.

(M7) $a \to \exists a = 1$.

(M8) $\forall(\exists a \to b) = \exists a \to \forall b$.

(M9) $\forall(a \to \exists b) = \exists a \to \exists b$.

(M10) $\forall 1 = 1$.

(M11) $\exists\forall a = \forall a$.

(M12) $\forall(\forall a \vee b) = \forall a \vee \forall b$.

(M13) $\forall 0 = 0$, $\exists 1 = 1$, *and* $\exists 0 = 0$.

(M14) $\exists\exists a = \exists a$ *and* $\forall\forall a = \forall a$.

(M15) $\forall(\exists a \to \exists b) = \exists a \to \exists b$.

(M16) $\exists(\exists a \to b) \to (\exists a \to \exists b) = 1$.

(M17) *If* $a \leq b$, *then* $\forall a \leq \forall b$ *and* $\exists a \leq \exists b$.

(M18) $\forall(\exists a \vee \exists b) = \exists a \vee \exists b$.

(M19) $\forall a = a$ *if and only if* $\exists a = a$.

(M20) $\exists(a \vee b) = \exists a \vee \exists b$.

(M21) $\exists(\exists a * \exists b) = \exists a * \exists b$.

(M22) $\forall(a \to b) \to (\forall a \to \forall b) = 1$.

(M23) $\forall(a \to b) \to (\exists a \to \exists b) = 1$.

(M24) $(\forall a * \exists b) \to \exists(a * b) = 1$.

(M25) $(\forall a * \forall b) \to \exists(a * b) = 1$.

(M26) $\exists(a * \exists b) = \exists a * \exists b$.

(M27) $\exists(a * \forall b) = \exists a * \forall b$.

(M28) $\exists(a \to \exists b) \to (\forall a \to \exists b) = 1$.

(M29) $\exists(\exists a \to \exists b) = \exists a \to \exists b$.

(M30) $\exists(\forall a \to \forall b) = \forall a \to \forall b$.

(M31) $\exists(\exists a \wedge \exists b) = \exists a \wedge \exists b$.

(M32) $\exists(a \wedge \exists b) = \exists a \wedge \exists b$.

(M33) $\forall(\forall a \to \forall b) = \forall a \to \forall b$.

(M34) $\exists(\forall a * \forall b) = \forall a * \forall b$.

(M35) $\forall(\forall a * \forall b) = \forall a * \forall b$.

(M36) $\forall(\forall a \wedge \forall b) = \forall a \wedge \forall b$.

(M37) $\forall(a \wedge b) = \forall a \wedge \forall b$.

Proof. (M6) From $\exists a = \exists a \vee \exists a$ and (M4), we have that $\forall\exists a = \forall(\exists a \vee \exists a) = \exists a \vee \forall\exists a$. But from (M1) we know that $\forall\exists a \leq \exists a$. So, $\forall\exists a = \exists a$.

(M7) From (M6), (M2) and (M1), we can write $1 = \exists a \to \forall\exists a = \forall(a \to \forall\exists a) \leq a \to \forall\exists a = a \to \exists a$. Thus, $a \to \exists a = 1$.

(M8) From (M6) and (M3) we have that $\forall(\exists a \to b) = \forall(\forall\exists a \to b) = \forall\exists a \to \forall b = \exists a \to \forall b$.

(M9) From (M6) and (M2) we have that $\forall(a \to \exists b) = \forall(a \to \forall\exists b) = \exists a \to \forall\exists b = \exists a \to \exists b$.

(M10) From (M1) and (M3), we have that $\forall 1 = \forall(\forall 1 \to 1) = \forall 1 \to \forall 1 = 1$.

(M11) We know that $\forall a \to \exists\forall a = 1$ by (M7). Furthermore, $\exists\forall a \to \forall a = \forall(\forall a \to \forall a) = \forall 1 = 1$ by (M2) and (M10). Then, $\exists\forall a = \forall a$.

(M12) Using (M4) and (M11), we have that $\forall(\forall a \vee b) = \forall(\exists\forall a \vee b) = \exists\forall a \vee \forall b = \forall a \vee \forall b$.

(M13) Clearly $\forall 0 = 0$ and $\exists 1 = 1$ by (M1) and (M7) respectively.

Since $\forall 0 = 0$, we have that $\exists 0 \to 0 = \exists 0 \to \forall 0 = \forall(0 \to \forall 0) = \forall(0 \to 0) = \forall 1 = 1$ by (M2) and (M10). So, $\exists 0 = 0$.

(M14) By (M7) we have that $\exists a \to \exists\exists a = 1$. On the other hand, from (M9) and (M10) we have that $\exists\exists a \to \exists a = \forall(\exists a \to \exists a) = \forall 1 = 1$. Thus, $\exists\exists a = \exists a$.

By (M3) and (M10), $\forall a \to \forall\forall a = \forall(\forall a \to \forall a) = \forall 1 = 1$. Since $\forall\forall a \to \forall a = 1$, then $\forall\forall a = \forall a$.

(M15) From (M9) and (M14), we have that $\forall(\exists a \to \exists b) = \exists\exists a \to \exists b = \exists a \to \exists b$.

(M16) Using (M15), (M2), and $\exists a \to b \leq \exists a \to \exists b$, we obtain $\exists(\exists a \to b) \to (\exists a \to \exists b) = \exists(\exists a \to b) \to \forall(\exists a \to \exists b) = \forall((\exists a \to b) \to \forall(\exists a \to \exists b)) = \forall((\exists a \to b) \to (\exists a \to \exists b)) = \forall 1 = 1$.

(M17) If $a \leq b$, then $\forall a \leq a \leq b \leq \exists b$. Thus, by (M3), $\forall a \to \forall b = \forall(\forall a \to b) = \forall 1 = 1$, and, by (M9), $\exists a \to \exists b = \forall(a \to \exists b) = \forall 1 = 1$.

(M18) Using (M4) and (M6), $\forall(\exists a \vee \exists b) = \exists a \vee \forall\exists b = \exists a \vee \exists b$.

(M19) If $a = \forall a$, then $\exists a = \exists\forall a = \forall a = a$ by (M11). The converse implication follows analogously using (M6).

(M20) Clearly $a \vee b \leq \exists a \vee \exists b$. Thus $\exists(a \vee b) \leq \exists(\exists a \vee \exists b)$, by (M17). But $\exists a \vee \exists b = \forall(\exists a \vee \exists b)$ by (M18). Then, taking (M19) into account, $\exists(\exists a \vee \exists b) = \exists a \vee \exists b$. Therefore, $\exists(a \vee b) \leq \exists a \vee \exists b$.

On the other hand, from $\exists a \leq \exists(a \vee b)$ and $\exists b \leq \exists(a \vee b)$, it is clear that $\exists a \vee \exists b \leq \exists(a \vee b)$.

(M21) Taking into account that $\exists a * \exists b \geq \forall(\exists a * \exists b)$, (M2) and (M8), we have that $\exists(\exists a * \exists b) \to (\exists a * \exists b) \geq \exists(\exists a * \exists b) \to \forall(\exists a * \exists b) = \forall((\exists a * \exists b) \to \forall(\exists a * \exists b)) = \forall(\exists a \to (\exists b \to \forall(\exists a * \exists b))) = \forall(\exists a \to \forall(\exists b \to (\exists a * \exists b))) = \forall\forall(\exists a \to (\exists b \to (\exists a * \exists b))) = \forall\forall 1 = 1$.

(M22) From $\forall a \leq a$, we have that $a \to b \leq \forall a \to b$. Then, using (M17) and (M3), $\forall(a \to b) \leq \forall(\forall a \to b) = \forall a \to \forall b$.

(M23) Since $b \leq \exists b$, we have that $a \to b \leq a \to \exists b$. Thus, $\forall(a \to b) \leq \forall(a \to \exists b)$. Then, from (M9), we have that $\forall(a \to b) \to (\exists a \to \exists b) = \forall(a \to b) \to \forall(a \to \exists b) = 1$.

(M24) From (M23), (M22) and (M10), we have that $(\forall a * \exists b) \to \exists(a * b) = \forall a \to (\exists b \to \exists(a * b)) \geq \forall a \to \forall(b \to (a * b)) \geq \forall(a \to (b \to (a * b))) = \forall 1 = 1$.

(M25) Using (M7), $(\forall a * \forall b) \to \exists(a * b) \geq (\forall a * \forall b) \to (a * b) = 1$.

(M26) Since $a \leq \exists a$, then $a*\exists b \leq \exists a*\exists b$. Consequently, $\exists(a*\exists b) \leq \exists(\exists a*\exists b) = \exists a*\exists b$ from (M21). On the other hand, from (M9) and (M8), we have that $(\exists a * \exists b) \to \exists(a * \exists b) = \exists b \to (\exists a \to \exists(a*\exists b)) = \exists b \to \forall(a \to \exists(a*\exists b)) = \forall(\exists b \to (a \to \exists(a*\exists b))) = \forall((a*\exists b) \to \exists(a*\exists b)) = \forall 1 = 1$.

(M27) Using (M11) and (M26), we have that $\exists(a * \forall b) = \exists(a * \exists\forall b) = \exists a * \exists\forall b = \exists a * \forall b$.

(M28) From (M2), $\exists(a \to \exists b) \to (\forall a \to \exists b) = \exists(a \to \exists b) \to \forall(\forall a \to \exists b) = \forall((a \to \exists b) \to \forall(\forall a \to \exists b)) = \forall((a \to \exists b) \to (\forall a \to \exists b)) = \forall 1 = 1$.

(M29) Clearly $\exists a \to \exists b \leq \exists(\exists a \to \exists b)$. Using (M28) and (M6), $\exists(\exists a \to \exists b) \leq \forall\exists a \to \exists b = \exists a \to \exists b$.

(M30) Using (M11) and (M29), $\exists(\forall a \to \forall b) = \exists(\exists\forall a \to \exists\forall b) = \exists\forall a \to \exists\forall b = \forall a \to \forall b$.

(M31) From (M7) we have that $\exists(\exists a \wedge \exists b) \geq \exists a \wedge \exists b$. Since $\exists a \wedge \exists b \leq \exists a$ and $\exists a \wedge \exists b \leq \exists b$, then $\exists(\exists a \wedge \exists b) \leq \exists\exists a = \exists a$ and $\exists(\exists a \wedge \exists b) \leq \exists\exists b = \exists b$. Thus, $\exists(\exists a \wedge \exists b) \leq \exists a \wedge \exists b$.

(M32) We know that $a \wedge \exists b \leq \exists a \wedge \exists b$, then $\exists(a \wedge \exists b) \leq \exists(\exists a \wedge \exists b) = \exists a \wedge \exists b$ by (M31). On the other hand, $(\exists a \wedge \exists b) \to \exists(a \wedge \exists b) = (\exists a * (\exists a \to \exists b)) \to \exists(a * (a \to \exists b)) = (\exists a * \forall(a \to \exists b)) \to \exists(a * (a \to \exists b)) = 1$ by (M9) and (M24).

(M33) From (M3), $\forall(\forall a \to \forall b) = \forall a \to \forall\forall b = \forall a \to \forall b$.

(M34) Using (M11) and (M21), we have that $\exists(\forall a * \forall b) = \exists(\exists\forall a * \exists\forall b) = \exists\forall a * \exists\forall b = \forall a * \forall b$.

(M35) Using (M34) and (M6), we have that $\forall a * \forall b = \exists(\forall a * \forall b) = \forall(\exists(\forall a * \forall b)) = \forall(\forall a * \forall b)$.

(M36) Using (M33) and (M35), we have that $\forall(\forall a \wedge \forall b) = \forall(\forall a * (\forall a \to \forall b)) = \forall(\forall a * \forall(\forall a \to \forall b)) = \forall a * \forall(\forall a \to \forall b) = \forall a * (\forall a \to \forall b) = \forall a \wedge \forall b$.

(M37) From (M36), $(\forall a \wedge \forall b) \to \forall(a \wedge b) = \forall(\forall a \wedge \forall b) \to \forall(a \wedge b) = \forall((\forall a \wedge \forall b) \to (a \wedge b)) = \forall 1 = 1$. Then, $\forall a \wedge \forall b \leq \forall(a \wedge b)$. Since $a \wedge b \leq a$ and $a \wedge b \leq b$, we have $\forall(a \wedge b) \leq \forall a \wedge \forall b$. $\square$

Remark 2.4. Observe that from (M6) and (M11), $x = \forall y$ for some y if and only if $x = \exists z$ for some z. Thus $\exists A = \{\exists a : a \in A\} = \{\forall a : a \in A\} = \forall A$. Using this fact, from (M2), (M3) and (M4), we deduce that:

- $\forall(a \to c) = \exists a \to c$,
- $\forall(c \to a) = c \to \forall a$,
- $\forall(c \vee a) = c \vee \forall a$,

for any $a \in A$ and $c \in \forall A$. Compare with the axioms of the logic $S5(\mathcal{C})$ in Section 4.

Remark 2.5. Note that the identity $\forall(x * x) \approx \forall x * \forall x$ holds in every monadic MV-algebra and trivially in any monadic Gödel algebra (cf. Section 5). However, this equation is not valid in any monadic BL-algebra. For example, let $\mathbf{L}_2$ and $\mathbf{L}_3$ be the 2-element and 3-element MV-chains, respectively, and consider the ordinal sum $\mathbf{A} = \mathbf{L}_3 \oplus \mathbf{L}_2$ (see Section 6 for the general definition of ordinal sum). If we define the quantifiers on A in the following way:

a	0_1	$\frac{1}{2}$	0_2	1
$\forall a$	0_1	$\frac{1}{2}$	$\frac{1}{2}$	1
$\exists a$	0_1	$\frac{1}{2}$	1	1

we get a monadic BL-chain $\langle \mathbf{A}, \exists, \forall \rangle$. In this algebra $\forall 0_2 * \forall 0_2 = \frac{1}{2} * \frac{1}{2} = 0_1$, but $\forall(0_2 * 0_2) = \forall 0_2 = \frac{1}{2}$.

Lemma 2.6. *If $\mathbf{A} \in \mathbb{MBL}$, then $\exists A = \forall A$ and $\exists\mathbf{A}$ is a BL-subalgebra of $\mathbf{A}$.*

Proof. We already showed that $\forall A = \exists A$. From (M7), (M13), (M20), (M31), (M21) and (M29), we obtain that $\exists\mathbf{A}$ is a BL-subalgebra of $\mathbf{A}$. $\square$

Given a monadic BL-algebra $\mathbf{A}$, a subset $F \subseteq A$ is a *monadic filter* of $\mathbf{A}$ if F is a filter and $\forall a \in F$ for each $a \in F$. In the following, we characterize the congruences of each MBL-algebra $\mathbf{A}$ by means of monadic filters in the standard way. More precisely, we establish an order isomorphism from the lattice $\mathbf{Con}_{\mathbb{MBL}}(\mathbf{A})$ of congruences of $\mathbf{A}$ onto the lattice $\mathbf{F}_m(\mathbf{A})$ of monadic filters of $\mathbf{A}$, both ordered by inclusion. Moreover, we prove that the lattice $\mathbf{F}_m(\mathbf{A})$ is isomorphic to the lattice $\mathbf{F}(\exists\mathbf{A})$ of filters of the BL-algebra $\exists\mathbf{A}$.

If $\mathbf{A} \in \mathbb{MBL}$ and $X \subseteq A$, $X \neq \emptyset$, the monadic filter generated by X is the set

$$\begin{aligned} \mathrm{MFg}(X) &= \{a \in A : \forall x_1 \to (\forall x_2 \to (\cdots(\forall x_n \to a)\cdots)) = 1,\ \text{where } x_1, x_2, \ldots, x_n \in X\} \\ &= \{a \in A : \forall x_1 * \forall x_2 * \cdots * \forall x_n \leq a,\ \text{where } x_1, x_2, \ldots, x_n \in X\}. \end{aligned}$$

In particular, if $X = \{b\}$ then $\mathrm{MFg}(\{b\}) = \mathrm{MFg}(b) = \{a \in A : (\forall b)^n \leq a \text{ for some } n \in \mathbb{N}\}$. Let us observe also that $\mathrm{MFg}(X) = \mathrm{Fg}(\forall X)$, the filter generated by $\forall X$.

Theorem 2.7. *Let $\mathbf{A} \in \mathbb{MBL}$. The correspondence $\mathbf{Con}_{\mathbb{MBL}}(\mathbf{A}) \to \mathbf{F}_m(\mathbf{A})$ defined by $\theta \mapsto 1/\theta$ is an order isomorphism whose inverse is given by $F \mapsto \theta_F = \big\{(a, b) \in A^2 : (a \to b) * (b \to a) \in F\big\}$.*

Proof. Let $\theta \in Con_{\mathbb{MBL}}(\mathbf{A})$. Let us consider the filter $1/\theta$ and let $a \in 1/\theta$, that is, $a \equiv 1 \pmod\theta$. Since θ is a congruence, we have that $\forall a \equiv \forall 1 \pmod\theta$ and from here we clearly obtain that $\forall a \in 1/\theta$. Consequently, $1/\theta$ is a monadic filter. Let $F \in F_m(\mathbf{A})$. Let us prove that $\theta_F \in Con_{\mathbb{MBL}}(\mathbf{A})$. Indeed, let $a, b \in A$ such that $(a \to b) * (b \to a) \in F$. Then, $a \to b \in F$ and $b \to a \in F$. So $\forall(a \to b) \in F$ and $\forall(b \to a) \in F$. Since F is increasing and from (M22), we obtain that $\forall a \to \forall b \in F$ and $\forall b \to \forall a \in F$. Thus, $(\forall a \to \forall b) * (\forall b \to \forall a) \in F$. Furthermore, from (M23) and considering again that $\forall(a \to b) \in F$, $\forall(b \to a) \in F$ and F is increasing, we have that $(\exists a \to \exists b) * (\exists b \to \exists a) \in F$. Thus, $\theta_F \in Con_{\mathbb{MBL}}(\mathbf{A})$. Now, it is straigthforward to see that the correspondence $\theta \mapsto 1/\theta$ is an order isomorphism whose inverse is given by $F \mapsto \theta_F$. $\square$

The following theorem is also routine.

Theorem 2.8. *Let $\mathbf{A} \in \mathbb{MBL}$. The correspondence $\mathbf{F}_m(\mathbf{A}) \to \mathbf{F}(\exists\mathbf{A})$ defined by $F \mapsto F \cap \exists A$ is an order isomorphism whose inverse is given by $M \mapsto \mathrm{MFg}(M)$.*

Corollary 2.9. *If $\mathbf{A} \in \mathbb{MBL}$ then*

$$\mathbf{Con}_{\mathbb{MBL}}(\mathbf{A}) \cong \mathbf{F}_m(\mathbf{A}) \cong \mathbf{F}(\exists\mathbf{A}) \cong \mathbf{Con}_{\mathbb{BL}}(\exists\mathbf{A}).$$

As an immediate consequence, we have the following results.

Corollary 2.10. *Let $\mathbf{A} \in \mathbb{MBL}$.*

(1) *$\mathbf{A}$ is subdirectly irreducible (simple) if and only if $\exists\mathbf{A}$ is a subdirectly irreducible (simple) BL-algebra.*

(2) *If $\mathbf{A}$ is subdirectly irreducible, then $\exists\mathbf{A}$ is totally ordered.*

(3) *$\mathbf{A}$ is a subdirect product of a family of MBL-algebras $\{\mathbf{A}_i : i \in I\}$ such that $\exists\mathbf{A}_i$ is totally ordered.*

To close this section of basic properties we will show an extension to monadic BL-algebras of a representation theorem for monadic MV-algebras showed by Rutledge in [23].

Lemma 2.11. *Let $\mathbf{A}$ be an MBL-algebra and F be a filter in $\mathbf{A}$. For any $x, y \in A$,*

$$F = \mathrm{Fg}(F \cup \{x \to y\}) \cap \mathrm{Fg}(F \cup \{y \to x\}).$$

Proof. The forward inclusion is straightforward. Now assume z is an element of both $\mathrm{Fg}(F \cup \{x \to y\})$ and $\mathrm{Fg}(F \cup \{y \to x\})$. Then, there are $f_1, f_2 \in F$, $n_1, n_2 \in \mathbb{N}_0$ such that $f_1 * (x \to y)^{n_1} \leq z$ and $f_2 * (y \to x)^{n_2} \leq z$. If we let $f = f_1 * f_2$ and $n = \max\{n_1, n_2\}$, it follows that $f * (x \to y)^n \leq z$ and $f * (y \to x)^n \leq z$. Using the residuation condition, $(x \to y)^n \leq f \to z$ and $(y \to x)^n \leq f \to z$. Thus we get $(x \to y)^n \vee (y \to x)^n \leq f \to z$. But $(x \to y)^n \vee (y \to x)^n = ((x \to y) \vee (y \to x))^n = 1$, so $f \leq z$ and $z \in F$. □

Recall that a filter F in a BL-algebra $\mathbf{A}$ is said to be *prime* if for every $a, b \in A$, either $a \to b \in F$ or $b \to a \in F$. Observe that F is prime if and only if $\mathbf{A}/F$ is totally ordered.

Lemma 2.12. *Let $\mathbf{A}$ be an MBL-algebra such that $\exists\mathbf{A}$ is totally ordered. Given $a \in A$, $a \neq 1$, there exists a prime filter P in $\mathbf{A}$ such that $a \vee \forall r \notin P$ for any $r \neq 1$.*

Proof. Consider $C = \{a \vee \forall r : r \neq 1\}$. Note that $1 \notin C$, since $a \vee \forall r = 1$ implies that $1 = \forall(a \vee \forall r) = \forall a \vee \forall r$ and this, in turn, would imply that $a = 1$ or $r = 1$.

Let $\mathcal{F}$ be the family of filters F in $\mathbf{A}$ such that $F \cap C = \emptyset$. The above paragraph shows that $\{1\} \in \mathcal{F}$, so that $\mathcal{F}$ is nonempty. In addition, it is straightfoward to verify that any chain in $\mathcal{F}$ has an upper bound in $\mathcal{F}$. Hence, by Zorn's Lemma, there exists a maximal filter P in $\mathcal{F}$.

We claim that P is prime. Indeed, let $x, y \in A$ and note that

$$P = \mathrm{Fg}(P \cup \{x \to y\}) \cap \mathrm{Fg}(P \cup \{y \to x\}).$$

If we assume that neither $\mathrm{Fg}(P \cup \{x \to y\})$ nor $\mathrm{Fg}(P \cup \{y \to x\})$ belongs to $\mathcal{F}$, then there are $r_1, r_2 \neq 1$ such that $a \vee \forall r_1 \in \mathrm{Fg}(P \cup \{x \to y\})$ and $a \vee \forall r_2 \in \mathrm{Fg}(P \cup \{y \to x\})$. Since $\forall r_1$ and $\forall r_2$ are comparable, it follows that one of them belongs to both filters. Hence one of them belongs to P, a contradiction. This shows that either $\mathrm{Fg}(P \cup \{x \to y\}) \in \mathcal{F}$ or $\mathrm{Fg}(P \cup \{y \to x\}) \in \mathcal{F}$.

Assume the first option is true. By the maximality of P, $P = \mathrm{Fg}(P \cup \{x \to y\})$, so $x \to y \in P$. Analogously, if $P = \mathrm{Fg}(P \cup \{y \to x\})$, $y \to x \in P$. □

Theorem 2.13. *Given an MBL-algebra $\mathbf{A}$, there exists a subdirect representation of the underlying BL-algebra $\mathbf{A} \leq \prod_{i \in I} \mathbf{A}_i$, where each $\mathbf{A}_i$ is a totally ordered BL-algebra and $\exists\mathbf{A}$ is embedded in $\mathbf{A}_i$ via the corresponding projection map.*

Proof. For each $a \in A$, $a \neq 1$, let P_a be one of the prime filters provided by the previous lemma. Clearly $\bigcap_{a \neq 1} P_a = \{1\}$ and we obtain a natural embedding $\mathbf{A} \to \prod_{a \neq 1} \mathbf{A}/P_a$. To close the proof we need only show that the natural map $\mathbf{A} \to \mathbf{A}/P_a$ is injective on $\exists A$. Indeed, suppose there were $r_1, r_2 \in A$ such that $\forall r_1 < \forall r_2$ and $\forall r_1/P_a = \forall r_2/P_a$. We have that $\forall r_2 \to \forall r_1 = \forall(\forall r_2 \to \forall r_1)$ and $\forall r_2 \to \forall r_1 \neq 1$. Hence, we know that $a \vee (\forall r_2 \to \forall r_1) \notin P_a$, which is a contradiction. □

3 Building MBL-algebras from m-relatively complete subalgebras

In this section we characterize those subalgebras of a given BL-algebra that may be the range of the quantifiers $\forall$ and $\exists$. As a consequence of this characterization, we build the most important examples of monadic BL-algebras, which we call *functional monadic BL-algebras*. This construction will allow us in the next section to prove the main result of this article: monadic BL-algebras are the equivalent algebraic semantics of Hájek's monadic fuzzy logic.

The intended characterization of the quantifiers was already achieved for monadic MV-algebras through the notion of m-relatively complete subalgebras (see [13]). We extend here this notion to the broader context of BL-algebras.

Given a BL-algebra $\mathbf{A}$, we say that a subalgebra $\mathbf{C} \leq \mathbf{A}$ is *m-relatively complete* if the following conditions hold:

(s1) For every $a \in A$, the subset $\{c \in C : c \leq a\}$ has a greatest element and $\{c \in C : c \geq a\}$ has a least element.

(s2) For every $a \in A$ and $c_1, c_2 \in C$ such that $c_1 \leq c_2 \vee a$, there exists $c_3 \in C$ such that $c_1 \leq c_2 \vee c_3$ and $c_3 \leq a$.

(s3) For every $a \in A$ and $c_1 \in C$ such that $a * a \leq c_1$, there exists $c_2 \in C$ such that $a \leq c_2$ and $c_2 * c_2 \leq c_1$.

Condition (s2) may be replaced by either of the following:

(s2′) For every $a \in A$ and $c_1, c_2 \in C$ such that $c_1 = c_2 \vee a$, there exists $c_3 \in C$ such that $c_1 = c_2 \vee c_3$ and $c_3 \leq a$.

(s2″) If $1 = c_1 \vee a$ for some $c_1 \in C$, $a \in A$, there exists $c_2 \in C$ such that $1 = c_1 \vee c_2$ and $c_2 \leq a$.

Indeed, (s2′) is an easy consequence of (s2) and (s2″) follows from (s2′) by setting $c_1 = 1$. Now assume (s2″). If $c_1 \leq c_2 \vee a$, it follows that $1 = (c_1 \to c_2) \vee (c_1 \to a)$. Hence, there exists $c_3 \in B$ such that $1 = (c_1 \to c_2) \vee c_3$ and $c_3 \leq c_1 \to a$. Thus $c_1 * c_3 \leq a$ and $c_1 \to (c_2 \vee (c_1 * c_3)) = (c_1 \to c_2) \vee (c_1 \to (c_1 * c_3)) \geq (c_1 \to c_2) \vee c_3 = 1$.

Furthermore observe that if $\mathbf{C}$ is totally ordered, condition (s2″) may be replaced by the following simpler equivalent form:

($\mathrm{s2}''_\ell$) If $1 = c \vee a$ for some $c \in C$, $a \in A$, then $c = 1$ or $a = 1$.

Theorem 3.1. *Given a BL-algebra $\mathbf{A}$ and an m-relatively complete subalgebra $\mathbf{C} \leq \mathbf{A}$, if we define on A the operations*

$$\exists a := \min\{c \in C : c \geq a\}, \qquad \forall a := \max\{c \in C : c \leq a\},$$

then $\langle \mathbf{A}, \exists, \forall \rangle$ is a monadic BL-algebra such that $\forall A = \exists A = C$. Conversely, if $\mathbf{A}$ is a monadic BL-algebra, then $\forall \mathbf{A} = \exists \mathbf{A}$ is an m-relatively complete subalgebra of $\mathbf{A}$.

Proof. Clearly condition (s1) from the definition of m-relatively complete subalgebra guarantees the existence of $\forall a$ and $\exists a$ for every $a \in A$. It remains to show that $\langle \mathbf{A}, \exists, \forall \rangle$ satisfies axioms (M1)–(M5). Let $a, b \in A$.

(M1) From the definition of $\forall a$, it is clear that $\forall a \leq a$. Thus $\forall a \to a = 1$.

(M2) Since, by definition, $a \leq \exists a$, it follows that $\exists a \to \forall b \leq a \to \forall b$. Then $\exists a \to \forall b \in \{c \in C : c \leq a \to \forall b\}$. Let us see that $\exists a \to \forall b = \max\{c \in C : c \leq a \to \forall b\}$. Indeed, if $c \in C$ and $c \leq a \to \forall b$, then $a \leq c \to \forall b$. Then, by definition of $\exists a$, $\exists a \leq c \to \forall b$. Thus $c \leq \exists a \to \forall b$. This shows that $\forall(a \to \forall b) = \exists a \to \forall b$.

(M3) From $\forall b \leq b$, we get $\forall a \to \forall b \leq \forall a \to b$. In addition, if $c \in C$ and $c \leq \forall a \to b$, then $c * \forall a \leq b$. Thus $c * \forall a \leq \forall b$ and $c \leq \forall a \to \forall b$. Hence we have shown that $\forall(\forall a \to b) = \forall a \to \forall b$.

(M4) Since $\forall b \leq b$, $\exists a \vee \forall b \leq \exists a \vee b$. Now, if $c \in C$ and $c \leq \exists a \vee b$, by condition (s2) in the definition of m-relatively complete subalgebra, there must be $c' \in C$ such that $c \leq \exists a \vee c'$ and $c' \leq b$. Then $c' \leq \forall b$ and $c \leq \exists a \vee \forall b$. Thus, we have shown that $\forall(\exists a \vee b) = \exists a \vee \forall b$.

(M5) We know that $a * a \leq \exists a * \exists a$. In addition, if $c \in C$ and $a * a \leq c$, by condition (s3), there is $c' \in C$ such that $c' * c' \leq c$ and $a \leq c'$. Then $\exists a \leq c'$ and $\exists a * \exists a \leq c' * c' \leq c$. We have thus proved that $\exists(a * a) = \exists a * \exists a$.

Conversely, let $\langle \mathbf{A}, \exists, \forall \rangle$ be a monadic BL-algebra. From Lemma 2.6, we already know that $\forall \mathbf{A}$ is a BL-subalgebra of $\mathbf{A}$. Let us now show that conditions (s1)–(s3) hold.

(s1) Using the properties from Lemma 2.3, we have that if $c \leq a$ for some $c \in \forall A$, then $c = \forall c \leq \forall a \leq a$. Thus $\forall a = \max\{c \in \forall A : c \leq a\}$. Analogously $\exists a = \min\{c \in \forall A : c \geq a\}$.

(s2) Assume $c_1 \leq c_2 \vee a$ for some $c_1, c_2 \in \forall A$, $a \in A$. Then, using the properties in Lemma 2.3 and the axioms for monadic BL-algebras, we get that $c_1 \leq c_2 \vee \forall a$ and $\forall a \leq a$.

(s3) Similarly to the previous paragraph, if $a * a \leq c$ for some $c \in \forall A$ and $a \in A$, then $\exists a * \exists a \leq c$ and $a \leq \exists a$, $\exists a \in \forall A$.

This completes the proof that $\forall \mathbf{A}$ is an m-relatively complete subalgebra of $\mathbf{A}$. □

The following is the most important example of monadic BL-algebras built according to the previous theorem.

Example 3.2. Consider a BL-chain $\mathbf{A}$ and a nonempty set X. We restrict our attention to those elements $f \in A^X$ such that $\inf\{f(x) : x \in X\}$ and $\sup\{f(x) : x \in X\}$ both exist in $\mathbf{A}$. We denote by S the subset of A^X of these "safe" elements. For every $f \in S$, we define: $(\forall_\wedge f)(x) = \inf\{f(y) : y \in X\}$ and $(\exists_\vee f)(x) = \sup\{f(y) : y \in X\}$, $x \in X$. Note that $\forall_\wedge f$ and $\exists_\vee f$ are constant maps.

Let $\mathbf{B}$ be a BL-subalgebra of $\mathbf{A}^X$ contained in S such that for every $f \in B$, $\forall_\wedge f, \exists_\vee f \in B$. We claim that $\mathbf{B}$ has a natural structure of monadic BL-algebra.

Let C be the subset of constant maps of A^X. We claim that $\mathbf{B} \cap \mathbf{C}$ is an m-relatively complete subalgebra of $\mathbf{B}$. Indeed, since $\mathbf{B}$ and $\mathbf{C}$ are subalgebras of $\mathbf{A}^X$, it is clear that $\mathbf{B} \cap \mathbf{C}$ is a subalgebra of $\mathbf{B}$.

If $f \in B$, then $\forall_\wedge f \in B$, so $\max\{c \in B \cap C : c \leq f\} = \forall_\wedge f \in B$. Analogously, $\min\{c \in B \cap C : c \geq f\} = \exists_\vee f \in B$. This shows that condition (s1) holds.

Since $\mathbf{B} \cap \mathbf{C}$ is totally ordered, we may check condition (s$2''_\ell$) instead of (s2). Assume $1 = c \vee f$ for some $f \in B$ and $c \in B \cap C$. Put $c(x) = c_0 \in A$, $x \in X$. Then $c_0 \vee f(x) = 1$ for every $x \in X$. As $\mathbf{A}$ is totally ordered, either $c_0 = 1$ or $f(x) = 1$ for every $x \in X$. Thus (s$2''_\ell$) holds.

Finally, let us show condition (s3). Assume $f * f \leq c$ for some $f \in B$ and $c \in B \cap C$. Then $f(x) * f(x) \leq c_0$ for every $x \in X$. Moreover, $f(x) * f(y) \leq c_0$ for every $x, y \in X$, since

$$f(x) * f(y) \leq (f(x) \vee f(y))^2 = f(x)^2 \vee f(y)^2 \leq c_0.$$

Hence, $f(x) \leq f(y) \to c_0$ for a fixed $y \in X$ and every $x \in X$. Thus $(\exists_\vee f)(x) \leq f(y) \to c_0$ (remember that $\exists_\vee f$ is a constant map). Now $f(y) \leq (\exists_\vee f)(x) \to c_0$ for every $y \in Y$. Then, $(\exists_\vee f)(x) \leq (\exists_\vee f)(x) \to c_0$ and $(\exists_\vee f)(x) * (\exists_\vee f)(x) \leq c_0$. This concludes the proof that $\mathbf{B} \cap \mathbf{C}$ is an m-relatively complete subalgebra of $\mathbf{B}$.

By the previous theorem, $\langle \mathbf{B}, \exists_\vee, \forall_\wedge \rangle$ is a monadic BL-algebra. Monadic BL-algebras of this form are called *functional monadic BL-algebras.*

Observe that if $\mathbf{A}$ is $|X|$-complete (the meet and join of every subset of A whose cardinality is less than or equal to $|X|$ exists in A), then $S = A^X$ and $\langle \mathbf{A}^X, \exists_\vee, \forall_\wedge \rangle$ is a functional monadic BL-algebra.

Remark 3.3. Observe that in the previous example the condition that $\mathbf{A}$ is totally ordered was only necessary to prove condition (s$2''_\ell$) in the definition of m-relatively complete subalgebras. In fact, there exist (non totally ordered) complete BL-algebras $\mathbf{A}$ for which the subalgebra $\mathbf{C}$ of constant maps in $\mathbf{A}^X$ is not m-relatively complete. For example, consider $\mathbf{A} = \langle \mathbb{N}_0, \vee, \wedge, *, \to, 0, 1 \rangle$, where $\mathbb{N}_0$ is the set of nonnegative integers, $\wedge$ is the least common multiple, $\vee$ is the greatest common divisor, $*$ is ordinary multiplication and its residuum is given by

$$a \to b = \begin{cases} 1 & \text{if } a = b, \\ \frac{b}{gcd(a,b)} & \text{if } a \neq b. \end{cases}$$

It is easy to check that $\mathbf{A}$ is a complete BL-algebra. Now consider the elements $c, f \in A^{\mathbb{N}}$ given by $c(n) = 2$, $f(n) = p_n$ (the n-th odd prime), for every $n \in \mathbb{N}$. Then $c \vee f = 1$. However, the only constant map below f is 0, and then $c \vee 0 = c \neq 1$. This shows that condition (s2$''$) does not hold. Observe that we may define an algebra $\langle \mathbf{A}^{\mathbb{N}}, \exists_{\vee}, \forall_{\wedge} \rangle$ that satisfies all conditions of monadic BL-algebras but (M4). In particular, this shows that axiom (M4) is independent of the rest of the axioms.

4 Hájek's modal and monadic fuzzy logic

In his monograph [17] Hájek defined the fuzzy modal logic S5(BL) as a modal expansion of his Basic Logic. Later, in [18], he presented an axiomatization for this logic and proved its strong completeness with respect to its generalized semantics. In this section we will show that monadic BL-algebras are the equivalent algebraic semantics of the logic S5(BL).

The modal logic S5(BL) is equivalent to the monadic fragment mBL$\forall$ of the fuzzy predicate calculus BL$\forall$, which contains only unary predicates and just one object variable x (without object constants). The propositional variable p_i is associated with the unary predicate $P_i(x)$ and the modalities $\Box$ and $\Diamond$ correspond to the quantifiers $(\forall x)$ and $(\exists x)$, respectively. For this reason, and to continue the algebraic tradition of naming *monadic* the algebraic semantics of monadic fragments of several logics (Boolean, intuitionistic, Łukasiewicz, etc.), we opted to call the algebras corresponding to the logic S5(BL) *monadic BL-algebras*. However, in this section we will work in the language of the modal logic S5(BL) instead of in the monadic fuzzy language.

We now recall the basic definition of S5(BL). The axiom schemata are the ones for the Basic Logic BL together with the following modal axiom schemata (ν stands for any propositional combination of formulas beginning with $\Box$ or $\Diamond$):

($\Box$1) $\Box\varphi \to \varphi$

($\Diamond$1) $\varphi \to \Diamond\varphi$

($\Box$2) $\Box(\nu \to \varphi) \to (\nu \to \Box\varphi)$

($\Diamond$2) $\Box(\varphi \to \nu) \to (\Diamond\varphi \to \nu)$

($\Box$3) $\Box(\nu \vee \varphi) \to (\nu \vee \Box\varphi)$

($\Diamond$3) $\Diamond(\varphi * \varphi) \equiv \Diamond\varphi * \Diamond\varphi$

where $\varphi \equiv \psi$ stands for $(\varphi \to \psi) \wedge (\psi \to \varphi)$.

The inference rules of S5(BL) are:

(MP) $\varphi, \varphi \to \psi \vdash \psi$

(Nec) $\varphi \vdash \Box\varphi$

The general semantics of S5(BL) is given by Kripke models. A *Kripke model* for S5(BL) is a triple $K = \langle X, e, \mathbf{A} \rangle$ where X is a nonempty set of worlds, $\mathbf{A}$ is a BL-chain and $e \colon \mathit{Prop} \times X \to A$ is an evaluation map, *Prop* being the set of propositional variables. The evaluation map extends to any formula:

- $e(0, x) = 0^{\mathbf{A}}$, $e(1, x) = 1^{\mathbf{A}}$.
- $e(\varphi \wedge \psi, x) = e(\varphi, x) \wedge^{\mathbf{A}} e(\psi, x)$, and the same for $\vee$, $\to$ and $*$.
- $e(\Box\varphi, x) = \inf\{e(\varphi, y) : y \in X\}$.
- $e(\Diamond\varphi, x) = \sup\{e(\varphi, y) : y \in X\}$.

Note that $e(\Box\varphi, x)$ and $e(\Diamond\varphi, x)$ may be undefined. We say that the Kripke model K is *safe* if $e(\Box\varphi, x)$ and $e(\Diamond\varphi, x)$ are defined for every formula φ.

We write $K \models \varphi$ if $e(\varphi, x) = 1$ for every $x \in X$. K is a model of a set of formulas Γ if $K \models \varphi$ for every $\varphi \in \Gamma$.

Theorem 4.1 (Hájek [18, Theorem 2])**.** *The modal logic* S5(BL) *is strongly complete with respect to its general semantics, i.e. the following are equivalent for every set of formulas* $\Gamma \cup \{\varphi\}$*:*

(1) $\Gamma \vdash \varphi$

(2) $K \models \varphi$ *for every safe model* K *of* Γ*.*

Remark 4.2. Consider a safe Kripke model $K = \langle X, e, \mathbf{A}\rangle$. Note that we can turn the map $e\colon Prop \times X \to A$ into a map $\overline{e}\colon Prop \to A^X$ given by the relation $\overline{e}(p)(x) = e(p, x)$. Since K is safe, $\overline{e}$ extends to formulas in the following way:

- $\overline{e}(0) = 0^{\mathbf{A}^X}$, $\overline{e}(1) = 1^{\mathbf{A}^X}$,
- $\overline{e}(\varphi \wedge \psi) = \overline{e}(\varphi) \wedge^{\mathbf{A}^X} \overline{e}(\psi)$, and the same for $\vee$, $\to$ and $*$,
- $\overline{e}(\Box\varphi) = \forall_\wedge \overline{e}(\varphi)$,
- $\overline{e}(\Diamond\varphi) = \exists_\vee \overline{e}(\varphi)$.

Thus, it is clear that $\{\overline{e}(\varphi) : \varphi \text{ formula}\} \subseteq A^X$ is the universe of a monadic functional BL-algebra (see Example 3.2).

Following the notation of the last remark, we can rewrite Hájek's completeness theorem as follows.

Theorem 4.3. *The following are equivalent for every set of formulas* $\Gamma \cup \{\varphi\}$*:*

(1) $\Gamma \vdash \varphi$

(2) $\overline{e}(\varphi) = 1$ *for every* $\overline{e}\colon Prop \to B$*, where* $\langle \mathbf{B}, \exists_\vee, \forall_\wedge\rangle$ *is any functional monadic BL-algebra and* $\overline{e}(\gamma) = 1$ *for every* $\gamma \in \Gamma$*.*

Theorem 4.4. *The variety* $\mathbb{MBL}$ *of monadic BL-algebras is the equivalent algebraic semantics for the logic* S5(BL) *(and* mBL$\forall$*).*

Proof. Following [6], it is enough to show the next two conditions for every set of formulas $\Gamma \cup \{\varphi, \psi\}$:

(ALG1) $\Gamma \vdash \varphi$ if and only if $\{\gamma \approx 1 : \gamma \in \Gamma\} \models_{\mathbb{MBL}} \varphi \approx 1$.

(ALG2) $\varphi \approx \psi \dashv\models_{\mathbb{MBL}} (\varphi \to \psi) \wedge (\psi \to \phi) \approx 1$.

Condition (ALG2) is trivially verified. We show condition (ALG1).

For the forward implication, note that if $\Gamma \vdash \varphi$, there exists a proof of φ from Γ and the axioms of S5(BL) by successive application of the inference rules (MP) and (Nec). Thus, it is enough to show that the equation $\varphi \approx 1$ is valid in $\mathbb{MBL}$ for every axiom φ of S5(BL) and that the inference rules preserve validity. The former statement follows from the definition of monadic BL-algebras and Lemma 2.3. The preservation of (MP) is trivial and the preservation of (Nec) follows from (M10).

For the converse implication, simply observe that, since $\langle \mathbf{B}, \exists_\vee, \forall_\wedge\rangle \in \mathbb{MBL}$, condition (2) of Theorem 4.3 holds. $\Box$

Thus, from the general theory of algebraic logic, we get the next corollary.

Corollary 4.5. *There is a one-one correspondence between axiomatic extensions of* S5(BL) *(or* mBL$\forall$*) and subvarieties of* $\mathbb{MBL}$.

From the same theorem we can also derive an important algebraic result for the variety $\mathbb{MBL}$.

Corollary 4.6. *The variety* $\mathbb{MBL}$ *is generated (as a variety) by the functional monadic BL-algebras.*

As a consequence of the algebraization of S5(BL) by monadic BL-algebras, we may give a simplified set of axioms for this calculus. We define a calculus S5'(BL) whose axiom schemata are all the ones for Basic Logic BL together with the following axiom schemata:

(A1) $\Box\varphi \to \varphi$

(A2) $\Box(\varphi \to \Box\psi) \equiv (\Diamond\varphi \to \Box\psi)$

(A3) $\Box(\Box\varphi \to \psi) \equiv (\Box\varphi \to \Box\psi)$

(A4) $\Box(\Diamond\varphi \vee \psi) \equiv (\Diamond\varphi \vee \Box\psi)$

(A5) $\Diamond(\varphi * \varphi) \equiv \Diamond\varphi * \Diamond\varphi$

and the same rules of inference: modus ponens and necessitation. It is easy to prove in the standard way (by means of a Lindenbaum-Tarski algebra) that this calculus is sound and complete with respect to a semantics based on monadic BL-algebras. Indeed, the only non-immediate results are the content of the following lemma.

Lemma 4.7. *In* S5'(BL)*:*

(1) $\varphi \equiv \psi \vdash \Box\varphi \equiv \Box\psi$.

(2) $\varphi \equiv \psi \vdash \Diamond\varphi \equiv \Diamond\psi$.

Proof. To prove (1), note that, by (A1), $\vdash \Box\varphi \to \varphi$, hence, by transitivity of implication, $\varphi \to \psi \vdash \Box\varphi \to \psi$. Using necessitation, $\varphi \to \psi \vdash \Box(\Box\varphi \to \psi)$, and using (A3), we get $\varphi \to \psi \vdash \Box\varphi \to \Box\psi$. By symmetry, we get (1).

To prove (2), first note that $\vdash \Diamond\varphi \equiv \Box\Diamond\varphi$. Indeed, using Basic Logic theorems, (1), (A4) and (A1), the following equivalences are valid: $\Box\Diamond\varphi \equiv \Box(\Diamond\varphi \vee \Diamond\varphi) \equiv \Diamond\varphi \vee \Box\Diamond\varphi \equiv \Diamond\varphi$.

Note also that $\vdash \psi \to \Diamond\psi$. Indeed, since $\vdash \Diamond\psi \equiv \Box\Diamond\psi$, we get that $\vdash \Diamond\psi \to \Box\Diamond\psi$. Using (A2), $\vdash \Box(\psi \to \Box\Diamond\psi)$, and again using that $\vdash \Diamond\psi \equiv \Box\Diamond\psi$ and (1), we get that $\vdash \Box(\psi \to \Diamond\psi)$. Finally, using (A1), we obtain that $\vdash \psi \to \Diamond\psi$.

Now, by transitivity, $\varphi \to \psi \vdash \varphi \to \Diamond\psi$. By necessitation, $\varphi \to \psi \vdash \Box(\varphi \to \Diamond\psi)$. Now, using the equivalences $\Box(\varphi \to \Diamond\psi) \equiv \Box(\varphi \to \Box\Diamond\psi) \equiv \Diamond\varphi \to \Box\Diamond\psi \equiv \Diamond\varphi \to \Diamond\psi$, we get that $\varphi \to \psi \vdash \Diamond\varphi \to \Diamond\psi$. By symmetry, we conclude (2). $\Box$

5 Main subvarieties

In this section we focus our study on three main subvarieties of $\mathbb{MBL}$: monadic MV-algebras, monadic Gödel algebras and monadic product algebras. These correspond naturally to the monadic expansions of the most important extensions of Hájek's Basic Logic: Łukasiewicz logic, Gödel logic and product logic, respectively. In each of these subvarieties we will also give explicit descriptions of their totally ordered structures.

5.1 Monadic MV-algebras

MV-algebras are the equivalent algebraic semantics of the infinite-valued Łukasiewicz logic (see [9]). It is widely known that they coincide with involutive BL-algebras. In other words, the variety of MV-algebras is term-equivalent to the subvariety of $\mathbb{BL}$ determined by the equation $\neg\neg x \approx x$.

In his Ph. D. thesis, Rutledge defined and studied monadic MV-algebras as a way to study the infinite-many-valued Łukasiewicz predicate calculus. More recently, in [12], we studied in depth the lattice of subvarieties of these algebras.

In this section we will show that the variety defined by Rutledge is term-equivalent to the subvariety of $\mathbb{MBL}$ determined by the equation $\neg\neg x \approx x$.

Let us recall the original definition of monadic MV-algebra.

Definition 5.1. An algebra $\mathbf{A} = \langle A, \oplus, \neg, \exists, 0\rangle$ of type $(2,1,1,0)$ is called a *monadic MV-algebra* (an MMV-algebra for short) if $\langle A, \oplus, \neg, 0\rangle$ is an MV-algebra and $\exists$ satisfies the following identities:

(MV1) $x \to \exists x \approx 1$.

(MV2) $\exists(x \vee y) \approx \exists x \vee \exists y$.

(MV3) $\exists\neg\exists x \approx \neg\exists x$.

(MV4) $\exists(\exists x \oplus \exists y) \approx \exists x \oplus \exists y$.

(MV5) $\exists(x * x) \approx \exists x * \exists x$.

(MV6) $\exists(x \oplus x) \approx \exists x \oplus \exists x$.

In an MMV-algebra $\mathbf{A}$, we define $\forall \colon A \to A$ by $\forall a = \neg\exists\neg a$, for every $a \in A$. Clearly, $\exists a = \neg\forall\neg a$. In the following lemma we collect some properties of MMV-algebras (see [12]). Recall that on each (monadic) MV-algebra $\mathbf{A}$ we can define the operations $*$, $\to$, $\wedge$ and $\vee$ as follows: $x * y := \neg(\neg x \oplus \neg y)$, $x \to y := \neg x \oplus y$, $x \wedge y := x * (x \to y)$ and $x \vee y := \neg(\neg x \oplus y) \oplus y$.

Lemma 5.2. *Let $\mathbf{A} = \langle A, \oplus, \neg, \exists, 0\rangle$ be an MMV-algebra. For every $a, b \in A$, the following properties hold:*

(1) $\forall a \to a = 1$.

(2) $\forall\neg\forall a = \neg\forall a$.

(3) $\forall(\exists a \vee b) = \exists a \vee \forall b$.

(4) $\forall(a \to b) \leq \forall a \to \forall b$ *or, equivalently,* $\forall(\neg a \oplus b) \leq \neg\forall a \oplus \forall b$.

(5) $\forall(a \wedge b) = \forall a \wedge \forall b$.

(6) $\forall(\forall a \oplus \forall b) = \forall a \oplus \forall b$.

(7) $\forall(a * a) = \forall a * \forall a$.

We collect here two properties of MV-algebras that will be useful later.

Lemma 5.3. *The following properties hold true in any MV-algebra $\mathbf{A} = \langle A, \oplus, \neg, 0\rangle$, where a, b denote arbitrary elements of A:*

(1) $a \oplus b = (a \to (a * b)) \to b$.

(2) $(a \to b)^2 \to (2a \to 2b) = 1$.

We now turn to derive some useful properties of involutive monadic BL-algebras. Some of the proofs are inspired by syntactic proofs given by Hájek in [17]. On each BL-algebra $\mathbf{A}$ we define the operations $\neg$ and $\oplus$ as follows: $\neg x := x \to 0$ and $x \oplus y := \neg x \to y$.

Lemma 5.4. *The following properties hold in any MBL-algebra $\mathbf{A}$ for every $a \in A$:*

(1) $\forall\neg\forall a = \neg\forall a$.

(2) $\neg\exists a = \forall\neg a$.

Moreover, if $\neg\neg a = a$ for every $a \in A$, the following also hold for arbitrary a, b in A:

(3) $\neg\forall a = \exists\neg a$.

(4) $a \oplus b = (\forall(a * b) \oplus \neg a) \to b$.

(5) $\forall(a * \exists b) = \forall a * \exists b$.

(6) $\exists(\exists a \to b) = \exists a \to \exists b$.

Proof. (1) By (M3), $\forall\neg\forall a = \forall(\forall a \to 0) = \forall a \to \forall 0 = \forall a \to 0 = \neg\forall a$.

(2) By (M9), $\neg\exists a = \exists a \to 0 = \exists a \to \exists 0 = \forall(a \to \exists 0) = \forall\neg a$.

(3) From (2), $\neg\forall a = \neg\forall\neg\neg a = \neg\neg\exists\neg a = \exists\neg a$.

(4) Since $\neg a \leq \forall(a * b) \oplus \neg a$, we have that $\neg a \to b \geq (\forall(a * b) \oplus \neg a) \to b$. On the other hand, using that $\forall(a * b) \oplus \neg a \leq (a * b) \oplus \neg a$, we get $(\forall(a * b) \oplus \neg a) \to b \geq ((a * b) \oplus \neg a) \to b$ and $(\neg a \to b) \to ((\forall(a * b) \oplus \neg a) \to b) \geq (\neg a \to b) \to (((a * b) \oplus \neg a) \to b) = 1$ from Lemma 5.3 (1).

(5) Since $\forall a * \exists b \leq a * \exists b$, then $\forall(\forall a * \exists b) \leq \forall(a * \exists b)$. Thus $\forall a * \exists b \leq \forall(a * \exists b)$. On the other hand, we intend to see that $\forall(a * \exists b) \to (\forall a * \exists b) = 1$. First observe that $(\neg\exists b \to a) \vee (a \to \neg\exists b) = 1$ then, by (4), $1 = ((\forall(a * \exists b) \oplus \neg\exists b) \to a) \vee (a \to \neg\exists b)$. Since $(a \to \neg\exists b) \to \neg\forall(a * \exists b) = \neg(a * \exists b) \to \neg\forall(a * \exists b) = 1$, we have that $((\forall(a * \exists b) \oplus \neg\exists b) \to a) \vee \neg\forall(a * \exists b) = 1$ and $1 = \forall(((\forall(a * \exists b) \oplus \neg\exists b) \to a) \vee \neg\forall(a * \exists b)) = \forall((\forall(a * \exists b) \oplus \neg\exists b) \to a) \vee \neg\forall(a * \exists b)$, from (M12).

We claim that $\forall((\forall(a * \exists b) \oplus \neg\exists b) \to a) \leq \forall(a * \exists b) \to (\forall a * \exists b)$. From this and the fact that $\neg\forall(a * \exists b) \leq \forall(a * \exists b) \to (\forall a * \exists b)$ it follows immediately that $\forall(a * \exists b) \to (\forall a * \exists b) = 1$.

It only remains to show that $\forall((\forall(a * \exists b) \oplus \neg\exists b) \to a) \leq \forall(a * \exists b) \to (\forall a * \exists b)$. Indeed, using (M8) and (M3), we have that $\forall((\forall(a * \exists b) \oplus \neg\exists b) \to a) = \forall((\exists b \to \forall(a * \exists b)) \to a) = \forall(\forall(\exists b \to (a * \exists b)) \to a) = \forall(\exists b \to (a * \exists b)) \to \forall a = (\exists b \to \forall(a * \exists b)) \to \forall a$. Since $\forall a \leq \exists b \to (\forall a * \exists b)$, it follows that $(\exists b \to \forall(a * \exists b)) \to \forall a \leq (\exists b \to \forall(a * \exists b)) \to (\exists b \to (\forall a * \exists b)) = (\exists b * (\exists b \to \forall(a * \exists b))) \to (\forall a * \exists b) = (\exists b \wedge \forall(a * \exists b)) \to (\forall a * \exists b) = \forall(a * \exists b) \to (\forall a * \exists b)$.

(6) By (2), (5), and (3), $\neg\exists(\exists a \to b) = \forall\neg(\exists a \to b) = \forall((\neg b \to (\exists a \to 0)) \to 0) = \forall(\neg b * \exists a) = \forall\neg b * \exists a = \neg((\forall\neg b * \exists a) \to 0) = \neg(\exists a \to (\forall\neg b \to 0)) = \neg(\exists a \to \neg\forall\neg b) = \neg(\exists a \to \exists b)$, then $\exists(\exists a \to b) = \exists a \to \exists b$. ☐

We now prove the main theorem of this subsection.

Theorem 5.5. *The subvariety of $\mathbb{MBL}$ determined by the equation $\neg\neg x \approx x$ is term-equivalent to the variety of MMV-algebras.*

Proof. Consider first an MBL-algebra $\mathbf{A} = \langle A, \vee, \wedge, *, \to, \exists, \forall, 0, 1\rangle$ that satisfies the equation $\neg\neg x \approx x$. We define $\mathbf{A}' = \langle A, \oplus, \neg, \exists, 0\rangle$ where $\neg x := x \to 0$ and $x \oplus y := \neg x \to y$. We claim that $\mathbf{A}'$ is an MMV-algebra. Indeed, conditions (MV1), (MV2), and (MV5) are precisely (M7), (M20), and (M5), respectively. Condition (MV3) follows from $\exists\neg\exists a = \exists(\exists a \to 0) = \exists(\exists a \to \exists 0) = \exists a \to \exists 0 = \exists a \to 0 = \neg\exists a$. From (MV3) and (M29) we have that $\exists(\exists a \oplus \exists b) = \exists(\neg\exists a \to \exists b) = \exists(\exists\neg\exists a \to \exists b) = \exists\neg\exists a \to \exists b = \neg\exists a \to \exists b = \exists a \oplus \exists b$; hence, (MV4) holds. Finally, we prove (MV6). On the one hand, since $a \oplus a \leq \exists a \oplus \exists a$, we get $\exists(a \oplus a) \leq \exists(\exists a \oplus \exists a) = \exists a \oplus \exists a$. On the other hand, from Lemma 5.3 (2), $(\exists a \to a)^2 \leq (\exists a \oplus \exists a) \to (a \oplus a)$. Then $\exists((\exists a \to a)^2) \leq \exists((\exists a \oplus \exists a) \to (a \oplus a))$. By (M5), we have that $(\exists(\exists a \to a))^2 \leq \exists((\exists a \oplus \exists a) \to (a \oplus a))$. From Lemma 5.4 (6), we conclude that $1 = \exists((\exists a \oplus \exists a) \to (a \oplus a)) = \exists(\exists(\exists a \oplus \exists a) \to (a \oplus a)) = (\exists a \oplus \exists a) \to \exists(a \oplus a)$.

Conversely, let $\mathbf{B} = \langle B, \oplus, \neg, \exists, 0\rangle$ be an MMV-algebra and define $\mathbf{B}' = \langle B, \vee, \wedge, *, \to, \exists, \forall, 0, 1\rangle$, where $1 := \neg 0$, $x \to y := \neg x \oplus y$, $\forall x := \neg\exists\neg x$, $x * y := \neg(\neg x \oplus \neg y)$, $x \vee y := \neg(\neg x \oplus y) \oplus y$, $x \wedge y := x * (x \to y)$. We claim that $\mathbf{B}'$ is an MBL-algebra that satisfies the equation $\neg\neg x \approx x$. Indeed, conditions (M1), (M4) and (M5) are found in Lemma 5.2 (1), Lemma 5.2 (3), and (MV5), respectively. To prove condition (M2), note that, since $a \leq \exists a$, $\exists a \to \forall b \leq a \to \forall b$. Thus $\exists a \to \forall b = \forall(\exists a \to \forall b) \leq \forall(a \to \forall b)$. On the other hand, $\forall(a \to \forall b) = \forall(\neg\forall b \to \neg a) \leq \forall\neg\forall b \to \forall\neg a = \neg\forall b \to \neg\exists a = \exists a \to \forall b$. It only remains to prove condition (M3). By Lemma 5.2, $\forall(\forall a \to b) \leq \forall a \to \forall b$. In addition, since $\forall a \to \forall b \leq \forall a \to b$, it follows that $\forall a \to \forall b = \forall(\forall a \to \forall b) \leq \forall(\forall a \to b)$. □

It would be interesting to characterize all monadic BL-chains and we will do this in the last section of this article. The problem for monadic MV-algebras is already solved (see [13]), but we give here an elementary proof.

Theorem 5.6. *Let $\mathbf{A}$ be a totally ordered MMV-algebra. Then $\exists a = a$ for every $a \in A$, that is, the quantifier on $\mathbf{A}$ is the identity.*

Proof. We show the result by way of contradiction. Let $\mathbf{A}$ be a totally ordered MMV-algebra and assume there exists $a \in A$ such that $\exists a \neq a$. Consider $b = \exists a \to a$ and note that $b \neq 1$ and $\exists b = \exists(\exists a \to a) = \exists a \to \exists a = 1$.

Suppose $b^2 \leq \forall b$. Then $1 = (\exists b)^2 = \exists b^2 \leq \exists\forall b = \forall b \leq b$, so $b = 1$, a contradiction. Since $\mathbf{A}$ is a chain, $\forall b < b^2$. Then $\forall b \leq \forall b^2 = (\forall b)^2$. Thus $\forall b$ is an idempotent element of $\mathbf{A}$ forcing $\forall b$ to be 0 or 1. The case $\forall b = 1$ implies $b = 1$, a contradiction. Now assume $\forall b = 0$, that is, $0 = \forall(\exists a \to a) = \exists a \to \forall a$. Since $\forall a \leq \exists a \to \forall a$, it follows that $\forall a = 0$. Moreover, $0 = \exists a \to \forall a = \neg\exists a = \forall\neg a$. Now, there are two possibilities for a: either $a \leq \neg a$ or $\neg a \leq a$. If $a \leq \neg a$, then $2(\neg a) = 1$. So $1 = \forall 2(\neg a) = 2\forall(\neg a) = 0$, a contradiction. Analogously, if $\neg a \leq a$, then $2a = 1$ and $1 = \forall 2a = 2\forall a = 0$, another contradiction. □

5.2 Monadic Gödel algebras

Gödel algebras are prelinear Heyting algebras, that is, they constitute the variety generated by totally ordered Heyting algebras. Concretely, Gödel algebras are the subvariety of Heyting algebras determined by the prelinearity equation $(x \to y) \vee (y \to x) \approx 1$. This variety is generated by the Gödel t-norm $[0, 1]_G$ and may be also identified with the subvariety of BL-algebras that satisfy the equation $x^2 \approx x$ (idempotence), since this equation is equivalent to $x * y \approx x \wedge y$. In this subsection we are interested in the subvariety of monadic BL-algebras determined by the equation $x^2 \approx x$. We call these algebras *monadic Gödel algebras.*

Monadic Heyting algebras were introduced by Monteiro and Varsavksy in [21] and later studied in depth by Bezhanishvili in [4]. As in the previous section we will study now the connection between monadic Gödel algebras and the variety of monadic Heyting algebras. More precisely, we will show that monadic Gödel algebras coincide with monadic prelinear Heyting algebras that satisfy the equation

$$\forall(\exists x \vee y) \approx \exists x \vee \forall y. \tag{1}$$

We recall the definition of monadic Heyting algebras from [4].

Definition 5.7. An algebra $\langle A, \vee, \wedge, \to, \exists, \forall, 0, 1\rangle$, shortened as $\langle \mathbf{A}, \exists, \forall\rangle$, is a *monadic Heyting algebra* if $\langle A, \vee, \wedge, \to, 0, 1\rangle$ is a Heyting algebra and $\forall, \exists$ are unary operators on A satisfying the following conditions for all $a, b \in A$.

(H1) $\forall a \leq a$, $a \leq \exists a$.

(H2) $\forall(a \wedge b) = \forall a \wedge \forall b$, $\exists(a \vee b) = \exists a \vee \exists b$.

(H3) $\forall 1 = 1$, $\exists 0 = 0$.

(H4) $\forall \exists a = \exists a$, $\exists \forall a = \forall a$.

(H5) $\exists(\exists a \wedge b) = \exists a \wedge \exists b$.

The following properties will be useful and their proofs may be found in [4].

Lemma 5.8. *The following properties hold in any monadic Heyting algebra* $\mathbf{A}$*, where* a, b *denote arbitrary elements of* A*.*

(1) *If* $a \leq b$ *then,* $\forall a \leq \forall b$ *and* $\exists a \leq \exists b$*.*

(2) $\forall(a \to b) \leq \forall a \to \forall b$.

(3) $\forall\forall a = \forall a$ *and* $\exists\exists a = \exists a$*.*

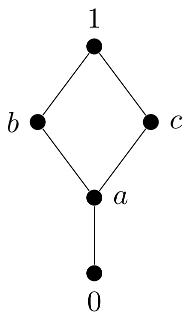
(4) $\forall(a \to \forall b) = \exists a \to \forall b$.

(5) $\forall(a \to \exists b) = \exists a \to \exists b$.

Theorem 5.9. *Monadic Gödel algebras coincide with monadic prelinear Heyting algebras that satisfy equation* (1)*.*

Proof. If $\mathbf{A}$ is a monadic Gödel algebra, from the definition of monadic BL-algebra and some of the properties of Lemma 2.3 it is immediate that $\mathbf{A}$ is also a monadic prelinear Heyting algebra and satisfies equation (1). Conversely, let $\mathbf{A}$ be a monadic prelinear Heyting algebra that satisfies equation (1). Clearly $\mathbf{A}$ satisfies conditions (M1), (M4) and (M5) in the definition of monadic BL-algebra. Condition (M2) follows from Lemma 5.8 (4). It remains to show condition (M3). From Lemma 5.8, $\forall(\forall a \to b) \leq \forall\forall a \to \forall b = \forall a \to \forall b$. In addition, from (H1) we know that $\forall b \leq b$. Then, using the properties from Lemma 5.8, we have that $\forall(\forall a \to \forall b) \leq \forall(\forall a \to b)$ and then $\exists \forall a \to \forall b = \forall a \to \forall b \leq \forall(\forall a \to b)$. $\square$

Remark 5.10. Note that monadic prelinear Heyting algebras may not satisfy equation (1). A counterexample is given by the monadic Heyting algebra depicted in the Hasse diagram below with the monadic operators defined as in the table.



x	0	a	b	c	1
$\exists x$	0	c	1	c	1
$\forall x$	0	0	0	c	1

Indeed, note that $\forall(b \vee \exists c) = \forall(b \vee c) = \forall 1 = 1$ whereas $\forall b \vee \exists c = 0 \vee c = c$.

Finally, we would like to make explicit all the possible monadic structures that we may define on a given totally ordered Gödel algebra. Using that $a * b = a \wedge b$ holds true in any Gödel algebra, the conditions stated in Theorem 3.1 reduce only to relative completeness. More precisely, we have the following result.

Theorem 5.11. *Given a totally ordered Gödel algebra* $\mathbf{A}$ *and a relatively complete subalgebra* $\mathbf{C} \leq \mathbf{A}$*, if we define on* A *the operations*

$$\forall a := \max\{c \in C : c \leq a\}, \qquad \exists a := \min\{c \in C : c \geq a\},$$

then $\langle \mathbf{A}, \exists, \forall \rangle$ *is a monadic Gödel algebra such that* $\forall A = \exists A = C$*.*

In addition, since in any totally ordered Gödel algebra $\mathbf{A}$ we have that

$$x \to y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise,} \end{cases}$$

any finite subset of A that contains 0 and 1 is a relatively complete subalgebra of $\mathbf{A}$ and, hence, defines a structure of monadic Gödel algebra on $\mathbf{A}$.

5.3 Monadic product algebras

In this subsection we introduce the subvariety of $\mathbb{MBL}$ that consists of those monadic BL-algebras whose underlying BL-structure is a product algebra. We name these algebras *monadic product algebras*. In particular, we will prove that only two monadic operators may be defined on any totally ordered product algebra, namely, the identity operators and the Monteiro-Baaz operators Δ and ∇ (see [20]). We think that this subvariety deserves a more detailed study, but we will intend to pursue this task in a forthcoming article.

Recall that a *product algebra* is a BL-algebra satisfying the following identities:

(P1) $\neg\neg z \to ((x * z \to y * z) \to (x \to y)) \approx 1$.

(P2) $x \wedge \neg x \approx 0$.

Note that the identity (P2) implies that every product algebra is pseudocomplemented, whereas the identity (P1) implies that in any chain the non-zero elements form a cancellative hoop. For basic properties of product algebras and cancellative hoops, see [17, 5].

We define a *monadic product algebra* to be a monadic BL-algebra that is also a product algebra.

Example 5.12. On any product algebra $\mathbf{A}$, we can define two sets of monadic operators, that we will henceforth call *trivial operators*:

- Identity operators: $\exists a = \forall a = a$ for every $a \in A$.
- Monteiro-Baaz operators:

  $$\forall a = \Delta a = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{if } a < 1, \end{cases} \qquad \text{and} \qquad \exists a = \nabla a = \begin{cases} 0 & \text{if } a = 0, \\ 1 & \text{if } a > 0, \end{cases}$$

  for every $a \in A$.

  Conditions (M1)-(M4) in the definition of monadic BL-algebras are easily checked for these operators. To verify condition (M5), note that if $a \neq 0$, $a^2 \neq 0$ and so $\exists a = (\exists a)^2 = \exists a^2 = 1$.

We now intend to prove that in any totally ordered product algebra we can only define the trivial quantifiers.

Lemma 5.13. *Let $\mathbf{A}$ be a non-trivial totally ordered monadic product algebra.*

(1) $\exists(\exists a \to a) = 1$ *for every* $a \in A$.

(2) *If* $a \in A \setminus \{0\}$ *and* $\forall a = 0$, *then* $\exists a = 1$.

(3) *If* $A' = \{u \in A : \forall u \neq 0\}$, *then* $\mathbf{A}' = \langle A', *, \to, 1 \rangle$ *is a cancellative hoop.*

Proof. To prove (1), note first that $\exists(\exists 0 \to 0) = 1$. Now let $a > 0$. Then $a = \exists a \wedge a = \exists a * (\exists a \to a)$. Thus,

$$\exists a = \exists(\exists a * (\exists a \to a)) = \exists a * \exists(\exists a \to a).$$

Since $\exists a > 0$, we can cancel out $\exists a$ in the previous equation and get that $1 = \exists(\exists a \to a)$.

To prove (2), let $a \in A \setminus \{0\}$ such that $\forall a = 0$. Since $\mathbf{A}$ is totally ordered then $\exists a \to a < \exists a$ or $\exists a \leq \exists a \to a$. Let us suppose that $\exists a \leq \exists a \to a$. Then $0 < a \leq \exists a = \forall \exists a \leq \forall(\exists a \to a) = \exists a \to \forall a = \exists a \to 0 = 0$, a contradiction. Then $\exists a \to a < \exists a$. From (1), we have that $1 = \exists(\exists a \to a) \leq \exists \exists a$, so $\exists a = 1$.

Finally we show (3). Since the non-zero elements of $\mathbf{A}$ form a cancellative hoop, it is enough to show that $1 \in A'$ and that A' is closed under $*$ and $\to$. Indeed, $1 \in A'$ since $\forall 1 = 1 > 0$. In adddition, if $u, v \in A'$, then

$$0 < \forall u * \forall v = \forall(\forall u * \forall v) \leq \forall(u * v).$$

Thus, $u * v \in A'$. From $0 < \forall v \leq \forall(u \to v)$, we have that $u \to v \in A'$. $\square$

From the last lemma, we know that $\mathbf{A}'$ is a cancellative hoop. In particular, $\mathbf{A}'$ is a Wajsberg hoop and hence, we may apply to this hoop the MV-closure construction given in [2]. We recall here this construction. We define the MV-closure $\mathbf{MV}(\mathbf{A}')$ of the cancellative hoop $\mathbf{A}' = \langle A', *, \to, 1\rangle$ as follows:

$$\mathbf{MV}(\mathbf{A}') = \langle A' \times \{0, 1\}, \oplus_{mv}, \neg_{mv}, 0_{mv}\rangle,$$

where $0_{mv} := (1, 0)$, $\neg_{mv}(a, i) := (a, 1 - i)$,

$$(a, i) \oplus_{mv} (b, j) := \begin{cases} (a \oplus b, 1) & \text{if } i = j = 1, \\ (b \to a, 1) & \text{if } i = 1 \text{ and } j = 0, \\ (a \to b, 1) & \text{if } i = 0 \text{ and } j = 1, \\ (a * b, 0) & \text{if } i = j = 0, \end{cases}$$

where $a \oplus b = (a \to (a * b)) \to b$. The other common MV-operations are defined as follows:

- $1_{mv} := \neg_{mv} 0_{mv} = (1, 1)$,
- $(a, i) *_{mv} (b, j) := \neg_{mv}(\neg_{mv}(a, i) \oplus_{mv} \neg_{mv}(b, j))$,
- $(a, i) \to_{mv} (b, j) := \neg_{mv}(a, i) \oplus_{mv} (b, j)$,
- $(a, i) \wedge_{mv} (b, j) := (a, i) *_{mv} ((a, i) \to_{mv} (b, j))$,
- $(a, i) \vee_{mv} (b, j) = ((a, i) \to_{mv} (b, j)) \to_{mv} (b, j)$.

If we identify a with $(a, 1)$ for every $a \in A'$, we can consider $\mathbf{A}'$ a subalgebra of the hoop-reduct of $\mathbf{MV}(\mathbf{A}')$. If $b = (a, 0)$ for some $a \in A'$, then $b = \neg_{mv}(a, 1)$ and we write $b = \neg_{mv} a$. Then we can consider the universe of $\mathbf{MV}(\mathbf{A}')$ as the disjoint union of A' and $\neg_{mv} A'$. The order relation on $\mathbf{MV}(\mathbf{A}')$ is given by: $(a, i) \leq (b, j)$ if and only if one of the following conditions holds:

- $a \leq b$ and $i = j = 1$,
- $a \oplus b = 1$, $i = 0$, and $j = 1$,
- $b \leq a$ and $i = j = 0$.

Observe that since $\mathbf{A}'$ is a cancellative hoop, then $a \oplus b = 1$ for every $a, b \in A'$, so in this case, the elements in A' are all above the ones in $\neg_{mv} A'$. Thus $\mathbf{MV}(\mathbf{A}')$ is a totally ordered MV-algebra. Let us define the following quantifiers:

$$\exists(a,i)=\begin{cases}(\exists a,1) & \text{if } i=1,\\ (\forall a,0) & \text{if } i=0,\end{cases} \qquad \forall(a,i)=\begin{cases}(\forall a,1) & \text{if } i=1,\\ (\exists a,0) & \text{if } i=0.\end{cases}$$

Then the following result may be easily checked.

Lemma 5.14. *Let* $\mathbf{A}$ *be a non-trivial totally ordered monadic product algebra and let* $A' = \{u \in A : \forall u \neq 0\}$*. Then* $\langle \mathbf{MV}(\mathbf{A}'), \exists, \forall\rangle$ *is an MMV-chain.*

As we saw in the previous subsection, on an MV-chain we can only define the identity quantifiers. Thus the next corollary is immediate.

Corollary 5.15. *Let* $\mathbf{A}$ *be a non-trivial totally ordered monadic product algebra, then* $\exists A' = \forall A' = A'$*.*

We are now ready to prove the main result of this subsection.

Theorem 5.16. *In any totally ordered monadic product algebra the quantifiers are trivial.*

Proof. Let $\mathbf{A}$ be a non-trivial totally ordered monadic product algebra. Let us suppose that there is $u \in A$ such that $\forall u = 0$ and $u > 0$. From Lemma 5.13 (2) we know that $\exists u = 1$. Let $v \in A$ and $v \neq 0, 1$. Since $\mathbf{A}$ is a chain we have that $v \leq u$ or $u \leq v$. If $v \leq u$, then $\forall v = 0$ and, again by Lemma 5.13 (2), $\exists v = 1$. If $u \leq v$ then $\exists v = 1$. If $\forall v \neq 0$ then $v \in A'$ and, from the previous corollary, $\forall v = v = \exists v = 1$, which is a contradiction. So, $\forall v = 0$. This proves that $\exists = \nabla$ and $\forall = \Delta$.

We now deal with the case in which $\forall u \neq 0$ for all $u \in A \setminus \{0\}$. Then $A' = \{u \in A : \forall u \neq 0\} = A \setminus \{0\}$ and, from the previous corollary, we have that $\exists A' = \forall A' = A'$. Therefore, $\exists = \forall = id$. □

6 Monadic BL-chains

The objective of this section is to characterize all MBL-chains. Based on the characterization of BL-chains as ordinal sums of totally ordered Wajsberg hoops given by Aglianò and Montagna in [3], later simplified by Busaniche in [7], we will present a way of building a monadic BL-chain as an ordinal sum of totally ordered Wajsberg hoops indexed on a monadic Heyting chain. Moreover, we will also show that any monadic BL-chain may be obtained using this construction.

First we need to recall the definition of ordinal sum of a family of Wajsberg hoops indexed on a totally ordered set I. Fix a bounded totally ordered set $(I, \leq, 0, 1)$ that will be used as an index set. Set $C_1 = \{1\}$ and, for each $i \in I \setminus \{1\}$, let C_i be a set such that $\langle C_i \cup \{1\}, *_i, \to_i, 1\rangle$ is a totally ordered Wajsberg hoop. Assume also that C_0 has a least element 0.

Let $C_I := \{(i, a) : i \in I, a \in C_i\} \subseteq I \times (\bigcup_{i\in I} C_i)$. We define a total order on C_I (lexicographic order) as follows:

$$(i, a) \leq (j, b) \text{ iff } i < j \text{ or } (i = j \text{ and } a \leq b).$$

We define on C_I the following operations:

$$(i,a) * (j,b) = \begin{cases}(i,a) & \text{if } i<j,\\ (i, a *_i b) & \text{if } i=j,\\ (j,b) & \text{if } i>j,\end{cases} \qquad (i,a)\to(j,b)=\begin{cases}(1,1) & \text{if } (i,a)\leq(j,b),\\ (i, a\to_i b) & \text{if } i=j \text{ and } a>b,\\ (j,b) & \text{if } i>j,\end{cases}$$

The algebra $\mathbf{C}_I = \langle C_I, \vee, \wedge, *, \to, (0,0), (1,1)\rangle$ is a BL-chain. This follows immediately from the construction given by Busaniche in [7].

In order to define a structure of monadic BL-algebra on $\mathbf{C}_I$, we will require that I be endowed with a monadic Heyting structure; we denote by $\mathbf{I}$ this monadic Heyting chain. We also need to assume that the sets C_i, $i \in I$, satisfy the following conditions:

- if $\forall i < i$, $C_{\forall i}$ has a greatest element $u_{\forall i}$,
- if $i < \exists i$, $C_{\exists i}$ has a least element $0_{\exists i}$.

Note that if $\exists i = 1$, then $0_{\exists i} = 0_1 = 1$.

Consider the subset $S \subseteq C_I$ given by: $(i, a) \in S$ iff $i \in \forall I$. We claim that $\mathbf{S}$ is an m-relatively complete subalgebra of $\mathbf{C}_I$. The fact that $\mathbf{S}$ is a subalgebra of $\mathbf{C}_I$ is immediate from the definition of the operations. We now show that the conditions for m-relative completeness hold for $\mathbf{S}$:

(1) Let (i, a) be any element of C_I. If $i \in \forall I$, then $(i, a) \in S$ and the conditions hold trivially. Suppose $i \notin \forall I$. Then $i < \exists i$ and it follows that $(\exists i, 0_{\exists i})$ exists and $(i, a) \leq (\exists i, 0_{\exists i})$. Moreover, if $(i, a) \leq (j, b)$ for some $(j, b) \in S$, i.e., $j \in \forall I$, then $i \leq j$, so $\exists i \leq j$ and then $(\exists i, 0_{\exists i}) \leq (j, b)$. The dual condition follows analogously.

(2) This condition is trivial for chains.

(3) Consider $(i, a) \in C_I$ and $(j, b) \in S$ (i.e. $j \in \forall I$) such that $(i, a)^2 = (i, a^2) \leq (j, b)$. If $i \in \forall I$, then $(i, a) \in S$ and the condition follows trivially. Suppose $i \notin \forall I$. Then $i < \exists i \leq j$, so $(i, a) \leq (\exists i, 0_{\exists i})$ and $(\exists i, 0_{\exists i})^2 = (\exists i, 0_{\exists i}) \leq (j, b)$.

We denote by $\mathbf{C_I}$ the monadic BL-algebra induced by S on $\mathbf{C}_I$.

We have thus shown the following theorem.

Theorem 6.1. $\mathbf{C_I}$ *is a monadic BL-chain and the monadic operators on* $\mathbf{C_I}$ *are given by:*

$$\exists(i, a) = \begin{cases} (i, a) & \text{if } i \in \forall I, \\ (\exists i, 0_{\exists i}) & \text{if } i \notin \forall I, \end{cases} \qquad \forall(i, a) = \begin{cases} (i, a) & \text{if } i \in \forall I, \\ (\forall i, u_{\forall i}) & \text{if } i \notin \forall I. \end{cases}$$

We will now show that *any* monadic BL-chain is isomorphic to $\mathbf{C_I}$ for a suitable monadic Heyting chain $\mathbf{I}$ and suitable Wajsberg chains $\{\mathbf{C}_i : i \in I\}$.

Fix a monadic BL-chain $\mathbf{A}$ and recall the representation of BL-chains as ordinal sums of totally ordered Wasjberg hoops given by Busaniche in [7].

For each $a \in A$, we consider the set $F_a = \{x \in A \setminus \{1\} : a \to x = x\}$. We can define the following equivalence relation on A: $a \sim b$ iff $F_a = F_b$. Each equivalence class C is a convex set and $\mathbf{C}' = \langle C \cup \{1\}, *, \to, 1\rangle$ is a totally ordered Wajsberg hoop. Let I be the set of equivalence classes ordered by: $C \preceq D$ iff either $C = D$, or, for all $a \in C$ and for all $b \in D$, $a \leq b$. We write $C \prec D$ when $C \preceq D$ and $C \neq D$. We also know that if $C \prec D$, $a \in C$ and $b \in D$, then $b \to a = a$ and $a * b = a$. We denote by C_0 the equivalence class that contains the element 0, and by C_1 the class that contains the element 1. Observe that $C_1 = \{1\}$. Then $\mathbf{A}$ is isomorphic as a BL-algebra to $\mathbf{C}_I$ as defined above.

For each equivalence class C we will show that either $C \subseteq \forall A$ or $C \cap \forall A = \emptyset$. Since this is trivially true for the class C_1, we assume $C \neq C_1$ in the sequel.

We now show that $C \cap \forall A$ is an increasing subset of C. Suppose there is $c \in C$ such that $\forall c = \exists c = c$ and consider $D = \{a \in C : a \geq c\} \cup \{1\}$. Since D is a bounded increasing subset of C', we can define an MV-structure on D. Indeed, $\mathbf{D} = \langle D, \vee, \wedge, *_c, \to, c, 1\rangle$ where $x *_c y := (x * y) \vee c$. Note that if $a \in D$, $a \neq 1$, then $c \leq a$ and $c = \forall c \leq \forall a \leq a$, and since C is convex, $\forall a \in D$. It follows that D is closed under $\forall$. We define a unary operation $\exists'$ on D by

$$\exists' x := \forall(x \to c) \to c = (\exists x \to c) \to c.$$

We claim that $\langle \mathbf{D}, \exists', \forall\rangle$ is a totally ordered monadic MV-algebra. To prove this, it is enough to check the identities (M1)-(M5).

Since the identities (M1) and (M3) involve only the operations $\to, \forall, 1$, and these are defined on $\langle \mathbf{D}, \exists', \forall\rangle$ as restrictions of the original operations of $\mathbf{A}$, these identities hold trivially on $\langle \mathbf{D}, \exists', \forall\rangle$.

Fix $a, b \in D$. If $\exists a \in D$, then $\exists' a = (\exists a \to c) \to c = \exists a \vee c = \exists a$ and it is clear that

$$\forall(a \to \forall b) = \exists' a \to \forall b, \tag{2}$$

$$\forall(\exists' a \vee b) = \exists' a \vee \forall b. \tag{3}$$

If $\exists a \notin D$, then $\exists' a = (\exists a \to c) \to c = c \to c = 1$. Then (3) holds trivially, and we also have that $\forall(a \to \forall b) = \exists a \to \forall b = \forall b = \exists' a \to \forall b$. This shows that $\langle \mathbf{D}, \exists', \forall \rangle$ satisfies (M2) and (M4).

It remains to show the validity of (M5). Take $a \in D$ and note that $\exists'(a *_c a) = \exists'(a^2 \vee c) = (\exists(a^2 \vee c) \to c) \to c = (\exists a^2 \to c) \to c$ and $(\exists' a) *_c (\exists' a) = ((\exists a \to c) \to c)^2 \vee c$. We distinguish the following cases:

- $\exists a \notin D$: In this case $\exists a \to c = c$, so $\exists' a = 1$. Moreover, $\exists a^2 = (\exists a)^2 \notin D$. Hence, $\exists' a *_c \exists' a = \exists'(a *_c a) = 1$.
- $\exists a \in D$ and $\exists a^2 \in D$: In this case, $\exists'(a *_c a) = \exists a^2 \vee c = \exists a^2$ and $\exists' a *_c \exists' a = (\exists a \vee c)^2 \vee c = (\exists a)^2 \vee c = \exists a^2$.
- $\exists a \in D$, but $\exists a^2 \notin D$: In this case $a \neq 1$, $\exists a \neq 1$, and $\exists a^2 = (\exists a)^2 < c$. Thus $\exists'(a *_c a) = c$ and $\exists' a *_c \exists' a = (\exists a)^2 \vee c = c$.

This concludes the verification that equation (M5) holds in $\langle \mathbf{D}, \exists', \forall \rangle$.

We have thus shown that $\langle \mathbf{D}, \exists', \forall \rangle$ is a totally ordered monadic MV-algebra. From Theorem 5.6, we deduce that $\exists'$, and also $\forall$ and $\exists$, are the identity quantifiers on D. Thus $D \subseteq \forall A$. This proves the claim that $C \cap \forall A$ is an increasing subset of C.

We now distinguish two possibilities for the totally ordered Wajsberg hoop $\mathbf{C}'$.

- $\mathbf{C}'$ is a bounded Wajsberg hoop, that is, there exists $0' \in C$ such that $0' \leq a$, for all $a \in C$.

  In this case we can endow $\mathbf{C}'$ with a natural MV-structure. We make a further distinction according to whether $0' \in \forall A$ or not.

  - Assume $0' \in \forall A$.
    In this case, since $C \cap \forall A$ is an increasing subset of C, it follows that $C \subseteq \forall A$.
  - Assume $0' \notin \forall A$.
    In this case $\forall 0' < 0' < \exists 0'$. We will show that $C \cap \forall A = \emptyset$. Indeed, assume there is $a \in C$ such that $\forall a = \exists a = a$. As $0' \leq a$, we have that $0' \leq \exists 0' \leq \exists a = a$, so $\exists 0' \in C$. Note that $\exists 0' \to 0' \neq 1$ and $\exists 0' \to 0' = \exists(0' * 0') \to 0' = (\exists 0' * \exists 0') \to 0' = \exists 0' \to (\exists 0' \to 0')$, hence $\exists 0' \to 0' \in F_{\exists 0'} = F_{0'}$, and then $0' \to (\exists 0' \to 0') = \exists 0' \to 0'$, which is not possible since $0' \to (\exists 0' \to 0') = 1$ and $\exists 0' \to 0' \neq 1$.
    Observe that, since $C \cap \forall A = \emptyset$, for every $a \in C$, $\forall a = \forall 0'$ and $\exists a = \exists 0'$.

- $\mathbf{C}'$ is an unbounded Wajsberg hoop, and hence a cancellative hoop.

  We claim that for any $a \in C$: if $\forall a \notin C$, then $\exists a \notin C$.

  By way of contradiction, assume that $\forall a \notin C$ and $\exists a \in C$. Note that $\forall a < a < \exists a$ and hence $\exists a \to a \in C$. Observe also that, since $a = a \wedge \exists a = \exists a * (\exists a \to a)$, we have that $\exists a = \exists a * \exists(\exists a \to a)$. If $\exists(\exists a \to a) \in C$, then, using the cancellative property, we get that $\exists(\exists a \to a) = 1$, a contradiction. This shows that $\exists(\exists a \to a) \notin C$ and, consequently, $\exists(\exists a \to a) > b$ for every $b \in C$.

  Suppose now that $\exists a \to a \leq \exists a$. Then $\exists(\exists a \to a) \leq \exists a$, which is a contradiction. On the other hand, suppose that $\exists a \leq \exists a \to a$. In this case, $\exists a^2 = (\exists a)^2 \leq a$ and, moreover, $\exists a^2 \leq \forall a$. However $\forall a \notin C$ and $\forall a < b$ for every $b \in C$. Thus $\forall a < a^2 \leq \exists a^2 \leq \forall a$, a contradiction.

  Consequently, if there is $a \in C$ such that $\forall a \notin C$, then $\exists a \notin C$ and it is clear that $C \cap \forall A = \emptyset$. Otherwise, if, for every $a \in C$, $\forall a \in C$, then, using the fact that $C \cap \forall A$ is an increasing subset of C, it follows that $C \subseteq \forall A$.

From what we have just proved, it follows that, for any given $a \in A$, there are two possibilities:

- if $C_a \subseteq \forall A$, then $\forall a = \exists a = a$;
- if $C_a \cap \forall A = \emptyset$, then $C_{\forall a} \prec C_a \prec C_{\exists a}$; moreover, since $C_{\forall a}, C_{\exists a}$ must be contained in $\forall A$, then $\forall a$ is the greatest element of $C_{\forall a}$ and $\exists a$ is the least element of $C_{\exists a}$.

Finally, note that we can define a monadic Heyting structure on $I = A/{\sim} = \{C_a : a \in A\}$. Indeed, since $(I, \preceq)$ is a bounded totally ordered set with least element C_0 and greatest element C_1, it can be turned into a totally ordered Heyting structure $\mathbf{I}$:

$$C_a \wedge C_b = C_{a \wedge b}, \qquad C_a \vee C_b = C_{a \vee b}, \qquad C_a \to C_b = \begin{cases} C_1 & \text{if } C_a \preceq C_b, \\ C_b & \text{if } C_b \prec C_a. \end{cases}$$

Now consider $S = \{C_a : a \in \forall A\}$. It is immediate that $\mathbf{S}$ is a subalgebra of $\mathbf{I}$. Moreover, $\mathbf{S}$ is a relatively complete subalgebra. Indeed, let C_a be an arbitrary element of I and $C_b \in S$ such that $C_b \preceq C_a$. There are two possible situations. If $C_b = C_a$, then $C_a \subseteq \forall A$, so $\forall a = a$ and $C_b = C_a = C_{\forall a}$. If $C_b \prec C_a$, then $b < a$, so $b \leq \forall a$ and $C_b \preceq C_{\forall a}$. This shows that $C_{\forall a}$ is the greatest element in S below C_a. In an analogous manner, it may be shown that $C_{\exists a}$ is the least element in S above C_a.

Since $\mathbf{S}$ is a relatively complete subalgebra of $\mathbf{I}$, by Theorem 5.11, S defines a monadic Heyting structure on $\mathbf{I}$.

If we consider now the map $\psi\colon A \to C_I$ given by $\psi(a) = (C_a, a)$, it is clear from what we have shown above that ψ is an isomorphism of monadic BL-algebras.

We have thus finished the proof of the characterization of all monadic BL-chains.

Theorem 6.2. *Any monadic BL-chain is isomorphic to some* $\mathbf{C_I}$.

To close this section, we would like to remark that the subvariety of $\mathbb{MBL}$ generated by monadic BL-chains may be axiomatized within $\mathbb{MBL}$ by a single identity:

$$\forall(x \vee y) \approx \forall x \vee \forall y.$$

Indeed, it is easily verified that any monadic BL-chain satisfies this identity. Conversely, consider a subdirectly irreducible algebra $\mathbf{A} \in \mathbb{MBL}$ that satisfies the identity. Given $x, y \in A$ we know that $(x \to y) \vee (y \to x) = 1$. Then $\forall(x \to y) \vee \forall(y \to x) = 1$, but, since $\forall A$ is totally ordered, either $\forall(x \to y) = 1$ or $\forall(y \to x) = 1$. It follows that $x \leq y$ or $y \leq x$. This proves that $\mathbf{A}$ is totally ordered.

Conclusions and further work

In this work we have presented an equivalent algebraic semantics for the monadic fragment of Hájek's fuzzy predicate logic. This class turned out to be a very interesting variety, whose algebras we called *monadic BL-algebras.*

We have started a study of some subvarieties of $\mathbb{MBL}$, but we think that a deeper study of many of its subvarieties is in order. For example, it would be interesting to know the lattice of subvarieties of monadic product algebras, Gödel algebras, and the variety generated by chains. In each case, we would like to determine whether they are generated by their finite members.

Another variety worth of research is the one generated by functional monadic BL-algebras defined over continuous t-norms. This is a proper variety because it satisfies the equation $\forall(x * \forall y) \approx \forall x * \forall y$.

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